Trees

Introduction to Data Structures

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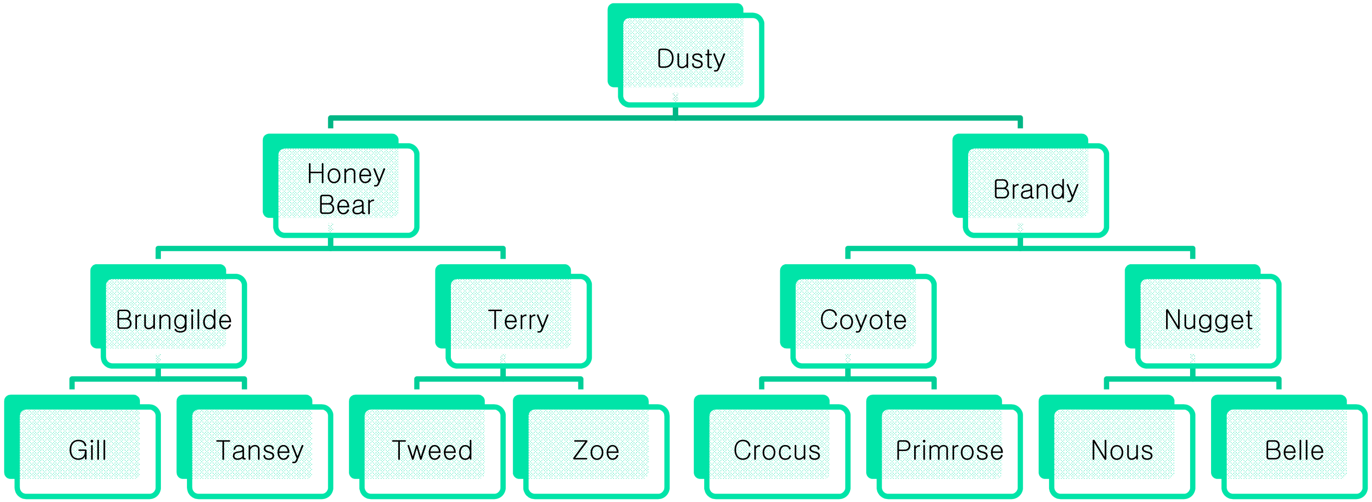
5.1 Introduction
5.1.1 Terminology

- Tree : a finite set of one or more nodes
  - there is a specially designated node called the root
  - the remaining nodes are partitioned into $n \geq 0$ disjoint sets $T_1, \ldots, T_n$, where each of these sets is a tree. $T_1, \ldots, T_n$ are called the subtrees of the root.
- Degree of a node : the number of subtrees of a node
- Leaf (terminal node) : a node that has degree zero
- Nonterminals : the other nodes
- Children, parent, siblings

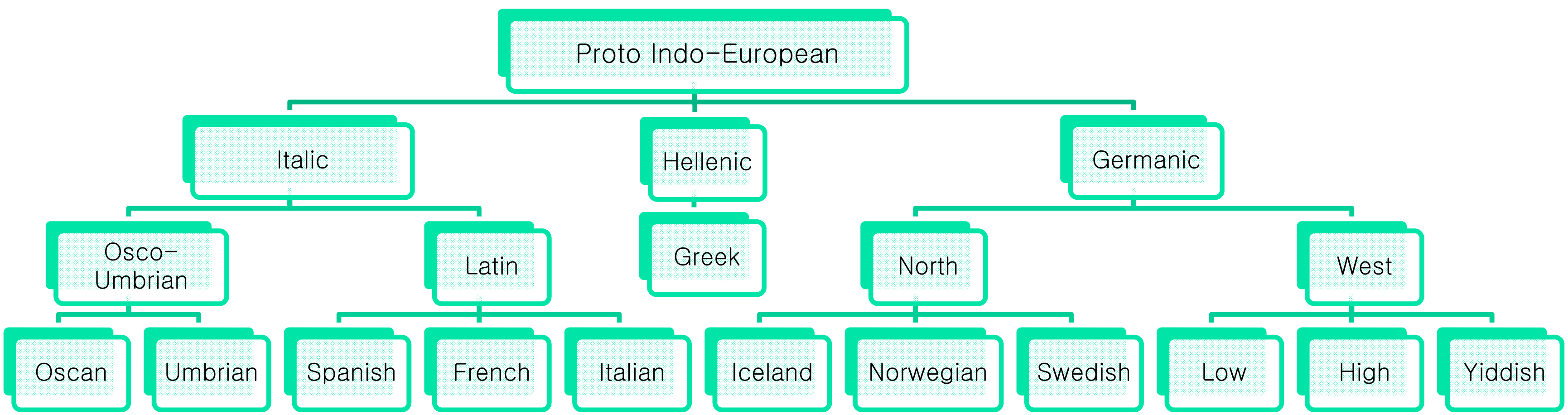
5.1.1 Terminology

- Degree of a tree : the maximum of degree of the nodes in the tree
- Ancestors of a node : all the nodes along the path from the root to that node
- Level of a node : the distance from the root+1
- Height (or depth) of a tree : the maximum level of any node in the tree

5.1.1 Terminology



(a) Pedigree



(b) Lineal

5.1.1 Terminology

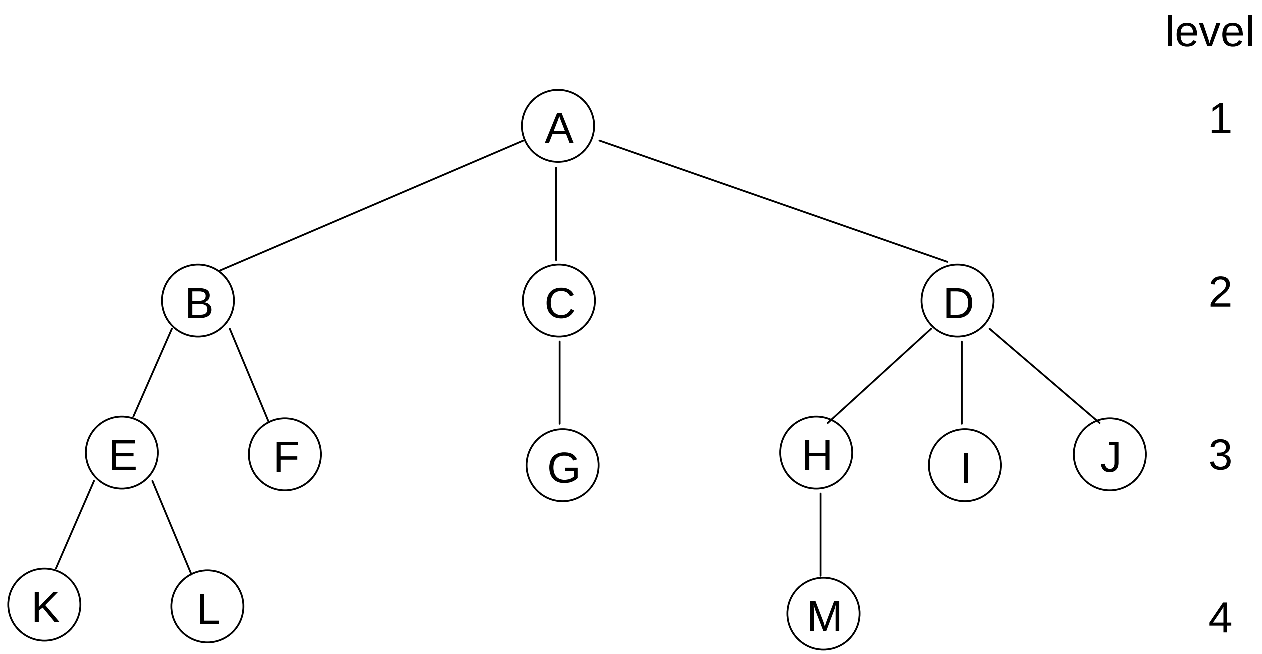


Figure 5.2 : A sample tree

5.1.2 Representation of Trees

- List representation
  - The tree of Figure 5.2
    - (A(B(E(K,L),F),C(G),D(H(M),I,J)))
  - The degree of each node may be different
    - possible to use memory nodes with a varying number of pointer fields
    - easier to write algorithms when the node size is fixed

5.1.2 Representation of Trees

- List representation
  - nodes of a fixed size
    - for a tree of degree k

DATA	$CHILD_1$	$CHILD_2$	…	$CHILD_k$

    - Lemma 5.1: If T is a k-ary tree with n nodes, each having a fixed size, then $n(k-1)+1$ of the nk child fields are 0, $n \geq 1$.
    - Proof: Since each non-zero child field points to a node and there is exactly one pointer to each node other than the root, the number of child fields in a k-ary tree with n nodes is nk. Hence, the number of zero fields is $nk - (n-1) = n(k-1)+1$

5.1.2 Representation of Trees

- Left child-right sibling representation
  - node structure

data	
left child	right sibling

  - the left child field of each node points to its left most child (if any), and the right sibling field points to its closest right sibling (if any)

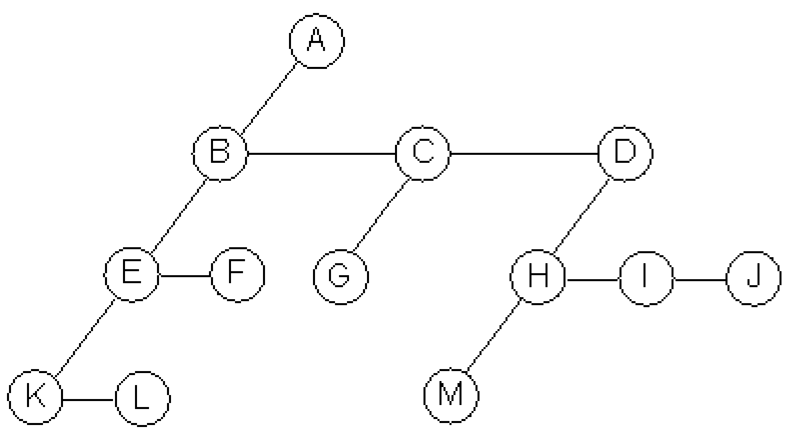
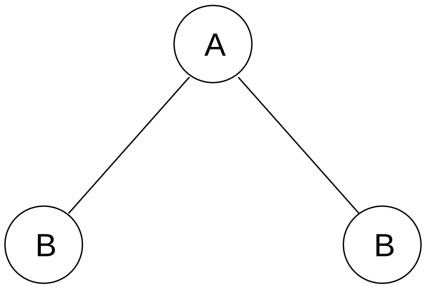


Figure 5.6 : Left child-right sibling representation of the tree of Figure 5.2

5.1.2 Representation of Trees

- Representation as a degree-two tree
  - rotate the right-sibling pointers clockwise by 45 degrees
  - the two children of a node are referred to as the left and right children



5.1.2 Representation of Trees

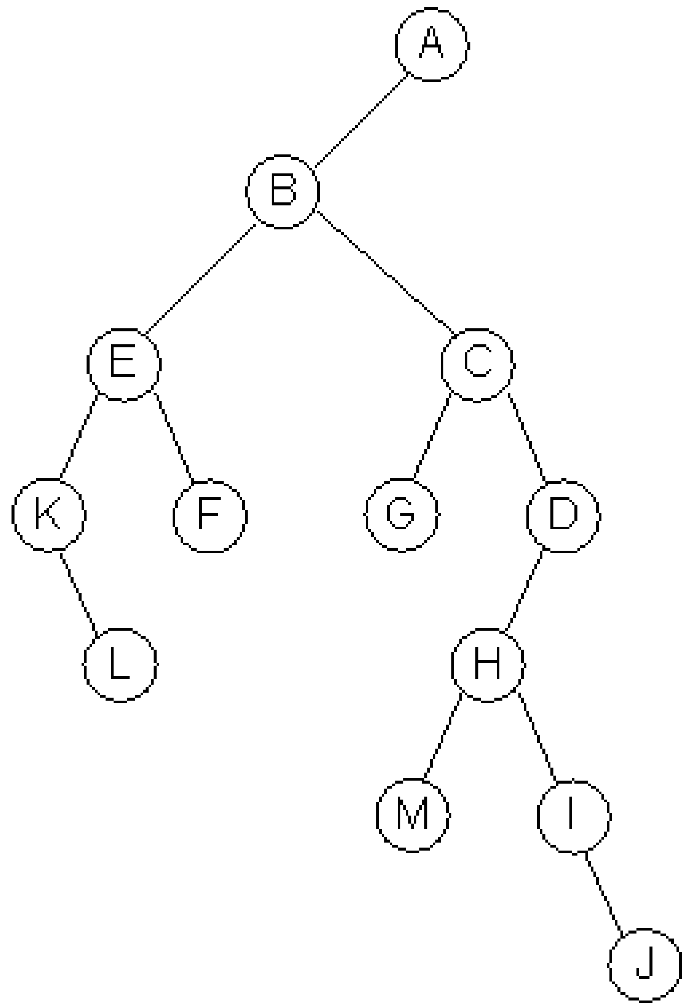


Figure 5.7 : Left child-right child tree representation

5.1.2 Representation of Trees

- Additional examples

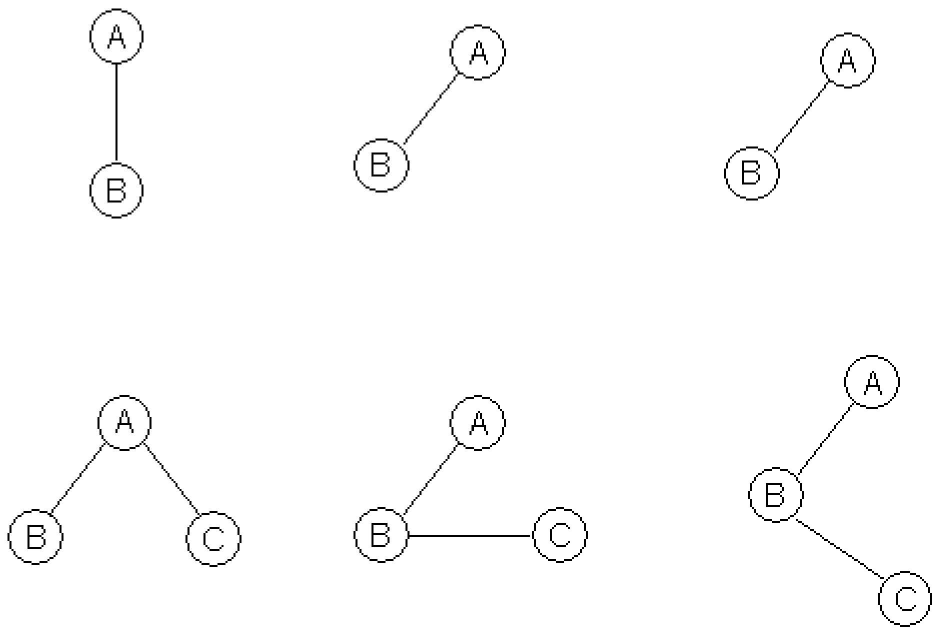


Figure 5.8 : Tree representations

  - left child-right child tree : binary tree
  - any tree can be represented as a binary tree

5.2 Binary Trees

- A binary tree
  - a finite set of nodes that either is empty or consists of a root and two disjoint binary trees called the left subtree and the right subtree
- Differences between a binary tree and a tree
  - there is an empty binary tree
  - the order of the children is distinguished in a binary tree

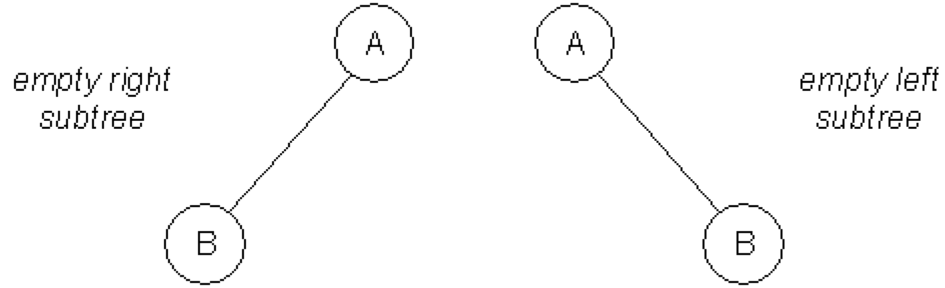


Figure 5.9 : Two different binary trees

5.2 Binary Trees

```
template <class T>
class BinaryTree
{ // objects: A finite set of nodes either empty or consisting of a
  //  root node, left BinaryTree and right BinaryTree.
public:
   BinaryTree();
   // creates an empty binary tree

   bool IsEmpty();
   // return true if the binary tree is empty

   BinaryTree(BinaryTree<T>& bt1,  T& item, BinaryTree<T>& bt2);
   // creates a binary tree whose left subtree is bt 1, whose right
   // subtree is bt 2, and whose root node contains item

   BinaryTree<T> LeftSubtree();
   // return the left subtree of *this

   BinaryTree<T> RightSubtree();
   // return the right subtree of *this

   T RootData();
   // return the data in root node of *this
};
--------------------------------------------------
```

