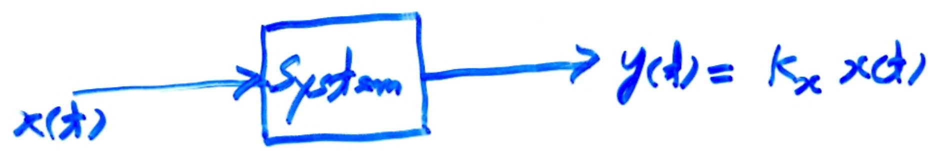
Chapter 3. Fourier Series Representation of Periodic Signals

§3.0, §3.1 — See the textbook

§3.2 The Response of LTI Systems to Complex Exponentials

<u>Eigenfunction and Eigenvalue</u>



$$\Rightarrow \begin{cases} x: & \text{an eigenfunction of the system} \\ K_x: & \text{the system's eigenvalue} \end{cases}$$

<u>Eigenfunctions of Continuous-time LTI systems</u>

Let $x(t) \triangleq e^{st}$. Then,

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = H(s) e^{st},$$

where

$$H(s) \triangleq \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \qquad (3.6)$$

$$\Rightarrow \quad y(t) = \sum_{k=-\infty}^{\infty} a_k H(s_k) e^{s_k t} \quad \text{if } x(t) = \sum_{k=-\infty}^{\infty} a_k e^{s_k t} \quad (3.13)$$

<u>Remark</u>

(3.6) and (3.13) are well-posed if $|e^{st}| \le B_x, \forall t \in \mathbb{R}$ and the system is stable

Eigenfunctions of Discrete-Time LTI Systems

Let $x[n] = z^n$. Then,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, z^{(n-k)} = H(z)\, z^n,$$

where

$$H(z) \triangleq \sum_{k=-\infty}^{\infty} h[k]\, z^{-k} \qquad (3.10)$$

$$\Rightarrow \quad y[n] = \sum_k a_k H(z_k) z_k^n \quad \text{if} \quad x[n] = \sum_k a_k z_k^n \qquad (3.15)$$

((3.10) and (3.15) are well posed if the system is stable) $|z^n| \le B_x,\ \forall n \in \mathbb{Z}$ and

Question

How broad a class of signals could be represented as a linear combination of complex exponentials?

<Example 3.1> p. 185

① Suppose that $y(t) = x(t-3)$. Then,

$$y(t) = e^{j2(t-3)} = e^{-j6} e^{j2t} \quad \text{if} \quad x(t) = e^{j2t}.$$

$$\Rightarrow \quad H(2j) = e^{-j6}.$$

Alternatively, this can be shown using the fact that

$$h(t) = \delta(t-3)$$

and hence that

$$H(s) = \int_{-\infty}^{\infty} \delta(\tau-3)\, e^{-s\tau}\, d\tau = e^{-3s}$$

② For $x(t) = \cos(4t) + \cos(9t)$, we have

$$y(t) = \frac{1}{2} e^{-j12} e^{j4t} + \frac{1}{2} e^{j12} e^{-j4t} + \frac{1}{2} e^{-j27} e^{j9t} + \frac{1}{2} e^{j27} e^{-j9t}$$

$$= \cos 4(t-3) + \cos 9(t-3)$$

since

$$x(t) = \frac{1}{2} e^{j4t} + \frac{1}{2} e^{-j4t} + \frac{1}{2} e^{j9t} + \frac{1}{2} e^{-j9t}$$

§3.3 Fourier Series Representation of Continuous-Time periodic Signals

Linear Combinations of Harmonically Related Complex Exponentials

If $x(t)$ has a Fourier series representation :

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} \quad (3.38),$$

then the coefficients (Fourier series) are given by

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_T x(t) e^{-jk(2\pi/T)t} dt \quad (3.39)$$

Optimal Approximation of a given periodic signal

To approximate $x(t)$ by $x_N(t) \triangleq \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ (3.47),

find a_k's minimizing

$$E_N \triangleq \int_T |x(t) - x_N(t)|^2 dt . \quad (3.49)$$

$$\Rightarrow \quad a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} \quad (3.50)$$

<proof>

Let $a_k \triangleq b_k + j c_k$. Then,

$$E_N = \int_T \Big\{ x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \Big\} \Big\{ \bar{x}(t) - \sum_{k=-N}^{N} \bar{a}_k e^{-jk\omega_0 t} \Big\} dt$$

$$= \int_T |x(t)|^2 dt + T \sum_{k=-N}^{N} (b_k^2 + c_k^2) - \sum_{k=-N}^{N} (b_k + j c_k) \int_T \bar{x}(t) e^{jk\omega_0 t} dt$$

$$- \sum_{k=-N}^{N} (b_k - j c_k) \int_T x(t) e^{-jk\omega_0 t} dt$$

Note that E_N is convex w.r.t. $z_k \triangleq [b_k \ c_k]^T$.

$\Rightarrow \dfrac{\partial E_N}{\partial b_k} = \dfrac{\partial E_N}{\partial c_k} = 0, \ k = 0, \pm 1, \dots, \pm N$ is the nec & suff.

conditions for minimal E_N.

$$\Rightarrow \begin{cases} b_k = \dfrac{1}{2T} \Big\{ \int_T \bar{x}(t) e^{jk\omega_0 t} dt + \int_T x e^{-jk\omega_0 t} dt \Big\} \\ c_k = \dfrac{j}{2T} \Big\{ \int_T \bar{x}(t) e^{jk\omega_0 t} dt - \int_T x(t) e^{-jk\omega_0 t} dt \Big\} \end{cases}$$

$$\Rightarrow a_k = b_k + j c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t}$$

$\square$

Convergence of the Fourier Series

If the Dirichlet conditions 1, 2, 3 are satisfied, then

(i) $x(t)$ equals its Fourier series representation, except at isolated values of t for which $x(t)$ is discontinuous.

(ii) At these points, the Fourier series converges to the average of the values on either side of the discontinuity.

Suppose : $x(t)$ is real. Then, $x^*(t) = x(t)$

(3.25) $\Rightarrow x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$ (3.25)

Replacing k by $-k$ in the summation, we have

$$x(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t},$$

which, by comparison with eq. (3.25), requires that $a_k = a_{-k}^*$, or equivalently, that

$$a_k^* = a_{-k}. \tag{3.29}$$

Note that this is the case in Example 3.2, where the a_k's are in fact real and $a_k = a_{-k}$.

