Course : Signal and Systems

Textbook : Signal & Systems, 2nd ed.
1997, Prentice-Hall

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Grading :
- Exam I (30%)
- Exam II (30%)
- Exam III (30%)
- HW, Attendence } 10%

- A : 30%
- B : 30%
- C : 30%
- D, E, F : 10%

Exams : Closed Book
- Chap. 1, 2, 3 : EXAM I
- Chap. 4, 5, 6 : EXAM II
- Chap. 7, (8), (9), 10, (11) : EAM III

Home Work : Solution manual

Chapter 1.

§1.1 Continuous-Time and Discrete-Time Signals

Two Basic Types of Signals

- continuous-time signals $x(t)$ (e.g.) Fig. 1.4 (a)
- discrete-time signals $x[n]$ (e.g.) Fig. 1.4 (b)

(e.g.) The weekly Dow-Jones Stock Market Index.
Fig. 1.6 : Inherently discrete

(e.g.) Successive samples of a continuous-time signal :

$$x[n] = x(nh)$$

Signal Energy and Power

- Total Energy over $[t_1, t_2]$:

$$\int_{t_1}^{t_2} |x(t)|^2 dt \quad , \quad \sum_{n=n_1}^{n_2} |x[n]|^2$$

- Total Energy over $(-\infty, \infty)$:

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \geq 0$$

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2 \geq 0$$

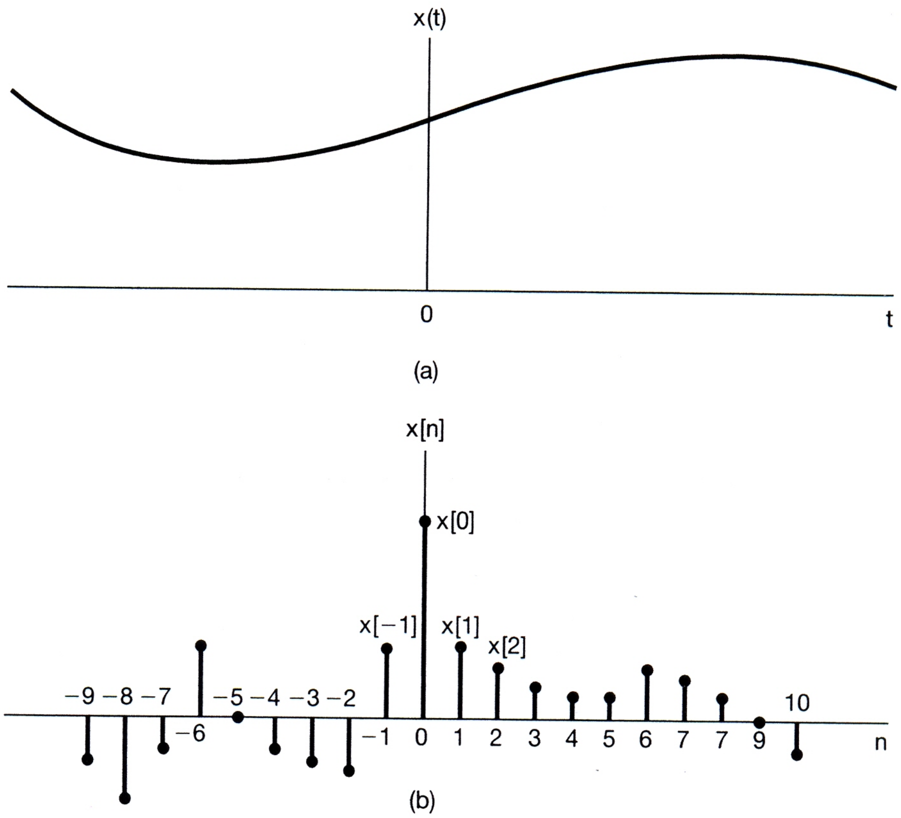


Figure 1.7 Graphical representations of (a) continuous-time and (b) discrete-time signals.

autopilot or speech and music for an audio system. Also, pictures in newspapers—or in this book, for that matter—actually consist of a very fine grid of points, and each of these points represents a sample of the brightness of the corresponding point in the original image. No matter what the source of the data, however, the signal $x[n]$ is defined only for integer values of n. It makes no more sense to refer to the $3\frac{1}{2}$th sample of a digital speech signal than it does to refer to the average budget for a family with $2\frac{1}{2}$ family members.

