4. Electrostatic Boundary-Value Problems

A. Poisson's and Laplace's Equations for Boundary-Value Problems (BVPs)

1) Poisson's equation

(3-67)

Ch 3-1, (10) or Ch 3-2, (3-63): $\nabla \cdot \mathbf{D} = \rho_v \quad \Rightarrow \quad \nabla \cdot \epsilon \mathbf{E} = \rho_v$ (3-124)

Ch 3-1, (3-4): $\nabla \times \mathbf{E} = 0 \quad \Rightarrow \quad \mathbf{E} = -\nabla V$ (3-26)

$$\Rightarrow \quad \nabla \cdot (\epsilon \nabla V) = \rho_v \quad \Rightarrow \quad \nabla^2 V = -\frac{\rho_v}{\epsilon} \quad (\text{V/m}^2) \qquad (3\text{-}126)$$

Ch 2-3, (54)　　Poisson's equation for a simple medium

where the Laplacian operator ∇^2 is given by (55) in Ch 2-3 D.

2) Laplace's equation

For no free charge ($\rho_v = 0$) in a simple medium,

$$(3\text{-}126) \quad \Rightarrow \quad \nabla^2 V = 0 \quad (\text{V/m}^2) \qquad (3\text{-}130)$$

Note) An electrostatic BVP for electrostatic potential V is set up by a governing equation of Poisson's or Laplace's type subject to the appropriate boundary conditions (BCs) specified at conductor/dielectric (or free space) interfaces to get a unique solution for electrostatics (Uniqueness Theorem).

B. Solutions of BVPs in Electrostatics

1) BVP for a parallel-plate capacitor in Cartesian coordinates

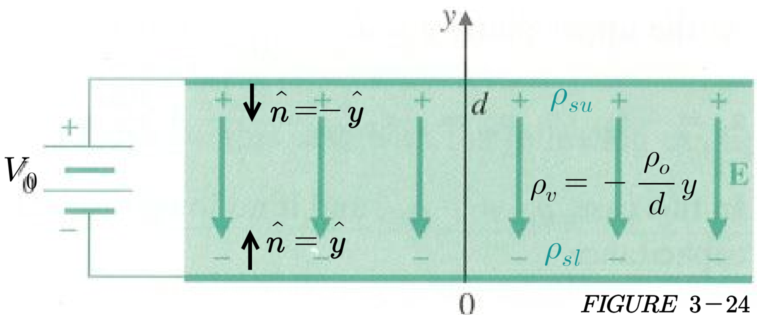


FIGURE 3-24

Negligible edge effect (width, depth $\ll d$):

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} = 0$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{d^2}{dy^2}$$

Poisson's equation:

$$\frac{d^2 V(y)}{dy^2} = -\frac{\rho_v}{\epsilon_o} = \frac{\rho_o}{\epsilon_o d} y, \quad 0 \le y \le d \qquad ①$$

BCs:

$$V(y)|_{y=0} = 0 \qquad ②$$

$$V(y)|_{y=d} = V_0 \qquad ③$$

Integrating ① twice, general solution: $V(y) = \frac{\rho_o}{6\epsilon_o d} y^3 + C_1 y + C_2$ ④

② in ④: $C_2 = 0$ ⑤

③, ⑤ in ④: $C_1 = \frac{V_0}{d} - \frac{\rho_o d}{6\epsilon_o}$ ⑥

∴ ⑥, ⑤ in ④: $V(y) = \frac{\rho_o}{6\epsilon_o d} y^3 + \left(\frac{V_0}{d} - \frac{\rho_o d}{6\epsilon_o}\right) y$ (3-134)

Consequently, $\boldsymbol{E}(y) = -\hat{\boldsymbol{y}} \frac{dV}{dy} = -\hat{\boldsymbol{y}} \left[\frac{\rho_o}{2\epsilon_o d} y^2 + \left(\frac{V_0}{d} - \frac{\rho_o d}{6\epsilon_o}\right)\right]$ (3-135)

Surface charge densities on the conducting plates:

Ch 3-2B (3-46) $\hat{n} \cdot \boldsymbol{E} = \rho_s/\epsilon_o \Rightarrow \rho_{sl} = \epsilon_o \hat{y} \cdot \boldsymbol{E}(0) = -\frac{\epsilon_o V_0}{d} + \frac{\rho_o d}{6}$

$\rho_{su} = \epsilon_o [-\hat{y} \cdot \boldsymbol{E}(d)] = +\frac{\epsilon_o V_0}{d} + \frac{\rho_o d}{3}$