To derive the alternative forms of the Fourier series, we first rearrange the summation in eq. (3.25) as

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}].$$

Substituting a_k^* for a_{-k} from eq. (3.29), we obtain

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}].$$

Since the two terms inside the summation are complex conjugates of each other, this can be expressed as

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\Re e\{a_k e^{jk\omega_0 t}\}. \tag{3.30}$$

If a_k is expressed in polar form as

$$a_k = A_k e^{j\theta_k},$$

then eq. (3.30) becomes

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\Re e\{A_k e^{j(k\omega_0 t + \theta_k)}\}.$$

That is,

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k). \tag{3.31}$$

Equation (3.31) is one commonly encountered form for the Fourier series of real periodic signals in continuous time. Another form is obtained by writing a_k in rectangular form as

$$a_k = B_k + jC_k,$$

where B_k and C_k are both real. With this expression for a_k, eq. (3.30) takes the form

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t]. \tag{3.32}$$

In Example 3.2 the a_k's are all real, so that $a_k = A_k = B_k$, and therefore, both representations, eqs. (3.31) and (3.32), reduce to the same form, eq. (3.28).

Bounded Variation

Let $I \triangleq [a,b]$. Let $P: a = t_0 < t_1 < \cdots < t_n = b$ be a partition of I. Any function $x: [a,b] \to \mathbb{R}$, for which

$$V(x) \triangleq \sup \left\{ \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| : P \text{ is a partition of } I \right\}$$

is finite, is said to be <u>a func. of bounded variation</u>, and $V(x)$ is said to be the total variation of x.

<u>Theorem 1.</u>: Let x be continuous on $I \triangleq (-\frac{T}{2}, \frac{T}{2}) \ni$ $x(-\frac{T}{2}) = x(\frac{T}{2})$. Let $\overset{x}{f}{}'$ be piecewise continuous. Then, the convergence of the Fourier series

$$(*) \qquad \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$\sum |a_k| < \infty$

to $x(t)$ on I is <u>absolute</u> and uniform w.r.t. t on I.

Furthermore, it is differentiable at each point t where $x^{\overset{\text{one}}{\prime\prime}}(t)$ exists. and

$$(*)' \qquad x'(t) = \sum_{k=-\infty}^{+\infty} jk\omega_0 a_k e^{jk\omega_0 t} \qquad \square$$

<u>Theorem 2</u>: Let x be piecewise continuous on I. Then, whether $(*)$ converges to x or not, the following equality is true for all $t \in I$.

$$(*)'' \qquad \int_{-\frac{T}{2}}^{t} x(\tau)\, d\tau = a_0\left(t + \frac{T}{2}\right) + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{a_k}{jk\omega_0} e^{jk\omega_0 t}$$

$\square$

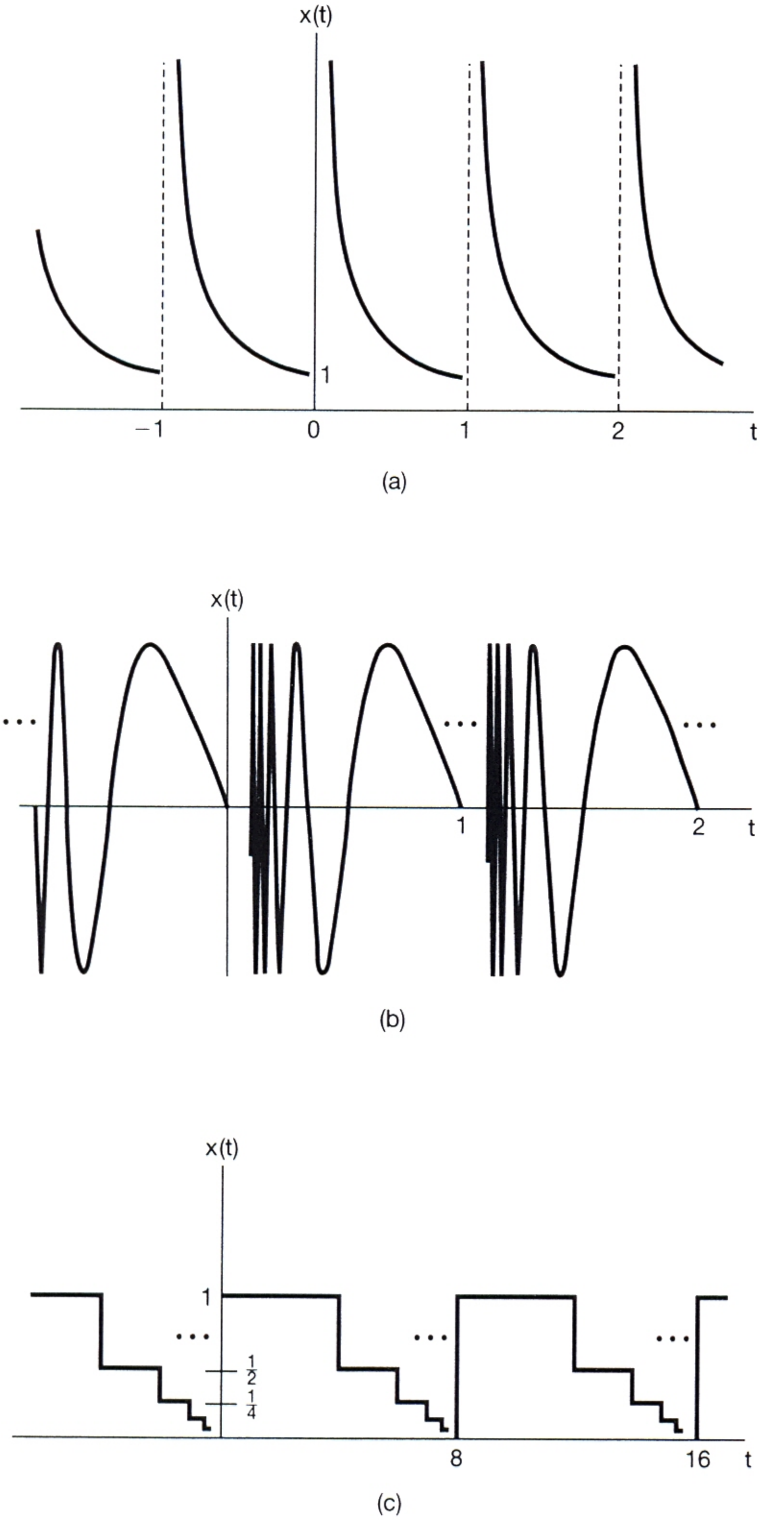
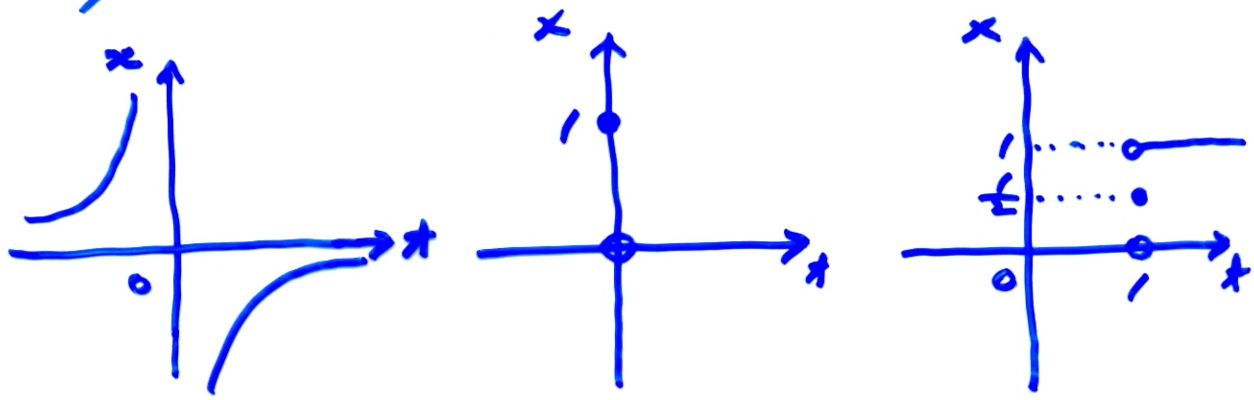