Throughout most of this book we will treat discrete-time signals and continuous-time signals separately but in parallel, so that we can draw on insights developed in one setting to aid our understanding of another. In Chapter 7 we will return to the question of sampling, and in that context we will bring continuous-time and discrete-time concepts together in order to examine the relationship between a continuous-time signal and a discrete-time signal obtained from it by sampling.

1.1.2 Signal Energy and Power

From the range of examples provided so far, we see that signals may represent a broad variety of phenomena. In many, but not all, applications, the signals we consider are directly related to physical quantities capturing power and energy in a physical system. For example, if $v(t)$ and $i(t)$ are, respectively, the voltage and current across a resistor with resistance R, then the instantaneous power is

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t). \tag{1.1}$$

– Time-averaged Power over $(-\infty, \infty)$

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt \quad \geq 0$$

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x[n]|^2 \quad \geq 0$$

Three Classes of signals

- signals with finite energy, i.e. $E_\infty < \infty \Rightarrow P_\infty = 0$
  (e.g.) $x(t) = e^{-t}$
- signals with finite average power, i.e. $P_\infty < \infty \Rightarrow E_\infty = \infty$
  (e.g.) $x(t) = \sin t$
- signals for which neither P_∞ nor E_∞ are finite

  (e.g.) $x(t) = t$

§1.2 Transformations of the independent variable

$$\underline{x(\alpha t + \beta)} \begin{cases} \alpha : \text{time scaling} \\ \beta : \text{time shift} \end{cases}$$

- linearly stretched if $|\alpha| < 1$
- linearly compressed if $|\alpha| > 1$
- reversed in time if $\alpha < 0$
- advanced in time if $\beta > 0$
- delayed in time if $\beta < 0$

(e.g.) Fig. 1.8 – 1.12, pp. 8–9

Periodic Continuous-Time Signals

- $x(t)$ is periodic with period T if $\exists T > 0 \ni$
  $x(t) = x(t+T), \ \forall t \in (-\infty, \infty)$.

- The smallest positive value of such T's is called the fundamental period T_0 of $x(t)$.

- $x(t)$ is an aperiodic signal if it is not periodic.
  (nonperiodic)
  (cf) almost periodic : (ex) $\sin 2t + \sin \sqrt{2} t$

Periodic Discrete-Time Signals

- $x[n]$ is periodic with period N if $\exists$ a positive integer $N \ni$
  $x[n] = x[n+N], \ \forall n = 0, \pm 1, \pm 2, \dots$

§1.3 Exponential and Sinusoidal Signals

Continuous-Time Complex Exponential Signals $x(t) = Ce^{at}$

- Real Exponential Signals if C, a are real

- Periodic Complex Exponential Signals : $x(t) = e^{j\omega_0 t}$

(1) $e^{j\omega_0 t} = e^{j\omega_0 (t+T)} \Rightarrow e^{j\omega_0 T} = 1$

$$\Rightarrow T = \frac{2\pi k}{|\omega_0|} \quad \Rightarrow T_0 = \frac{2\pi}{|\omega_0|}$$

$\Rightarrow |\omega_0|$ is called the fundamental frequency.

(2) $\begin{cases} P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |e^{j\omega_0 t}|^2 dt = 1 \\ E_\infty = \infty \end{cases}$ (1.32)

(3) A <u>Set of harmonically related Complex exponentials</u> is a set of periodic exponentials, all of which are periodic with a common period T_0.

