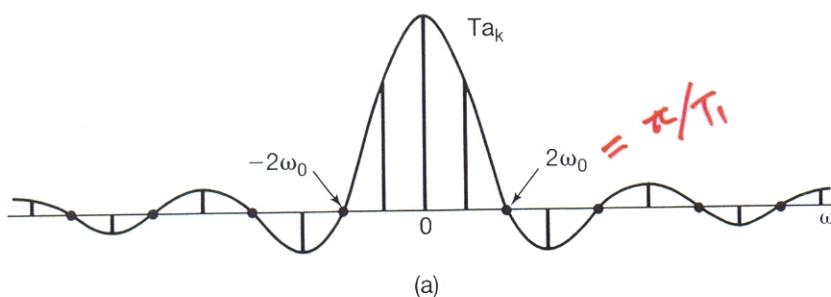


Chapter 4. The Continuous-Time Fourier Transform

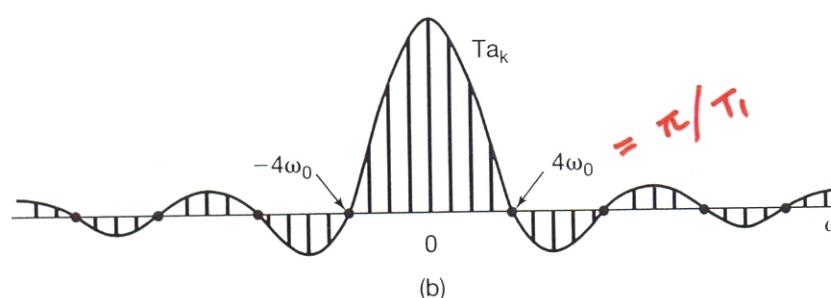
§4.1 Development of Fourier Transform Representation

The Continuous-Time Fourier Transform

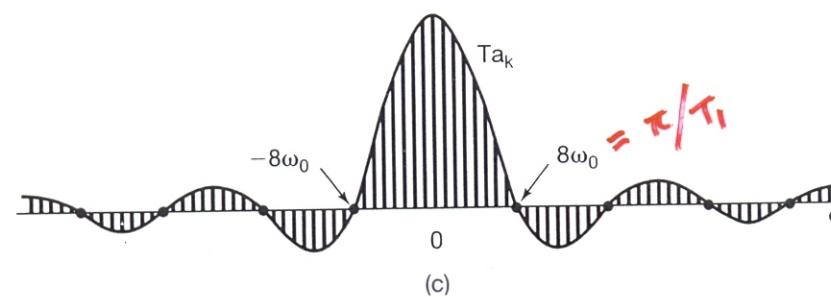
Chap. 4



$$T = 4T_1$$

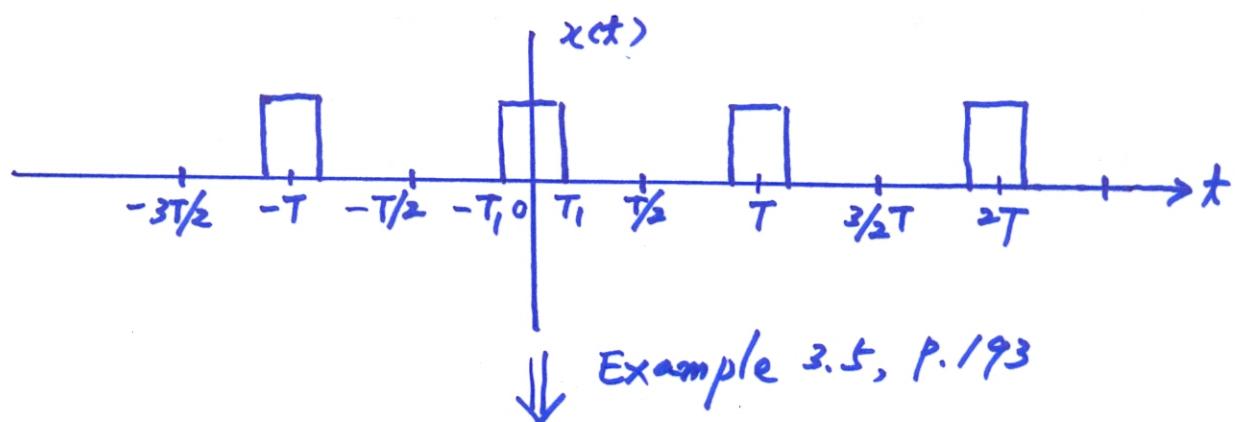


$$T = 8T_1$$



$$T = 16T_1$$

Figure 4.2 The Fourier series coefficients and their envelope for the periodic square wave in Figure 4.1 for several values of T (with T_1 fixed): (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$.