Figure 3.8 Signals that violate the Dirichlet conditions: (a) the signal $x(t) = 1/t$ for $0 < t \leq 1$, a periodic signal with period 1 (this signal violates the first Dirichlet condition); (b) the periodic signal of eq. (3.57), which violates the second Dirichlet condition; (c) a signal periodic with period 8 that violates the third Dirichlet condition [for $0 \leq t < 8$, the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor of 2; that is, $x(t) = 1$, $0 \leq t < 4$, $x(t) = 1/2$, $4 \leq t < 6$, $x(t) = 1/4$, $6 \leq t < 7$, $x(t) = 1/8$, $7 \leq t < 7.5$, etc.].

where except at the isolated points of discontinuity, at which the series converges to the average value of the signal on either side of the discontinuity. In this case the difference between the original signal and its Fourier series representation contains no energy, and consequently, the two signals can be thought of as being the same for all practical pur-

<Fact 1> If x is a fun. of bounded variation, it has only countable jump discontinuities.

(x has a jump-discontinuity at $t = c$ if $\exists\, x(c^+)$ and $x(c^-)$ but it is discontinuous at $t = c$)



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<Fact 2> continuous $\not\Rightarrow$ bounded variation

e.g. :
$$x(t) = \begin{cases} t \sin 1/t, & 0 < t \le 1,\ -1 \le t < 0 \\ 0, & t = 0 \end{cases}$$

Choose a partition of $[0,1] \ni t_i \triangleq \frac{1}{(i+\frac{1}{2})\pi}$

$$\Rightarrow |x(t_i) - x(t_{i-1})| = \frac{1}{(i+\frac{1}{2})\pi} + \frac{1}{(i-\frac{1}{2})\pi} > \frac{2}{i\pi}$$

$$\Rightarrow V(x) = |x(t_n) - x(0)| + \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| + |x(1) - x(t_0)| > \sum_{i=1}^{n} \frac{2}{i\pi} \text{ for all } n$$

$$\Rightarrow V(x) \to \infty$$

<Fact 3> If x is a func. of bounded variation, it is equal to the difference of two increasing functions

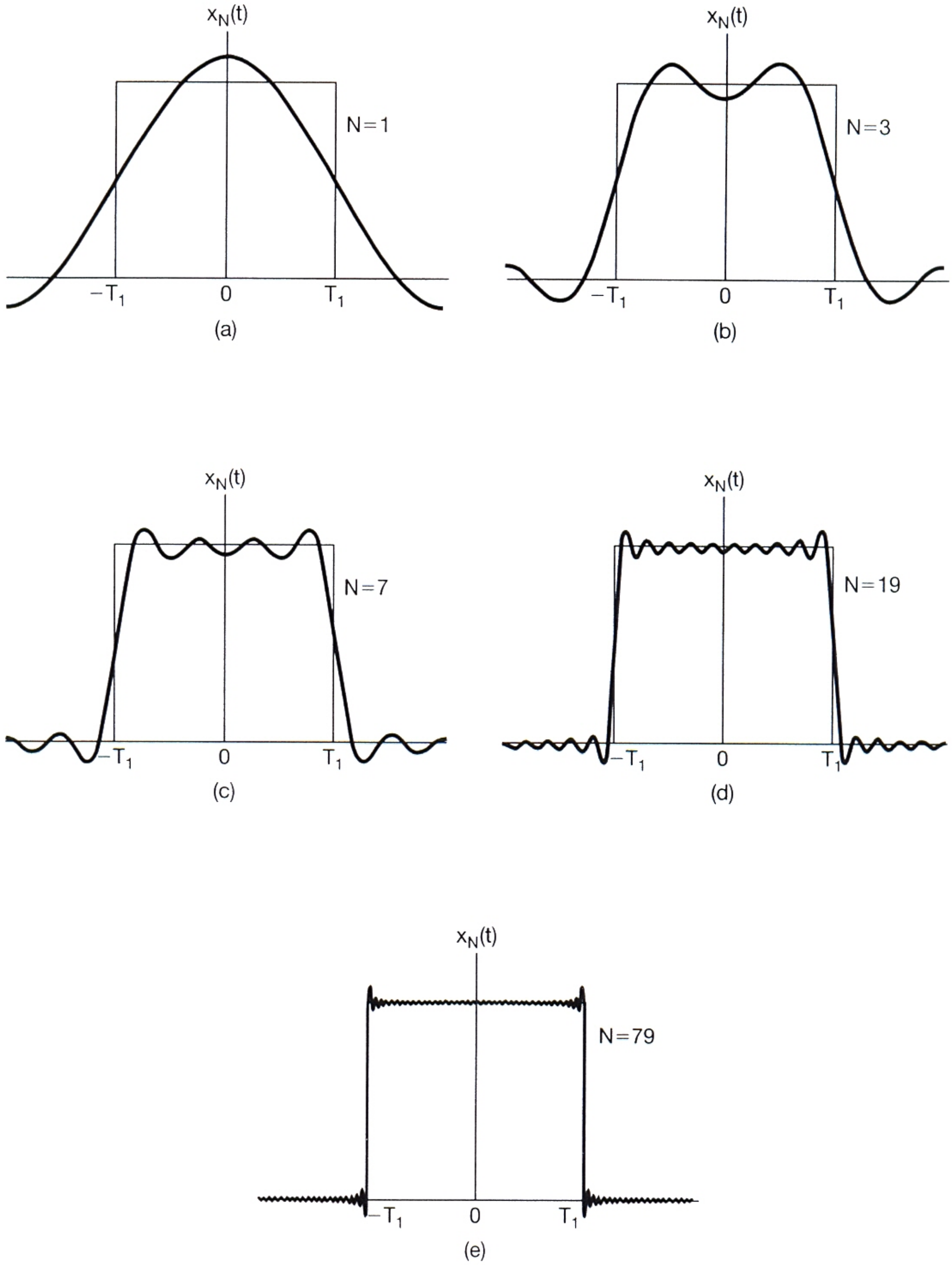


Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(\tau)\, d\tau$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t-1) - 1/2. \tag{3.69}$$

Remark

$$\int_{-\infty}^{t+T} x(\tau)\, d\tau = \int_{-\infty}^{t} x(\tau)\, d\tau \Leftrightarrow \int_{t}^{t+T} x(\tau)\, d\tau = 0 \Leftrightarrow \text{No d.c. Component} \Leftrightarrow a_0 = 0$$

<Example 3.9>, p. 210

Suppose that a signal $x(t)$ has the following properties.