$e^{j\omega t} = e^{j\omega(t+T_0)} \Rightarrow e^{j\omega T_0} = 1 \Rightarrow \omega = k\omega_0$ where $\omega_0 \triangleq \frac{2\pi}{T_0}$

$\{ \phi_k(t) \triangleq e^{jk\omega_0 t},\ k = 0, \pm 1, \pm 2, \cdots \}$

(4) $\phi_k(t)$: the kth harmonic

$\begin{cases} \text{fundamental freq. } |k|\omega_0 \\ \text{fundamental period } \frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|} \end{cases}$

– General Complex Exponential signals

$C = |C| e^{j\theta}$, $a = r + j\omega_0$

$\Rightarrow x(t) = C e^{at}$

$= |C| e^{rt} \{ \cos(\omega_0 t + \theta) + j \sin(\omega_0 t + \theta) \}$

See (Fig. 1.23)

Discrete-Time Complex Exponential Signals

$$x[n] = C\alpha^n \text{ or } Ce^{\beta n} \text{ with } \alpha = e^{\beta}$$

– Real Exponential Signals

(Fig. 1.24)

– General Complex Exponential Signals

$$C = |C|e^{j\theta}, \quad \alpha = |\alpha|e^{j\omega_0}$$

$$\Rightarrow x[n] = C\alpha^n = \underbrace{|C||\alpha|^n}_{|C|e^{\beta n}}\{\cos(\omega_0 n+\theta) + j\sin(\omega_0 n+\theta)\}$$

(Fig. 1.26)

Periodic Properties of Discrete-Time Complex Exponential Signals

$$x[n] = e^{j\omega_0 n}$$

① $x[n] = x[n+N]$

$\Rightarrow e^{j\omega_0 N} = 1 \Leftrightarrow \omega_0 N = 2\pi m$

$\Rightarrow$ Periodic only if $\omega_0 = 2\pi m/N$ for some $N>0$ and m.

$\Rightarrow$ The fundamental period N_0 of $x[n] = e^{j\left(\frac{2\pi m}{N}\right)n}$ is N ($\frac{2\pi m}{N} = \omega_0$)

if $\omega_0 \neq 0$ and if N and m have no common factors.