$$\left\{ \begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 t} \\ a_k &= \frac{2 \sin(k \omega_0 T_1)}{k \omega_0 T} \end{aligned} \right. , \quad \omega_0 \triangleq \frac{2\pi}{T}$$

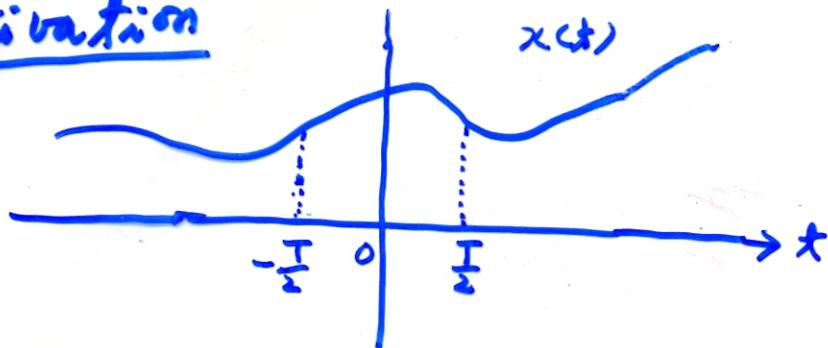
$$a_k = \frac{2 \sin(k \omega_0 T_1)}{k \omega_0 T} = \left. \frac{2 \sin \omega T_1}{T \omega} \right|_{\omega=k \omega_0}$$

Note that

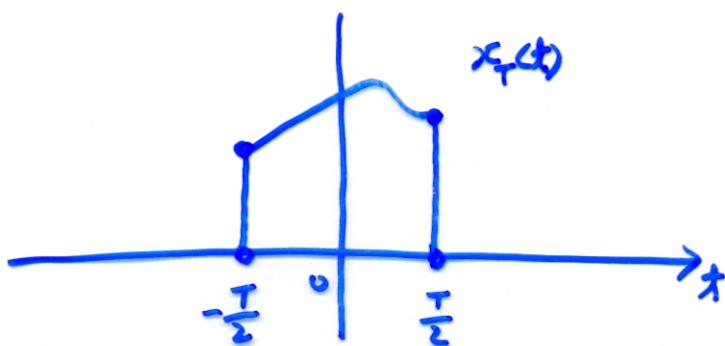
$$\omega_0 \rightarrow 0 \text{ as } T \rightarrow +\infty$$

Motivation

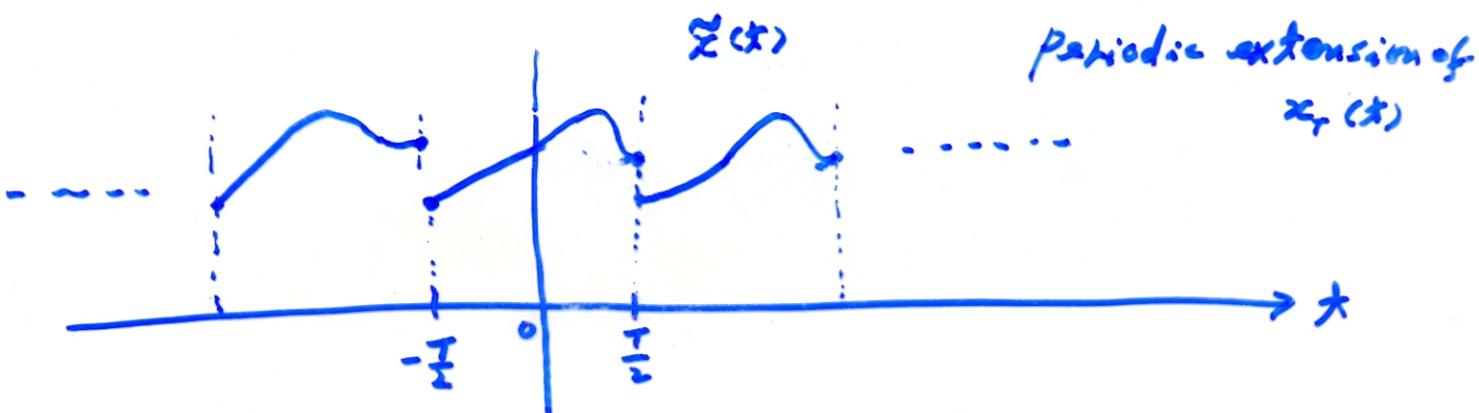
53.