Notes) For no volume charge ($\rho_v = \rho_o = 0$),

$$V(y) = \frac{V_0}{d} y, \quad \boldsymbol{E}(y) = -\hat{\boldsymbol{y}} \frac{V_0}{d}, \quad \rho_{sl} = -\frac{\epsilon_o V_0}{d}, \quad \rho_{su} = +\frac{\epsilon_o V_0}{d}$$

2) BVPs in Cylindrical coordinates

a) A long ($(b-a) \ll L$) coaxial cable with no volume charge ($\rho_v = 0$)

Axially symmetric:

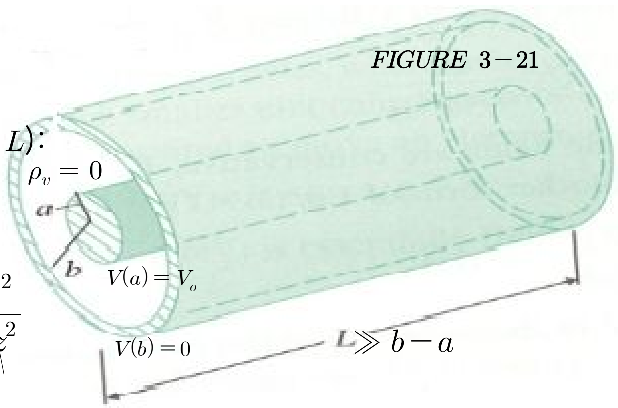
$$\frac{\partial}{\partial \phi} = 0$$

Negligible edge effect ($(b-a) \ll L$):

$$\frac{\partial}{\partial z} = 0$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

$$= \frac{1}{r}\frac{d}{dr}\left(r\frac{\partial}{dr}\right)$$



Laplace's equation:

$$\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0 \quad ①$$

BCs:

$$V(r)|_{r=a} = V_o \quad ②$$

$$V(r)|_{r=b} = 0 \quad ③$$

Integrating ① twice,

$$r\frac{dV}{dr} = C_1 \quad \Rightarrow \quad dV = C_1\frac{dr}{r} \quad \Rightarrow \quad V(r) = C_1 \ln r + C_2 \qquad ④$$

② in ④: $\qquad C_2 = V_o - C_1 \ln a \qquad ⑤$

③, ⑤ in ④: $\qquad C_1 = -V_o/\ln(b/a), \quad C_2 = V_o \ln b/\ln(b/a) \qquad ⑥$

∴ ⑥ in ④: $\qquad V(r) = \dfrac{V_o \ln(b/r)}{\ln(b/a)}$

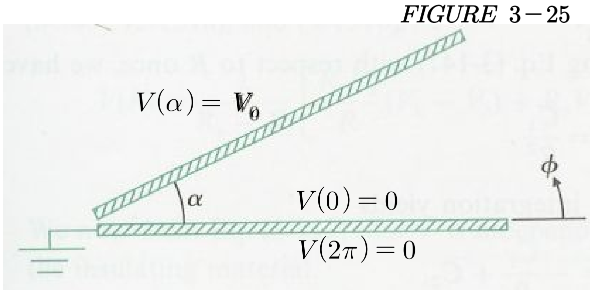
Consequently, $\qquad \boldsymbol{E}(r) = -\hat{\boldsymbol{r}}\dfrac{dV}{dr} = \hat{\boldsymbol{r}}\dfrac{V_o}{r\ln(b/a)}$

b) Two infinite insulated conducting plates maintained at constant V_o

Infinite plates : $\quad 0 < r < \infty,$

$\qquad -\infty < z < \infty$

FIGURE 3−25



$$\Rightarrow \quad \frac{\partial}{\partial r} = 0 \quad \text{and} \quad \frac{\partial}{\partial z} = 0$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

$$= \frac{1}{r^2}\left(\frac{d^2}{d\phi^2}\right)$$

Laplace's equation:

$$\frac{d^2 V}{d\phi^2} = 0 \ , \quad 0 \le \phi \le \alpha \ \text{ and } \ \alpha \le \phi \le 2\pi \qquad ①$$

BCs: $\quad V(\phi)|_{\phi = 0, 2\pi} = 0 \qquad ②$

$\qquad V(\phi)|_{\phi = \alpha} = V_o \qquad ③$

Integrating ① twice, general solution: $V(\phi) = K_1\phi + K_2 \qquad ④$

(1) For $0 \le \phi \le \alpha$,

② at $\phi = 0$, ③ in ④: $\qquad K_2 = 0 \ , \quad K_1 = V_o/\alpha \qquad ⑤$

∴ ⑤ in ④: $\quad V(\phi) = \dfrac{V_o}{\alpha}\phi \quad \& \quad \boldsymbol{E}(\phi) = -\hat{\boldsymbol{\phi}}\dfrac{1}{r}\dfrac{dV}{d\phi} = -\hat{\boldsymbol{\phi}}\dfrac{V_o}{\alpha r} \qquad$ (3−143)