ADT 5.1 : Abstract data type BinaryTree

5.2 Binary Trees

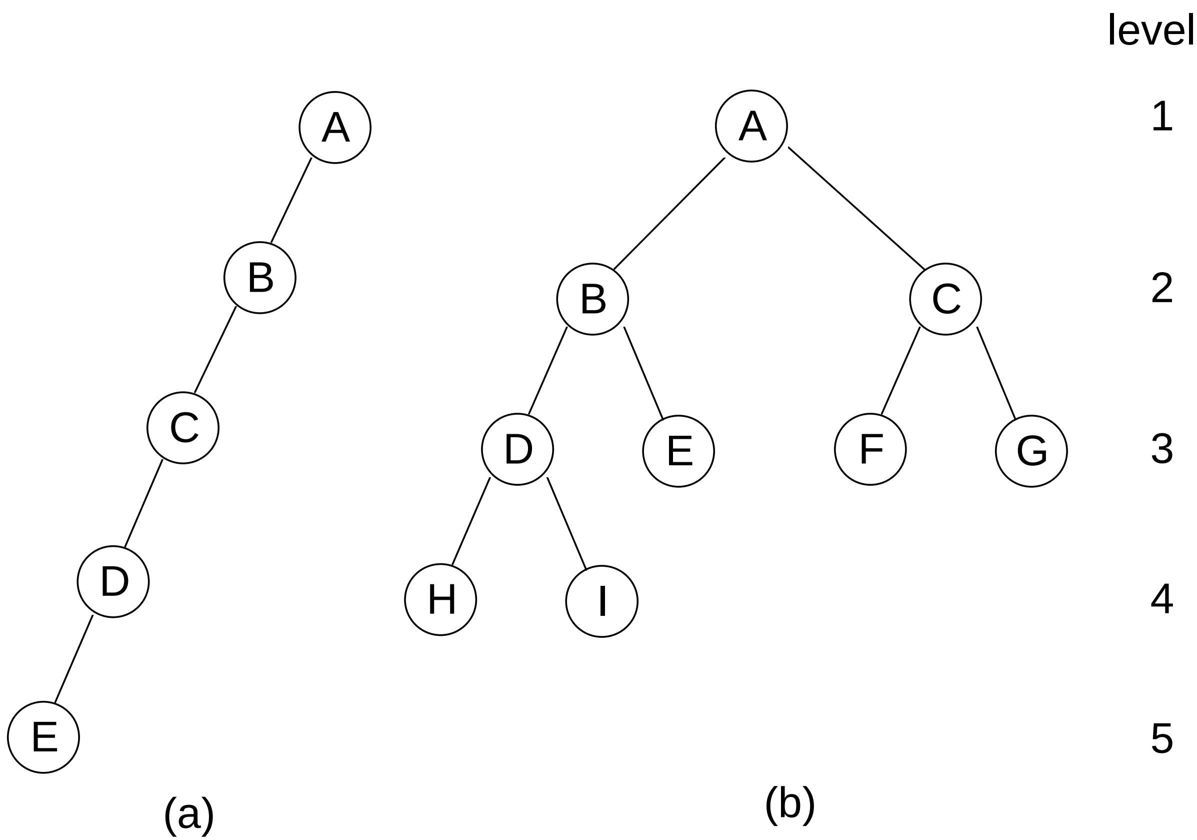


Figure 5.10 : Skewed and complete binary trees

5.2.2 Properties of Binary Trees

- Lemma 5.2 [Maximum number of nodes]:

  (1) The max number of nodes on level i of a binary tree is 2^{i-1}, $i \geq 1$

  (2) The max number of nodes in a binary tree of depth k is 2^k-1, $k \geq 1$

5.2.2 Properties of Binary Trees

- Proof:
  - (1) The proof is by induction on i.
    - Induction Base : The root is the only node on level i = 1. Hence, the maximum number of nodes on level i = 1 is $2^{i-1}=2^0=1$
    - Induction Hypothesis : Let i be an arbitrary positive integer greater than 1. Assume that the maximum number of nodes on level i−1 is 2^{i-2}
    - Induction Step : The maximum number of nodes on level i−1 is 2^{i-2} by the induction hypothesis. Since each node in a binary tree has a maximum degree of 2, the maximum number of nodes on level i is two times the maximum number of nodes on level i−1, or 2^{i-1}
  - (2) The maximum number of nodes in a binary tree of depth k is

$$\sum_{i=1}^{k}(\max imum\ number\ of\ nodes\ on\ level\ i) = \sum_{i=1}^{k} 2^{i-1} = 2^k - 1$$

5.2.2 Properties of Binary Trees

- Lemma 5.3 [Relation between number of leaf nodes and degree-2 nodes]: For any non-empty binary tree, T, if n_0 is the number of leaf nodes and n_2 the number of nodes of degree 2, then $n_0=n_2+1$

5.2.2 Properties of Binary Trees

- Proof : Let n_1 be the number of nodes of degree one and n the total number of nodes. Since all nodes in T are at most of degree two, we have $n=n_0+n_1+n_2$ If we count the number of branches in a binary tree, we see that every node except the root has a branch leading into it. If B is the number of branches, then $n=B+1$. All branches stem from a node of degree one or two. Thus, $B=n_1+2n_2$. Hence, we obtain $n = B+1 = n_1+2n_2+1$. We get $n_0 = n_2 + 1$
- Def : *A full binary tree* of depth k
  - a binary tree of depth k having 2^k-1 nodes, $k\geq 0$

5.2.2 Properties of Binary Trees

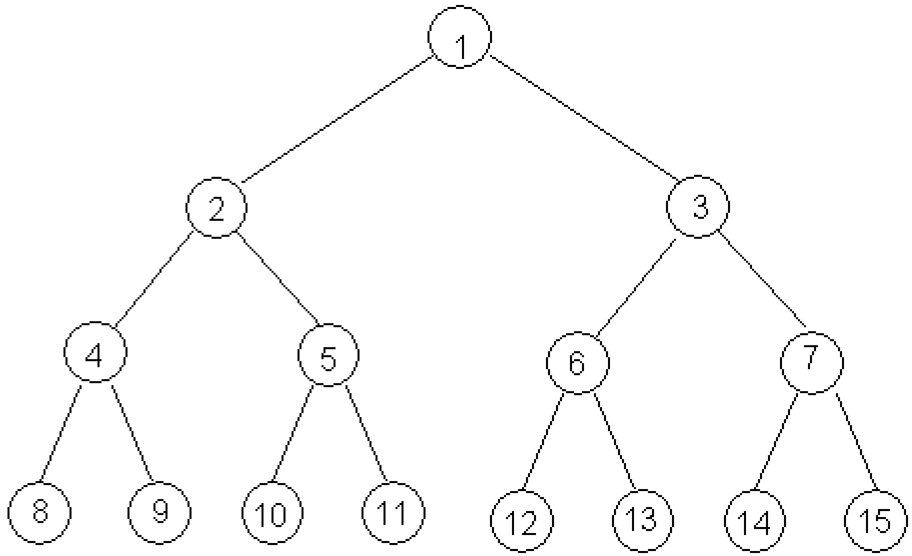


Figure 5.11 : Full binary tree of depth 4 with sequential node numbers

- A binary tree with n nodes and depth k is complete
  - its nodes correspond to the nodes numbered from 1 to n in the full binary tree of depth k
- The height of a complete binary tree with n nodes is $\lceil \log_2(n+1) \rceil$

5.2.3 Binary Tree Representation
5.2.3.1 Array Representations

- Lemma 5.4 : If a complete binary tree with n nodes is represented sequentially, then for any node with index i, $1 \leq i \leq n$, we have
  (1) parent(i) is at $\lfloor i/2 \rfloor$ if $i \neq 1$. If $i=1$, i is at the root and has no parent
  (2) leftChild(i) is at 2i if $2i \leq n$. If $2i > n$, then i has no left child
  (3) rightChild(i) is at 2i+1 if $2i+1 \leq n$. If $2i+1 > n$, then i has no right child

5.2.3 Binary Tree Representation
5.2.3.1 Array Representations

- Proof : We prove (2). (3) is an immediate consequence of (2) and the numbering of nodes on the same level from left to right. (1) follows from (2) and (3). We prove (2) by induction on i.
  For i=1, clearly the left child is at 2 unless 2>n, in which chase i has no left child. Now assume that for all j, $1 \leq j \leq i$, leftChild(j) is at 2j.
  Then the two nodes immediately preceding leftChild(i+1) are the right and left children of i. The left child is at 2i. Hence, the left child of i+1 is at 2i+2=2(i+1) unless 2(i+1)>n, in which case i+1 has no left child

5.2.3.1 Array Representations

(a)

[0]	-
[1]	A
[2]	B
[3]	-
[4]	C
[5]	-
[6]	-
[7]	-
[8]	D
[9]	-
.	.
.	.
.	.
[16]	E

(b)

[0]	-
[1]	A
[2]	B
[3]	C
[4]	D
[5]	E
[6]	F
[7]	G
[8]	H
[9]	I

Figure 5.12 : Array representation of the binary trees of Figure 5.10

5.2.3.2 Linked representation

- Classes to define a tree

```
class Tree;  //forward declaration
class TreeNode {
    friend class Tree;
    private:
        TreeNode *LeftChild;
        char data;
        TreeNode *RightChild;
};
class Tree {
    public:
        // Tree operations
        ...
    private:
        TreeNode *root;
};
```

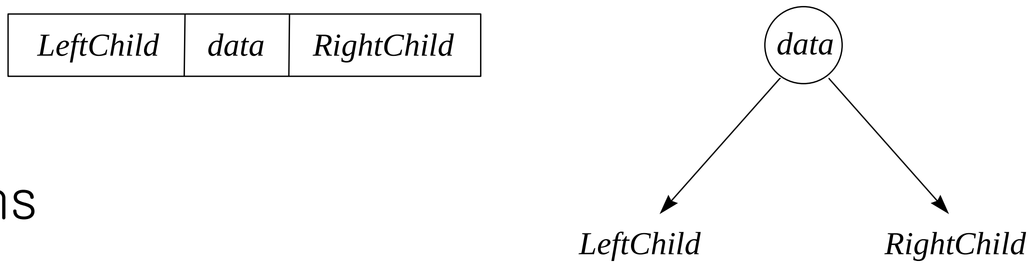


Figure 5.13 : Node representations

5.2.3.2 Linked representation

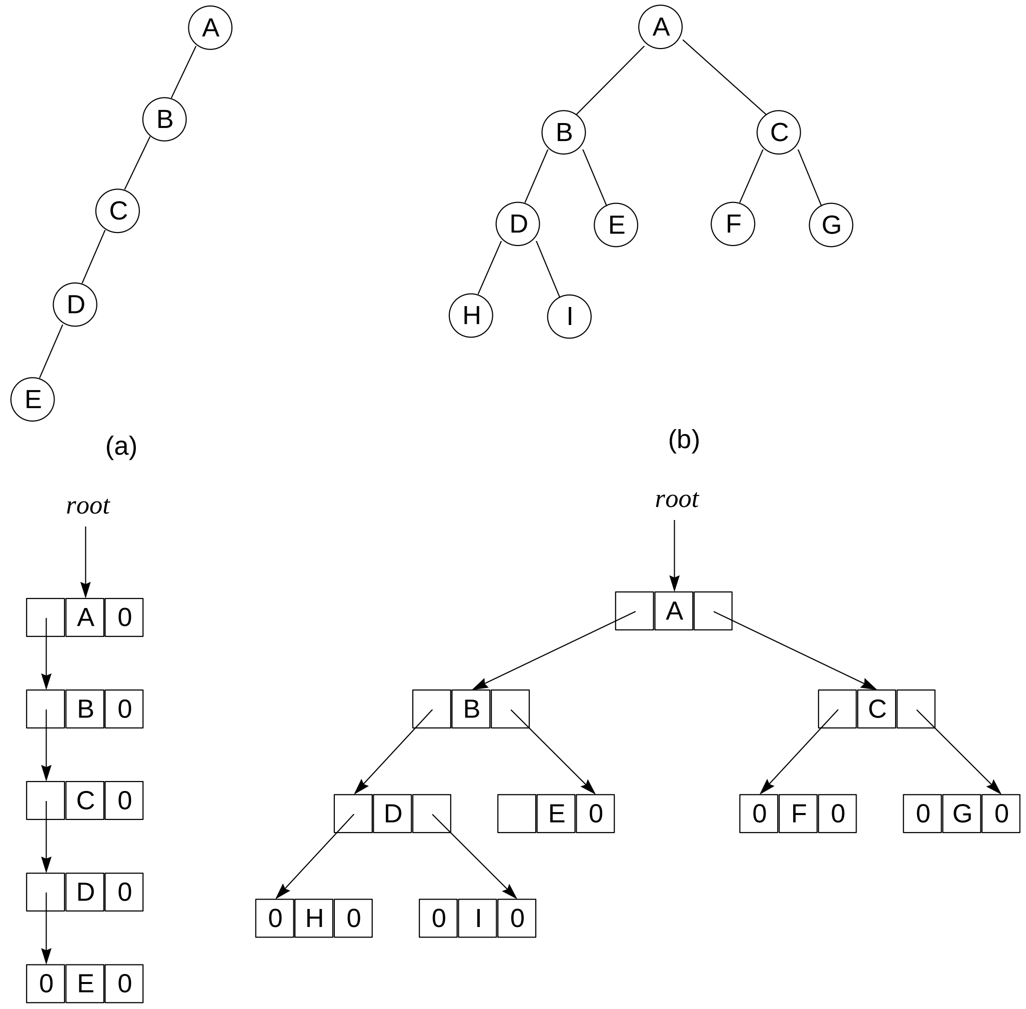


Figure 5.14 : Linked representation for the binary trees of Figure 5.10

5.3 Binary Tree Traversal and Tree Iterators
5.3.1 Introduction

- Tree traversal
  - visiting each node in the tree exactly once
  - a full traversal produces a linear order for the nodes
- Order of node visit

  L : move left

  V : visit node

  R : move right
  - possible combinations : LVR, LRV, VLR, VRL, RVL, RLV
  - traverse left before right
- LVR : inorder
- LRV : postorder
- VLR : preorder

5.3.1 Introduction

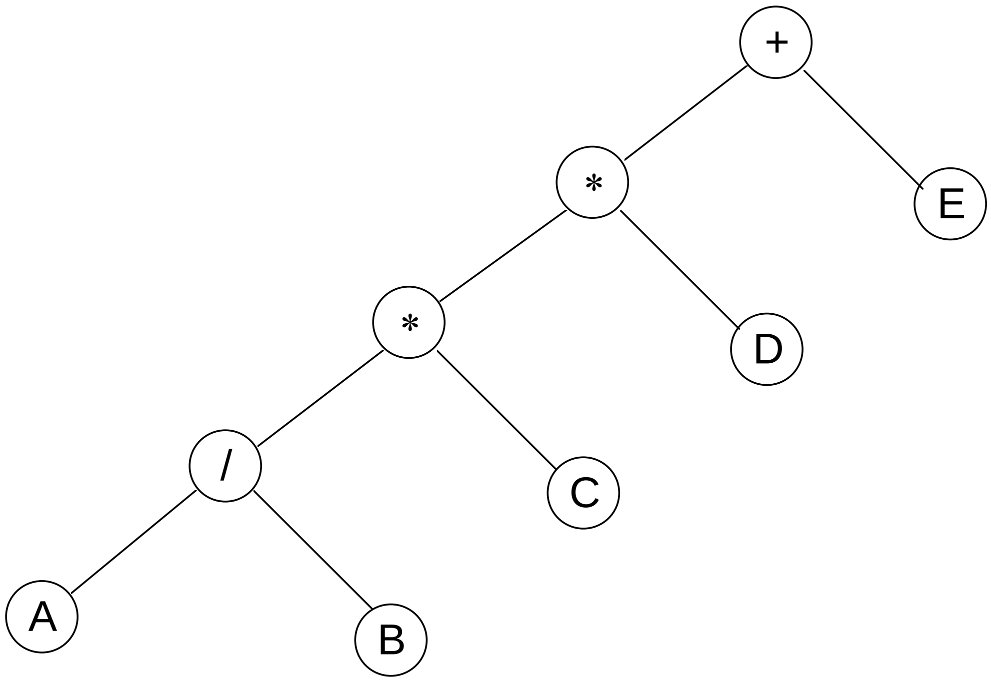


Figure 5.16 : Binary tree with arithmetic expression

5.3.2 Inorder Traversal

- LVR

```
※ Visit(TreeNode<T> *CurrentNode) {
        cout << currentNode->data
   }
```

```
template<class T>
void Tree::inorder()
{ // Driver call workhorse for traversal of entire tree. The driver is
  // declared as a public member function of Tree.
       inorder(root);
}
template<class T>
void Tree<T>::inorder(TreeNode<T> *CurrentNode)
{ // Workhorse traverses the subtree rooted at CurrentNode
  // The workhorse is declared as a private member function of Tree.
       if (CurrentNode) {
              inorder(CurrentNode->leftChild);
              Visit(currentNode);
              inorder(CurrentNode->rightChild);
       }
}
```

Program 5.1 : Inorder traversal of a binary tree

5.3.2 Inorder Traversal

Call of *inorder*	Value in *CurrentNode*	Action	Call of *inorder*	Value in *CurrentNode*	Action
Driver	+		10	C	
1	*		11	0	
2	*		10	C	cout<<'C'
3	/		12	0	
4	A		1	*	cout<<'*'
5	0		13	D	
4	A	cout<<'A'	14	0	
6	0		13	D	cout<<'D'
3	/	cout<<'/'	15	0	
7	B		Driver	+	cout<<'+'
8	0		16	E	
7	B	cout<<'B'	17	0	
9	0		16	E	cout<<'E'
2	*	cout<<'*'	18	0	

Figure 5.17 : Trace of Program 5.1

- Output : A/B*C*D+E
  - infix form of the expression

5.3.3 Preorder Traversal

- VLR

```
template <class T>
void Tree<T>::preorder()
{ // Driver
        preorder(root);
}
template <class T>
void Tree<T>::preorder(TreeNode<T> *CurrentNode)
{ // Workhorse
       if (CurrentNode) {
              Visit(CurrentNode);
              preorder(CurrentNode->leftChild);
              preorder(CurrentNode->rightChild);
       }
}
-----------------------------------------
Program 5.2 : Preorder traversal of a binary tree
```

· Output : +**/ABCDE
  - prefix form of the expression

5.3.4 Postorder Traversal

- LRV

```
template <class T>
void Tree<T>::Postorder()
{ // Driver
        postorder(root);
 }
template <class T>
void Tree::postorder(TreeNode *CurrentNode)
{ // Workhorse
        if (CurrentNode) {
            postorder(CurrentNode->LeftChild);
            postorder(CurrentNode->RightChild);
            Visit(currentNode);
        }
}
-----------------------------------------------------------
Program 5.3 : Postorder traversal of a binary tree
```

· Output : AB/C*D*E+

– postfix form of the expression

5.3.5 Iterative Inorder Traversal

- Tree is a container class
  - may implement a tree traversal algorithm by using iterators
  - the algorithm needs to be non-recursive
  - use template Stack class
- Definition
  - a data object of Type A USES-A data object of Type B if a Type A object uses a Type B object to perform a task
  - this relationship is typically expressed by emplo ying the Type B object in a member function of Type A

5.3.5 Iterative Inorder Traversal

```
template <class T>
void Tree<T>::NonrecInorder()
{ // nonrecursive inorder traversal using a stack
        Stack<TreeNode<T> *> s; // declare and initialize stack
        TreeNode<T> *currentNode = root;
        while(1) {
            while(currentNode) {  // move down LeftChild fields
                s.Push(currentNode);    // add to stack
                currentNode = currentNode->leftChild;
            }
            if (s.IsEmpty()) return;
            currentNode = s.Top();
            s.Pop();
            Visit(currentNode);
            currentNode = currentNode->rightChild;
       }
}
---------------------------------------------------------------
```