(1) $x(t) \in R$

(2) $x(t+4) = x(t)$, i.e. $T = 4$, $\omega_0 = \frac{\pi}{2}$

(3) $a_k = 0$ for $|k| > 1$

(4) The signal with Fourier coeff. $b_k = e^{-j\pi k/2} a_{-k}$ is odd.

(5) $\frac{1}{4}\int_4 |x(t)|^2 dt = \frac{1}{2}$

$$\left.\begin{matrix}(2)\\(3)\end{matrix}\right\} \Rightarrow x(t) = a_0 + a_1 e^{j\pi t/2} + a_{-1} e^{-j\pi t/2}$$

$$(1) \Rightarrow a_k = a_{-k}^*$$

Recall that $x(-t) = \sum_{k=-\infty}^{\infty} a_{-k} e^{jk\omega_0 t}$ (Time Reversal)

$$\Rightarrow x(-(t-1)) = \sum_{k=-\infty}^{\infty} (a_{-k} e^{-jk\omega_0}) e^{jk\omega_0 t} \quad \text{(Time Shifting)}$$

$$= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} = b_0 + b_1 e^{j\pi t/2} + b_{-1} e^{-j\pi t/2}$$

$\Rightarrow$ the signal in (4) is real

$\Rightarrow$ b_k purely imaginary and odd (by (4))

$\Rightarrow$ $b_0 = 0$, $b_1 + b_1^* = 0$

$\Rightarrow$ $a_0 = 0$, $e^{-j\pi/2} a_{-1} + e^{j\pi/2} \overset{a_{-1}^*}{\overset{\shortparallel}{a_1}} = 0$

$\Rightarrow$ $a_0 = 0$, $a_1 = a_{-1}$, $a_1, a_{-1} \in R$

(5) $\Rightarrow 2|a_1|^2 = \frac{1}{2}$ (Parseval's Relation)

Conclusion: $x(t) = \frac{1}{2}(e^{j\pi t/2} + e^{-j\pi t/2}) = \cos \pi t/2$

or

$x(t) = -\frac{1}{2}(e^{j\pi t/2} + e^{-j\pi t/2}) = -\cos \pi t/2$

§3.6 Fourier Series Representation of Discrete-time Signals

Linear Combinations of Harmonically Relaxed Complex Exponentials

If $x[n]$ has a discrete-time Fourier series representation:

$$x[n]\left(=\sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}\right)=\sum_{k\in\langle N\rangle'} a_k e^{jk\omega_0 n}, \quad \omega_0 \triangleq \frac{2\pi}{N} \qquad (3.94)$$

then the Fourier series coefficients are given by

$$a_k\left(=\frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\omega_0 n}\right)=\frac{1}{N}\sum_{n\in\langle N\rangle} x[n] e^{-jk\omega_0 n} \qquad (3.95)$$

(The equalities in (3.94), (3.95) follow from the fact (P.6):

$$e^{jk\left(\frac{2\pi}{N}\right)n} = e^{j(k+rN)\left(\frac{2\pi}{N}\right)n}$$

<proof> First note that

$$\sum_{n=\langle N\rangle} e^{jk\left(\frac{2\pi}{N}\right)n} = e^{jk\left(\frac{2\pi}{N}\right)n_0}\left[\frac{1-e^{jk\left(\frac{2\pi}{N}\right)N}}{1-e^{jk\left(\frac{2\pi}{N}\right)}}\right]$$

$$= 0 \qquad \text{if } k \neq mN \qquad (3.90)$$

However, note that

$$\sum_{n=\langle N\rangle} e^{jk\left(\frac{2\pi}{N}\right)n} = N \qquad \text{if } k = mN \qquad (3.90)$$

Now, multiplying both sides of (3.94) by $e^{-jr\left(\frac{2\pi}{N}\right)n}$ and summing over N terms, we obtain

$$\sum_{n\in\langle N\rangle} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{n\in\langle N\rangle}\sum_{k\in\langle N\rangle'} a_k e^{j(k-r)\left(\frac{2\pi}{N}\right)n}$$

$$= \sum_{k\in\langle N\rangle'} a_k\left[\sum_{n\in\langle N\rangle} e^{j(k-r)\left(\frac{2\pi}{N}\right)n}\right]$$

$$\begin{cases} k = N_0+\bar{k}, & 0\le \bar{k}\le N-1 \\ r = N_0+\bar{r}, & 0\le \bar{r}\le N-1 \end{cases}$$

$$= N a_r \qquad \text{if } r\in\langle N\rangle' \quad (\text{by } (3.90)) \qquad \square$$

Remarks

(1) The a_k are called the spectral coefficient of $x[n]$

(2) (3.95) $\Rightarrow$ (3.94) : Inverse Discrete-Time Fourier Series

Similarly, if k ranges from 1 to N, we obtain

$$x[n] = a_1\phi_1[n] + a_2\phi_2[n] + \ldots + a_N\phi_N[n]. \tag{3.97}$$

From eq. (3.86), $\phi_0[n] = \phi_N[n]$, and therefore, upon comparing eqs. (3.96) and (3.97), we conclude that $a_0 = a_N$. Similarly, by letting k range over any set of N consecutive integers and using eq. (3.86), we can conclude that

(3) (3.95) $\Longrightarrow$

$$a_k = a_{k+N}. \quad \text{or } a_k = a_{k+rN} \tag{3.98}$$

That is, if we consider more than N sequential values of k, the values a_k repeat periodically with period N. It is important that this fact be interpreted carefully. In particular, since there are only N distinct complex exponentials that are periodic with period N, the discrete-time Fourier series representation is a finite series with N terms. Therefore, if we fix the N consecutive values of k over which we define the Fourier series in eq. (3.94), we will obtain a set of exactly N Fourier coefficients from eq. (3.95). On the other hand, at times it will be convenient to use different sets of N values of k, and consequently, it is useful to regard eq. (3.94) as a sum over any *arbitrary* set of N successive values of k. For this reason, it is sometimes convenient to think of a_k as a sequence defined for all values of k, but where only N successive elements in the sequence will be used in the Fourier series representation. Furthermore, since the $\phi_k[n]$ repeat periodically with period N as we vary k [eq. (3.86)], so must the a_k [eq. (3.98)]. This viewpoint is illustrated in the next example.

