§6.6 First-order and Second-order Discrete-time Systems

§6.6.1 First-order Discrete-time Systems

$$y[n] - a\,y[n-1] = x[n] \Longrightarrow \begin{cases} h[n] = a^n u[n], \quad s[n] = h[n] * u[n] = \dfrac{1-a^{n+1}}{1-a}\, u[n] \\ H(e^{j\omega}) = \dfrac{1}{1-a e^{-j\omega}} \end{cases}$$

$|a| < 1$

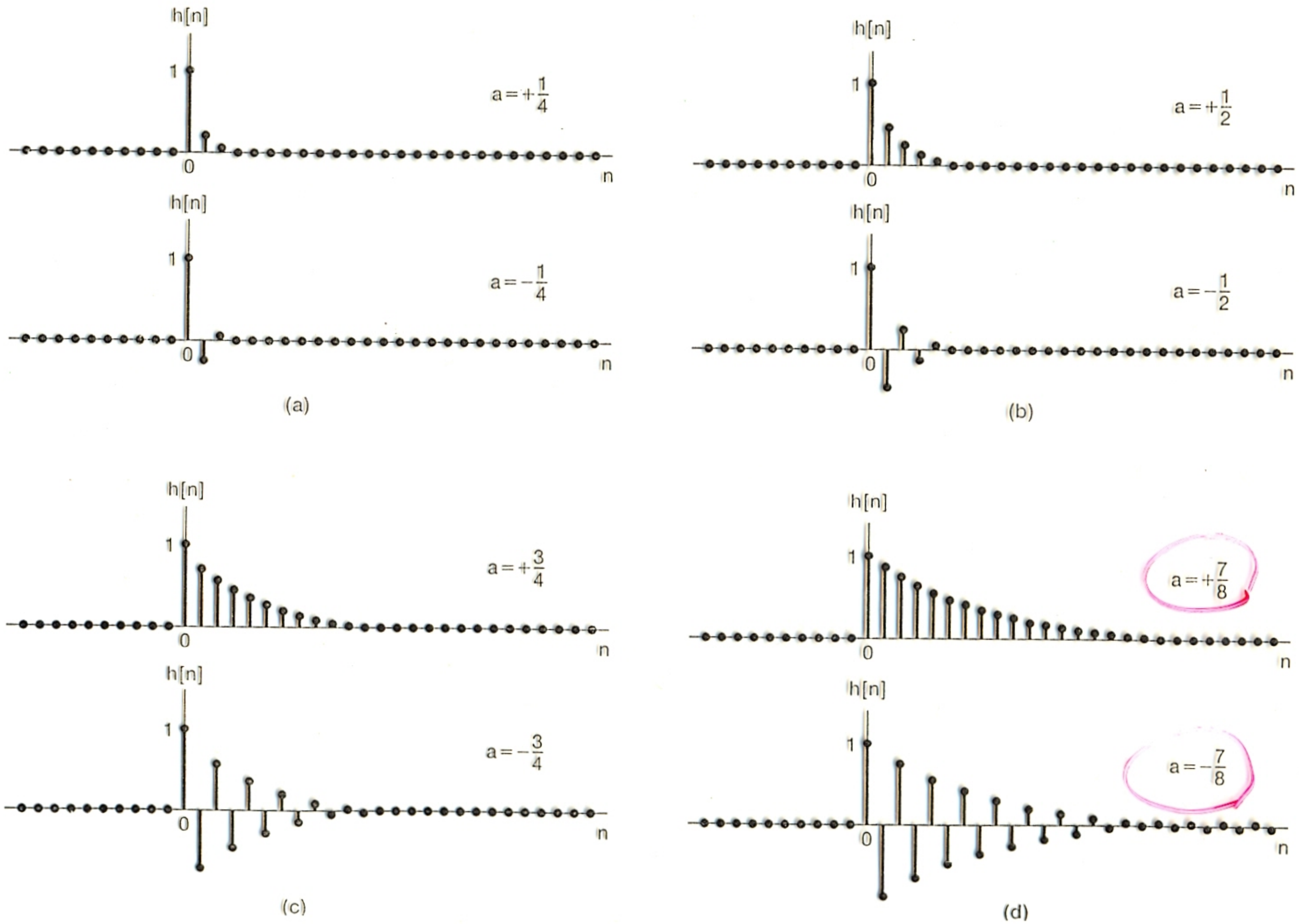


Figure 6.26 Impulse response $h[n] = a^n u[n]$ of a first-order system: (a) $a = \pm 1/4$; (b) $a = \pm 1/2$; (c) $a = \pm 3/4$; (d) $a = \pm 7/8$.