Aperiodic signal



$$\lim_{T \rightarrow \infty} x_T(t) = x(t)$$



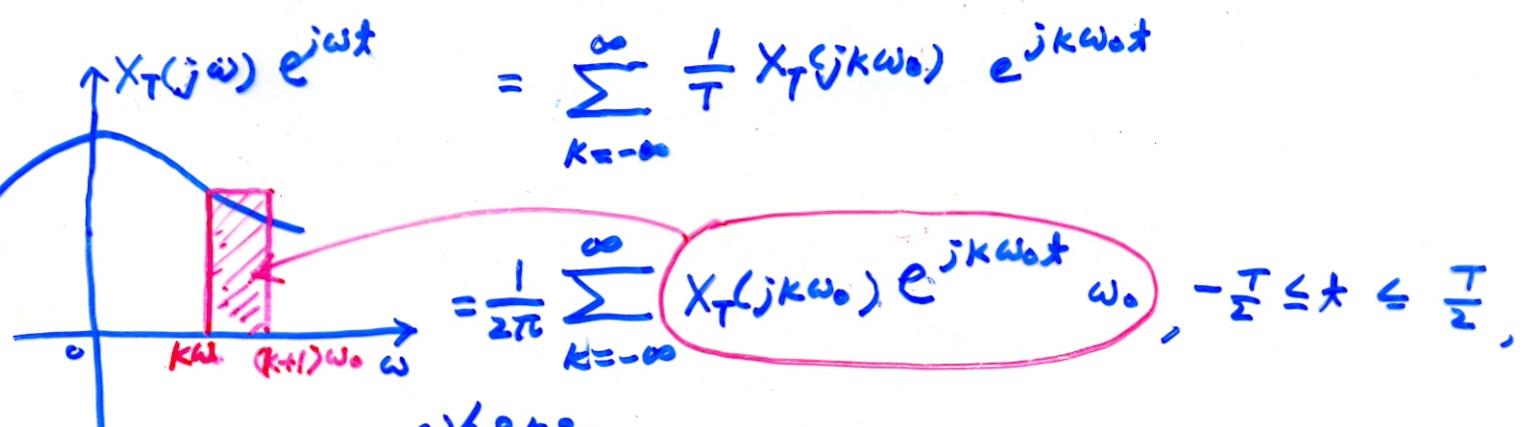
Periodic extension of
 $x_T(t)$

$$\text{Let } X_T(j\omega) \triangleq \int_{-\infty}^{\infty} x_T(t) e^{-j\omega t} dt.$$

Then,

$$x_T(t) = \tilde{x}(t) = \sum_{k=-\infty}^{\infty} \left[\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t) e^{-jk\omega_0 t} dt \right] e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T} X_T(jk\omega_0) e^{jk\omega_0 t}$$



where

$$\omega_0 \triangleq \frac{2\pi}{T}$$

Therefore, we may write

$$\text{Fourier Transform } x(t) = \lim_{T \rightarrow \infty} X_T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad (\text{K.8})$$

if we define X by

$$\text{Pair } X(j\omega) \stackrel{?}{=} \lim_{T \rightarrow \infty} X_T(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad (\text{K.9})$$

$\Rightarrow X(j\omega)$ is referred to as the Fourier Transform or Fourier Integral of $x(t)$.

It is also commonly referred to as the spectrum of $x(t)$.

Remark 1.

$$\left\{ \begin{array}{l} \tilde{x}(t): \text{a periodic signal with period } T \text{ and Fourier coeff. } a_k \\ x(t) \stackrel{?}{=} \begin{cases} \tilde{x}(k), & s \leq t \leq s+T \\ 0, & \text{otherwise} \end{cases} \end{array} \right.$$

Then,

$$a_k = \frac{1}{T} \int_s^{s+T} x(t) e^{-jk\omega_0 t} dt, \text{ where } X(j\omega) \text{ is given by (4.9)}$$

<Proof>

$$a_k = \frac{1}{T} \int_s^{s+T} \tilde{x}(k) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_s^{s+T} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} X(jk\omega_0)$$

□

Remark 2.