Example 3.10

Consider the signal

$$x[n] = \sin\omega_0 n, \tag{3.99}$$

which is the discrete-time counterpart of the signal $x(t) = \sin\omega_0 t$ of Example 3.3. $x[n]$ is periodic only if $2\pi/\omega_0$ is an integer or a ratio of integers. For the case when $2\pi/\omega_0$ is an integer N, that is, when

$$\omega_0 = \frac{2\pi}{N},$$

$x[n]$ is periodic with fundamental period N, and we obtain a result that is exactly analogous to the continuous-time case. Expanding the signal as a sum of two complex exponentials, we get

$$x[n] = \frac{1}{2j}e^{j(2\pi/N)n} - \frac{1}{2j}e^{-j(2\pi/N)n}. \tag{3.100}$$

Comparing eq. (3.100) with eq. (3.94), we see by inspection and by the uniqueness of Fourier coefficients that

$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}, \tag{3.101}$$

and the remaining coefficients over the interval of summation are zero. As described previously, these coefficients repeat with period N; thus, a_{N+1} is also equal to $(1/2j)$ and a_{N-1} equals $(-1/2j)$. The Fourier series coefficients for this example with $N = 5$ are illustrated in Figure 3.13. The fact that they repeat periodically is indicated. However, only one period is utilized in the synthesis equation (3.94).

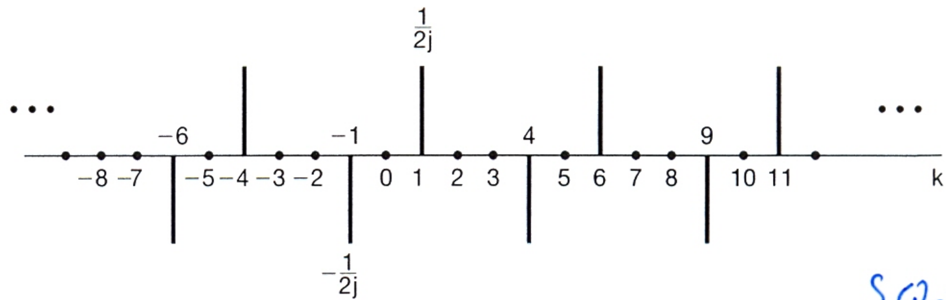


Figure 3.13 Fourier coefficients for $x[n] = \sin(2\pi/5)n$.

$\left\{ \begin{array}{l} \omega_0 = \frac{2\pi}{N} \\ N = 5 \end{array} \right.$

Consider now the case when $2\pi/\omega_0$ is a ratio of integers—that is, when

$$\omega_0 = \frac{2\pi M}{N}.$$

Assuming that M and N do not have any common factors, $x[n]$ has a fundamental period of N. Again expanding $x[n]$ as a sum of two complex exponentials, we have

$$x[n] = \frac{1}{2j}e^{jM(2\pi/N)n} - \frac{1}{2j}e^{-jM(2\pi/N)n},$$ $\Rightarrow$ may be not consequtive

from which we can determine by inspection that $a_M = (1/2j)$, $a_{-M} = (-1/2j)$, and the remaining coefficients over one period of length N are zero. The Fourier coefficients for this example with $M = 3$ and $N = 5$ are depicted in Figure 3.14. Again, we have indicated the periodicity of the coefficients. For example, for $N = 5$, $a_2 = a_{-3}$, which in our example equals $(-1/2j)$. Note, however, that over any period of length 5 there are only two nonzero Fourier coefficients, and therefore there are only two nonzero terms in the synthesis equation.

$M=3$, $-M=-3$ are not consequtive numbers if $N=5$ $\Longrightarrow$ $a_M = a_3$, $a_{-M} = a_{-3} = a_2$

Now, 2, 3 are consequtive numbers

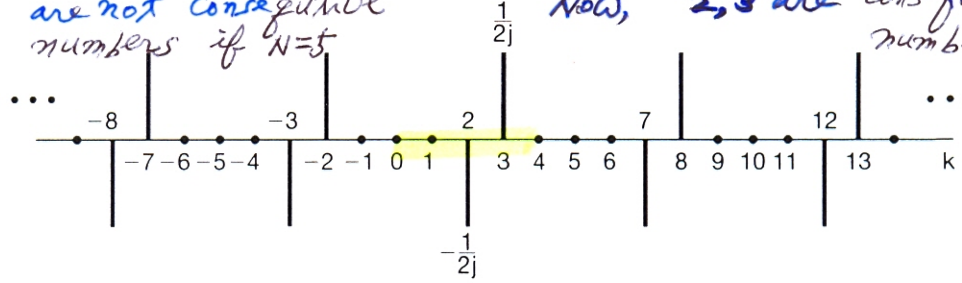


Figure 3.14 Fourier coefficients for $x[n] = \sin 3(2\pi/5)n$.

$$\Rightarrow x[n] = \frac{-1}{2j}e^{j2(2\pi/5)n} + \frac{1}{2j}e^{j3(2\pi/5)n}$$

Example 3.12

In this example, we consider the discrete-time periodic square wave shown in Figure 3.16. We can evaluate the Fourier series for this signal using eq. (3.95). Because $x[n] = 1$ for $-N_1 \le n \le N_1$, it is particularly convenient to choose the length-N interval of summation in eq. (3.95) so that it includes the range $-N_1 \le n \le N_1$. In this case, we can express eq. (3.95) as

$$(3.95)\quad a_k = \frac{1}{N}\sum_{n\in\langle N\rangle} x[n]\, e^{-jk\omega_0 n} \Rightarrow$$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}. \tag{3.102}$$

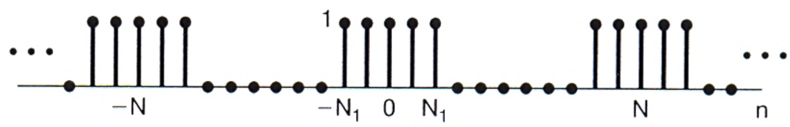


Figure 3.16 Discrete-time periodic square wave.

Letting $m = n + N_1$, we observe that eq. (3.102) becomes

$$\begin{aligned} a_k &= \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)} \\ &= \frac{1}{N} e^{jk(2\pi/N)N_1} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}. \end{aligned} \tag{3.103}$$

The summation in eq. (3.103) consists of the sum of the first $2N_1 + 1$ terms in a geometric series, which can be evaluated using the result of Problem 1.54. This yields