(2) For $\alpha \le \phi \le 2\pi$,

③ in ④ at $\phi = \alpha$: $\quad V(\alpha) = \alpha K_1 + K_2 = V_o \qquad ③^*$

② in ④ at $\phi = 2\pi$: $\quad V(2\pi) = 2\pi K_1 + K_2 = 0 \qquad ②^*$

$$\Rightarrow \quad K_1 = -\frac{V_o}{2\pi - \alpha} \ , \quad K_2 = \frac{2\pi V_o}{2\pi - \alpha} \qquad ⑤^*$$

∴ ⑤* in ④: $\quad V(\phi) = \dfrac{V_o}{2\pi - \alpha}(2\pi - \phi) \quad \& \quad \boldsymbol{E}(\phi) = \hat{\boldsymbol{\phi}}\dfrac{V_o}{(2\pi - \alpha)r} \qquad$ (3−146)

3) BVPs in spherical coordinates

a) Two concentric conducting shells maintained at constant potentials

Spherically symmetric:

$$\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\phi} = 0$$

$$\nabla^2 = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$= \frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{d}{dR}\right)$$

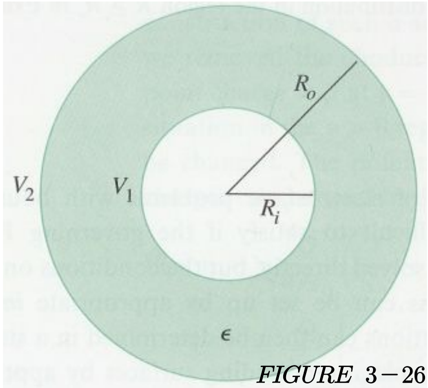


FIGURE 3−26

Laplace's equation:

$$\frac{d}{dR}\left(R^2\frac{dV}{dR}\right) = 0, \qquad R_i \le R \le R_o \qquad ①$$

BCs:

$$V(R)|_{R=R_i} = V_1 \qquad ②$$

$$V(R)|_{R=R_o} = V_2 \qquad ③$$

Integrating ① twice,

$$R^2\frac{dV}{dR} = C_1 \quad \Longrightarrow \quad dV = \frac{C_1}{R^2}dR \quad \Longrightarrow \quad V(R) = -\frac{C_1}{R} + C_2 \qquad ④$$

② in ④: $\quad C_2 = V_1 + C_1/R_i \qquad ⑤$

③, ⑤ in ④: $\quad C_1 = -\frac{R_oR_i(V_1-V_2)}{R_o-R_i}, \quad C_2 = \frac{R_oV_2-R_iV_i}{R_o-R_i} \qquad ⑥$

∴ ⑥ in ④:

$$V(R) = \frac{1}{R_o-R_i}\left[\frac{R_oR_i}{R}(V_1-V_2)+R_oV_2-R_iV_1\right] \qquad (3\text{-}152)$$

$$= \left(\frac{V_1-V_2}{1/R_i-1/R_o}\right)\left[\frac{1}{R}+\left(\frac{V_2}{V_1-V_2}\right)\frac{1}{R_i}-\left(\frac{V_1}{V_1-V_2}\right)\frac{1}{R_o}\right]$$

Consequently,

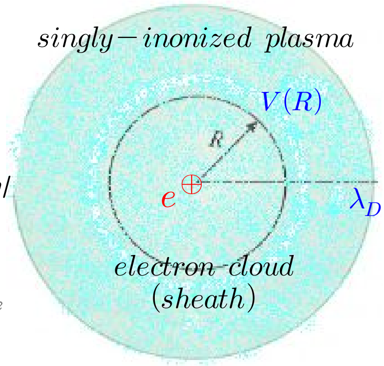
$$\mathbf{E}(R) = -\hat{\mathbf{R}}\frac{dV}{dR} = \hat{\mathbf{R}}\left(\frac{V_1-V_2}{1/R_i-1/R_o}\right)\frac{1}{R^2} \qquad (3\text{-}152)^*$$

Note) $V(R)$ and $\mathbf{E}(R)$ are independent of ϵ.

(cf) Two concentric conducting shells maintained at constant charges

b) Screened Coulomb potential in plasmas (Debye shielding)

Consider a positive test charge e in a singly-ionized plasma.