$\boxed{\text{Remarks}}$ $\xrightarrow[y[-1]=0]{x[n]=\delta[n]} \boxed{h[n]} \longrightarrow y[n]$, $\quad y[n] = \sum_{k=-\infty}^{n} h[n-k]\, x[k] = \sum_{k=0}^{n} h[n-k] = \sum_{k=0}^{n} h[k]$

— The responses are slower for $|a|$ near to 1

— The step responses exhibits both overshoot and ringing when $a<0$.

$$s[n] = \frac{1-a^{n+1}}{1-a}\, u[n]$$

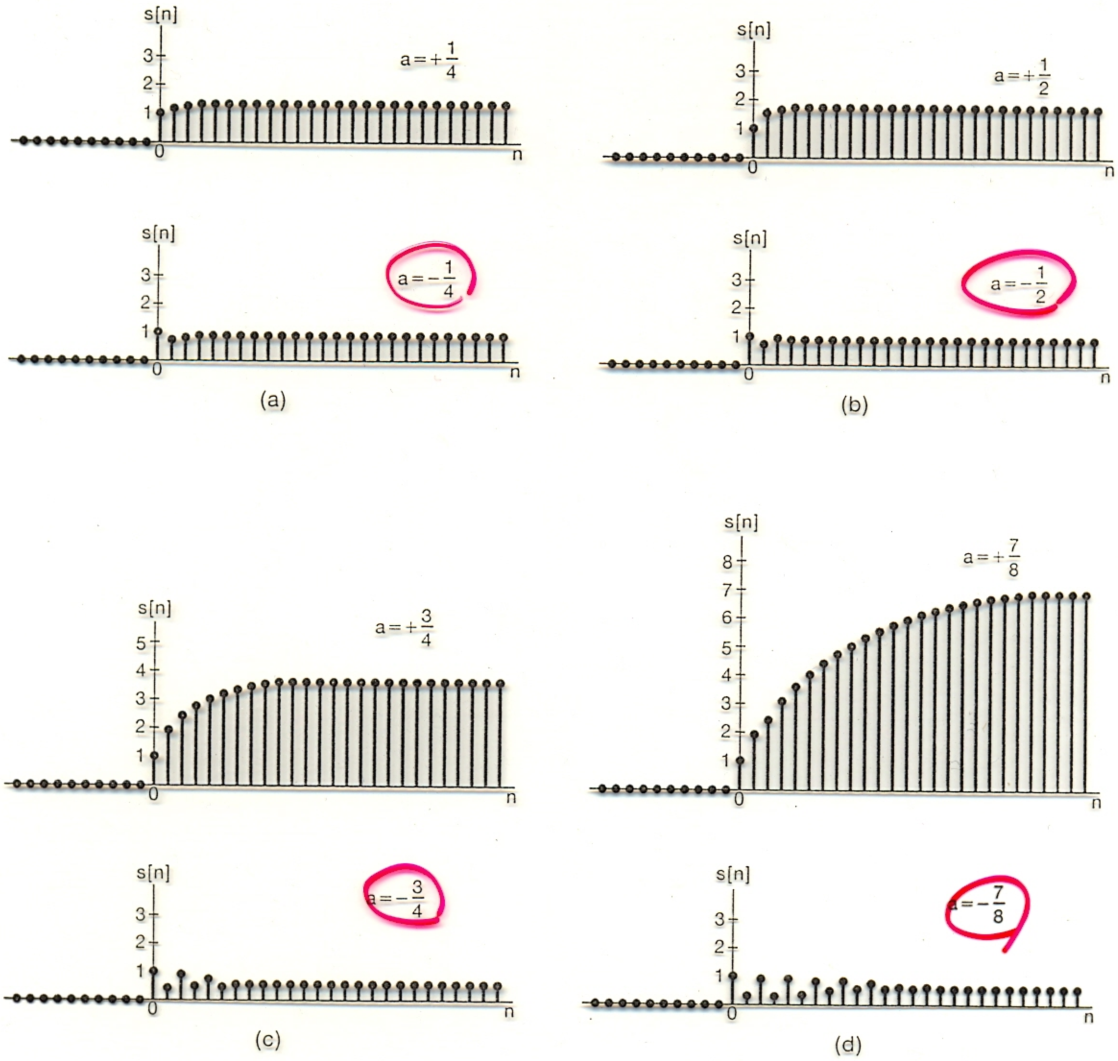


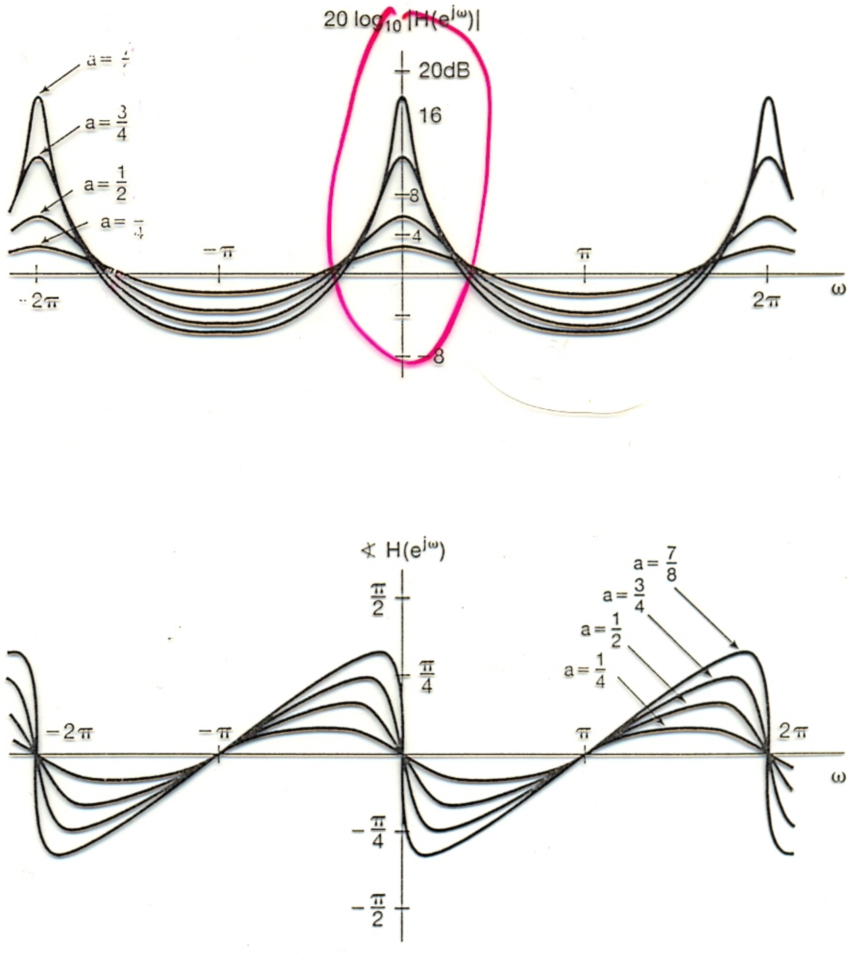
Figure 6.27 Step response $s[n]$ of a first-order system: (a) $a = \pm 1/4$; (b) $a = \pm 1/2$; (c) $a = \pm 3/4$; (d) $a = \pm 7/8$.

$$H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} \Rightarrow \left\{ \begin{array}{l} |H(e^{j\omega})| = \dfrac{1}{(1+a^2-2a\cos\omega)^{1/2}} \\ \sphericalangle H(e^{j\omega}) = -\tan^{-1}\left[\dfrac{a\sin\omega}{1-a\cos\omega}\right]. \end{array}\right. \quad (6.56)$$

In Figure 6.28(a), we have plotted the log magnitude and the phase of the frequency response in eq. (6.52) for several values of $a > 0$. The case of $a < 0$ is illustrated in