The contents in pp. 52 - 53 are to explain the motivation for the expressions in (4.8), (K.9)

□

§ 4.1.2 Convergence of Fourier Transforms

Theorem 1 : If $x \in L_2(-\infty, \infty)$ (i.e. $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$: square-integrable), then $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ (4.9) is well-defined, and furthermore, the equality:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad (4.8) \text{ is valid in the } L_2 \text{ sense. } \square$$

Theorem 2 : If (1) $x \in L_1(-\infty, \infty)$ (i.e. $\int_{-\infty}^{\infty} |x(t)| dt < \infty$: absolutely integrable), (2) $x \in BV(-\infty, \infty)$, and (3) $x(t)$ has a finite number of jump discontinuities within any finite interval, then $X(j\omega)$ in (4.9) is well defined (by (1)), and furthermore, the equality in (4.8) is valid (by (2) and (3)) for any t except at discontinuities where the RHS of (4.8) is equal to the average of the values on either side of the discontinuity. \square

Remarks

Theorems 1 and 2 present only the sufficient conditions for the existence of a Fourier transform, for instance, unit impulse, unit step, or periodic signals not satisfying the hypotheses of Theorems 1 and 2 can be considered to have Fourier transforms if impulse functions are permitted in the transform, as shown in § 4.2 and in Example 4.11. \square

§4.13 Examples of Continuous-Time Fourier Transform 56

< Example 4.2 >

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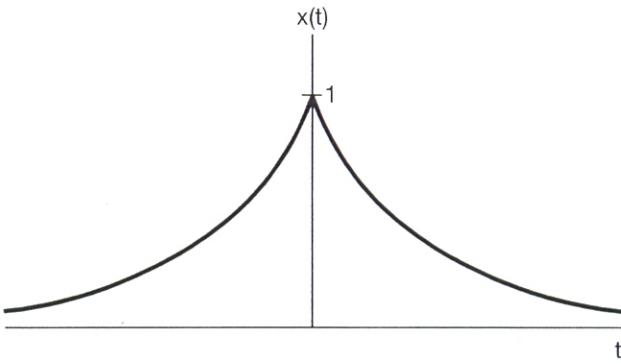
$$x(t) = e^{-a|t|} u(t)$$

where $a > 0$

This signal is sketched in Figure 4.6. The Fourier transform of the signal is

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{+\infty} e^{-a|t|} e^{-j\omega t} dt = \int_{-\infty}^0 e^{at} e^{-j\omega t} dt + \int_0^{\infty} e^{-at} e^{-j\omega t} dt \\ &= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} \\ &= \frac{2a}{a^2 + \omega^2}. \end{aligned}$$

In this case $X(j\omega)$ is real, and it is illustrated in Figure 4.7.



It is shown in P.18 that

Figure 4.6 Signal $x(t) = e^{-a|t|} u(t)$ of Example 4.2.

$$x(t) = x(t) + s(t) \text{ for any } x \quad (2.138)$$

$$\Leftrightarrow g(\omega) = \int_{-\infty}^{\infty} g(\gamma) s(\gamma) d\gamma \quad (2.139)$$

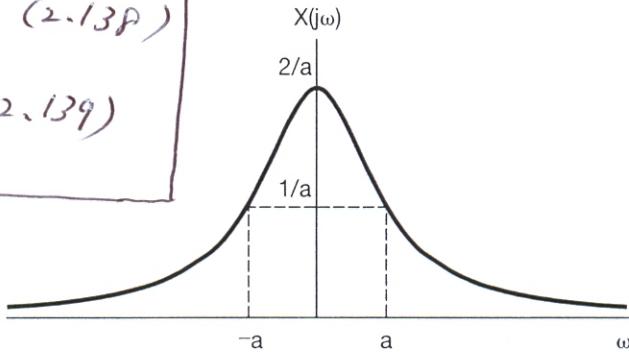


Figure 4.7 Fourier transform of the signal considered in Example 4.2 and depicted in Figure 4.6.

< Example 4.3 >

Now let us determine the Fourier transform of the unit impulse

$$x(t) = \delta(t). \quad (4.14)$$

Substituting into eq. (4.9) yields

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1. \quad (4.15)$$

(by (2.139))

That is, the unit impulse has a Fourier transform consisting of equal contributions at all frequencies.

(Remark) $s(t)$ does not satisfy the hypotheses of neither theorem 1 nor theorem 2.

Example 4.4

- Consider the rectangular pulse signal

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}, \quad (4.16)$$

as shown in Figure 4.8(a). Applying eq. (4.9), we find that the Fourier transform of this signal is

$$X(j\omega) = \int_{-T_1}^{T_1} \frac{e^{-j\omega t}}{2T_1} dt = 2 \frac{\sin \omega T_1}{\omega T_1}, \quad (4.17)$$

as sketched in Figure 4.8(b).