Assume $n_i = n_o = const$: *fixed* $(\because\ m_i/m_e \gg 1)$

$n_e(R)|_{R\to\infty} = n_o$: *not affected by potential*

Electron distribution → Boltzmann distribution

$\rightarrow\ n_e(R) = n_o\, e^{eV(R)/kT_e}$

Poisson's equation :

$$\nabla^2 V(R) = -\frac{\rho_v}{\varepsilon_o} = \frac{e[n_e(R) - n_i]}{\varepsilon_o}, \quad 0 \le R < \infty \qquad ①$$

BCs: $V(R)|_{R\to 0} = \dfrac{e}{4\pi\varepsilon_o R} \equiv V_o$: Coulomb potential ②

$V(R)|_{R\to\infty} = 0$ ③

$n_e(R) = n_o\, e^{e\Phi(R)/kT_e}$ in ① :

$$\nabla^2 V = \frac{e n_o}{\varepsilon_o}\left(e^{eV/kT_e} - 1\right) \qquad ①^*$$

Assuming $|eV| \ll kT_e$ over most of space which is usually the case.

$$e^{eV/kT_e} \approx 1 + \frac{eV}{kT_e}$$

Then $\nabla^2 V = \dfrac{1}{\lambda_D^2} V$ where $\lambda_D^2 \equiv \dfrac{\varepsilon_o kT_e}{e^2 n}$ $①^{**}$

In spherical coordinates,

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dV}{dR}\right) - \frac{1}{\lambda_D^2}V = 0 \;\Rightarrow\; V'' + \frac{2}{R}V' - \frac{1}{\lambda_D^2}V = 0 \;\Rightarrow\; RV'' + 2V' - \frac{1}{\lambda_D^2}RV = 0$$

Let $\Phi(r) \equiv RV(R)$ then $\Phi' = V + RV'$, $\Phi'' = RV'' + 2V'$

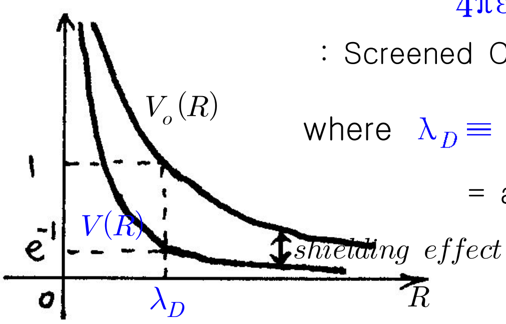
$$\Phi'' - \lambda_D^{-2}\Phi = 0 \quad \Rightarrow \quad \Phi(R) = C_1 e^{-R/\lambda_D} + C_2 e^{+R/\lambda_D}$$

$$\therefore\ V(R) = \frac{C_1}{R}e^{-R/\lambda_D} + \frac{C_2}{R}e^{R/\lambda_D} \qquad ④$$

③ in ④ : $C_2 = 0$ ⑤

② in ④ : $C_1 = e/4\pi\varepsilon_o$ ⑥

⑤,⑥ in ④: $V(R) = \dfrac{e}{4\pi\varepsilon_o R}e^{-R/\lambda_D} \equiv V_o e^{-R/\lambda_D}$ ⑦



: Screened Coulomb potential (Yukawa-type potential)

where $\lambda_D \equiv \left(\dfrac{\varepsilon_o kT_e}{ne^2}\right)^{1/2}$: Debye length ⑧