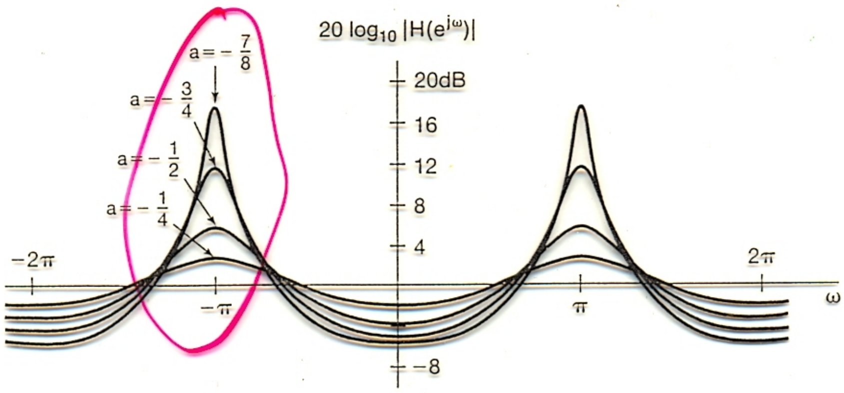
Figure 6.28(b). From these figures, we see that for $a > 0$, the system attenuates high frequencies [i.e., $|H(e^{j\omega})|$ is smaller for ω near $\pm\pi$ than it is for ω near 0], while when $a < 0$, the system amplifies high frequencies and attenuates low frequencies. Note also that for $|a|$ small, the maximum and minimum values, $1/(1+a)$ and $1/(1-a)$, of $|H(e^{j\omega})|$ are close together in value, and the graph of $|H(e^{j\omega})|$ is relatively flat. On the other hand, for $|a|$ near 1, these quantities differ significantly, and consequently $|H(e^{j\omega})|$ is more sharply peaked, providing filtering and amplification that is more selective over a narrow band of frequencies.

$1 > a > 0$

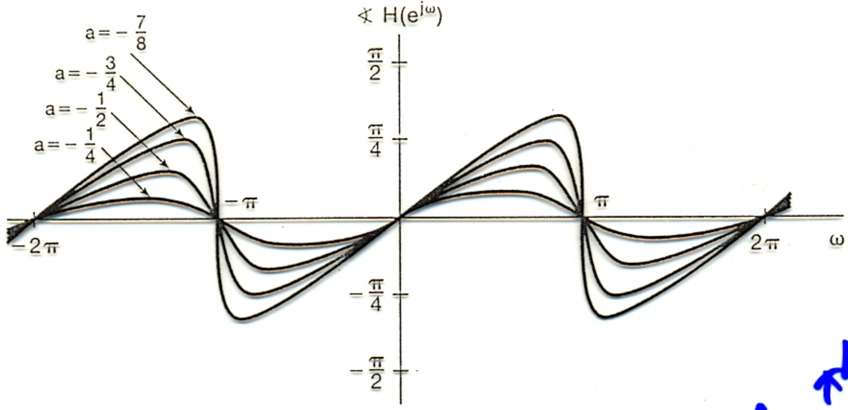


(a)

Figure 6.28 Magnitude and phase of the frequency response of eq. (6.52) for a first-order system: (a) plots for several values of $a > 0$; (b) plots for several values of $a < 0$.



$-1 < a < 0$



(b)

Figure 6.28 *Continued*

Recall that if $x, h \in \ell_1(-\infty,\infty)$, $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$ is well defined

6.6.2 Second-Order Discrete-Time Systems

Consider next the second-order causal LTI system described by

$$y[n] - 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n], \tag{6.57}$$

with $\boxed{0 < r < 1}$ and $0 \leq \theta \leq \pi$. The frequency response for this system is

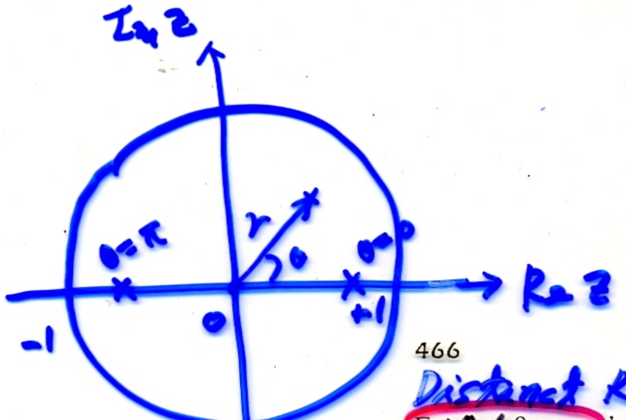
$$H(e^{j\omega}) = \frac{1}{1 - 2r\cos\theta e^{-j\omega} + r^2 e^{-j2\omega}}. \tag{6.58}$$

The denominator of $H(e^{j\omega})$ can be factored to obtain

$$H(e^{j\omega}) = \frac{1}{[1 - (re^{j\theta})e^{-j\omega}][1 - (re^{-j\theta})e^{-j\omega}]} \tag{6.59}$$

Remark (shown in chapter 10)