Program 5.4 : Nonrecursive inorder traversal

5.3.5 Iterative Inorder Traversal

```
class InorderIterator {
public:
        InorderIterator(){ CurrentNode = root;};
        T* Next();
private:
        Stack <TreeNode<T> *> s;
        TreeNode<T>* currentNode;
};
----------------------------------------------------------
```

Program 5.5 : Definition of inorder iterator class

```
T* InorderIterator::Next()
{
         while(currentNode) {
                s.Push(currentNode);
                currentNode = currentNode->leftChild;
        }
        if (s.IsEmpty()) return 0;
        currentNode = s.Top();
        s.Pop();
        T& temp = currentNode->data;
        currentNode = currentNode->rightChild; // update
        return &temp;
   }
----------------------------------------------------------
```

Program 5.6 : Code for obtaining the next inorder element

5.3.6 Level-Order Traversal

- Root-left child-right child
  - requires a queue

```
template <class T>
void Tree<T>::LevelOrder()
{ // Traverse the binary tree in level order
      Queue<TreeNode<T>*> q;
      TreeNode<T> *currentNode = root;
      while(currentNode) {
            Visit(currentNode);
            if (currentNode->leftChild) q.Push(currentNode->leftChild);
            if (currentNode->rightChild) q.Push(currentNode->rightChild);
            currentNode = *q.Front();
            q.Pop();
      }
}
---------------------------------------------------------------
Program 5.7 : Level-order traversal of a binary tree
```

- Output : +*E*D/CAB
  - used the circular queue

5.3.7 Traversal without a Stack

- Is binary tree traversal possible without the use of extra space for stack?
  - Add a parent field to each node
  - Another solution represents binary tree as threaded binary trees in Section 5.5

5.4 Additional Binary Tree Operations
5.4.1 Copying Binary Trees

- Implement a copy constructor
  - Using modified postorder traversal algorithm
  - Assume TreeNode has a constructor that sets all three data members of a tree node

5.4.1 Copying Binary Trees

```
template <class T>
bool Tree<T>::Tree(const Tree<T>& s) //driver
{ // Copy constructor
        root = Copy(s.root);
}

template <class T>
TreeNode<T>* Tree<T>::Copy(TreeNode<T>* origNode) // Workhorse
{ // Return a pointer to an exact copy of the binary tree rooted at origNode.
        if(!origNode) return 0;
        return  new TreeNode<T>( origNode->data,
                                 Copy(origNode->leftChild);
                                 Copy(origNode->rightChild);
}
```

Program 5.9 : Copying a binary tree

5.4.2 Testing Equality

- Determining the equivalence of two binary trees
  - They have the same topology and data in corresponding node is identical
  - Function operator==() calls workhorse function Equal()

5.4.2 Testing Equality

```
template <class T>
bool Tree<T>::operator==(const Tree<T>& s) const//driver
{
            return Equal(root, t.root);
}

template <class T>
bool Tree<T>::Equal(TreeNode<T>* a, TreeNode<T>* b)
{// Workhorse
            if((!a)&&(!b)) return true; // both a and b are 0
            return ( a&&b                // both a and and are non-zero
                        && (a->data==b->data) // data is the same
                        && Equal(a->leftChild, b->leftChild) // left subtrees equal
                        && Equal(a->rightChild, b->rightChild)); // right subtrees equal
}
---------------------------------------------------------------------
Program 5.10 : Binary tree equivalence
```

5.4.3 The Satisfiability Problem

- Consider the operations $\wedge$(and), $\vee$(or), $\neg$(not)
  - The variables can hold only *true* or *false*
- Example : $x_1 \vee (x_2 \wedge \neg x_3)$
  - $x_1 = x_3 =$ false, $x_2 =$ true
  - false $\vee$(true $\wedge$ $\neg$false)

    = false $\vee$ true = true
- Let assume our formula is
  - $(x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_3) \vee \neg x_3$

5.4.3 The Satisfiability Problem

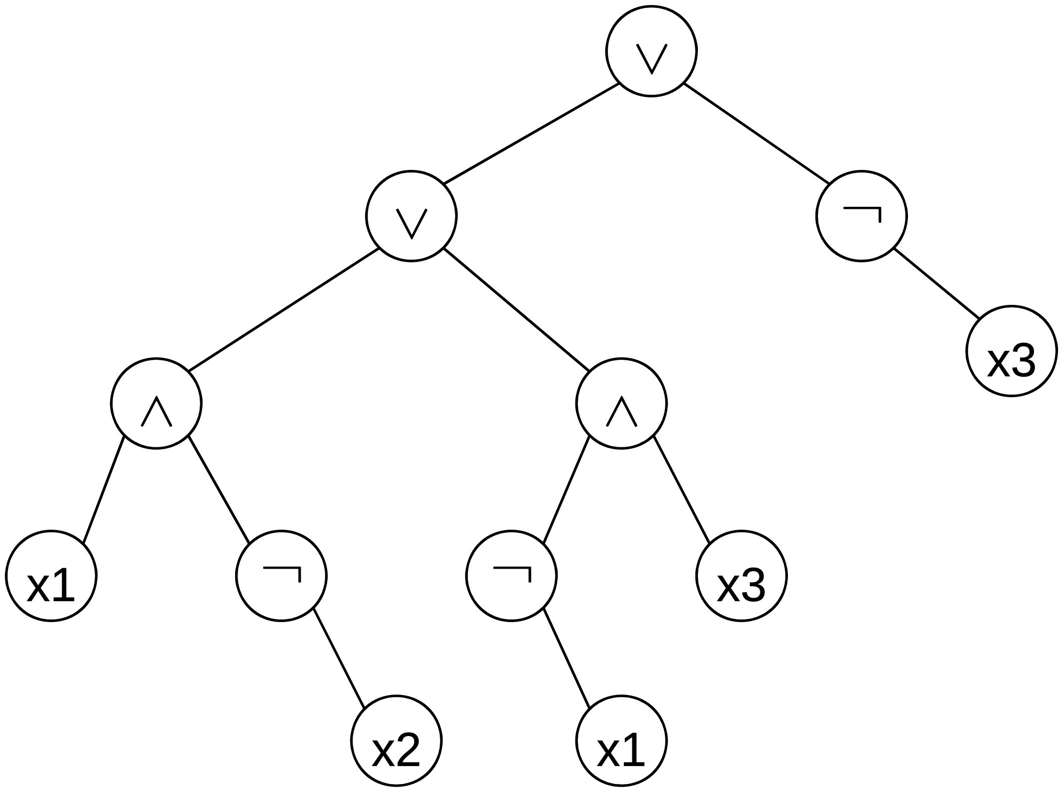


Figure 5.18 : Propositional formula in a binary tree

5.4.3 The Satisfiability Problem

- There are 2^n possible combination
  - if n=3, true=t, false=f

    (t,t,t), (t,t,f), (t,f,t), (t,f,f), (f,t,t), (f,t,f) (f,f,t), (f,f,f)

    $O(2^n)$ time complexity
- To evaluate an expression, using postorder traversal
  - $(x_1 \land \neg x_2) \lor (\neg x_1 \land x_3) \lor \neg x_3$

  => $x_2 \neg x_1 \land x_1 \neg x_3 \land \lor x_3 \neg \lor$

5.4.3 The Satisfiability Problem

- Define new data type
  - $T = \text{pair}<\overbrace{\text{Operator}}^{\text{first}}, \overbrace{\text{bool}}^{\text{second}}>$
  - enum Operator{Not, And, Or, True, False}
- Program 5.11
  - n is the number of variables in formula
  - *formula* is the binary tree that represents the formula
- Program 5.12
  - Assume every leaf node's *data.first* filed has been set either True or false

5.4.3 The Satisfiability Problem

```
for each of the 2^n possible truth value combinations for the n variables
{
        replace the variables by their values in the current truth value
        combination evaluate the formula by traversing the tree it points
        to in postorder;
        if (formula.Data().second()){ cout << current combination; return;}
}
cout << "no satisfiable combination";
```

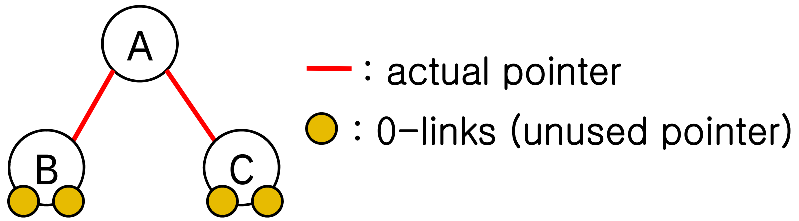
Program 5.11 : First version of satisfiability algorithm

```
// visit the node pointed at by p
switch (p->data.first) {
    case Not: p->data.second = !p->rightChild->data.second; break;
    case And: p->data.second =
                p->leftChild->data.second&&p->rightChild->data.second;
                break;
    case Or: p->data.second =
                p->leftChild->data.second||p->rightChild->data.second;
                break;
    case True: p->data.second=true; break;
    case False: p->data.second=false;
}
```

Program 5.12: Visiting a node in an expression tree

5.5 Threaded Binary Trees
5.5.1 Threads

- There are more 0-links than actual pointers



- Replace the 0-links to pointers, threads
  (1) A 0 rightChild field in node p is replaced by a pointer to the node that would be visited after p when traversing the tree in inorder. That is, it is replaced by the inorder successor of p
  (2) A 0 leftChild field in node p is replaced by a pointer to the node that immediately precedes node p in inorder

5.5.1 Threads

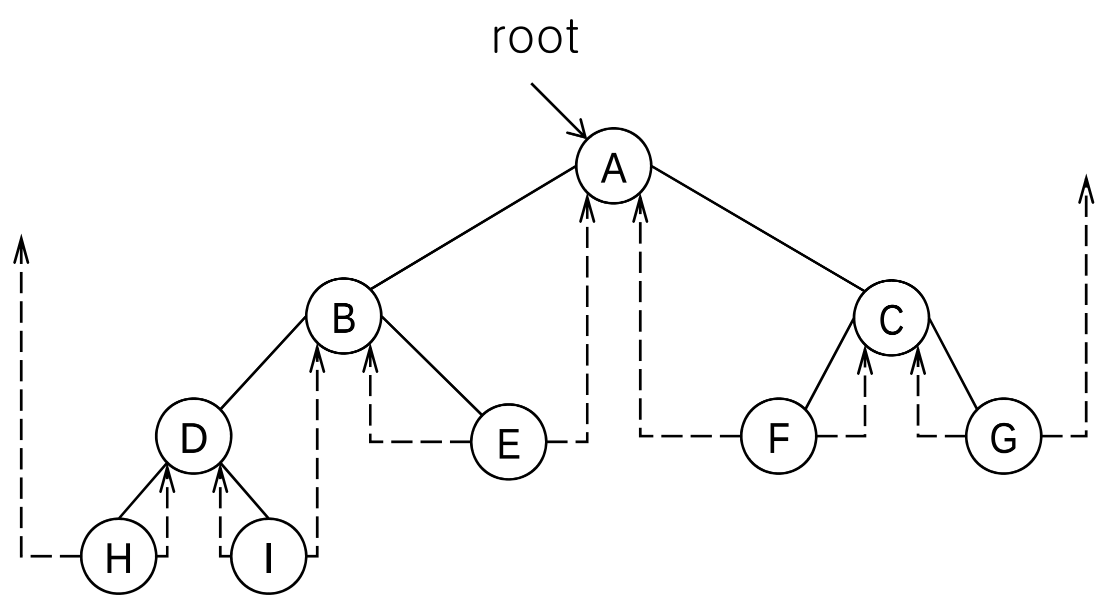


Figure 5.20 : Threaded tree corresponding to Figure 5.10(b)

- 9 nodes 10 0-links which replaced by threads
- Visit H, D, I, B, E, A, F, C, G
- e.g.) Node E has a predecessor thread points B and a successor thread points to A

5.5.1 Threads

- New node structure considering threads

Figure 5.21 : An empty threaded binary tree

- bool leftTread, rightThread
  - If leftThread==true, leftChild contains a thread otherwise, righThread==false, NO thread

5.5.1 Threads

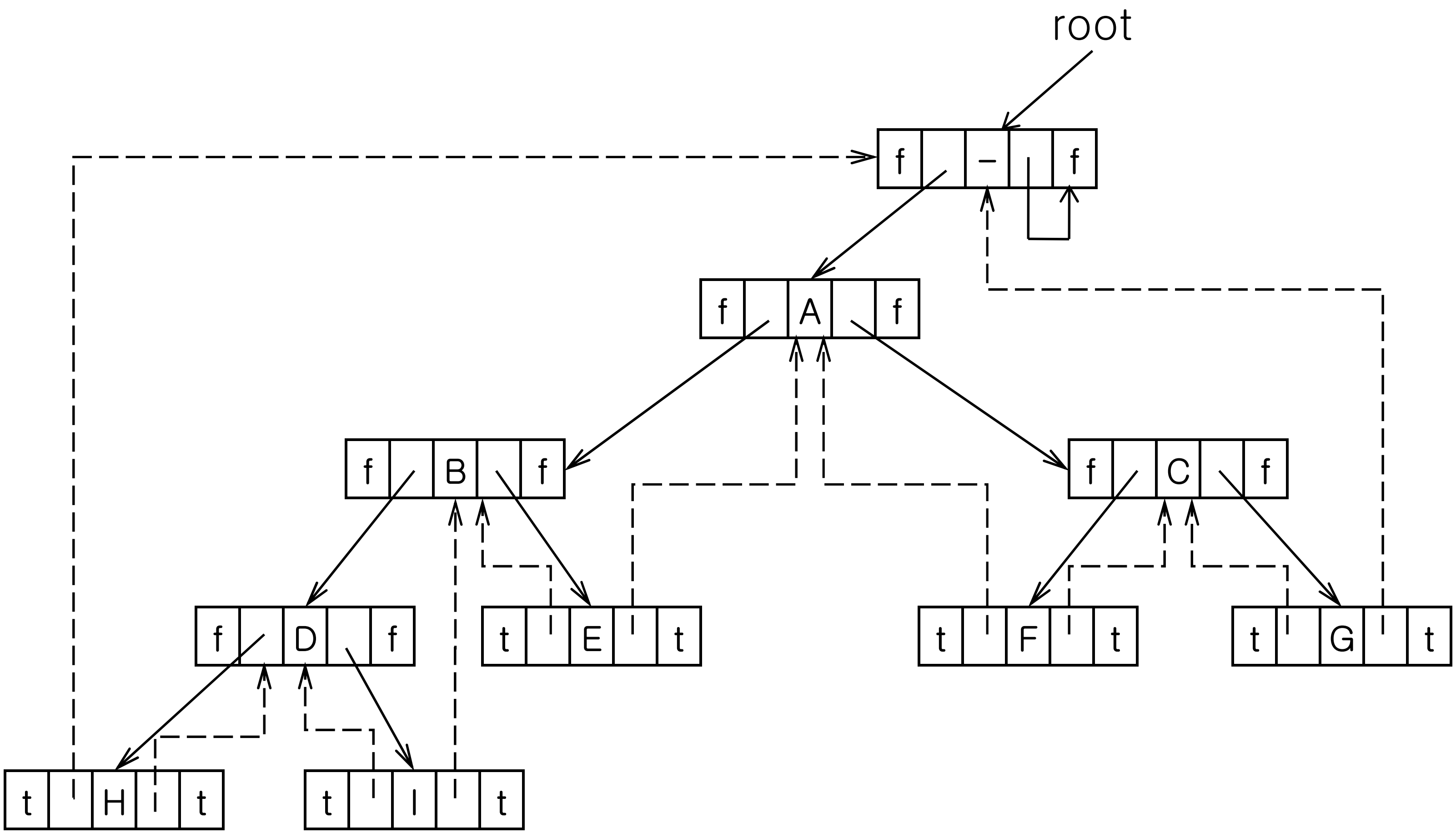


Figure 5.22 : Memory representation of threaded tree

5.5.2 Inorder Traversal of a Threaded Binary Tree

- Inorder traversal without a stack
  - If rightThread==true, next is rightChild
  - Otherwise follow the right child until reaching a node with leftThread==true

```
T* ThreadeInorderIterator::Next()
{ // Return the inorder successor of currentNode in a thread binary tree
        ThreadedNode<T>* temp = currentNode->rightChild;
        if(!currentNode->rightThread)
                while(!temp->leftThread) temp = temp->leftChild;
        currentNode = temp;
        if( currentNode == root ) return 0;
        else return &currentNode->data;
}
---------------------------------------------------------------
Program 5.13 : Finding the inorder successor in a threaded binary tree
```

5.5.3 Inserting a Node into a Threaded Binary Tree

- Consider only inserting r as the right child of a node s

  (1) If s has an empty right subtree, then the insertion is simple

  (2) If the right subtree of s is not empty, then this right subtree is made the right subtree of r after insertion. When this is done, r becomes the inorder predecessor of a node that has a leftThread==true field, and consequently there is a thread which has to be updated to point to r. The node containing this thread was previously the inorder successor of s.