= a measure of shielding distance by the sheath

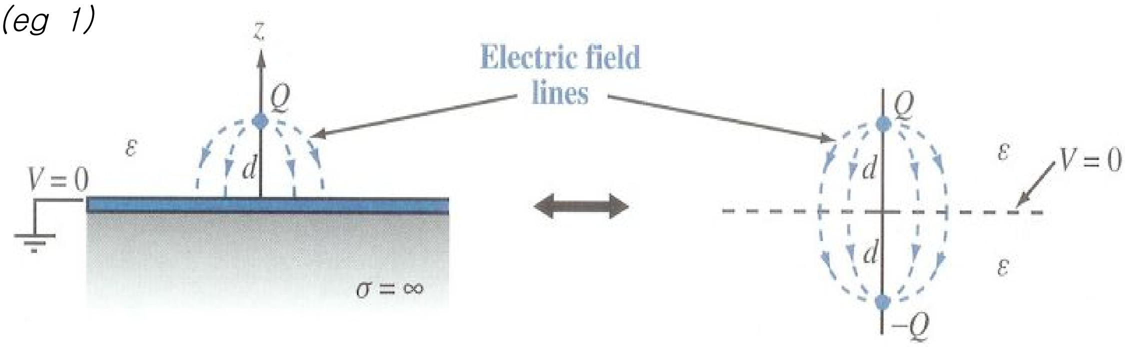
C. Image Method

1) Theory of images

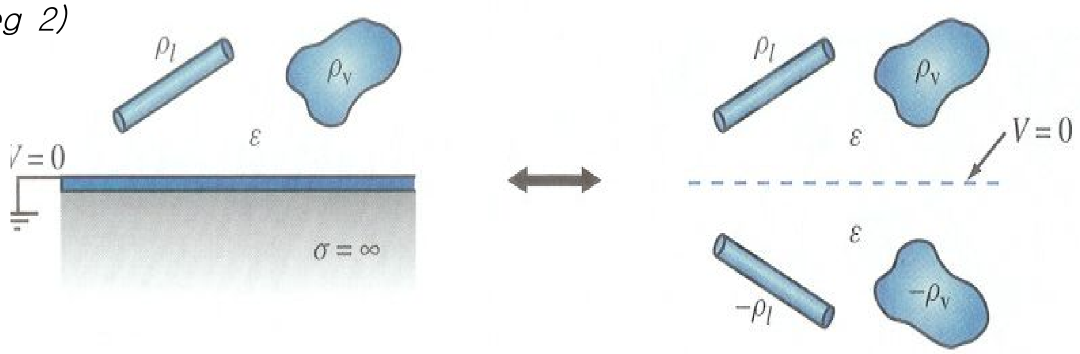
Any charge distribution above an infinite conducting plane

⟷ Combination of the given and its image charge distributions with the conducting plane removed

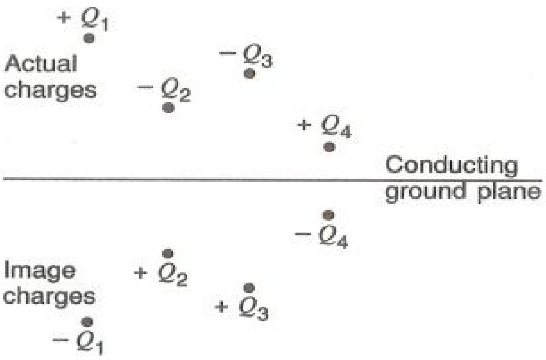
(eg 1)



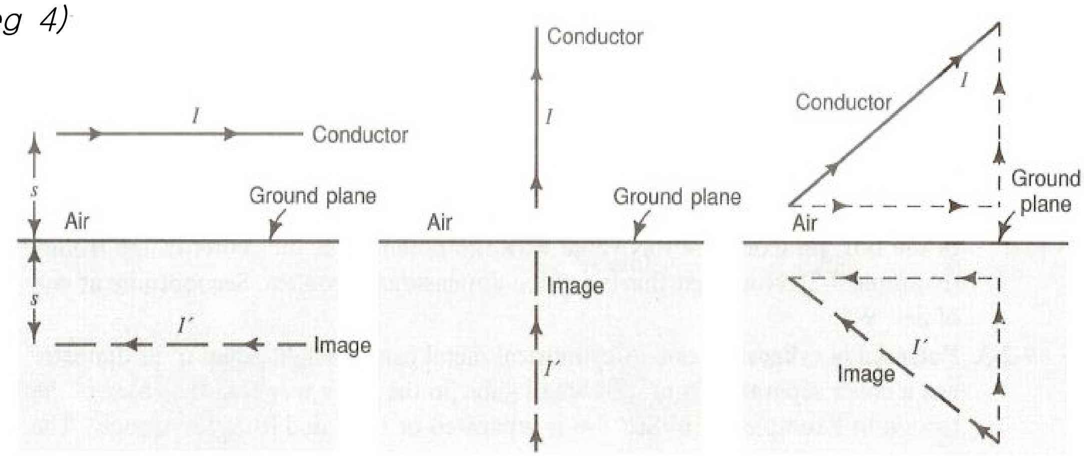
(eg 2)



(eg 3)



(eg 4)