That is, $N_0 = N = \dfrac{2\pi m}{\omega_0}$

(e.g.) $x[n] = \cos(2\pi n/12)$: periodic

$x[n] = \cos(n/6)$: not periodic

② $e^{j\omega_0 n} = e^{j(\omega_0 + 2\pi)n}$

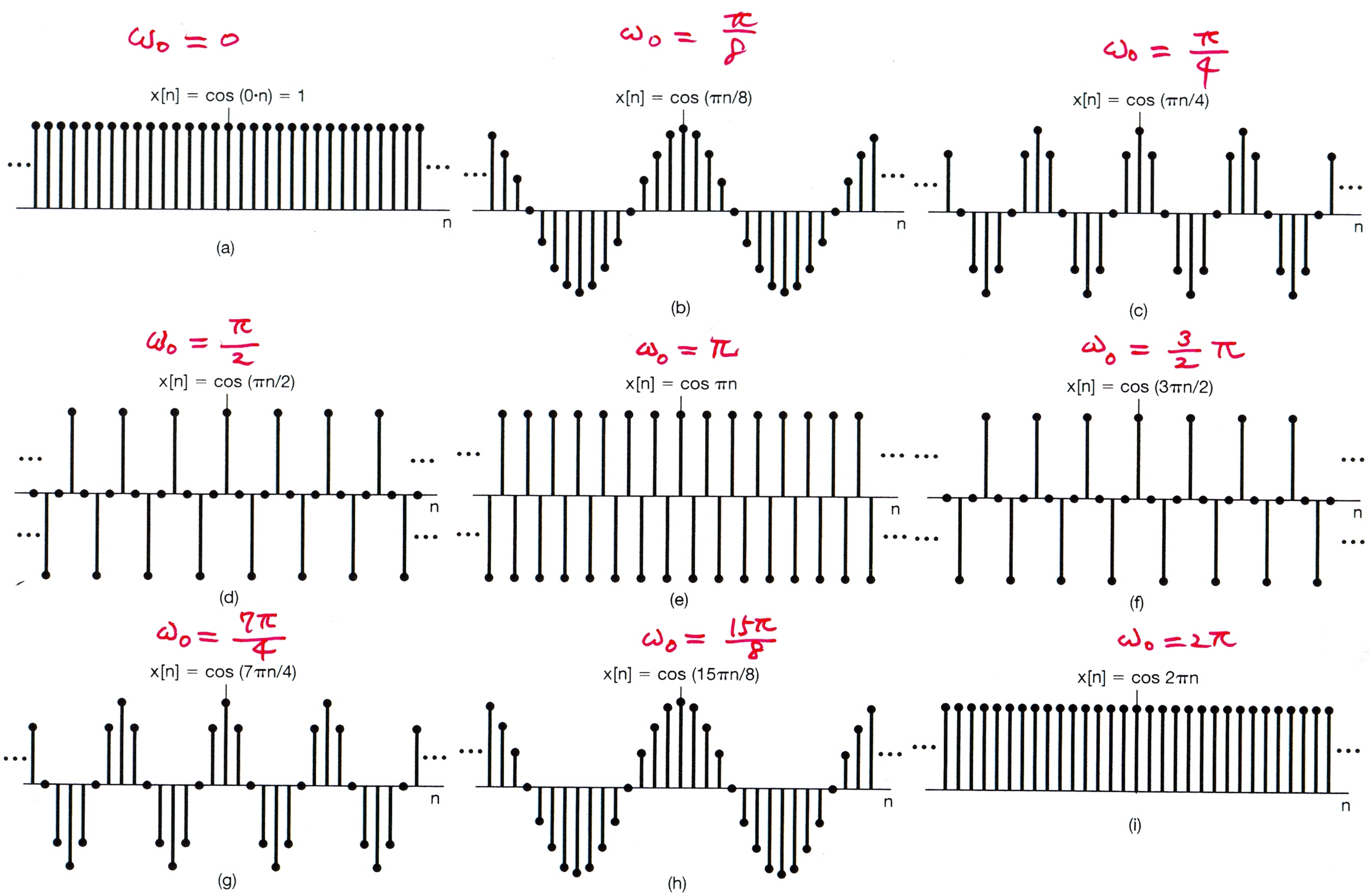
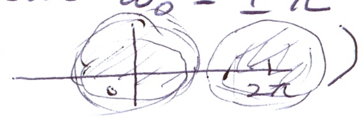


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

② $e^{j\omega_0 n} = e^{j(\omega_0 + 2\pi)n}$ ⟹ The low-frequency discrete-time exponentials have values of ω_0 near $0, 2\pi$, and any other even multiples of π, while the high frequencies are located near $\omega_0 = \pm\pi$ and other odd multiples of π (∵ Consider $e^{\pm j\omega_0 n}$ and )

According to eq. (1.56), the signal $e^{j\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a rational number and is not periodic otherwise. These same observations also hold for discrete-time sinusoids. For example, the signals depicted in Figure 1.25(a) and (b) are periodic, while the signal in Figure 1.25(c) is not.

Using the calculations that we have just made, we can also determine the fundamental period and frequency of discrete-time complex exponentials, where we define the fundamental frequency of a discrete-time periodic signal as we did in continuous time. That is, if $x[n]$ is periodic with fundamental period N, its fundamental frequency is $2\pi/N$. Consider, then, a periodic complex exponential $x[n] = e^{j\omega_0 n}$ with $\omega_0 \neq 0$. As we have just seen, ω_0 must satisfy eq. (1.56) for some pair of integers m and N, with $N > 0$. In Problem 1.35, it is shown that if $\omega_0 \neq 0$ and if N and m have no factors in common, then the fundamental period of $x[n]$ is N. Using this fact together with eq. (1.56), we find that the fundamental frequency of the periodic signal $e^{j\omega_0 n}$ is