5.5.3 Inserting a Node into a Threaded Binary Tree

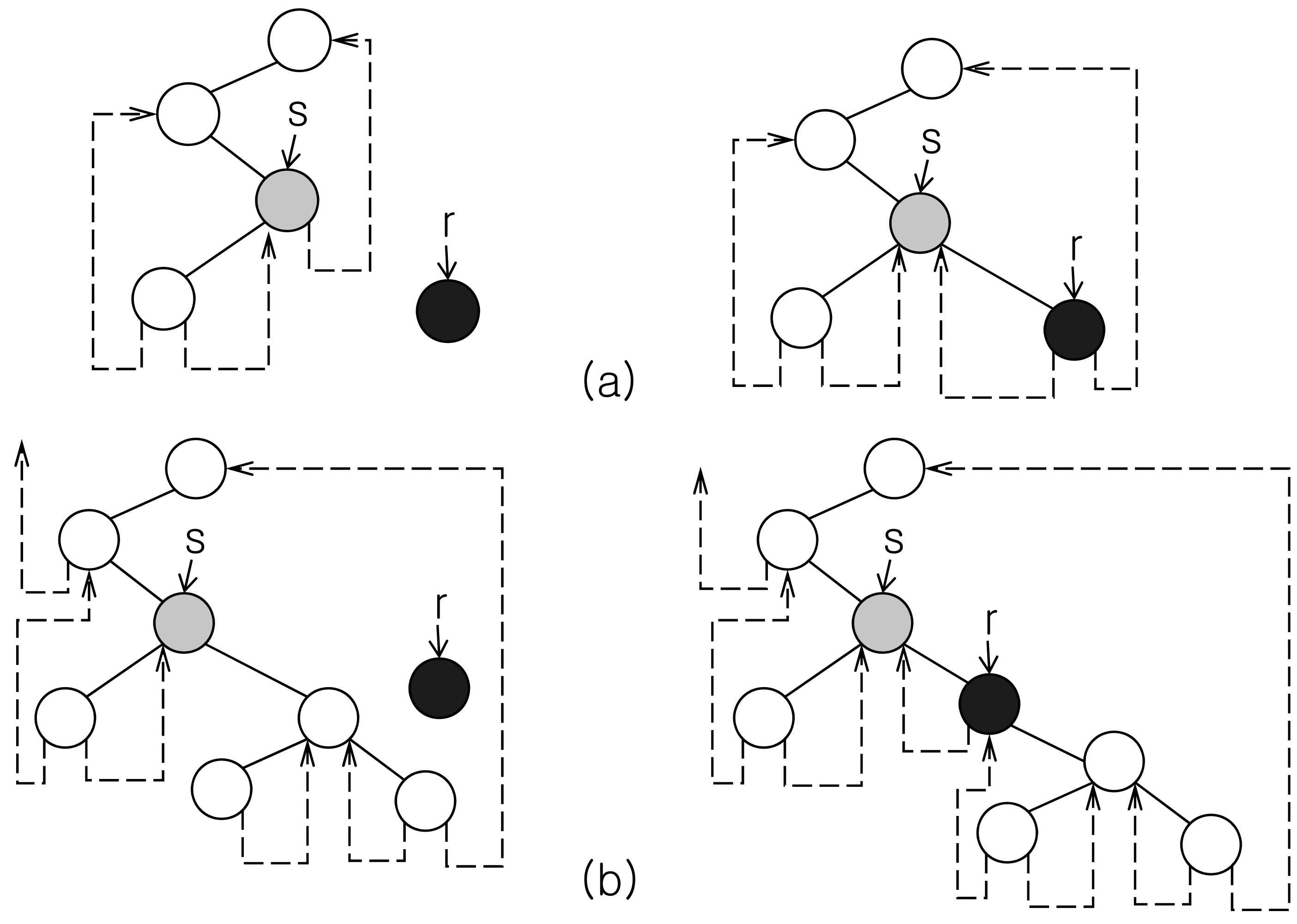


Figure 5.23 : Insertion of r as a right child of s in a threaded binary tree

5.5.3 Inserting a Node into a Threaded Binary Tree

```
template <class T>
void ThreadedTree<T>::InsertRight(ThreadedNode<T> *s,
                                  ThreadedNode<T> *r)
{ // Insert r as the right child of s.
        r->rightChild = s->rightChild;
        r->rightThread = s->rightThread;
        r->leftChild = s;
        r->leftThread = true; // leftChid is a thread
        s->rightChild = r;
        s->rightThread = false;
        if(!r->rightThread) {
                ThreadedNode<T> *temp = InorderSucc(r);
                                // returns the in order successor of r
                temp->leftChild = r;
        }
}
```

Program 5.14 : Inserting r as the right child of s

5.6 Heap
5.6.1 Priority Queues

- Max(min) priority queue
  - element with highest(lowest) priority is deleted
  - element with arbitrary priority can be inserted
  - frequently implemented using max(min) heap

5.6.1 Priority Queues

- Abstract class in C++

```
template <class T>
class MaxPQ {
public:
    virtual ~MaxPQ(){}
            // virtual destructor
    virtual bool IsEmpty() const = 0;
            // return true if the priority queue is empty
    virtual const T& Top() const = 0;
            // return reference to max element
    virtual void Push(const T&) = 0;
            // add an element to the priority queue
    virtual void Pop() = 0;
            // delete element with max priority
 };
```

5.6.2 Definition of a Heap

- Max(min) tree
  - a tree in which the key value in each node is no smaller (larger) than the key values in its children (if any)
  - the key in the root is the largest (smallest)
- Max(min) heap
  - a complete binary tree that is also a max(min) tree

5.6.2 Definition of a Heap

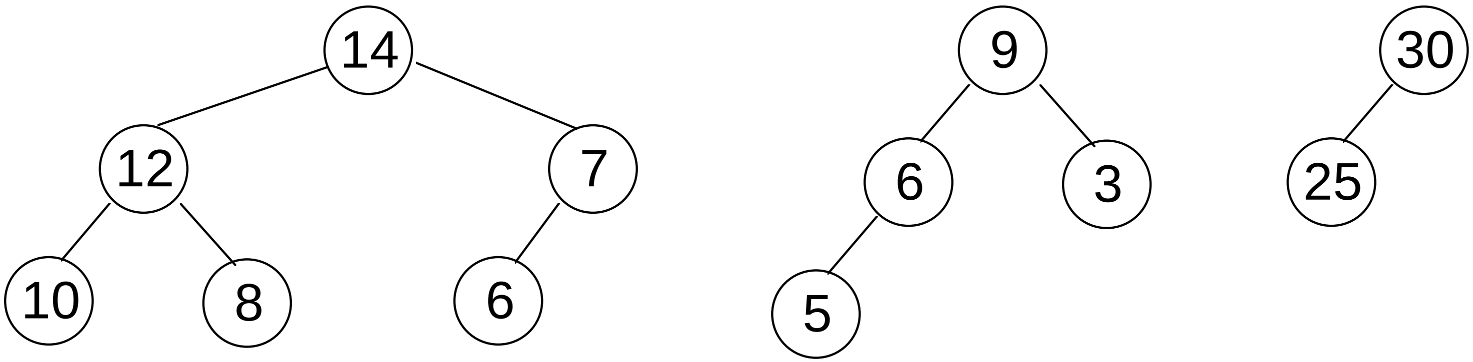


Figure 5.24 : Max heaps

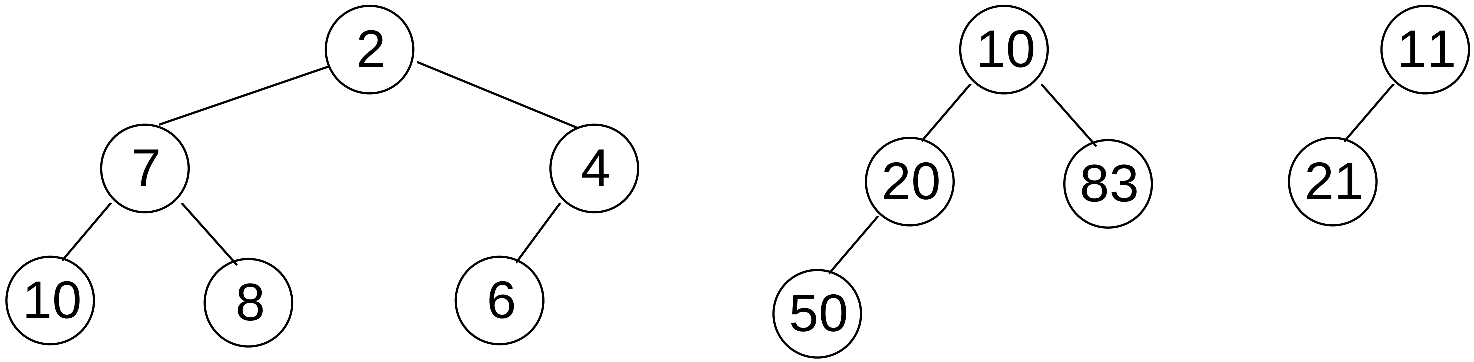


Figure 5.25 : Min heaps

5.6.2 Definition of a Heap

- Basic operations of a max heap
  - creation of an empty heap
  - insertion of a new element into the heap
  - deletion of the largest element from the heap
- Private data members of class MaxHeap

```
private:
        T *heap;            // element array
        int heapSize;       // number of elements in heap
        int capacity;       // size of the array heap
```

5.6.2 Definition of a Heap

```
template <class T>
MaxHeap<T>::MaxHeap(int theCapacity = 10)
{
        if (theCapacity < 1) throw "Capacity must be >= 1";
        capacity = theCapacity;
        heapSize = 0;
        heap = new T[capacity+1]; // heap[0] is not used
}
----------------------------------------
Program 5.15 : Max heap constructor
```

5.6.3 Insertion into Max Heap

- Examples

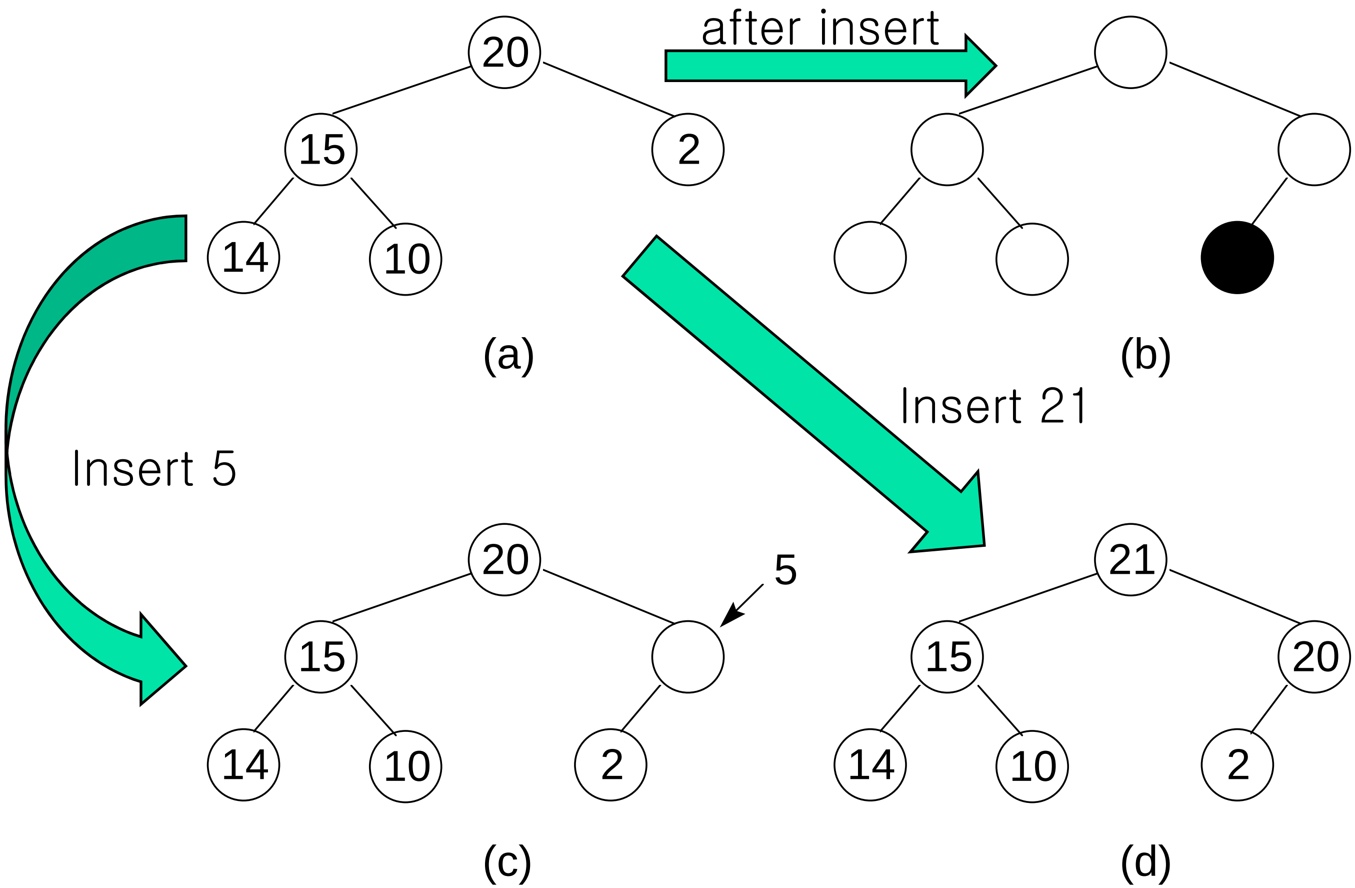


Figure 5.26 : Insertion into a max heap

5.6.3 Insertion into Max Heap

- Implementation
  - need to move from child to parent
  - heap is complete binary tree
    - use formula-based representation
    - Lemma 5.4 : parent(i) is at $\lfloor i/2 \rfloor$ if $i \neq 1$
  - Complexity is O(log n)

5.6.3 Insertion into Max Heap

```
template <class T>
void MaxHeap<T>::Push(const T& e)
{ // Insert e into the max heap
        if (heapSize == capacity) { // double the capacity
                ChangeSize1D(heap, capacity, 2*capacity);
                capacity *= 2;
        }
        int currentNode = ++heapSize;
        while (currentNode != 1 && heap[currentNode / 2] < e)
        { // bubble up
                heap[currentNode] = heap[currentNode / 2];
                                                    // move parent down
                currentNode /= 2;
        }
        heap[currentNode] = e;
}
----------------------------------------------------------
Program 5.16 : Insertion into a max heap
```

5.6.4 Deletion from Max Heap

- Example

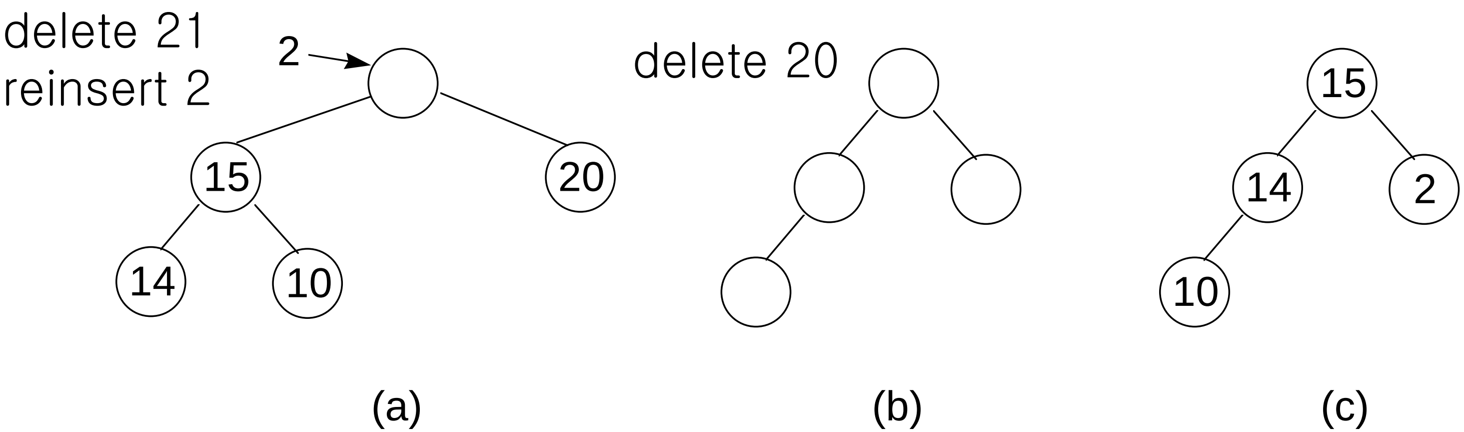


Figure 5.27 : Deletion from a heap

5.6.4 Deletion from Max Heap

```
template <class T>
void MaxHeap<T>::Pop()
{ // Delete max element
        if (IsEmpty()) throw "Heap is empty. Cannot delete.";
        heap[1].~T(); // delete max element

        // remove last element from heap
        T lastE = heap[heapSize--];

        // trickle down
        int currentNode = 1;        // root
        int child = 2;               // a child of currentNode
        while (child <= heapSize)
        {
                // set child to larger child of currentNode
                if (child<heapSize && heap[child]<heap[child+1]) child++;
                // can we put lastE in current Node?
                if (lastE>=heap[child]) break; // yes

                // no
                heap[currentNode] = heap[child];
                currentNode = child; child *= 2;
        }
        heap[currentNode] = lastE;
}
----------------------------------------------------------
Program 5.17 : Deletion from a max heap
```

5.7 Binary Search Trees
5.7.1 Definition

- Binary tree which may be empty
  - if not empty
    - (1) every element has a distinct key
    - (2) keys in left subtree < root key
    - (3) keys in right subtree > root key
    - (4) left and right subtrees are also binary search trees

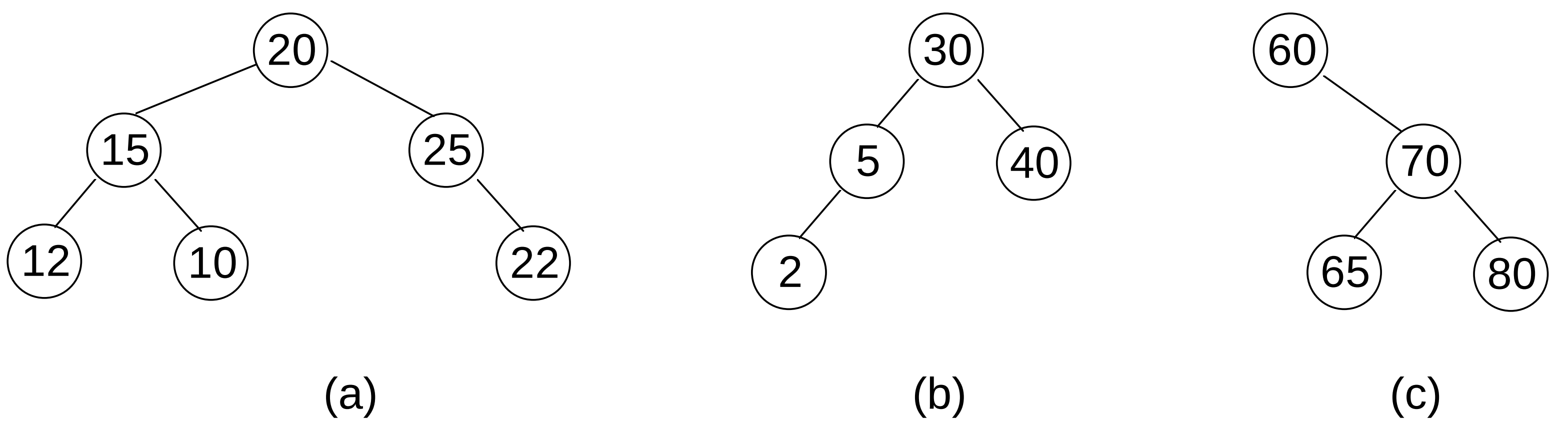


Figure 5.28 : Binary trees

5.7.2 Searching Binary Search Tree

- Recursive search by key value
  - definition of binary search tree is recursive
  - key(element) = x
    - x=root key : element=root
    - x<root key : search left subtree
    - x>root key : search right subtree