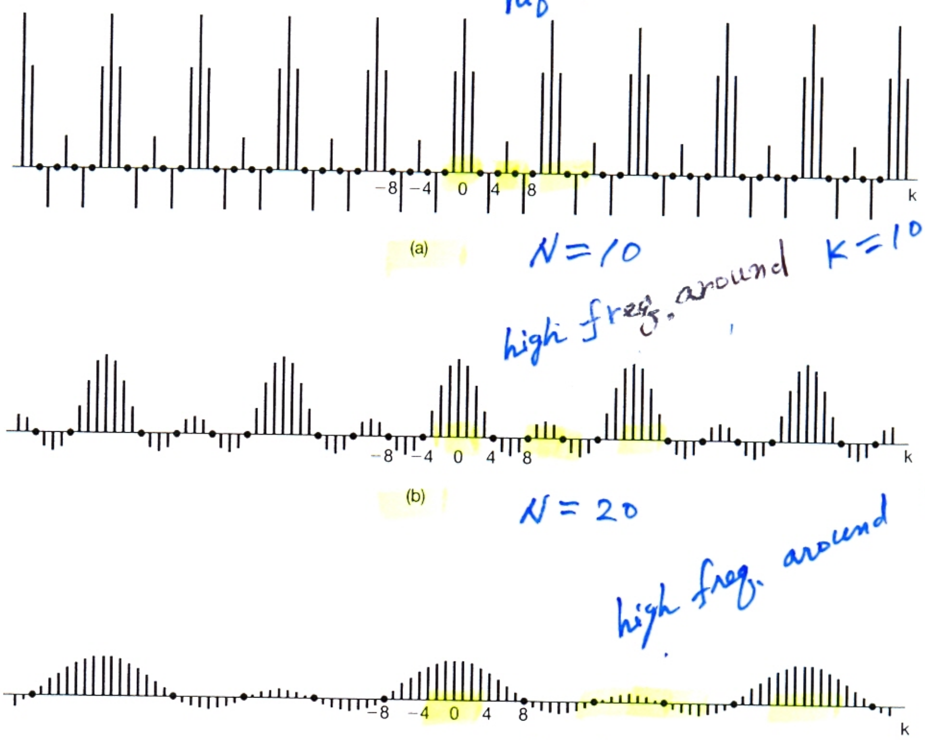
$$\begin{aligned} a_k &= \frac{1}{N} e^{jk(2\pi/N)N_1}\left(\frac{1 - e^{-jk2\pi(2N_1+1)/N}}{1 - e^{-jk(2\pi/N)}}\right) \\ &= \frac{1}{N}\frac{e^{-jk(2\pi/2N)}[e^{jk2\pi(N_1+1/2)/N} - e^{-jk2\pi(N_1+1/2)/N}]}{e^{-jk(2\pi/2N)}[e^{jk(2\pi/2N)} - e^{-jk(2\pi/2N)}]} \\ &= \frac{1}{N}\frac{\sin[2\pi k(N_1+1/2)/N]}{\sin(\pi k/N)}, \quad k \ne 0, \pm N, \pm 2N, \ldots \end{aligned} \tag{3.104}$$

and

$$a_k = \frac{2N_1 + 1}{N}, \quad k = 0, \pm N, \pm 2N, \ldots. \tag{3.105}$$

The coefficients a_k for $2N_1 + 1 = 5$ are sketched for $N = 10$, 20, and 40 in Figures 3.17(a), (b), and (c), respectively.

In discussing the convergence of the continuous-time Fourier series in Section 3.4, we considered the example of a symmetric square wave and observed how the finite sum in eq. (3.52) converged to the square wave as the number of terms approached infinity. In parhigh freq. around k=5



N=10

high freq. around k=10

N = 20

high freq. around k=20

N=40

Figure 3.17 Fourier series coefficients for the periodic square wave of Example 3.12; plots of Na_k for $2N_1 + 1 = 5$ and (a) $N = 10$; (b) $N = 20$; and (c) $N = 40$.

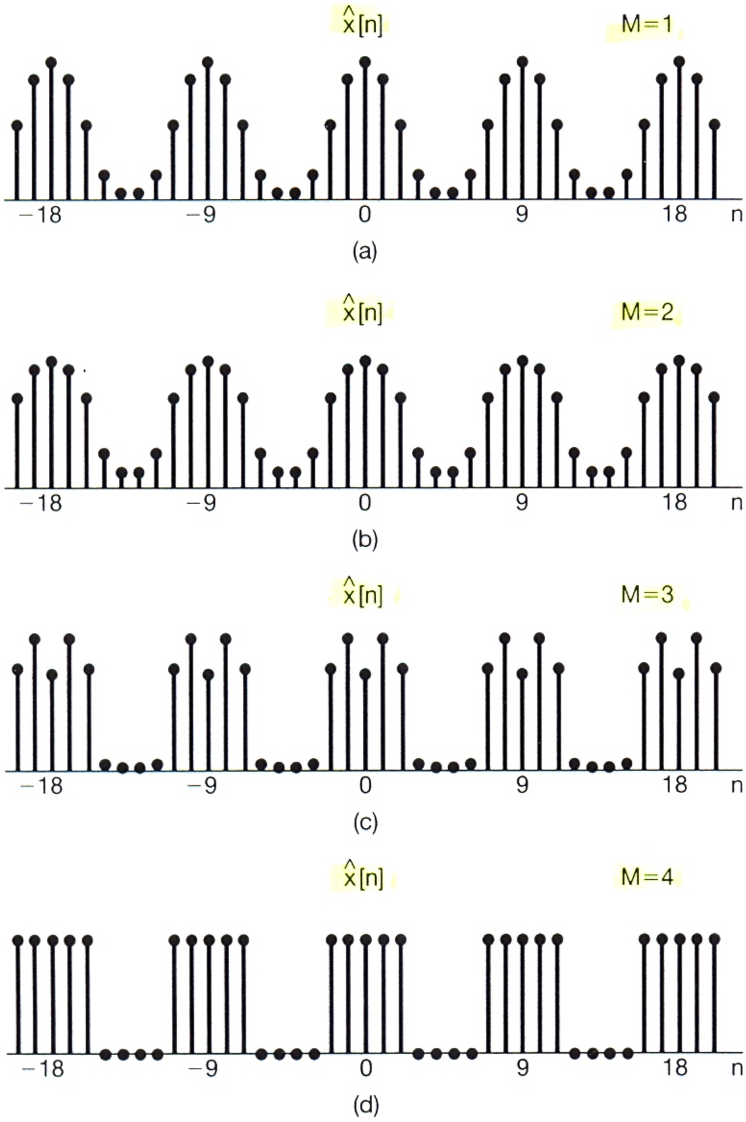
$N_1 = 2$ fixed

ticular, we observed the Gibbs phenomenon at the discontinuity, whereby, as the number of terms increased, the ripples in the partial sum (Figure 3.9) became compressed toward the discontinuity, with the peak amplitude of the ripples remaining constant independently of the number of terms in the partial sum. Let us consider the analogous sequence of partial sums for the discrete-time square wave, where, for convenience, we will assume that the period N is odd. In Figure 3.18, we have depicted the signals

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk(2\pi/N)n} \tag{3.106}$$

for the example of Figure 3.16 with $N = 9$, $2N_1 + 1 = 5$, and for several values of M. For $M = 4$, the partial sum exactly equals $x[n]$. We see in particular that in contrast to the continuous-time case, there are no convergence issues and there is no Gibbs phenomenon. In fact, there are no convergence issues with the discrete-time Fourier series in general. The reason for this stems from the fact that any discrete-time periodic sequence $x[n]$ is completely specified by a *finite* number N of parameters, namely, the values of the sequence over one period. The Fourier series analysis equation (3.95) simply transforms this set of N parameters into an equivalent set—the values of the N Fourier coefficients—and