$$\frac{2\pi}{N} = \frac{\omega_0}{m}. \tag{1.57}$$

Note that the fundamental period can also be written as

$$N = m\left(\frac{2\pi}{\omega_0}\right). \tag{1.58}$$

These last two expressions again differ from their continuous-time counterparts. In Table 1.1, we have summarized some of the differences between the continuous-time signal $e^{j\omega_0 t}$ and the discrete-time signal $e^{j\omega_0 n}$. Note that, as in the continuous-time case, the constant discrete-time signal resulting from setting $\omega_0 = 0$ has a fundamental frequency of zero, and its fundamental period is undefined.

TABLE 1.1 Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

*Assumes that m and N do not have any factors in common.

To gain some additional insight into these properties, let us examine again the signals depicted in Figure 1.25. First, consider the sequence $x[n] = \cos(2\pi n/12)$, depicted in Figure 1.25(a), which we can think of as the set of samples of the continuous-time sinusoid $x(t) = \cos(2\pi t/12)$ at integer time points. In this case, $x(t)$ is periodic with fundamental period 12 and $x[n]$ is also periodic with fundamental period 12. That is, the values of $x[n]$ repeat every 12 points, exactly in step with the fundamental period of $x(t)$.

③ $$\begin{cases} P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| e^{j\omega_0 n} \right|^2 = 1 \\ E_\infty = \infty \end{cases}$$

④ A set of hamonically related complex exponentials.

$$e^{j\omega(n+N)} = e^{j\omega n} \Rightarrow e^{j\omega N} = 1 \Rightarrow \omega N = 2\pi m$$

$$\Rightarrow \omega = \frac{2\pi}{N} k, \quad k = 0, \pm 1, \cdots$$

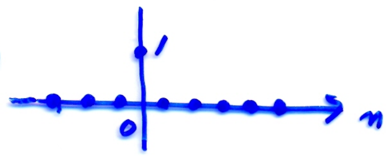
$$\Rightarrow \left\{ \phi_k[n] \triangleq e^{j\frac{2\pi}{N}kn}, \quad k = 0, 1, \cdots, (N-1) \right\}$$

Also, see Fig. 1.27, p. 27

§1.4 The unit impulse and unit step functions

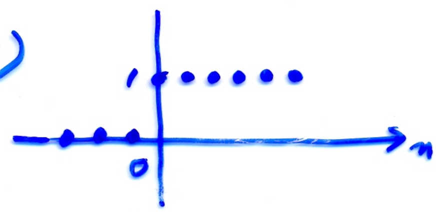
The discrete-time unit impulse (or unit sample)

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$ (Fig. 1.28)



The discrete-time unit step

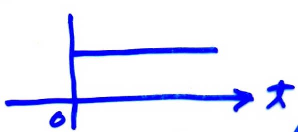
$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$ (Fig. 1.29)



$$\Rightarrow \begin{cases} \delta[n] = u[n] - u[n-1] \\ u[n] = \sum_{m=-\infty}^{n} \delta[m] = \sum_{k=0}^{\infty} \delta[n-k] \end{cases}$$

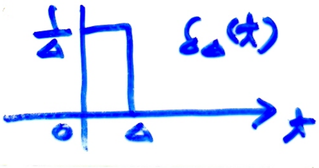
The continuous-time unit step function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



The continuous-time unit impulse (Dirac delta func)

$$\delta(t) \triangleq \lim_{\Delta \to 0} \delta_\Delta(t)$$

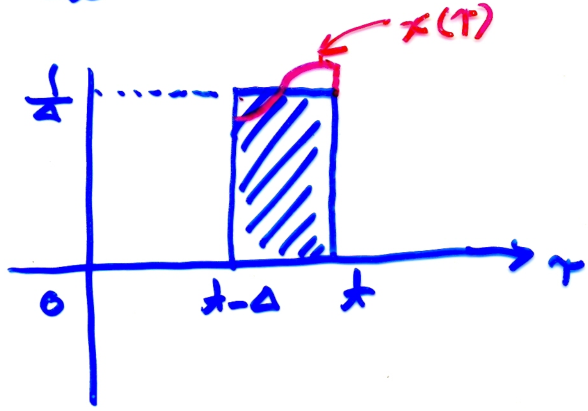


Theorem

Let $t>0$. Suppose that x is integrable on $(-\infty,\infty)$, and continuous at t. Then,