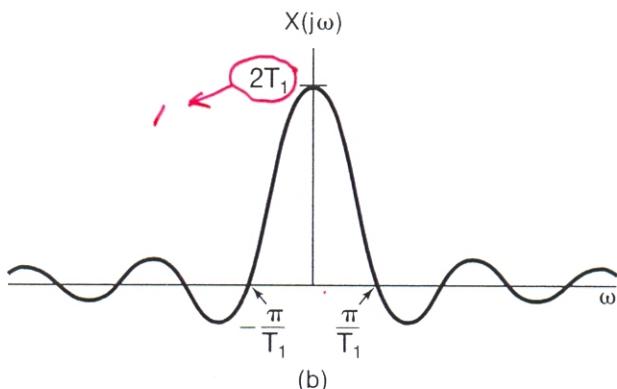
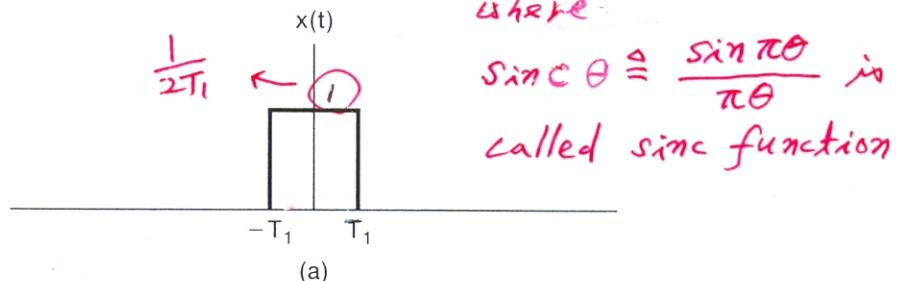
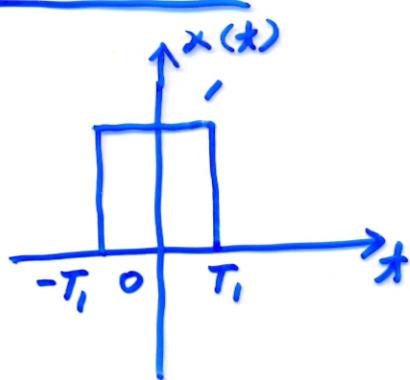


Figure 4.8 (a) The rectangular pulse signal of Example 4.4 and (b) its Fourier transform.

* As we discussed at the beginning of this section, the signal given by eq. (4.16) can be thought of as the limiting form of a periodic square wave as the period becomes arbitrarily large. Therefore, we might expect that the convergence of the synthesis equation for this signal would behave in a manner similar to that observed in Example 3.5 for the square wave. This is, in fact, the case. Specifically, consider the inverse Fourier transform for the rectangular pulse signal:

$$\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} 2 \frac{\sin \omega T_1}{\omega T_1} e^{j\omega t} d\omega.$$

Then, since $x(t)$ is square integrable, i.e. $x \in L_2(-\infty, \infty)$.

Remark 1

$$\xrightarrow{\mathcal{F}} X(j\omega) = \frac{2\sin \omega T_1}{\omega} \\ = 2\pi \left[\frac{T_1}{\pi} \text{sinc}\left(\frac{T_1 \omega}{\pi}\right) \right]$$

$$\lim_{T_1 \rightarrow \infty} x(t)$$

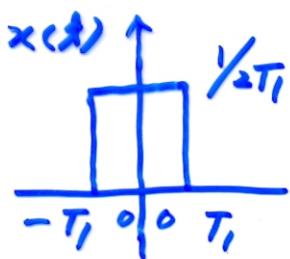
$$\xrightarrow{\mathcal{F}} \lim_{T_1 \rightarrow \infty} 2\pi \left[\frac{T_1}{\pi} \text{sinc}\left(\frac{T_1 \omega}{\pi}\right) \right] \\ \parallel 2\pi \delta(\omega)$$

Since

$$\lim_{W \rightarrow \infty} \frac{W}{\pi} \text{sinc}\left(\frac{Wt}{\pi}\right) = \delta(t)$$

in the L_2 sense

$$\Rightarrow \boxed{u(t) \xrightarrow{\mathcal{F}} 2\pi \delta(\omega)}$$

Remark 2

$$\xrightarrow{\mathcal{F}} X(j\omega) = \text{sinc}\left(\frac{T_1 \omega}{\pi}\right)$$

$$\lim_{T_1 \rightarrow 0} x(t)$$

$$\xrightarrow{\mathcal{F}} \lim_{T_1 \rightarrow 0} \text{sinc} \frac{T_1 \omega}{\pi}$$

\parallel

$\delta(t)$

$\xrightarrow{\mathcal{F}}$

\parallel

1

Hence, (4.15) can be shown in a limit sense

$(x(t) = \hat{x}(t) \text{ in the } L_2 \text{ sense})$ i.e. $\int_{-\infty}^{+\infty} |x(t) - \hat{x}(t)|^2 dt = 0.$