2) Applications of image method

a) Point charge above a grounded plane conductor

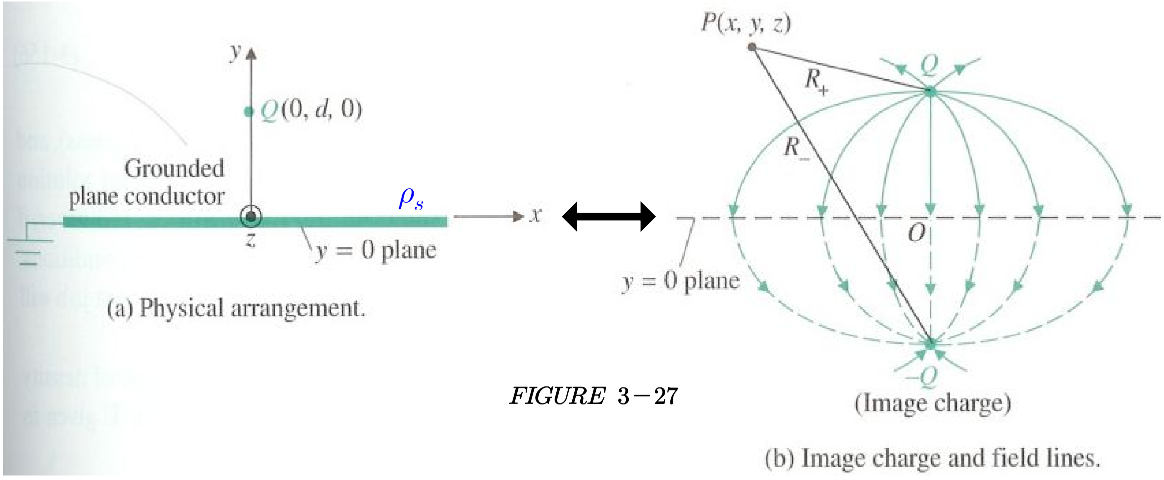


FIGURE 3−27

<u>BVP</u> $\Rightarrow$ complicated to solve!

Poisson's equation:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)V(x,y,z) = Q\,\delta(y-d), \qquad -\infty < x,y,z < \infty$$

BCs:

$$V(x,y,z)|_{y=0} = 0, \qquad -\infty < x,z < \infty$$

$$V(x,y,z)|_{y\to\infty} = 0, \qquad -\infty < x,z < \infty$$

<u>Image method</u>

Potential at a point $P(x,y,z)$

due to an actual charge Q and its image charge $-Q$:

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_o}\left(\frac{1}{R_+}-\frac{1}{R_-}\right)$$

$$= \frac{Q}{4\pi\epsilon_o}\left[\frac{1}{\sqrt{x^2+(y-d)^2+z^2}}-\frac{1}{\sqrt{x^2+(y+d)^2+z^2}}\right], \quad y \geq 0$$

$$V(x,y,z) = 0, \quad y \leq 0 \qquad \text{(3−153)}$$

Electric field intensity :

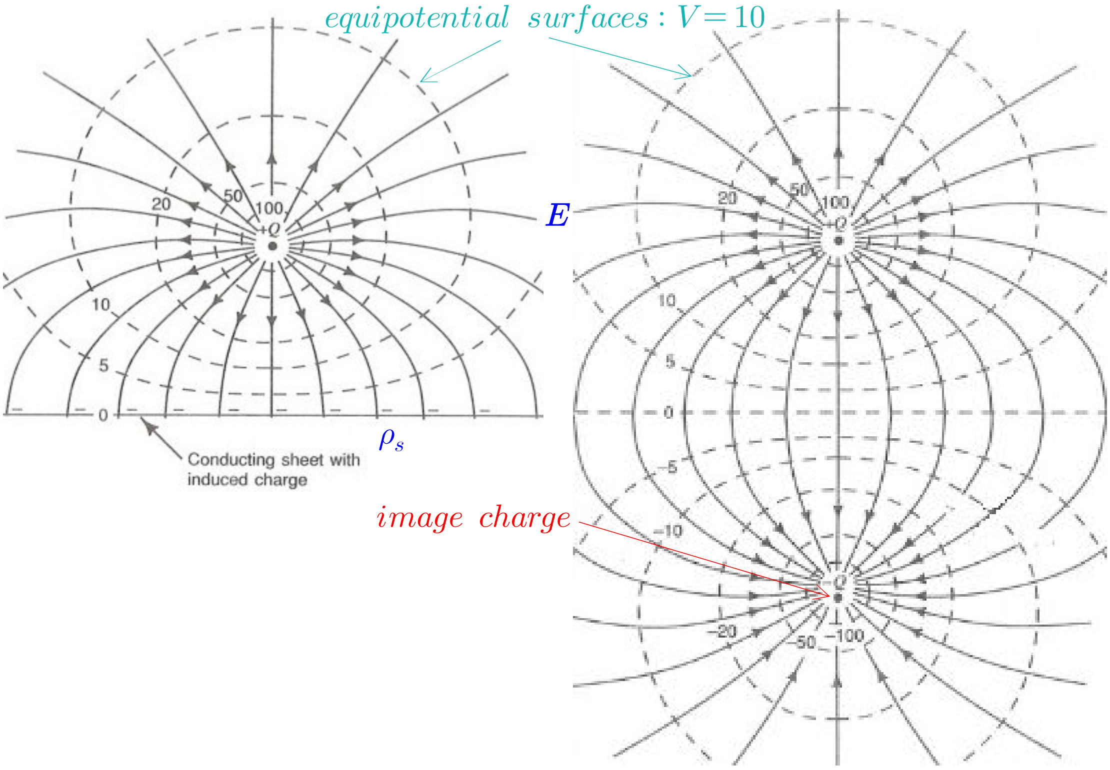
$$\boldsymbol{E}(x,y,z) = -\nabla V = -\left(\hat{\boldsymbol{x}}\frac{\partial V}{\partial x}+\hat{\boldsymbol{y}}\frac{\partial V}{\partial y}+\hat{\boldsymbol{z}}\frac{\partial V}{\partial z}\right)$$