$0 < r < 1 \Rightarrow$ BIBO stable



$z = re^{j\theta}$

$a^n u[n] \xleftrightarrow{\mathcal{F}} \dfrac{1}{1-ae^{-j\omega}}$

From Fig. (5.18)

Distinct Roots

For $\theta \neq 0$ or π, the two factors in the denominator of $H(e^{j\omega})$ are different, and a partial-fraction expansion yields

$$H(e^{j\omega}) = \frac{A}{1-(re^{j\theta})e^{-j\omega}} + \frac{B}{1-(re^{-j\theta})e^{-j\omega}}, \tag{6.60}$$

where

(5.51)

$$A = \frac{e^{j\theta}}{2j\sin\theta}, \quad B = \frac{e^{-j\theta}}{2j\sin\theta}. \tag{6.61}$$

In this case, the impulse response of the system is

$$\begin{aligned} h[n] &= [A(re^{j\theta})^n + B(re^{-j\theta})^n]u[n] \\ &= r^n \frac{\sin[(n+1)\theta]}{\sin\theta} u[n]. \end{aligned} \tag{6.62}$$

Double Roots

For $\theta = 0$ or π, the two factors in the denominator of eq. (6.58) are the same. When $\theta = 0$,

$$H(e^{j\omega}) = \frac{1}{(1-re^{-j\omega})^2} \tag{6.63}$$

(5.54)

and

$$h[n] = (n+1)r^n u[n]. \tag{6.64}$$

When $\theta = \pi$,

$$H(e^{j\omega}) = \frac{1}{(1+re^{-j\omega})^2} \tag{6.65}$$

(5.54)

and

$$h[n] = (n+1)(-r)^n u[n]. \tag{6.66}$$

Recall:

$n x[n] \xleftrightarrow{\mathcal{F}} j \dfrac{dX(e^{j\omega})}{d\omega}$

$(n+1)x[n+1] \xleftrightarrow{\mathcal{F}} j e^{j\omega} \dfrac{dX(e^{j\omega})}{d\omega}$

The impulse responses for second-order systems are plotted in Figure 6.29 for a range of values of r and θ. From this figure and from eq. (6.62), we see that the rate of decay of $h[n]$ is controlled by r—i.e., the closer r is to 1, the slower is the decay in $h[n]$. Similarly, the value of θ determines the frequency of oscillation. For example, with $\theta = 0$ there is no oscillation in $h[n]$, while for $\theta = \pi$ the oscillations are rapid. The effect of different values of r and θ can also be seen by examining the step response of eq. (6.57). For $\theta \neq 0$ or π,

$s[n] = \sum_{k=0}^{n} h[k] \Rightarrow$

(zero-state response)

$$s[n] = h[n] * u[n] = \left[A\left(\frac{1-(re^{j\theta})^{n+1}}{1-re^{j\theta}}\right) + B\left(\frac{1-(re^{-j\theta})^{n+1}}{1-re^{-j\theta}}\right)\right]u[n]. \tag{6.67}$$

Also, using the result of Problem 2.52, we find that for $\theta = 0$,

$$s[n] = \left[\frac{1}{(r-1)^2} - \frac{r}{(r-1)^2}r^n + \frac{r}{r-1}(n+1)r^n\right]u[n], \tag{6.68}$$

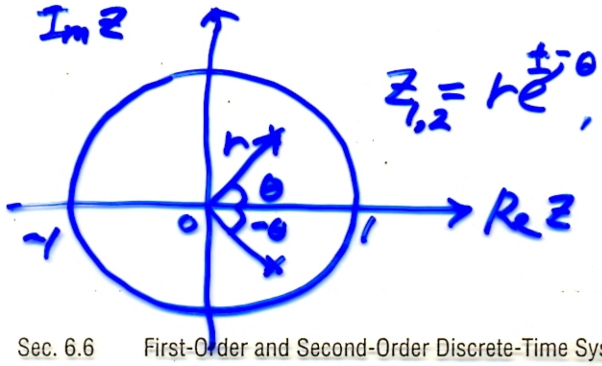
while for $\theta = \pi$,

$$s[n] = \left[\frac{1}{(r+1)^2} + \frac{r}{(r+1)^2}(-r)^n + \frac{r}{r+1}(n+1)(-r)^n\right]u[n]. \tag{6.69}$$

The step response is plotted in Figure 6.30, again for a range of values of r and θ