$x[0], \dots, x[N-1] \longleftrightarrow a_0, a_1, \dots, a_{N-1}$



$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk(2\pi/N)n} \quad (3.106)$$

$\hat{x}[n] = x[n]$ if $M = 4 \Rightarrow N = 9$

Figure 3.18 Partial sums of eqs. (3.106) and (3.107) for the periodic square wave of Figure 3.16 with $N = 9$ and $2N_1 + 1 = 5$: (a) $M = 1$; (b) $M = 2$; (c) $M = 3$; (d) $M = 4$.

the synthesis equation (3.94) tells us how to recover the values of the original sequence in terms of a *finite* series. Thus, if N is odd and we take $M = (N - 1)/2$ in eq. (3.106), the sum includes exactly N terms, and consequently, from the synthesis equations, we have $\hat{x}[n] = x[n]$. Similarly, if N is even and we let

$$\hat{x}[n] = \sum_{k=-M+1}^{M} a_k e^{jk(2\pi/N)n}, \tag{3.107}$$

then with $M = N/2$, this sum consists of N terms, and again, we can conclude from eq. (3.94) that $\hat{x}[n] = x[n]$.

In contrast, a continuous-time periodic signal takes on a continuum of values over a single period, and an infinite number of Fourier coefficients are required to represent it.

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
(1) Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n - r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n - 1]$	$(1 - e^{-jk(2\pi/N)})a_k$
(2) Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$, $k \neq 0$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

(1) $a_{(m)k} = \frac{1}{mN} \sum_{n=0}^{mN-1} x_{(m)}[n] e^{-j\frac{2\pi}{mN}n} = \frac{1}{mN} \sum_{k=0}^{N-1} x[k] e^{-j\frac{2\pi}{N}k}$ (letting $n = mk$)

(2) $\sum_{k=-\infty}^{n+N} x[k] = \sum_{k=-\infty}^{n} x[k] \Leftrightarrow \sum_{k=n+1}^{n+N} x[k] = 0 \Leftrightarrow a_{0+mN} = 0$

Parseval's Relation for Discrete-Time Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2 \qquad (3.110)$$

<Proof>

By (3.94), we can write the LHS of (3.110) as follows.

$$\frac{1}{N}\sum_{n=\langle N\rangle}\left(\sum_{p=\langle N\rangle}\sum_{q=\langle N\rangle} a_p a_q^* e^{j(p-q)\omega_0 n}\right)$$

$$= \frac{1}{N}\sum_{p=\langle N\rangle}\sum_{q=\langle N\rangle} a_p a_q^* \left(\sum_{n=\langle N\rangle} e^{j(p-q)\omega_0 n}\right) \qquad (3.110)'$$

where $\omega_0 \triangleq \frac{2\pi}{N}$

Note that

$$\sum_{n=\langle N\rangle} e^{j(p-q)\omega_0 n} = \begin{cases} N & \text{if } p-q = mN \\ 0 & \text{otherwise} \end{cases}$$

Let I_N be the set of any N consecutive values of k. Then, it is clear that the number of the pairs (p, q) satisfying $p-q = mN$ and $p, q \in I_N$ is only one and $p=q$. Therefore, $(3.110)'$ is reduced to (3.110).

Periodic Convolution

If $x[n] \xrightarrow{FS} a_k$ and $y[n] \xrightarrow{FS} b_k$, then

$$w[n] = \sum_{r=\langle N\rangle} x[r]\, y[n-r] \xrightarrow{FS} N a_k b_k$$

<proof>

$$c_k = \frac{1}{N}\sum_{n=\langle N\rangle}\left(\sum_{r=\langle N\rangle} x[r]\,y[n-r]\right)e^{-jk\omega_0 n}$$

$$= \frac{1}{N}\sum_{n=\langle N\rangle}\sum_{r=\langle N\rangle}\left(x[r]\,e^{-jk\omega_0 r}\right)\left(y[n-r]\,e^{-jk\omega_0(n-r)}\right)$$

$$= \sum_{r=\langle N\rangle}\left(x[r]\,e^{-jk\omega_0 r}\right)\left(\frac{1}{N}\sum_{n=\langle N\rangle} y[n-r]\,e^{-jk\omega_0(n-r)}\right)$$

$$=\left(\frac{1}{N}\sum_{r=\langle N\rangle} x[r]\,e^{-jk\omega_0 r}\right) N b_k$$

$$= N a_k b_k$$

§ 3.8 Fourier Series and LTI Systems

$$y[n] = \sum_{k=-\infty}^{\infty} a_k H(z_k)\, z_k^n \quad \text{if } x[n] = \sum_{k=-\infty}^{\infty} a_k z_k^n \qquad (3.15)$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(s_k)\, e^{s_k t} \quad \text{if } x(t) = \sum_{k=-\infty}^{\infty} a_k e^{s_k t} \qquad (3.13)$$

where $H(s) \triangleq \int_{-\infty}^{\infty} h(t)\,e^{-j\omega t}\,dt$, $H(z) \triangleq \sum_{k=-\infty}^{\infty} h[k]\,z^{-k}$

Let $a_k = k\omega_0$. Then, we have

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \ (3.123) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)\, e^{jk\omega_0 t} \qquad (3.124)$$

$$x[n] = \sum_{k\in\langle N\rangle} a_k e^{jk\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k\in\langle N\rangle} a_k H(e^{jk\omega_0})\, e^{jk\omega_0 n} \qquad (3.131)$$

Remarks: (1) Valid if the system is BIBO stable
(2) steady-state responses

Example 3.16 : $h(t) = e^{-t} u(t) \Rightarrow$ BIBO stable

$x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$, $a_0 = 1$, $a_1 = a_{-1} = \frac{1}{4}$

$a_2 = a_{-2} = \frac{1}{2}$, $a_3 = a_{-3} = \frac{1}{3}$

To calculate the Fourier series coefficients of the output $y(t)$, we first compute the frequency response: $\Rightarrow x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$

$$\begin{aligned} H(j\omega) &= \int_0^{\infty} e^{-\tau} e^{-j\omega\tau}\, d\tau \\ &= -\frac{1}{1+j\omega} e^{-\tau} e^{-j\omega\tau}\Big|_0^{\infty} \\ &= \frac{1}{1+j\omega}. \end{aligned} \tag{3.125}$$