$$= \frac{Q}{4\pi\epsilon_o}\left[\frac{\hat{\boldsymbol{x}}x+\hat{\boldsymbol{y}}(y-d)+\hat{\boldsymbol{z}}z}{[x^2+(y-d)^2+z^2]^{3/2}}-\frac{\hat{\boldsymbol{x}}x+\hat{\boldsymbol{y}}(y+d)+\hat{\boldsymbol{z}}z}{[x^2+(y+d)^2+z^2]^{3/2}}\right], \quad y \geq 0 \qquad \text{(3−154)}$$

Induced surface charge density on the plane conductor :

$$\text{(3−46)}\quad E_n = \rho_s/\epsilon_o \quad \Rightarrow \quad \rho_s = \epsilon_o E_y|_{y=0} = -\frac{Qd}{2\pi(x^2+d^2+z^2)^{3/2}}$$

(3−155)



b) Line charge near a parallel conducting cylinder

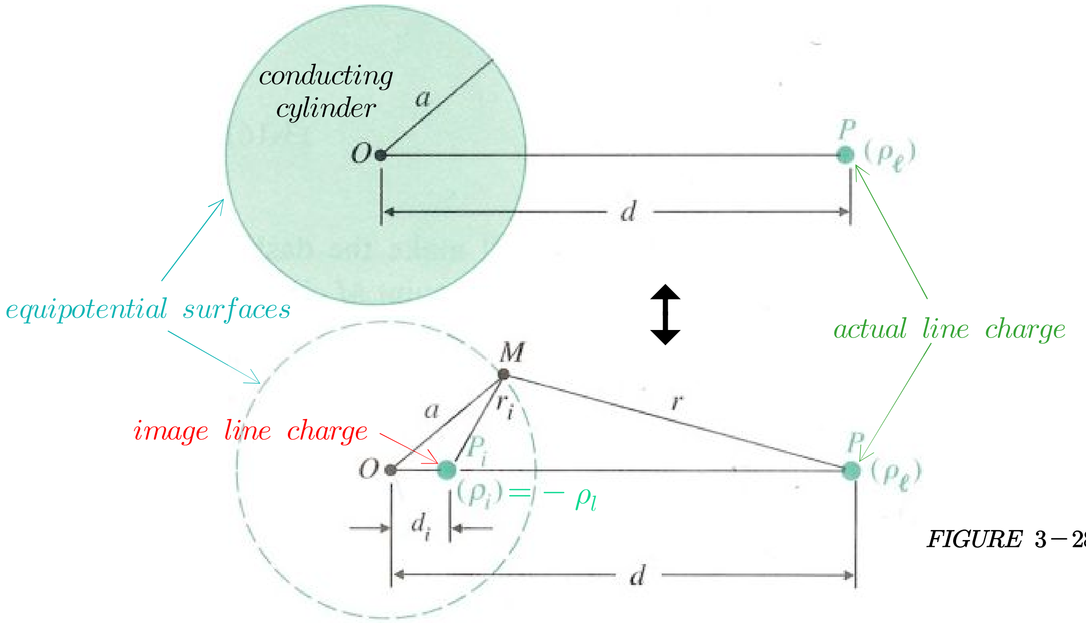


FIGURE 3−28

Electric potential : (3-23) $\boldsymbol{E} = \hat{\boldsymbol{r}}\dfrac{\rho_l}{2\pi\epsilon_o r}$ in $V(\boldsymbol{r}) = -\displaystyle\int_{ref.pt.}^{r} \boldsymbol{E}\cdot dl$

$$V(\boldsymbol{r}) = -\frac{\rho_l}{2\pi\epsilon_o}\int_{r_o}^{r}\frac{dr}{r} = \frac{\rho_l}{2\pi\epsilon_o} ln\frac{r_o}{r} \tag{3-157}$$

where r_o is the reference point.