(Prob. 2.52)

$$\sum_{k=0}^{n} (k+1) r^k = \frac{d}{dr} \sum_{k=0}^{n+1} r^k = \frac{d}{dr}\left(\frac{1-r^{n+1}}{1-r}\right)$$



$$z_{1,2} = re^{\pm j\theta}, \quad 0<r<1$$

$$H(e^{j\omega}) = \frac{A}{1-(re^{j\theta})e^{-j\omega}} + \frac{B}{1-(re^{-j\theta})e^{-j\omega}}$$

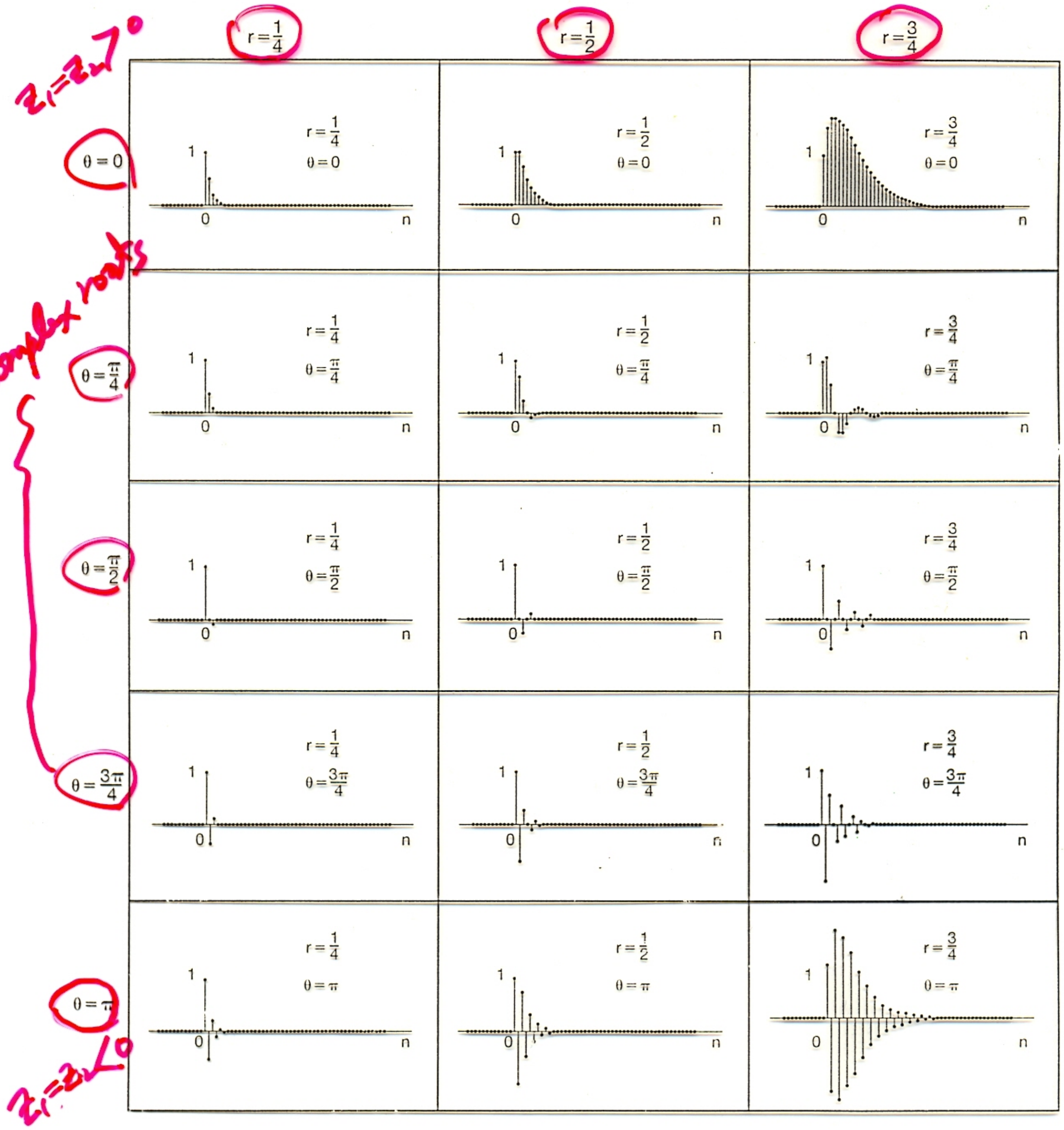
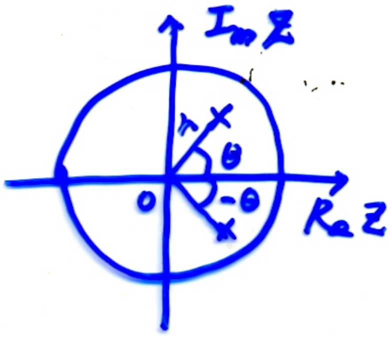


Figure 6.29 Impulse response of the second-order system of eq. (6.57) for a range of values of r and θ.