Furthermore, because $x(t)$ satisfies the Dirichlet conditions, $\hat{x}(t) = x(t)$, except at the points of discontinuity, $t = \pm T_1$, where $\hat{x}(t)$ converges to $1/2$, which is the average of the values of $x(t)$ on both sides of the discontinuity. In addition, the convergence of $\hat{x}(t)$ to $x(t)$ exhibits the Gibbs phenomenon, much as was illustrated for the periodic square wave in Figure 3.9. Specifically, in analogy with the finite Fourier series approximation, eq. (3.47), consider the following integral over a finite-length interval of frequencies:

$$\frac{1}{2\pi} \int_{-W}^W 2 \frac{\sin \omega T_1}{\omega} e^{j\omega t} d\omega.$$

As $W \rightarrow \infty$, this signal converges to $x(t)$ everywhere, except at the discontinuities. Moreover, the signal exhibits ripples near the discontinuities. The peak amplitude of these ripples does not decrease as W increases, although the ripples do become compressed toward the discontinuity, and the energy in the ripples converges to zero.

< Example 4.5 >

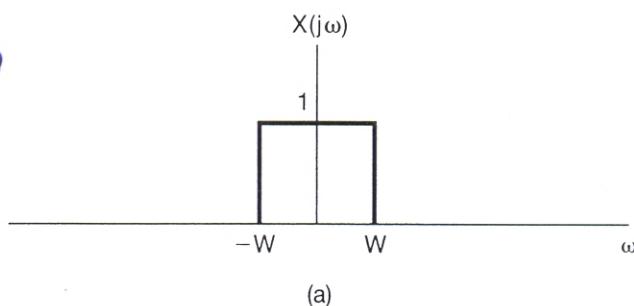
Consider the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}. \quad (4.18)$$

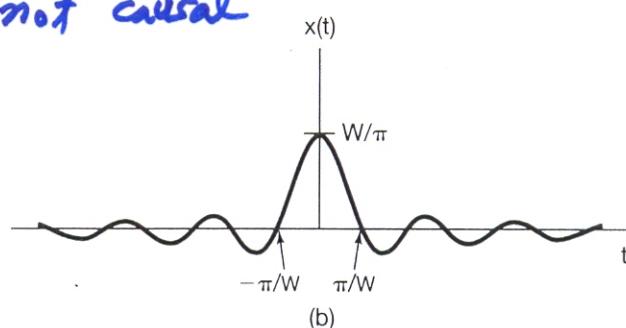
This transform is illustrated in Figure 4.9(a). Using the synthesis equation (4.8), we can

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-W}^W e^{j\omega t} d\omega \\ &= \frac{\sin Wt}{\pi t} \end{aligned}$$

$$= \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wt}{\pi}\right) : \text{not causal}$$



(a)



(b)

Figure 4.9 Fourier transform pair of Example 4.5: (a) Fourier transform for Example 4.5 and (b) the corresponding time function.

Remark : From Examples 4.4 & 4.5, observe the inverse relationship between the time and the frequency domains.