5.7.2 Searching Binary Search Tree

```
template <class K, class E> // Driver
pair<K, E>* BST<K, E>::Get(const K& k)
{ // Search the binary search tree (*this) for a pair with key k
  // If such a pair is found, return a pointer to this pair; otherwise, return 0
        return Get(root, k);
}

template <class K, class E> // Workhorse
pair<K, E>* BST<K, E>::Get(TreeNode<pair<K, E> >* p, const K& k)
{
        if(!p) return 0;
        if(k<p->data.first) return Get(p->leftChild, k);
        if(k>p->data.first) return Get(p->rightChild, k);
        return &p->data;
}
-----------------------------------------------------------
```

Program 5.18 : Recursive search of a binary search tree

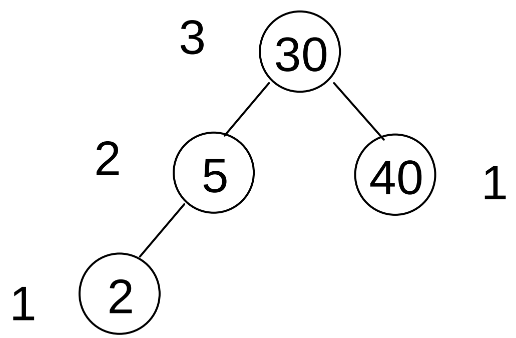
5.7.2 Searching Binary Search Tree

```
template <class K, class E> // Driver
pair<K, E>* BST<K, E>::Get(const K& k)
{
        TreeNode<pair<K,E> > *currentNode = root;
        while(currentNode)
        {
                if (k<currentNode->data.first)
                        currentNode = currentNode->leftChild;
                else if ( k>currentNode->data.first)
                        currentNode = currentNode->rightChild;
                else return &currentNode->data;
        }
        //no matching pair
        return 0;
}
```

Program 5.19 : Iterative search of a binary search tree

5.7.2 Searching Binary Search Tree

- Search by rank
  - node needs LeftSize field
  - LeftSize=1 + #elements in left subtree

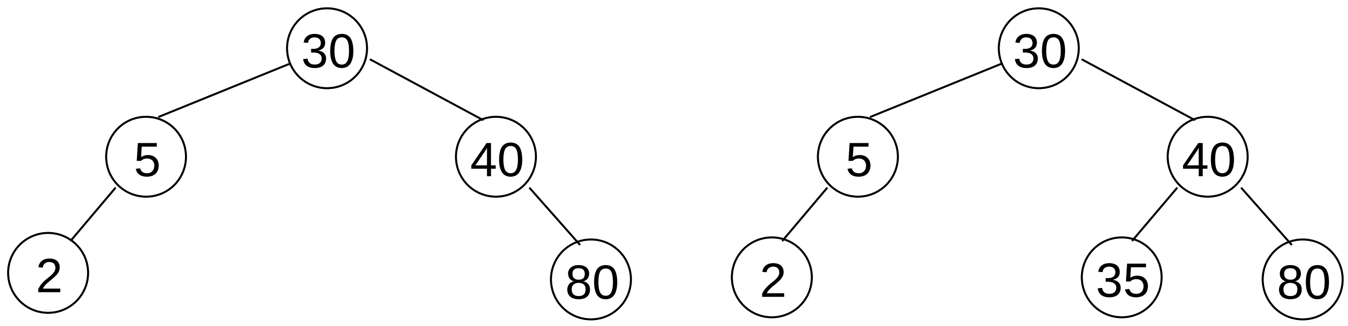


5.7.2 Searching Binary Search Tree

```
template <class K, class E> // search by rank
pair<K,E>* BST<K,E>::RankGet(int r)
{ // Search the binary search tree for the rth smallest pair
        TreeNode<pair<K,E> > *currentNode = root;
        while (currentNode)
                if(r<currentNode->leftSize)
                        currentNode = currentNode->leftChild;
                else if (r>currentNode->leftSize)
                {
                        r -= currentNode->leftSize;
                        currentNode = currentNode->rightChild;
                }
                else return &currentNode->data;
        return 0;
}
---------------------------------------------------------------
Program 5.20 : Searching a binary search tree by rank
```

5.7.3 Insertion into Binary Search Tree

- New element x
  - search x in the tree
    - success : x is in the tree
    - fail : insert x at the point the search terminated



(a) Insert 80 (b) Insert 35

Figure 5.29 : Inserting into a binary search tree

5.7.3 Insertion into Binary Search Tree

```
template <class K, class E>
void BST<K, E>::Insert(const pair<K, E>& thePair)
{ // Insert thePair into the binary search tree.
        // search for thePair.first, pp is parent of p
        TreeNode<pair<K, E> > *p = root, *pp = 0;
        while(p) {
                pp = p;
                if (thePair.first < p->data.first) p = p->leftChild;
                else if(thePair.first > p->data.first) p = p->rightChild;
                else // duplicate, update associated element
                    { p->data.second = thePair.second; return; }
        }
        //perform insertion
        p = new TreeNode<pair<K, E> >(thePair);
        if(root) // tree not empty
                if (thePair.first<pp->data.first) pp->leftChild=p;
                else pp->rightChild = p;
        else root = p;
}
```

Program 5.21 : Insertion into a binary search tree

5.7.4 Deletion from Binary Search Tree

- Leaf node
  - corresponding child field of its parent is set to 0
  - the node is disposed
- Nonleaf node with one child
  - the node is disposed
  - child takes the place of the node
- Nonleaf node with two children
  - node is replaced by either
    - the largest node in its left subtree
    - the smallest node in its right subtree
  - delete the replacing node from the subtree

5.7.4 Deletion from Binary Search Tree

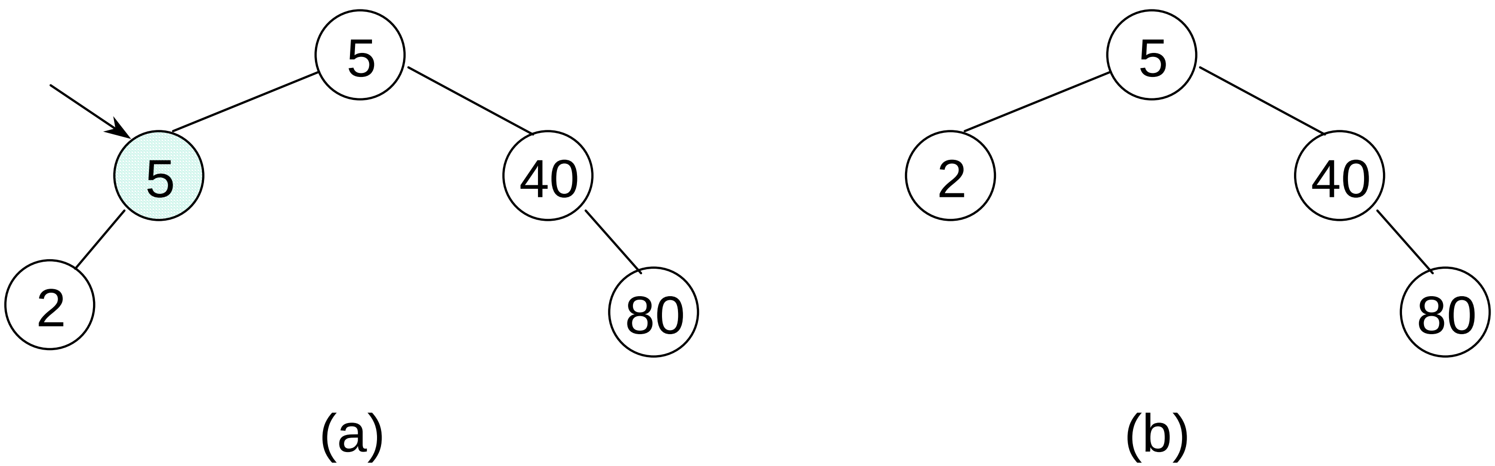


Figure 5.30 : Deletion from a binary search tree

5.7.5 Joining and Splitting Binary Trees

- ThreeWayJoin(*small, mid, big*)
  - new BST ← BST *small* + node *mid* + BST *big*
  - each node in *small* has smaller key than *mid.first*
  - each node in *big* has larger key than *mid.first*
- TwoWayJoin(small, big)
  - new BST ← BST *small* + BST *big*
  - all keys of *small* are smaller than all keys of *big*
- Split(*k, small, mid, big*)
  - BST → BST *small* + node *mid* + BST *big*
  - all keys of *small* < *k*
  - all keys of *big* > *k*
  - if A contains a node with key=*k*, the node is copied into *mid*

5.7.5 Joining and Splitting Binary Trees

```
template <class K, class E>
void BST<K,E>::Split(const K& k, BST<K,E>& small,
                      pair<K,E>*& mid, BST<K,E>& big)
{ // Split the binary search tree with respect to key k
    if(!root){ small.root=big.root=0; return;} // empty tree
    // create header nodes for small and big
    TreeNode<pair<K,E> > *sHead = new TreeNode<pair<K, E> >,
                      *s = sHead;
                      *bHead = new TreeNode<pair<K, E> >,
                      *b = bHead;
                      *currentNode = root;
    while (currentNode)
        if (k<currentNode->data.first) { // add to big
            b->leftChild = currentNode;
            b = currentNode; currentNode = currentNode->leftChild;
        }
        else if (k>currentNode->data.first) { // add to small
            s->rightChild = currentNode;
            s = currentNode; currentNode = currentNode->rightChild;
        }
        else { //split at currentNode
            s->rightChild = currntNode->leftChild;
            b->leftChild = currntNode->rightChild;
            small.root = sHead->rightChild; delete sHead;
            big.root = bHead->leftChild; delete bHead;
            mid=new pair<K,E>(currentNode->data.first,
                              currentNode->data.second);
            delete currentNode;
            return;
        }
    // no pair with key k
    s->rightChild=b->leftChild = 0;
    small.root = sHead->rightChild; delete sHead;
    big.root = bHead->leftChild; delete bHead;
    mid = 0;
    return;
}
---------------------------------------
```

Program 5.22 : Splitting a binary search tree

5.7.6 Height of Binary Search Tree

- Height of BST with n nodes
  - worst-case : n
  - average : O(log n)
- Balanced search trees
  - worst-case height : O(log n)
  - some perform search, insert, delete in O(h)

- 5.8 SELECTION TREES
- 5.9 FOREST
- 5.10 REPRESENTATION OF DISJOINT SETS
- 5.11 COUNTING BINARY TREES

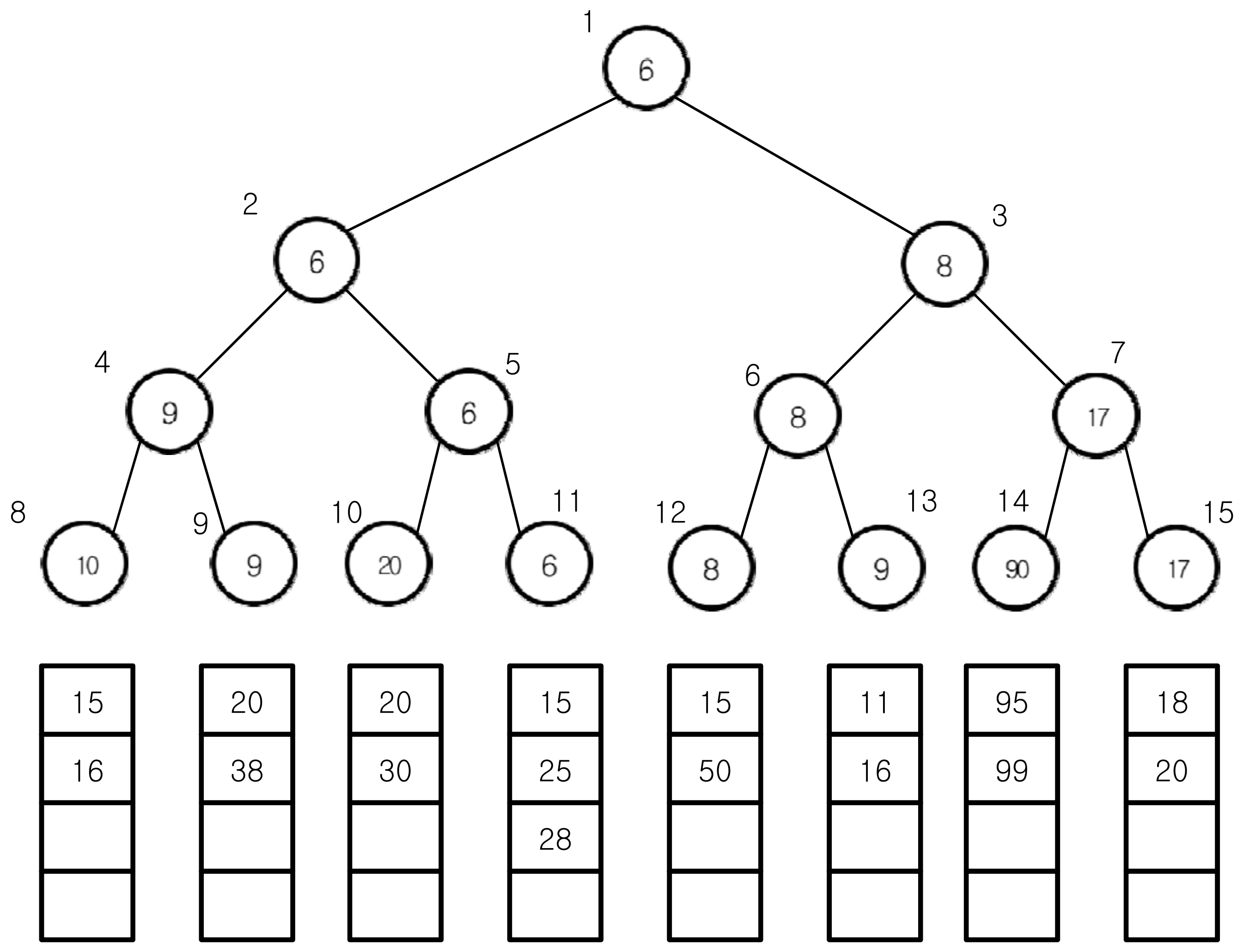
Selection Trees

- k ordered sequences(runs) -> merge -> single ordered sequence.
- Each run
  - consists of some records
  - In nondecreasing order of a designated field (key)

Winner Trees

- A Complete binary tree
- Each node represents the smaller of its two children.
- The root node represents the smallest node in the tree.

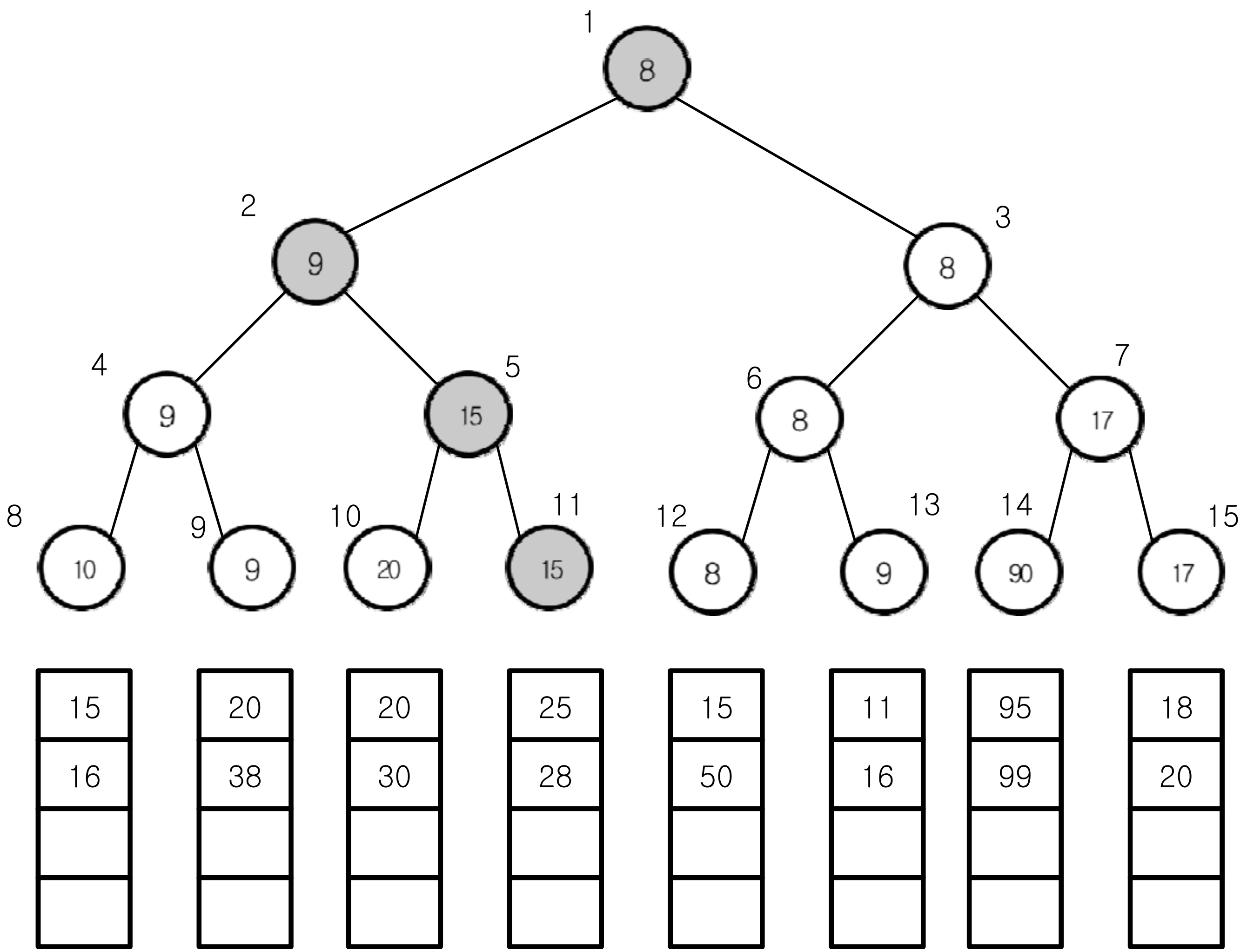
Figure 5.31 : Winner tree for k=8, showing the first three keys in each of the eight runs



Winner Trees (cont.)