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t) \quad \left(\text{i.e. } x(t) * \delta(t) = x(t)\right)$$

<Proof>



$$\int_{-\infty}^{\infty} x(\tau)\,\delta_\Delta(t-\tau)\,d\tau = \frac{1}{\Delta}\int_{t-\Delta}^{t} x(\tau)\,d\tau \qquad (1)$$

By the mean value theorem for integrals, $\exists$ $t_0 \in [t-\Delta, t] \ni$

$$\int_{t-\Delta}^{t} x(\tau)\,d\tau = \Delta\, x(t_0) \qquad (2)$$

By (1) and (2),

$$\int_{-\infty}^{\infty} x(\tau)\,\delta_\Delta(t-\tau)\,d\tau = x(t_0) \qquad (3)$$

Take the limit of both sides of eq. (3) as $\Delta \to 0^+$. Then,

$$\begin{cases} \lim\limits_{\Delta\to 0^+} t_0 = t & \text{since } t_0 \in [t-\Delta, t] \\ \lim\limits_{\Delta\to 0^+} x(t_0) = x(t) & \text{since } x \text{ is continuous at } t \\ \lim\limits_{\Delta\to 0^+} \delta_\Delta(t-\tau) = \delta(t-\tau) & \text{by definition.} \end{cases}$$

///

$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = \lim_{\Delta \to 0} \int_{-\infty}^{t} \delta_\Delta(\tau)\,d\tau = \begin{cases} 0 & \text{if } t \le 0 \\ 1 & \text{if } t > 0 \end{cases}$$

$$\Rightarrow \quad u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = \int_{0}^{\infty} \delta(t-\tau)\,d\tau \qquad \text{(Fig. 1.37)} \quad \text{(Fig. 1.38)}$$

(Theorem, p.9 is not applicable)

§1.5 Continuous-time and Discrete-time systems

Process — A system can be viewed as a process in which input signals are transformed by the system

$$x(t) \longrightarrow \boxed{\text{Continuous-time System } H} \longrightarrow y(t): \quad y(t) = Hx(t)$$

$$(H: x(t) \rightarrow y(t))$$

$$x[n] \longrightarrow \boxed{\text{Discrete-time system } H} \longrightarrow y[n]: \quad y[n] = Hx[n]$$

$$(H: x[n] \rightarrow y[n])$$

(1.85) $\dot{y}(t) + a\,y(t) = b\,x(t)$: ODE

(1.89) $y[n] + a\,y[n-1] = b\,x[n]$: Difference Equation

<u>Example 1.11</u>

A digital simulation of an ODE $\Rightarrow$ diff. eq.