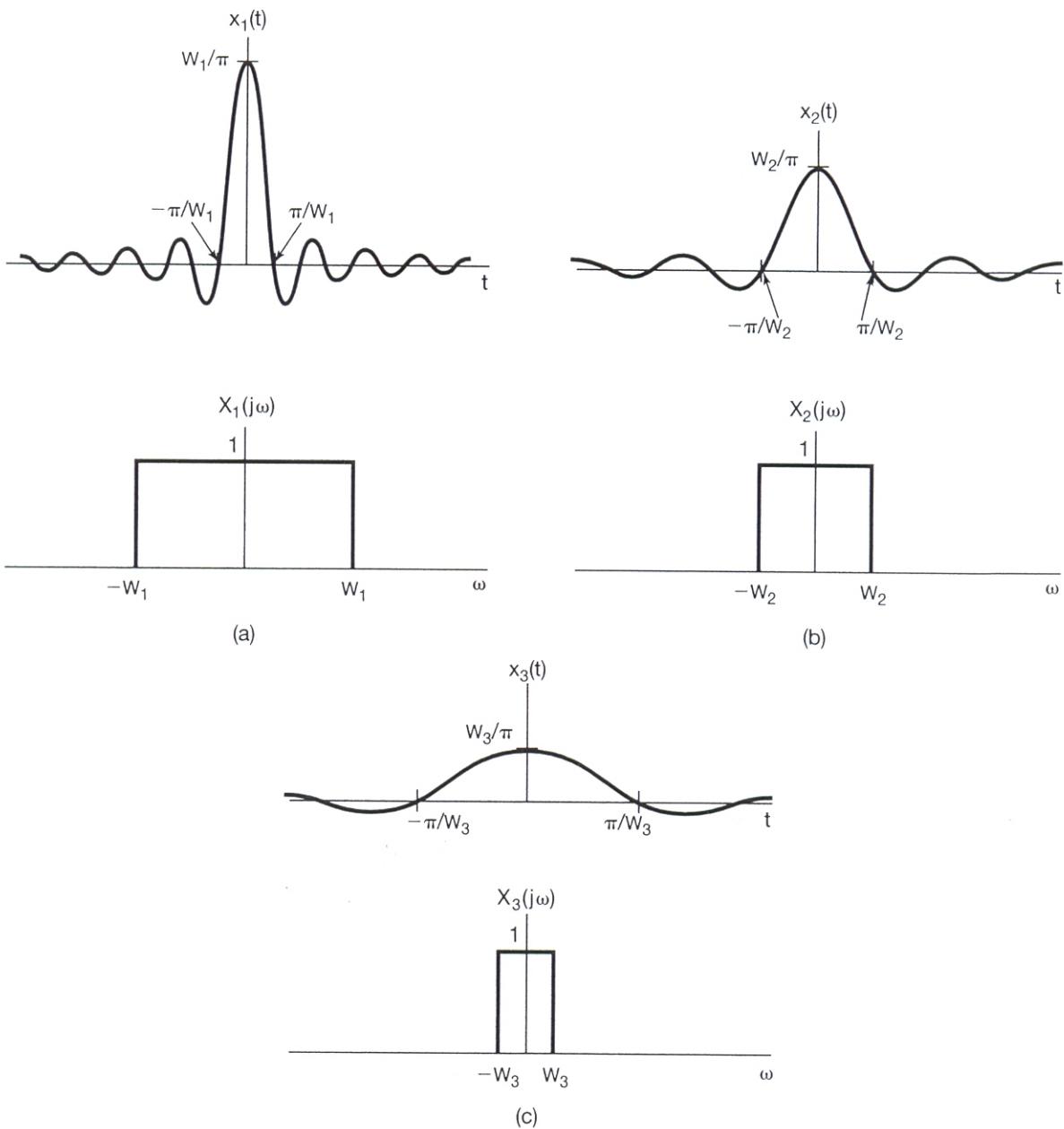


Figure 4.11 Fourier transform pair of Figure 4.9 for several different values of W .

and we can see a similar effect in Figure 4.8, where an increase in T_1 broadens $x(t)$ but makes $X(j\omega)$ narrower. In Section 4.3.5, we provide an explanation of this behavior in the context of the scaling property of the Fourier transform.

4.2 THE FOURIER TRANSFORM FOR PERIODIC SIGNALS

In the preceding section, we introduced the Fourier transform representation and gave several examples. While our attention in that section was focused on aperiodic signals, we can also develop Fourier transform representations for periodic signals, thus allowing us to

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§4.2 The Fourier Transform For Periodic Signals

- Both periodic and aperiodic signals can be considered within a unified context.

Let $x(t) \triangleq \sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 t}$ be well-defined.

Nevertheless, $x \notin L_2(-\infty, \infty)$, $x \notin L_1(-\infty, \infty)$

Let $X(j\omega) \triangleq \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$ (K.22)

$$\text{Then, } \boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = x(t)}$$

⇒ The Fourier transform of a periodic signal with Fourier series coeff. $\{a_k\}$ can be interpreted as a train of impulses occurring at $\omega = k\omega_0$ and for which the area of the impulse at the k th harmonic freq. $k\omega_0$ is $2\pi a_k$.

<Example 4.8>

$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$: a periodic signal with period T.

$$\Rightarrow a_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \text{ where } \omega_0 \triangleq \frac{2\pi}{T}$$

$$\Rightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - \frac{2\pi k}{T})$$

Also, observe the inverse relationship between the time and the frequency domains for the spacing between the impulses. □

§ 4.3 Properties of the continuous-time Fourier Transform

§ 4.3.1 Linearity

$$\left. \begin{array}{l} x(t) \xrightarrow{\mathcal{F}} X(j\omega) \\ y(t) \xrightarrow{\mathcal{F}} Y(j\omega) \end{array} \right\} \Rightarrow ax(t)+by(t) \xrightarrow{\mathcal{F}} aX(j\omega)+bY(j\omega)$$

§ 4.3.2 Time Scaling

$$x(t) \xrightarrow{\mathcal{F}} X(j\omega) \Rightarrow x(t-t_0) \xrightarrow{\mathcal{F}} e^{-j\omega t_0} X(j\omega)$$