- Construction
  - tournament in which the winner is the record with the smaller key.
  - Each nonleaf node – winner of a tournament
  - Root node – the overall winner.(smallest key)
  - Each leaf node – first record in the corresponding run
- Each node contain only pointer to record.

Winner tree of Figure 5.31 after one record has been output and the tree restructured (nodes that were changed are shaded)



Winner Trees (cont.)

- Analysis of merging runs using winner trees
  - Level in the tree : $\log_2(k+1)$
  - time required to restructure the tree : $O(\log_2 k)$
  - time required to merge all n records : $O(n\log_2 k)$
  - Set up the selection tree : $O(k)$
  - Total time : $O(n\log_2 k)$

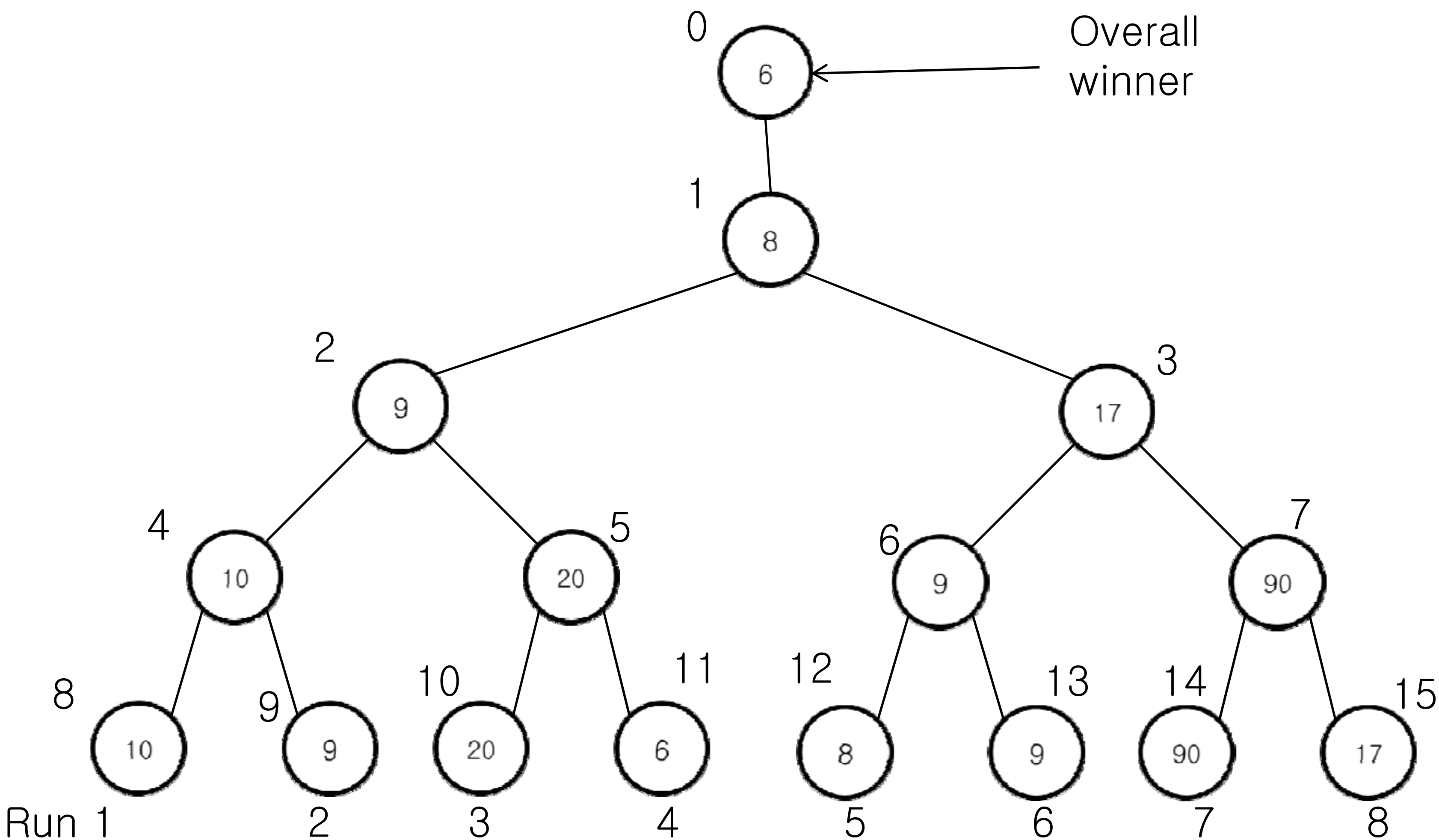
Loser Trees

- A complete binary tree added node 0 over the root node.
  - leaf node – element having smallest key value of each run.
  - internal node – loser of a tournament
  - root node(1) – loser of final tournament
  - node 0 – the overall winner

Loser Trees (cont.)

- Construction
  - Leaf node is smallest key value of each run
  - Children nodes have tournament in parent node
    - loser – remain parent node
    - winner – go to parent's parent node and perform another tournament
  - Tournament of node 1
    - loser – remain root node.
    - winner – up to node 0 and printed order sequence

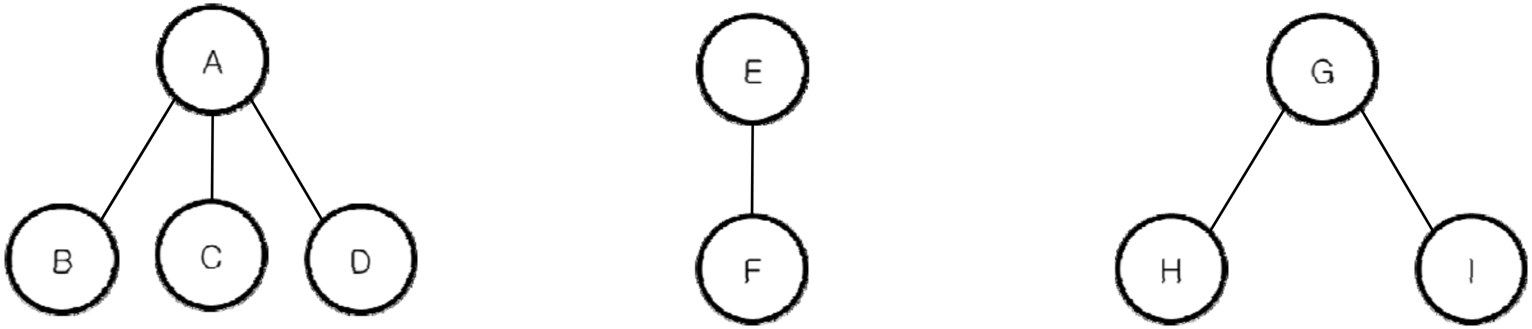
Looser Trees corresponding to winner tree of Figure 5.31



- 5.8 SELECTION TREES
- 5.9 FOREST
- 5.10 REPRESENTATION OF DISJOINT SETS
- 5.11 COUNTING BINARY TREES

Forests

- Definition : A forest is a set of n≥0 disjoint trees.



5.34 : Three-tree forest

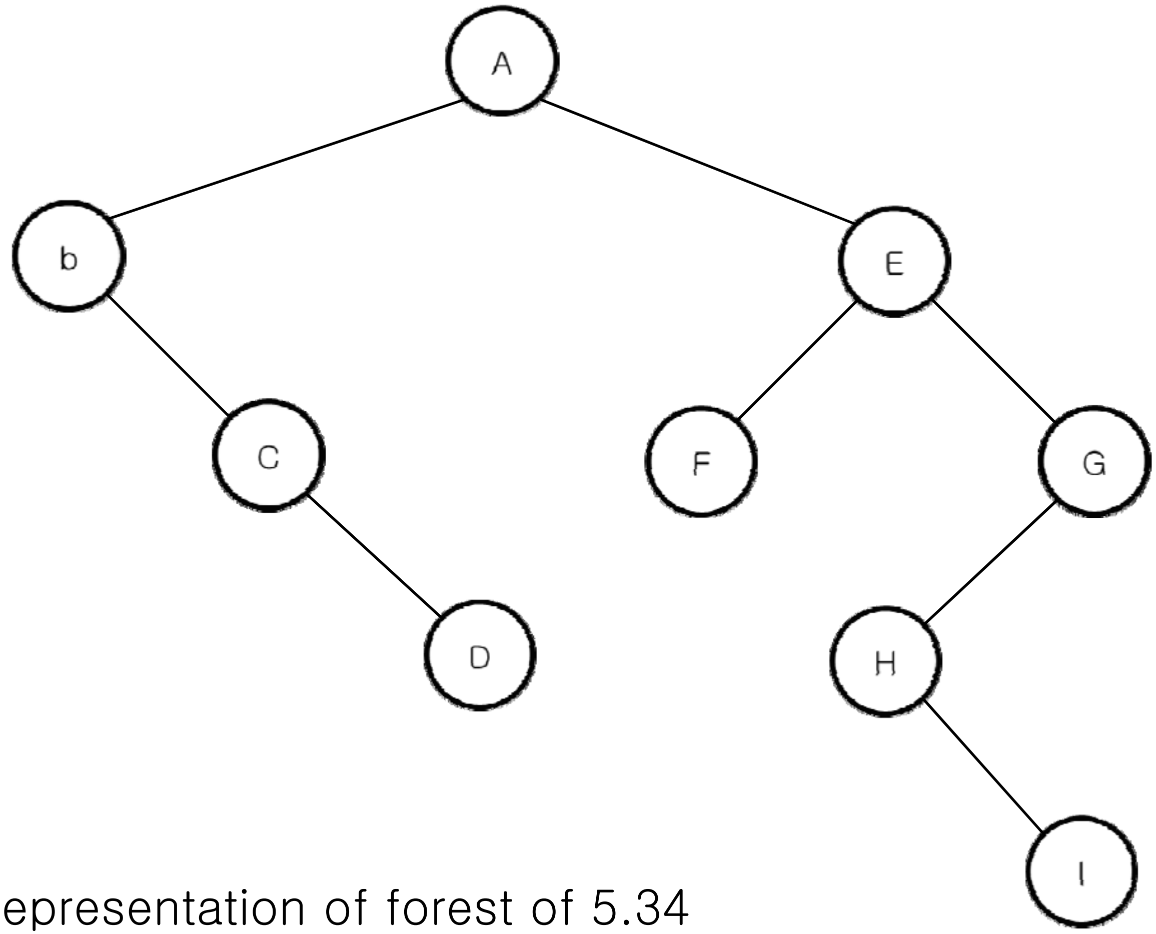
Transforming a Forest into a Binary Tree (cont.)

- Definition : If $T_1,\cdots,T_n$ is a forest of trees, then the binary tree corresponding to this forest, denoted by $B(T_1,\cdots,T_n)$,

  (1) is empty if $n=0$

  (2) has root equal to root(T_1); has left subtree equal to $B(T_{11}, T_{12}, \cdots, T_{1m})$, where $T_{11},\cdots,T_{1m}$ are the subtrees of root(T_1); and has right subtree $B(T_2,\cdots,T_n)$.

Transforming a Forest into a Binary Tree



5.35 : Binary tree representation of forest of 5.34

Forest Traversals

- Preorder and inorder traversals of the corresponding binary tree T of a forest F have a natural correspondence to traversals on F.
- No natural analog for postorder traversal of the corresponding binary tree of a forest.

Forest Traversals – preorder

1) If F is empty then return.
2) Visit the root of the first tree of F.
3) Traverse the subtrees of the first tree in forest preorder.
4) Traverse the remaining trees of F in forest preorder.

Forest Traversals – inorder

1) If F is empty then return.
2) Traverse the subtrees of the first tree in forest inorder.
3) Visit the root of the first tree.
4) Traverse the remaining trees in forest inorder.

Forest Traversals – postorder

1) If F is empty then return.
2) Traverse the subtrees of the first tree of F in forest postorder.
3) Traverse the remaining trees of F in forest postorder.
4) Visit the root of the first tree of F.

Forest Traversals (cont.)

- The level-order traversal of a forest and that of its associated binary tree do not necessarily yield the same result.

- 5.8 SELECTION TREES
- 5.9 FOREST
- 5.10 REPRESENTATION OF DISJOINT SETS
- 5.11 COUNTING BINARY TREES

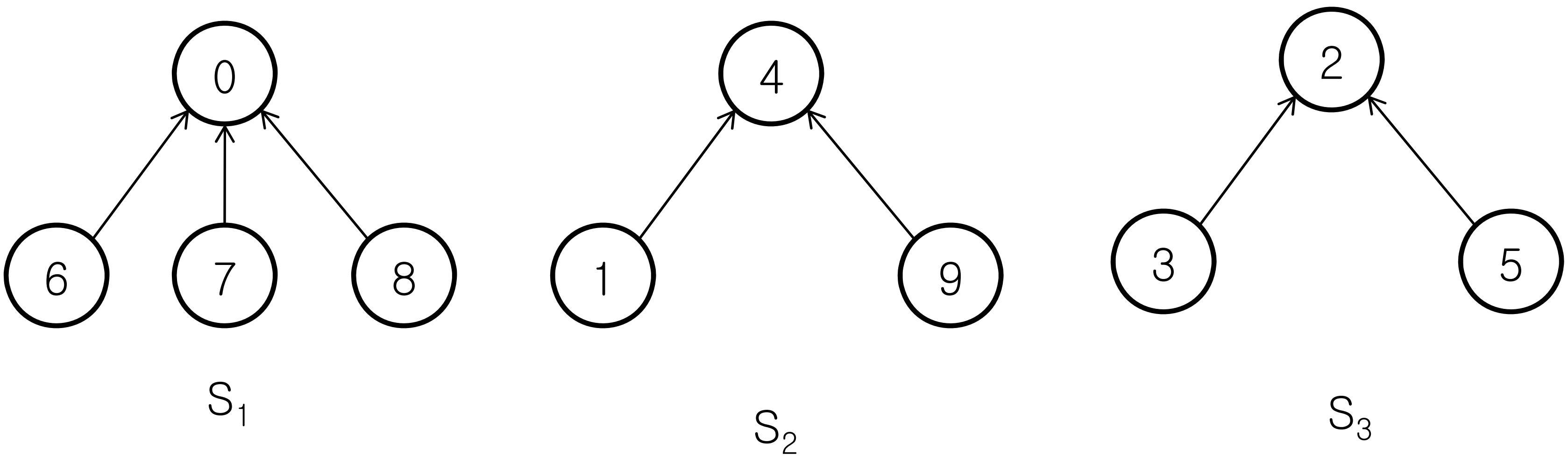
Introduction

- Use of trees in the representation of sets.
- Assume
  - Elements of the sets are the numbers $0,1,2,3,\cdots,n-1$
  - Pairwise disjoint (S_i and S_j, $i \neq j$, there is no element that is in both S_i and S_j)

Introduction (cont.)

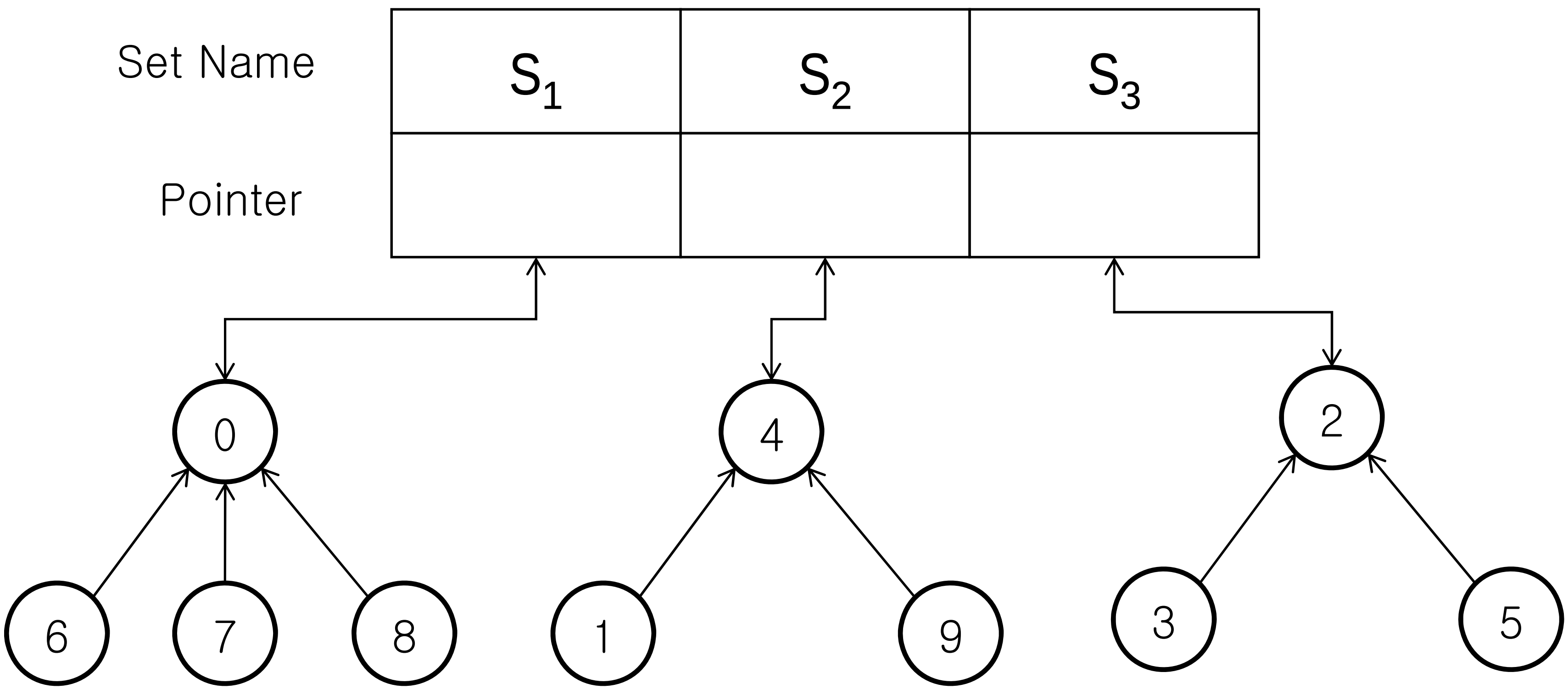
- Operation
  1) Disjoint set union. If S_i and S_j are two disjoint sets, then their union $S_i \cup S_j$ = { all elements x such that x in in S_i or S_j }
  2) Find(i). Find the set containing element i.