§1.6 Basic System Properties

<u>The Identity System</u>

$y(t) = x(t)$

$y[n] = x[n]$

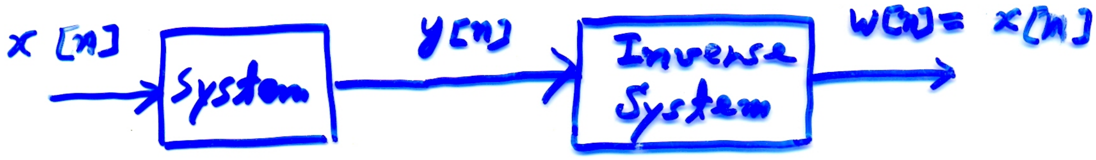
<u>A Memoryless System</u>

$y[n] = f(x[n], n)$, $\quad y(t) = f(x(t), t)$

Delay

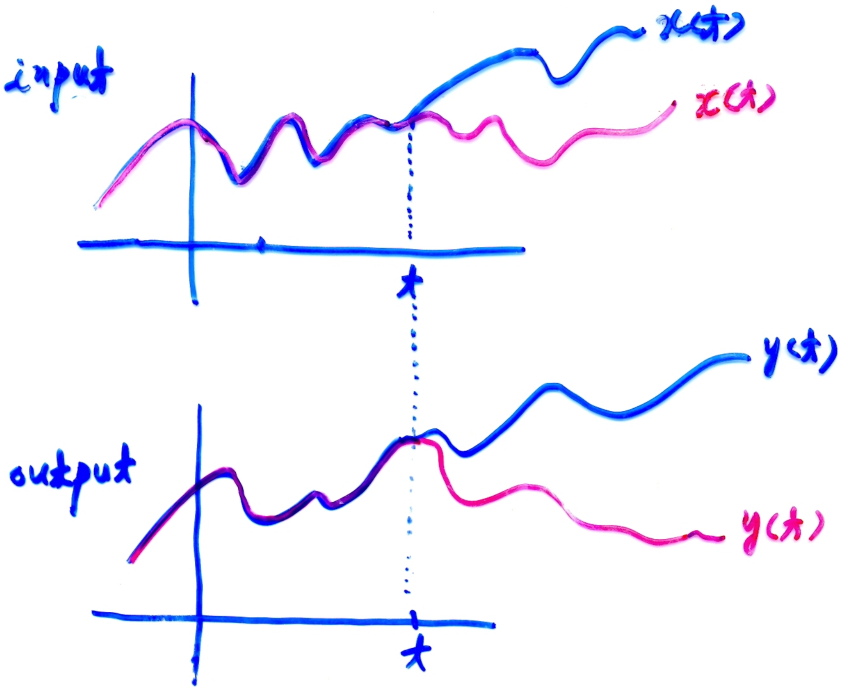
$y[n] = x[n-1]$, $y(t) = x(t-a)$, $a>0$

Invertibility and Inverse Systems



Causality

The output of a causal system at any time $\underline{t}$ depends only on values of the input on $\underline{(-\infty, t]}$



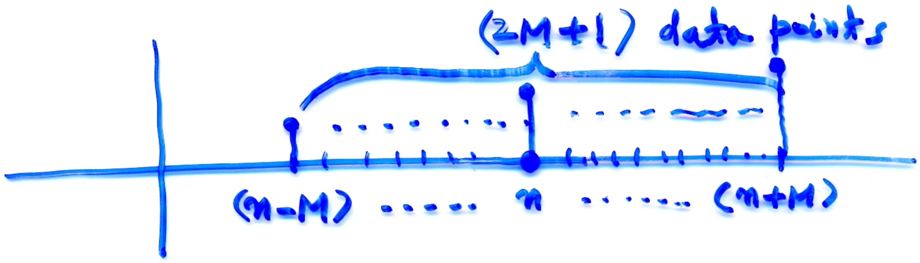
$$\begin{cases} x \equiv x & \text{on } (-\infty, t] \\ x \not\equiv x & \text{on } [t, +\infty) \end{cases}$$

$$\Downarrow$$

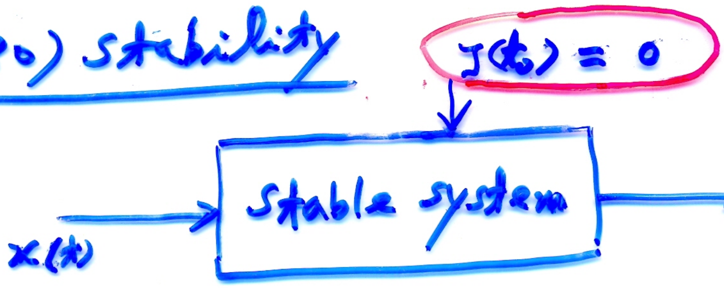
$$y \equiv y \quad \text{on } (-\infty, t]$$