§ 4.3.3 Conjugation

$$x(t) \xrightarrow{\mathcal{F}} X(j\omega) \Rightarrow x^*(t) \xrightarrow{\mathcal{F}} X^*(-j\omega)$$

If $x(t)$ is real, this implies that
 $\{X(-j\omega) = X^*(j\omega)\}$: conjugate symmetry

$$\left\{ \begin{array}{l} \operatorname{Re}\{X(j\omega)\} = \operatorname{Re}\{X(-j\omega)\} \\ \operatorname{Im}\{X(j\omega)\} = -\operatorname{Im}\{X(-j\omega)\} \end{array} \right.$$

§ 4.3.3' Integration

$$y(t) = \int_{-\infty}^t x(\tau) d\tau \xrightarrow{\mathcal{F}} \frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega) \quad (4.32)$$

<Proof>

$$e^{-at} \operatorname{sgn} t \xrightarrow{\mathcal{F}} \frac{-2j\omega}{(\omega^2 + a^2)} \xrightarrow{a \gg 0} \operatorname{sgn} t \xrightarrow{\mathcal{F}} \frac{2}{j\omega}$$

$$\Rightarrow u(t) = \frac{1}{2} + \frac{1}{\pi} \operatorname{sgn} t \xrightarrow{\mathcal{F}} \pi \delta(\omega) + \frac{1}{j\omega}$$

Note that $\int_{-\infty}^t x(\tau) d\tau = x(t) * u(t)$. By Convolution Property,

$$\int_{-\infty}^t x(\tau) d\tau \xrightarrow{\mathcal{F}} X(\omega) \left\{ \pi \delta(\omega) + \frac{1}{j\omega} \right\} = \pi X(0) \delta(\omega) + \frac{X(\omega)}{j\omega}$$

□

Remark :

$$\int_{-\infty}^{\infty} e^{-j\omega t} \left[\int_{-\infty}^t x(\tau) d\tau \right] dt = -\frac{e^{-j\omega t}}{j\omega} \int_{-\infty}^t x(\tau) d\tau \Big|_{-\infty}^{\infty} + \frac{x(j\omega)}{j\omega}$$

The first term on the right will be zero if f.p.c component,
 $x(0) = \int_{-\infty}^{\infty} x(\tau) d\tau = 0$ (*)

In fact, if (*) is not true, the Fourier transform does not exist formally. □

< Example 4.11 > (nonsense)

Let us determine the Fourier transform $X(j\omega)$ of the unit step $x(t) = u(t)$, making use of eq. (4.32) and the knowledge that

$$g(t) = \delta(t) \xrightarrow{\mathcal{F}} G(j\omega) = 1.$$

Noting that

$$x(t) = \int_{-\infty}^t g(\tau) d\tau$$

and taking the Fourier transform of both sides, we obtain

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega),$$

where we have used the integration property listed in Table 4.1. Since $G(j\omega) = 1$, we conclude that

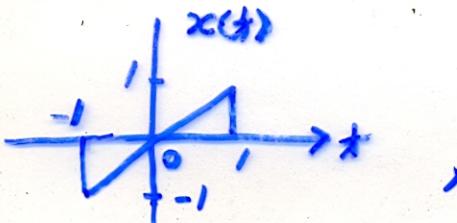
$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega). \quad (4.33)$$

Observe that we can apply the differentiation property of eq. (4.31) to recover the transform of the impulse. That is,

$$\delta(t) = \frac{du(t)}{dt} \xrightarrow{\mathcal{F}} j\omega \left[\frac{1}{j\omega} + \pi\delta(\omega) \right] = 1,$$

where the last equality follows from the fact that $\omega\delta(\omega) = 0$.

< Example 4.12 >



$$g(t) = \frac{dx(t)}{dt}$$

$$x(t) = \int_{-\infty}^t g(\tau) d\tau$$

$$\Rightarrow g(t) = \begin{cases} 1 & -1 \leq t < 0 \\ -1 & 0 \leq t < 1 \\ 0 & \text{else} \end{cases} + \begin{cases} 1 & -1 \leq t < 0 \\ -1 & 0 \leq t < 1 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow G(j\omega) = \frac{2\sin\omega}{\omega} - e^{j\omega} - e^{-j\omega}$$

$$\Rightarrow X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega) = \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}$$

§.4.3.5 Time and Frequency Scaling

$$x(t) \xrightarrow{\mathcal{F}} X(j\omega) \Rightarrow x(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Inverse relationship!