The second-order system given by eq. (6.57) is the counterpart of the *underdamped* second-order system in continuous time, while the special case of $\theta = 0$ is the critically damped case. That is, for any value of θ other than zero, the impulse response has a damped

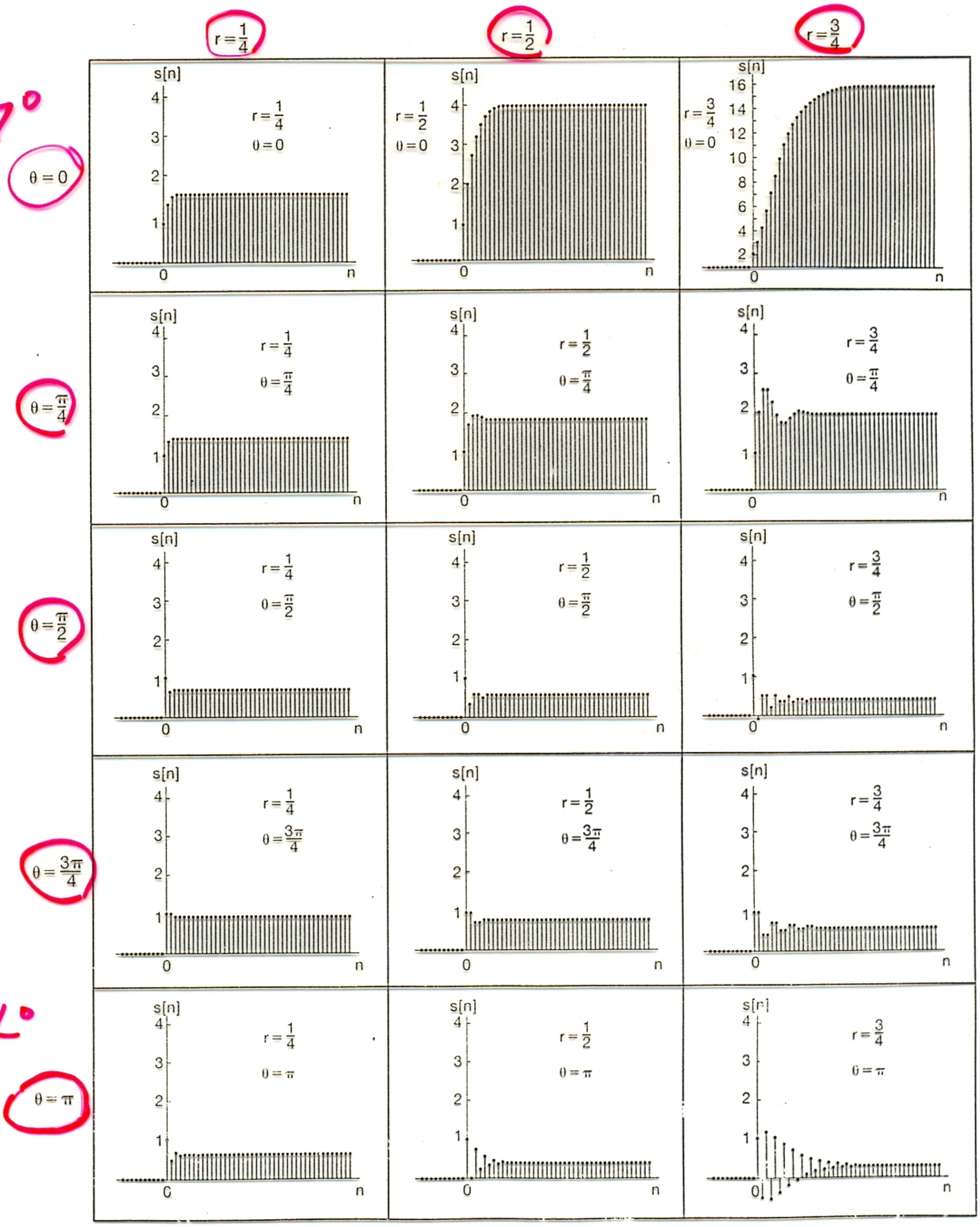
$z_{1,2} = re^{\pm j\theta}$, $0<r<1$



$z_1 = z_2 > 0$

complex roots

$z_1 = z_2 < 0$



* Note: The plot for $r=\frac{3}{4}$, $\theta=0$ has a different scale from the others.