- An Example of a noncausal averaging system

$$y[n] = \frac{1}{2M+1}\sum_{k=-M}^{+M} x[n-k]$$



(BIBO) stability



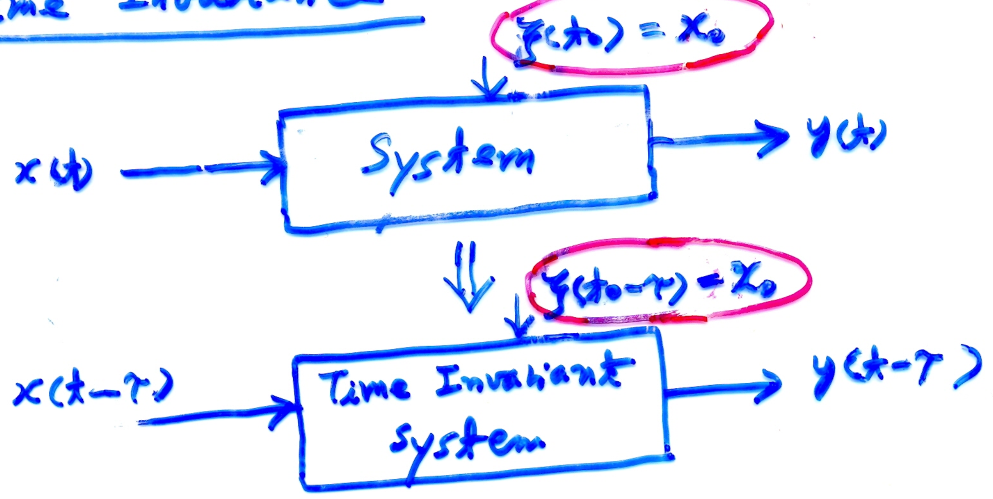
relaxed initially at $t = t_0$

$$|x(t)| \le B_x \; \forall t \in [t_0, \infty) \Rightarrow |y(t)| \le B_y, \; \forall t \in [t_0, \infty)$$

- unstable systems : $y(t) = t\, x(t)$
- stable systems : ① $y(t) = \sin x(t)$, $y(t) = x(t-a)$

  ② $\begin{cases} \dot{x}_1 = -x_1 + x \\ \dot{x}_2 = x_2 \end{cases}$, $y = x_1 + x_2$. $(x_2(0) = 0)$

Time Invariance



- Time-Varying systems : $y[n] = n\, x[n]$ (Example 1.15)

  Let $y_1[n] = n\, x[n]$ and $y_2[n] = n\, x[n-N]$.

  Then, $y_1[n-N] = [n-N]\; x[n-N]$

  Hence, $y_2[n] \ne y_1[n-N]$.

Superposition Property

$$a x_1(t) + b\, x_2(t) \longrightarrow a y_1(t) + b y_2(t)$$
$$a x_1[n] + b\, x_2[n] \longrightarrow a y_1[n] + b y_2[n]$$

in case of zero-state response (i.e. initially rests)

- Example 1.20, p.55 : $y[n] = 2x[n] + 3$
- Continuous-Time Systems :

$$\ddot{y}(t) + a\dot{y}(t) + b\,y(t) = x(t), \quad \dot{y}(0) = y(0) = 0$$

- Discrete-Time Systems :

$$y[n] + a\,y[n-1] + b\,y[n-2] = x[n], \quad y[-1] = y[-2] = 0$$

- The system is <u>linear</u> if the above property holds.

HW#1

3, 10, 11, 17, 18, 26, 27, 28, 30, 32, 34, 44

Chapter 2. Linear TI Systems

§2.1 Discrete-Time LTI Systems: The Convolution Sum

The representation of Discrete-Time Signals in terms of Impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \qquad (2.2)$$

See Fig. 2.1, p. 76

Eq. (2.2) is called the "shifting property" of the discrete-time unit impulse.