Therefore, using eqs. (3.124) and (3.125), together with the fact that $\omega_0 = 2\pi$ in this example, we obtain

$$y(t) = \sum_{k=-3}^{+3} b_k e^{jk2\pi t}, \tag{3.126}$$

with $b_k = a_k H(jk2\pi)$, so that

$$\begin{aligned} &b_0 = 1, \\ &b_1 = \frac{1}{4}\left(\frac{1}{1+j2\pi}\right), \quad b_{-1} = \frac{1}{4}\left(\frac{1}{1-j2\pi}\right), \\ &b_2 = \frac{1}{2}\left(\frac{1}{1+j4\pi}\right), \quad b_{-2} = \frac{1}{2}\left(\frac{1}{1-j4\pi}\right), \\ &b_3 = \frac{1}{3}\left(\frac{1}{1+j6\pi}\right), \quad b_{-3} = \frac{1}{3}\left(\frac{1}{1-j6\pi}\right). \end{aligned} \tag{3.127}$$

Note that $y(t)$ must be a real-valued signal, since it is the convolution of $x(t)$ and $h(t)$, which are both real. This can be verified by examining eq. (3.127) and observing that $b_k^* = b_{-k}$. Therefore, $y(t)$ can also be expressed in either of the forms given in eqs. (3.31) and (3.32); that is,

$$(3.31) \Rightarrow \quad y(t) = 1 + 2\sum_{k=1}^{3} D_k \cos(2\pi k t + \theta_k), \tag{3.128}$$

or

$$(3.32) \Rightarrow \quad y(t) = 1 + 2\sum_{k=1}^{3} [E_k \cos 2\pi k t - F_k \sin 2\pi k t], \tag{3.129}$$

where

$$b_k = D_k e^{j\theta_k} = E_k + jF_k, \quad k = 1, 2, 3. \tag{3.130}$$

These coefficients can be evaluated directly from eq. (3.127). For example,

$$D_1 = |b_1| = \frac{1}{4\sqrt{1+4\pi^2}}, \qquad \theta_1 = \sphericalangle b_1 = -\tan^{-1}(2\pi),$$

$$E_1 = \Re e\{b_1\} = \frac{1}{4(1+4\pi^2)}, \qquad F_1 = \Im m\{b_1\} = -\frac{\pi}{2(1+4\pi^2)}.$$

In discrete time, the relationship between the Fourier series coefficients of the input and output of an LTI system exactly parallels eqs. (3.123) and (3.124). Specifically, let $x[n]$ be a periodic signal with Fourier series representation given by

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}.$$

If we apply this signal as the input to an LTI system with impulse response $h[n]$, then, as in eq. (3.16) with $z_k = e^{jk(2\pi/N)}$, the output is

$$y[n] = \sum_{k=\langle N\rangle} a_k H(e^{j2\pi k/N}) e^{jk(2\pi/N)n}. \tag{3.131}$$

Thus, $y[n]$ is also periodic with the same period as $x[n]$, and the kth Fourier coefficient of $y[n]$ is the product of the kth Fourier coefficient of the input and the value of the frequency response of the LTI system, $H(e^{j2\pi k/N})$, at the corresponding frequency.

Example 3.17

Consider an LTI system with impulse response $h[n] = \alpha^n u[n]$, $-1 < \alpha < 1$, and with the input

$|\alpha|^n$

//

$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

$$x[n] = \cos\left(\frac{2\pi n}{N}\right). \tag{3.132}$$

As in Example 3.10, $x[n]$ can be written in Fourier series form as

$$x[n] = \frac{1}{2}e^{j(2\pi/N)n} + \frac{1}{2}e^{-j(2\pi/N)n}.$$

or

Also, from eq. (3.122),

$\underbrace{|\alpha e^{-j\omega}|}_{=|\alpha|} < 1$

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty}\left(\alpha e^{-j\omega}\right)^n. \tag{3.133}$$

This geometric series can be evaluated using the result of Problem 1.54, yielding

$\Downarrow$

$$H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}. \tag{3.134}$$

Using eq. (3.131), we then obtain the Fourier series for the output:

BIBO stable

$$\begin{aligned} y[n] &= \frac{1}{2}H\left(e^{j2\pi/N}\right)e^{j(2\pi/N)n} + \frac{1}{2}H\left(e^{-j2\pi/N}\right)e^{-j(2\pi/N)n} \\ &= \frac{1}{2}\left(\frac{1}{1-\alpha e^{-j2\pi/N}}\right)e^{j(2\pi/N)n} + \frac{1}{2}\left(\frac{1}{1-\alpha e^{j2\pi/N}}\right)e^{-j(2\pi/N)n}. \end{aligned} \tag{3.135}$$

If we write

$$\frac{1}{1 - \alpha e^{-j2\pi/N}} = re^{j\theta},$$

then eq. (3.135) reduces to

$$y[n] = r\cos\left(\frac{2\pi}{N}n + \theta\right). \tag{3.136}$$

For example, if $N = 4$,

$$\frac{1}{1 - \alpha e^{-j2\pi/4}} = \frac{1}{1 + \alpha j} = \frac{1}{\sqrt{1 + \alpha^2}} e^{j(-\tan^{-1}(\alpha))},$$

and thus,

$$y[n] = \frac{1}{\sqrt{1 + \alpha^2}} \cos\left(\frac{\pi n}{2} - \tan^{-1}(\alpha)\right).$$

We note that for expressions such as eqs. (3.124) and (3.131) to make sense, the frequency responses $H(j\omega)$ and $H(e^{j\omega})$ in eqs. (3.121) and (3.122) must be well defined and finite. As we will see in Chapters 4 and 5, this will be the case if the LTI systems under consideration are stable. For example, the LTI system in Example 3.16, with impulse response $h(t) = e^{-t}u(t)$, is stable and has a well-defined frequency response given by eq. (3.125). On the other hand, an LTI system with impulse response $h(t) = e^{t}u(t)$ is unstable, and it is easy to check that the integral in eq. (3.121) for $H(j\omega)$ diverges for any value of ω. Similarly, the LTI system in Example 3.17, with impulse response $h[n] = \alpha^n u[n]$, is stable for $|\alpha| < 1$ and has frequency response given by eq. (3.134). However, if $|\alpha| > 1$, the system is unstable, and then the summation in eq. (3.133) diverges.

3.9 FILTERING — will be discussed after Chap. 4

In a variety of applications, it is of interest to change the relative amplitudes of the frequency components in a signal or perhaps eliminate some frequency components entirely, a process referred to as *filtering*. Linear time-invariant systems that change the shape of the spectrum are often referred to as *frequency-shaping filters*. Systems that are designed to pass some frequencies essentially undistorted and significantly attenuate or eliminate others are referred to as *frequency-selective filters*. As indicated by eqs. (3.124) and (3.131), the Fourier series coefficients of the output of an LTI system are those of the input multiplied by the frequency response of the system. Consequently, filtering can be conveniently accomplished through the use of LTI systems with an appropriately chosen frequency response, and frequency-domain methods provide us with the ideal tools to examine this very important class of applications. In this and the following two sections, we take a first look at filtering through a few examples.

HW #3

11, 12, 27, 30, 31, 32, 37, 38, 43, 52, 55, 57, 58, 70