Introduction (cont.)



5.36 : Possible tree representation of sets

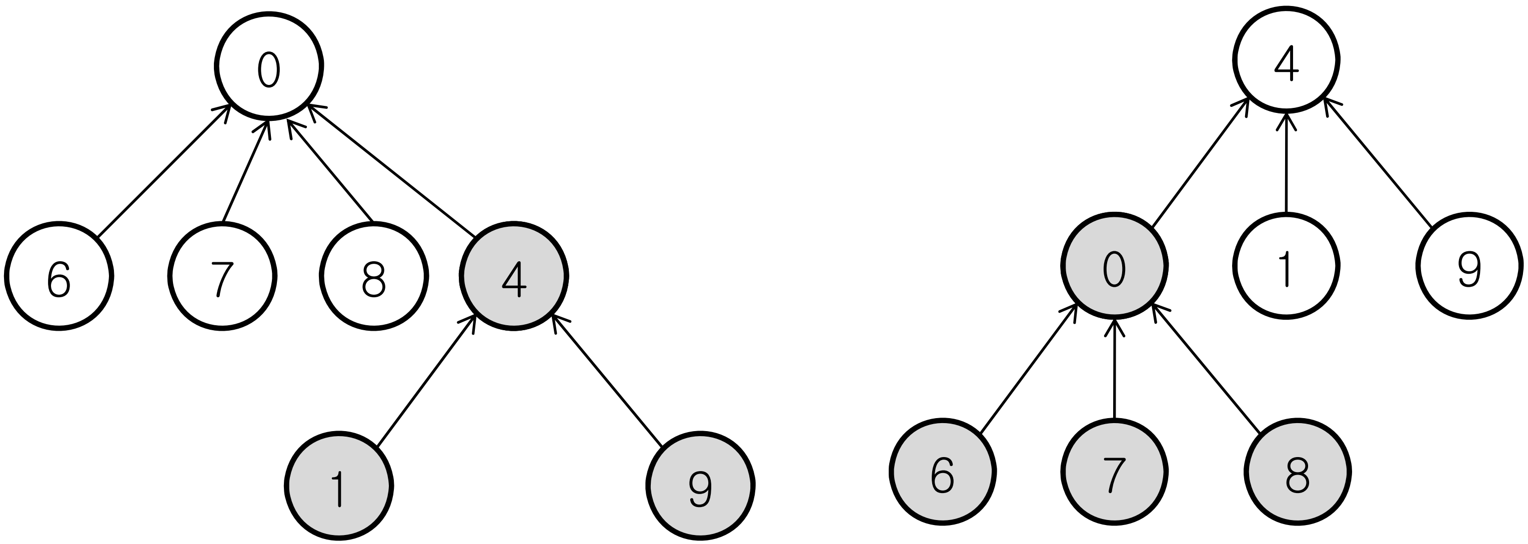
Introduction (cont.)



Data representation for S_1, S_2, and S_3

Union and Find Operations

- Union
  - Union of S_1 and S_2



Possible representations of $S_1 \cup S_2$

Union and Find Operations

- Union
  - Set parent field of one of the roots to the other root.

Union and Find Operations

- Since set elements are numbered 0 through n−1, we represent the tree nodes using an array parent[n].
- This array element gives the parent pointer of the corresponding tree node.

i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-1	4	-1	2	-1	2	0	0	0	4

Array Representation of S_1, S_2, and S_3 of 5.36

Union and Find Operations

- Find(i)
  - Ex. Find(5)
  - Start at 5 -> moves to 5's parent, 2 -> parent[2]=-1, we have reached root.
- Union(i,j)
  - We pass in two trees with roots i and j, $i \neq j$ -> parent[i]=j

i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-1	4	-1	2	-1	2	0	0	0	4

Class definition and constructor for Sets

```
class Sets{
pulib:
        //set operations follow
        .
        .
private:
        int *parent;
        int n; //number of set elements
};

Sets::Sets(int numberOfElements)
{
        if (numberOfElements<2) throw "Must have at least 2 elements";
        n=numberOfElements;
        parent=new int[n];
        fill(parent,parent+n,-1);
}
```

Simple function for union and fine

```
void Sets::SimpleUnion(int i, int j)
{//Replace the disjoint sets with roots i and j, i!=j with their union.
            parent[i]=j;
}

int Sets::SimpleFind(int i)
{//Find the root of the tree containing element i.
            while (parent[i]>=0) i=parent[i];
            return i;
}
```

Union and Find operations

- Analysis of SimpleUnion and SimpleFind
  - Start off with n elements each in a set of its own(i.e., $S_i=\{i\}$, $0 \le i < n$) -> Initial configuration consists of a forest with n nodes, and $parent[i]=-1$, $0 \le i < n$

Union and Find operations

- Process the following sequence of operations:

  Union(0,1), Union(1,2),⋯,Union(n−2,n−1)

  Find(0),Find(1),⋯,Find(n−1)

- Time Taken for a union is constant : n−1 unions in time O(n).
- Each find operation requires following a sequence of parent pointers from the element to be found to the root.

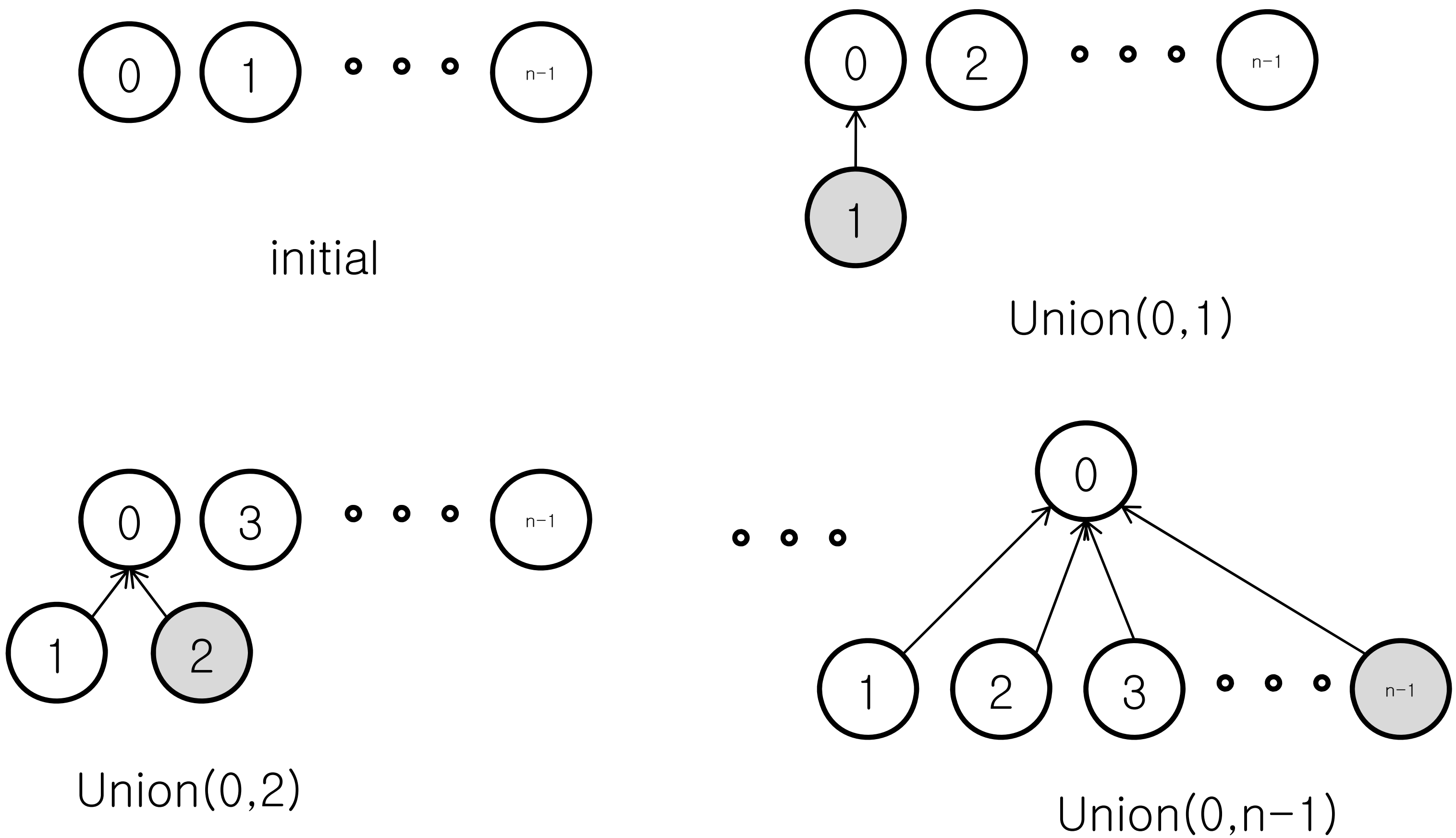
$$O\left(\sum_{i=1}^{n} i\right) = O(n^2)$$

Union and Find operations

- Avoiding the creation of degenerate trees.
- Definition [Weighting rule for Union(i,j)]:

  If the number of nodes in the tree with root i is less than the number in the tree with root j, then make j the parent of i; otherwise make i the parent of j

Union and Find operations



Trees obtained using the weighting rule

Union and Find operations

- Unions have been modified so that the input parameter values correspond to the roots of the trees to be combined.

Union function with weighting rule

```
void Sets::WeightedUnion(int i, int j)
//Union sets with roots i and j, i≠j using the weighting rule.
//parent[i]=-count[i] and parent[j]= -count[j]
{
        int temp=parent[i]+parent[j];
        if (parent[i]>parent[j]){//i has fewer nodes
                parent[i]=j;
                parent[j]=temp;
        }
        else {//j has fewer nodes(or i and j have the same
             //                 number of nodes)
                parent[j]=i;
                parent[i]=temp;
        }
}
```

Union and Find operations

- Analysis of WeightedUnion and Simple Find
  - The time required to perform a union has increased somewhat but is still bounded by a constant.
  - The maximum time to perform a find is determined by Lemma 5.5
- Lemma 5.5: Assume that we start with a forest of trees, each having one node. Let T be a tree with m nodes created as a result of a swquence of unions each performed using function WeightedUnion. The Height of T is no greater than $\lfloor \log_2 m + 1 \rfloor$

Union and Find operations (cont.)

- Proof :
  - true for m=1
  - Assume true for all trees with i nodes, i≤m−1
    −> show that also true for i=m
  - Let T be a tree with m nodes created by function WeightedUnion.
  - Consider the last union operation performed, Union(k,j).
  - Let a be the number of nodes in tree j and m−a the number in k
  - Without loss of generality we may assume 1≤a≤m/2
  - Height of T is either the same as that of k or is one more than that of j
  - former : $T \le [\log_2(m-a)]+1 \le [\log_2 m]+1$
  - latter : $T \le [\log_2 a]+2 \le [\log_2 m/2]+2 \le [\log_2 m]+1$

Union and Find operations

- Definition [Collapsing rule]：

  If j is a node on the path from i to its root and parent[i]≠root(i), then set parent[j] to root(i).

```
int Sets::CollasingFind(int i)
{//Find the root of the tree containing element i.
//Use of collapsing rule to collapse all nodes from i to the root.
        for (int r=i; parent[r]>=0; r=parent[r]);//find foot
        while (i!=r){//collapse
                int s=parent[i];
                parent[i]=r;
                i=s;
        }
        return r;
}
```

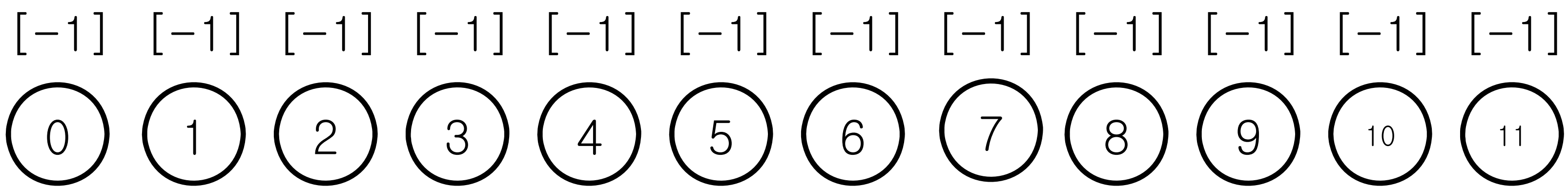
Union and Find operations

- Analysis of WeightedUnion and CollapsingFind
  - Use of the collapsing rule roughly doubles the time for an individual find.
  - It reduce worst case time over a sequence of finds.
  - The worst-case complexity of processing a swquence of unions and finds is stated in Lemma 5.6.

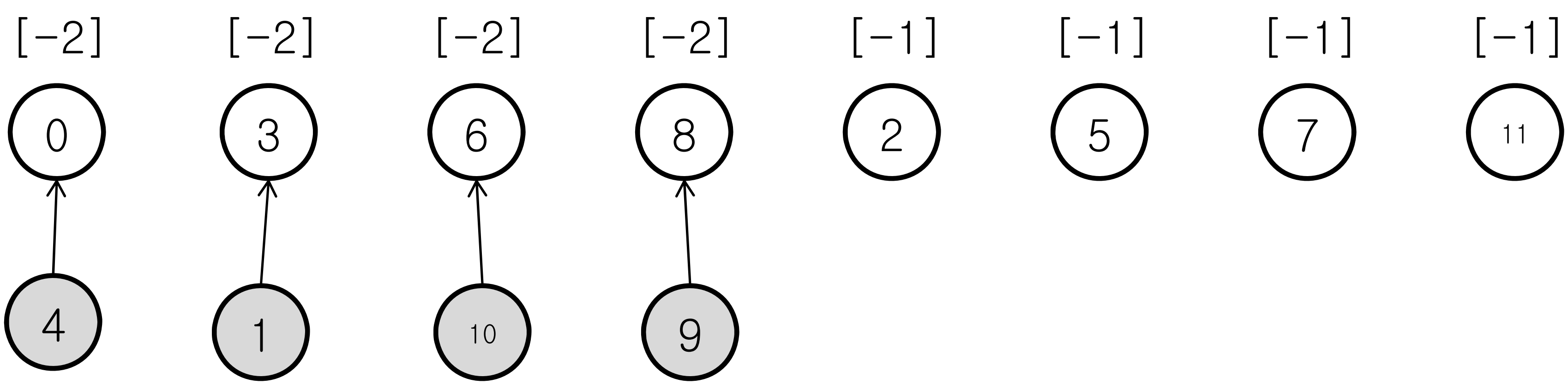
- Lemma 5.6[Tarjan and Van Leeuwed]: Assume that we start with a forest of trees, each having one node. Let T(f,u) be the maximum time required to process any intermixed sequence of f finds and u unions. Assum that $u \geq n/2$.

  Then $k_1(n+f\alpha(f+n,n)) \leq T(f,u) \leq k_2(n+f\alpha(f+n,n))$ for some positive constants k_1 and k_2.