Potential at a point M due to actual and image line charges :

$$V_M = \frac{\rho_l}{2\pi\epsilon_o}\left(\ln\frac{r_o}{r} - \ln\frac{r_o}{r_i}\right) = \frac{\rho_l}{2\pi\epsilon_o} ln\frac{r_i}{r} \tag{3-158}$$

If $V_M = const$, then $\frac{r_i}{r} = const$. The equipotential surface becomes the cylindrical surface of radius a. In this case, $\triangle OMP_i \propto \triangle OPM$

$$\Rightarrow \quad \frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = const \equiv K = e^{2\pi\epsilon_o V_M/\rho_l} \qquad (3\text{-}160)$$

Position of point P_i = Inverse point of P w.r.t a circle of radius a :

$$(3\text{-}160) \quad \Rightarrow \quad d_i = \frac{a^2}{d} \qquad (3\text{-}161)$$

c) Two-wire transmission line

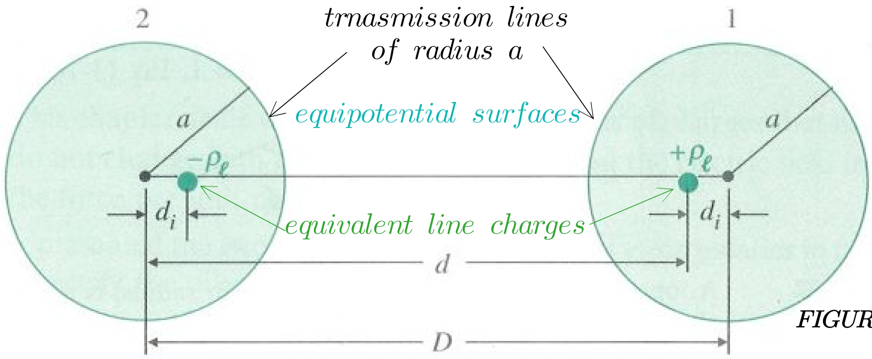


FIGURE 3−29

(3-158), (3-160) :

$$V_2 = \frac{\rho_l}{2\pi\epsilon_o} ln\frac{a}{d} \quad \text{and} \quad V_1 = -\frac{\rho_l}{2\pi\epsilon_o} ln\frac{a}{d}$$

$$\Rightarrow \quad V_1 - V_2 = -\frac{\rho_l}{\pi\epsilon_o} ln\frac{a}{d} = \frac{Q}{\pi\epsilon_o L} ln(d/a)$$

$$\therefore \; C = \frac{Q}{V_1 - V_2} = \frac{\pi\epsilon_o L}{\ln(d/a)} \quad \text{or} \quad \frac{C}{L} = \frac{\pi\epsilon_o}{\ln(d/a)} \qquad (3\text{-}162)$$

$$d = D - d_i = D - a^2/d \quad \Rightarrow \quad d^2 - Dd + a^2 = 0$$

$$\Rightarrow \quad d = \frac{1}{2}(D + \sqrt{D^2 - 4a^2}) \qquad (3\text{-}163)$$

(3-163) in (3-162) :

$$\frac{C}{L} = \frac{\pi\epsilon_o}{\ln[(D/2a) + \sqrt{(D/2a)^2 - 1}]} = \frac{\pi\epsilon_o}{\cosh^{-1}(D/2a)} \quad \text{(F/m)} \qquad (3\text{-}165)$$

$\ln(x + \sqrt{x^2 - 1}) = \cosh^{-1}x$ for $x > 1$

For the usual two-wire transmission line ($2a \ll D$) ,

$$\frac{C}{L} = \frac{\pi\epsilon_o}{\ln(D/a)} \quad \text{(F/m)} \qquad (3\text{-}166)$$

(cf) For a coaxial transmission line of radii a and b

$$\frac{C}{L} = \frac{2\pi\epsilon}{\ln(b/a)} \quad \text{(F/m)} \qquad (3\text{-}90)$$

Homework Set 4

1) P.3-16
2) P.3-17 c)
3) P.3-19
4) P.3-21
5) P.3-27
6) P.3-31
7) P.3-32
8) P.3-35