Application to Equivalence Class

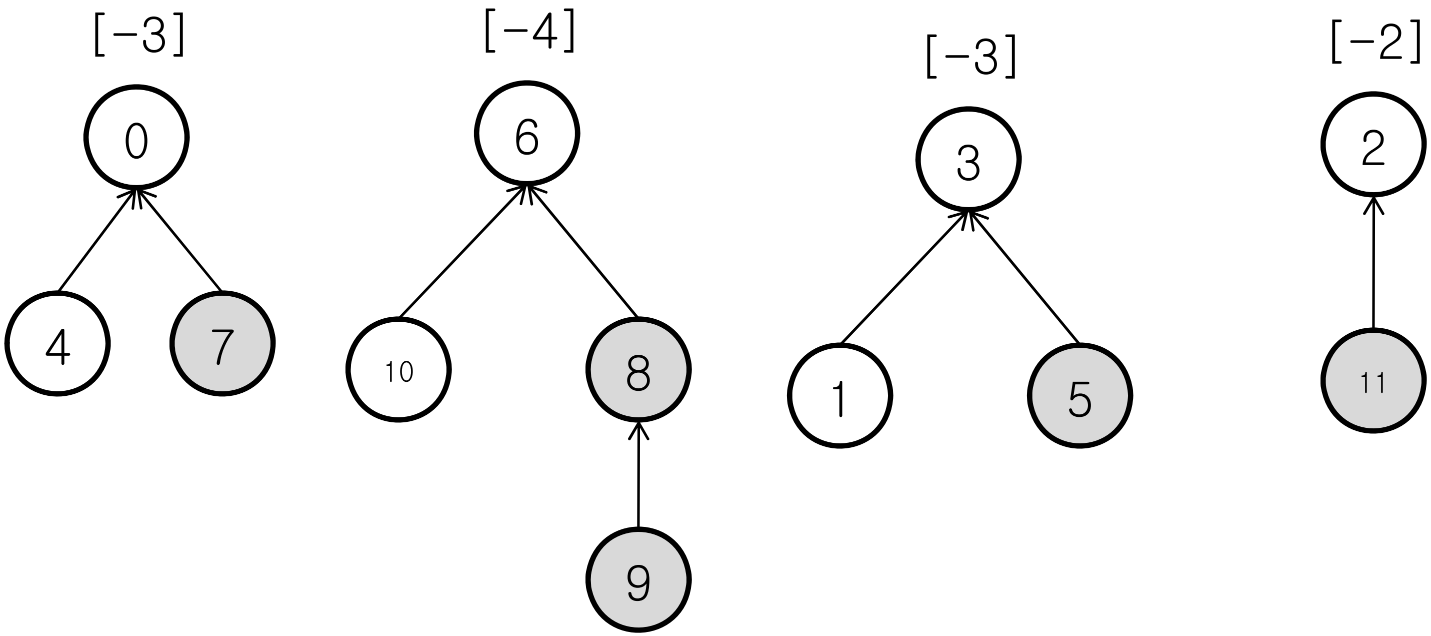


Initial Trees



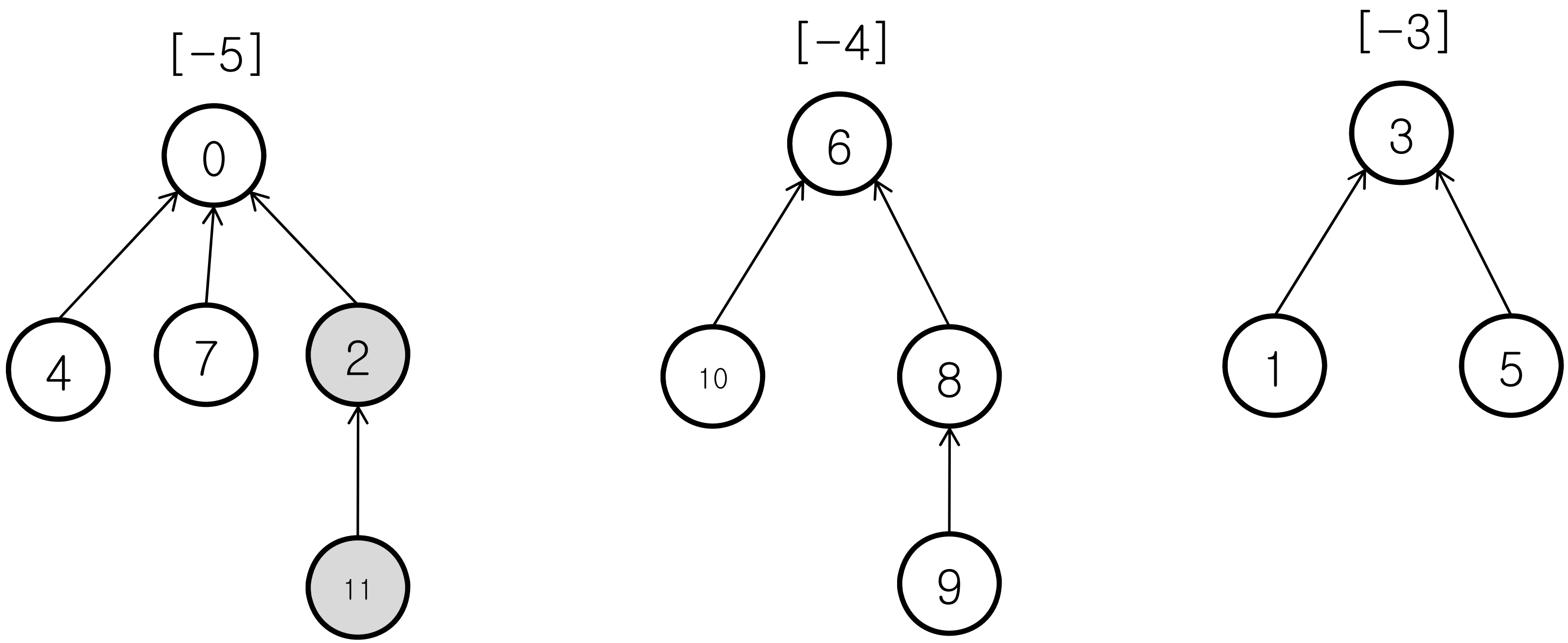
Height−2 trees following 0≡4, 3≡1, 6≡10, 8≡9

Application to Equivalence Class



Trees following 7≡4, 3≡5, 6≡8, 2≡11

Application to Equivalence Class



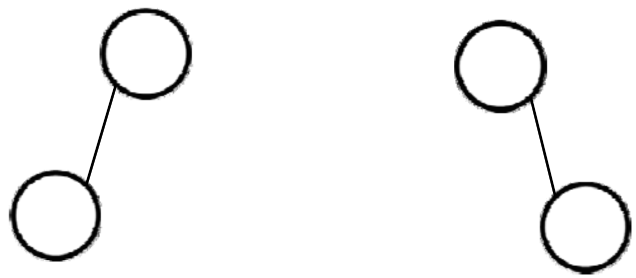
Trees following 11≡0

- 5.8 SELECTION TREES
- 5.9 FOREST
- 5.10 REPRESENTATION OF DISJOINT SETS
- 5.11 COUNTING BINARY TREES

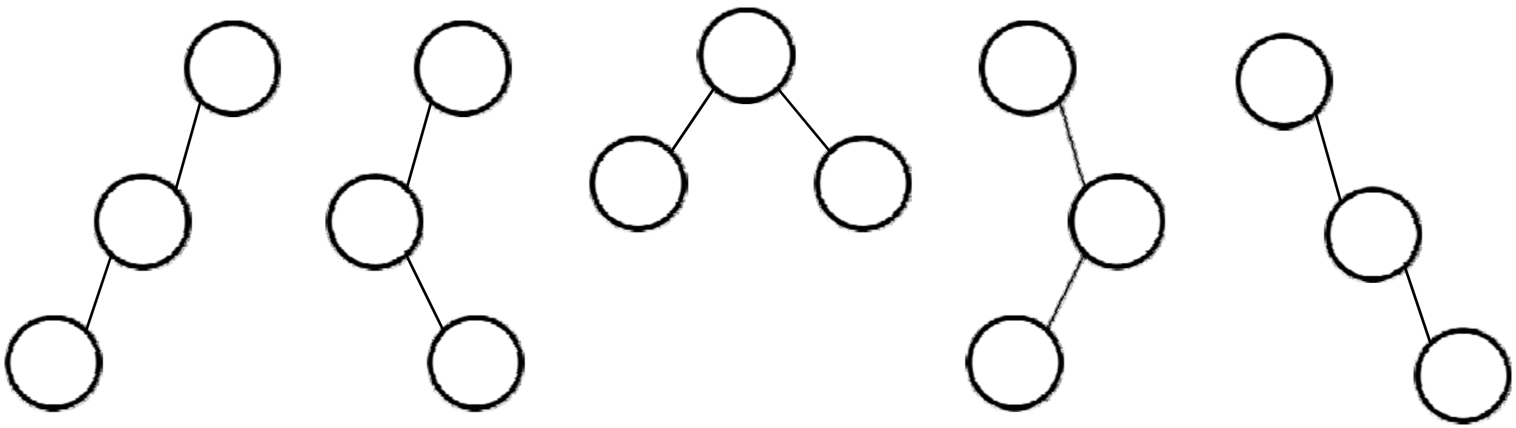
Counting binary trees

- Determine the number of distinct binary trees having n nodes.
  - Number of distinct permutations of the numbers from 1 through n obtainable by a stack.
  - Number of distinct ways of multiplying n+1 matrices.

Distinct Binary Trees



Distinct binary trees with n=2



Distinct binary trees with n=3

Stack Permutations

- Suppose we have the preorder and inorder sequence of the same binary tree
  - Preorder sequence :

    A B C D E F G H I
  - Inorder sequence :

    B C A E D G H F I

- A – root of the tree by VLR
- (B C) – left subtree by LVR
- (E D G H F I) – right subtree

Stack Permutations (cont.)

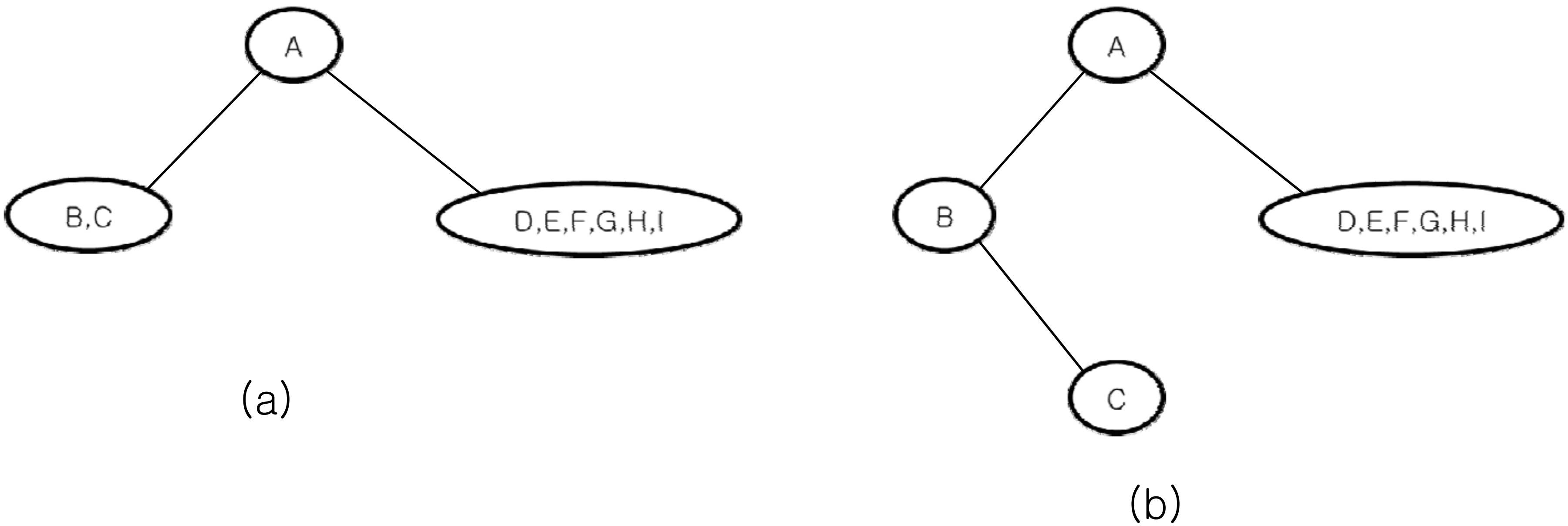
- Right in the preorder sequence
- B – next root
- No node precedes B in inorder :

  B has an empty left subtree,

  C is right subtree of B

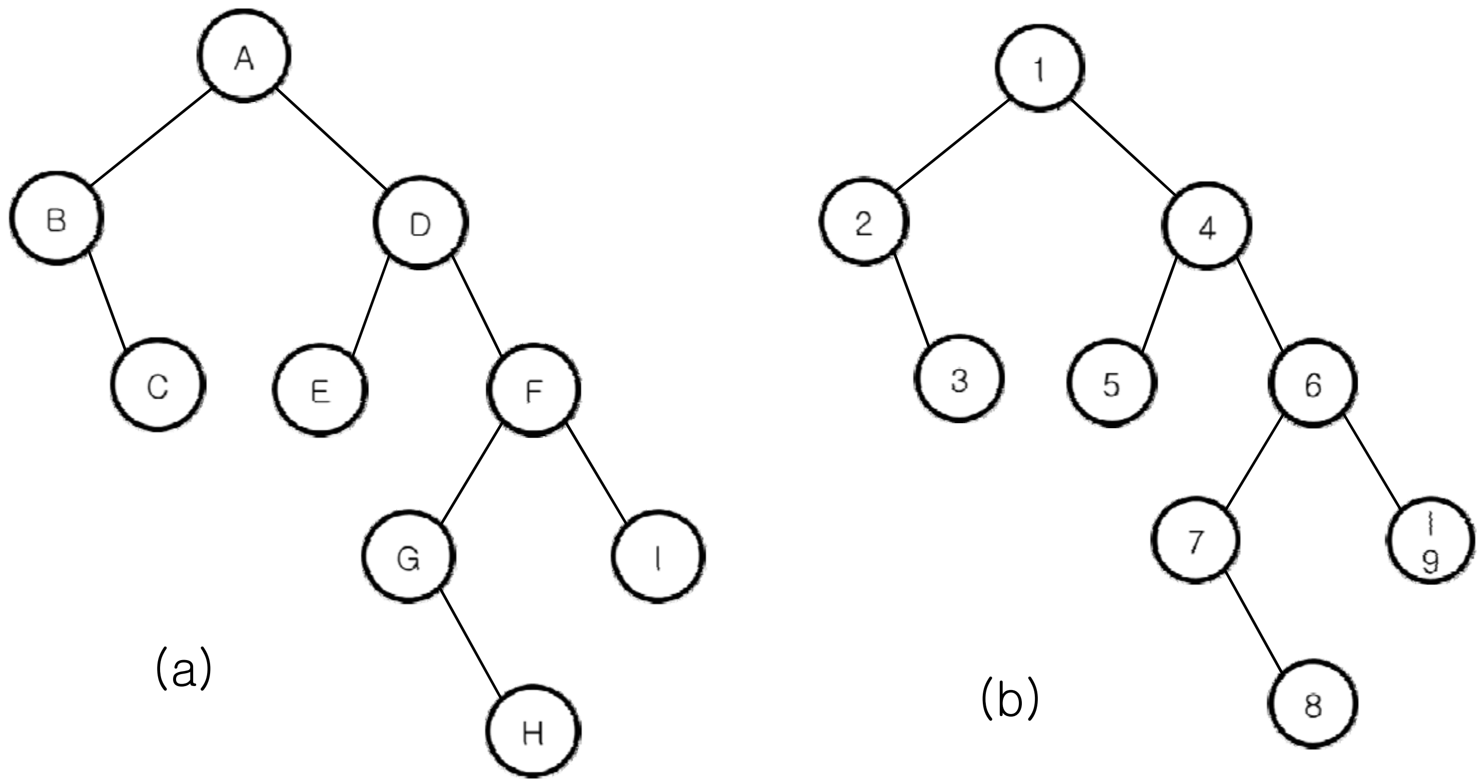
…………

Stack Permutations (cont.)



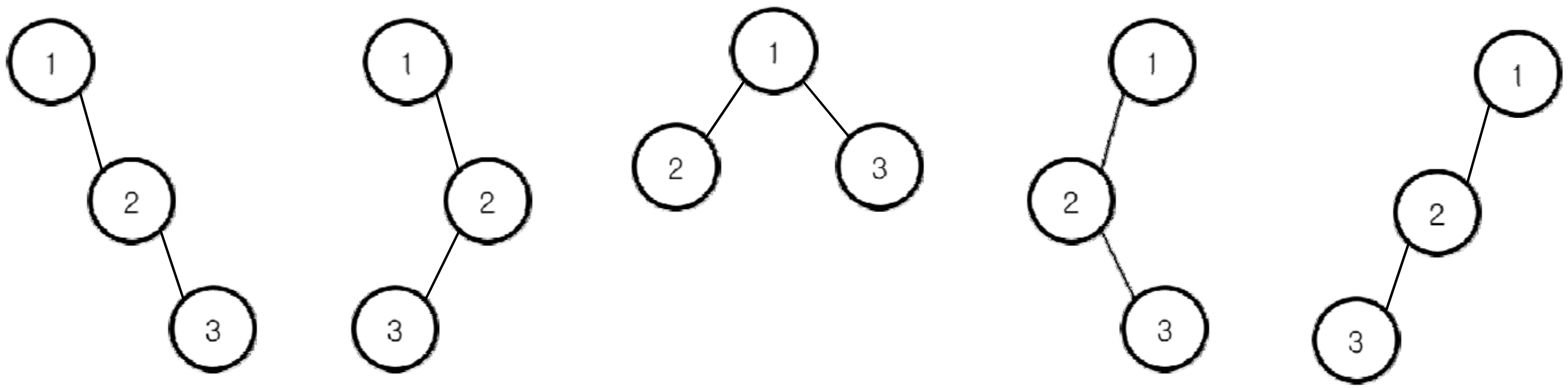
Constructing a binary tree from its inorder and preorder sequences

Stack Permutations (cont.)



Binary tree constructed from its inorder and preorder sequences

Stack Permutations (cont.)



Binary trees corresponding to five permutations

Stack Permutations (cont.)

- Number of distinct binary trees is equal to the number of distinct inorder permutations obtainable from binary trees having the preorder permutation, 1, 2, ⋯, n.

Stack Permutations (cont.)

- Distinct permutations obtainable by passing the numbers 1 through n though a stack and deleting in all possible ways = the number of distinct binary trees with n nodes.
- {1,2,3} possible permutation obtainable by a stack

  (1,2,3)(1,3,2)(2,1,3)(2,3,1)(3,2,1)
- (3,1,2) – impossible

Matrix Multiplication

- Compute the product of n matrices:

$$M_1 * M_2 * M_3 * \cdots * M_n$$

- n=3:

$$(M_1 * M_2) * M_3$$
$$M_1 * (M_2 * M_{3)}$$

- n=4:

$$((M_1 * M) * M_3) * M_4$$
$$((M_1 * M) * M_3)\ ) * M_4$$
$$M_1 * ((M * M_3) * M_4)$$
$$(M_1 * (M) * (M_3 * M_4)$$
$$((M_1 * M) * (M_3 * M_4)\ )$$

Matrix Multiplication (cont.)

- b_n : different ways to compute the product of n matrices.
- $b_2=1$, $b_3=2$, $b_4=5$
- M_{ij}, $i \le j$: product $M_i * M_{i+1} * \cdots * M_j$.
- M_{1n} : $M_{1i} * M_{i+1,n}$, $1 \le i \le n$
- Distinct ways to obtain M_{1i}:b_i, $M_{i+1,n}$:b_{n-i},

$$b_n = \sum_{i=0}^{n-1} b_i b_{n-i-1}, n \ge 1, \text{and } b_0 = 1$$

Number of Distinct Binary Trees

- Solve the recurrence of

$$b_n = \sum_{i=0}^{n-1} b_i b_{n-i-1}, n \geq 1, \text{and } b_0 = 1$$

- Begin we let

$B(x) = \sum_{i \geq 0} b_i x^i$ – generating function for the number of binary trees.

$xB^2(x) = B(x) - 1$ –by the recurrence relation

$B(x) = \dfrac{1 - \sqrt{1-4x}}{2x}$ –$B(0)=b_0=1$

$$B(x) = \frac{1}{2x}\left(1 - \sum_{n \geq 0} \binom{1/2}{n} (-4x)^n\right) = \sum_{m \geq 0} \binom{1/2}{m+1} (-1)^m 2^{2m+1} x^m$$

– by binomial theorem to expand $(1-4x)^{1/2}$

Number of Distinct Binary Trees(cont.)

- Comparing

$$B(x) = \sum_{i \geq 0} b_i x^i$$

$$B(x) = \frac{1}{2x}\left(1 - \sum_{n \geq 0} \binom{1/2}{n} (-4x)^n\right) = \sum_{m \geq 0} \binom{1/2}{m+1} (-1)^m 2^{2m+1} x^m$$

b_n = coefficient of x^n

$$b_n = \frac{1}{n+1}\binom{2n}{n}$$

$$b_n = O\left(\frac{4^n}{n^{\frac{3}{2}}}\right)$$