Figure 6.30 Step response of the second-order system of eq. (6.57) for a range of values of r and θ.

oscillatory behavior, and the step response exhibits ringing and overshoot. The frequency response of this system is depicted in Figure 6.31 for a number of values of r and θ. From Figure 6.31, we see that a band of frequencies is amplified, and r determines how sharply peaked the frequency response is within this band.

θ r

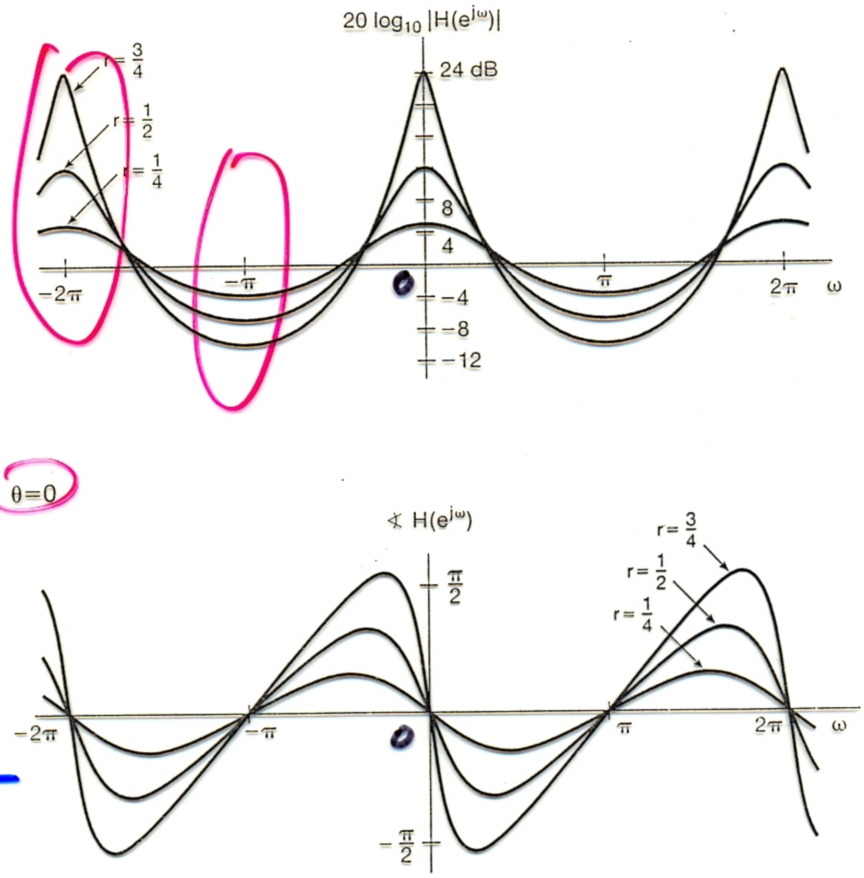
As we have just seen, the second-order system described in eq. (6.59) has factors with complex coefficients (unless $\theta = 0$ or π). It is also possible to consider second-order systems having factors with real coefficients. Specifically, consider

(6.59) $\theta = 0$ or π $\overset{d_1 = d_2}{\Longrightarrow}$

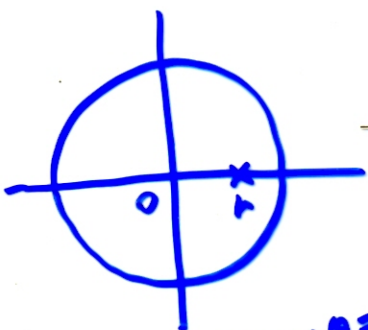
$$H(e^{j\omega}) = \frac{1}{(1 - d_1 e^{-j\omega})(1 - d_2 e^{-j\omega})}, \tag{6.70}$$

z_1, z_2 distinct real roots

where d_1 and d_2 are both real numbers with $|d_1|, |d_2| < 1$. Equation (6.70) is the frequency response for the difference equation



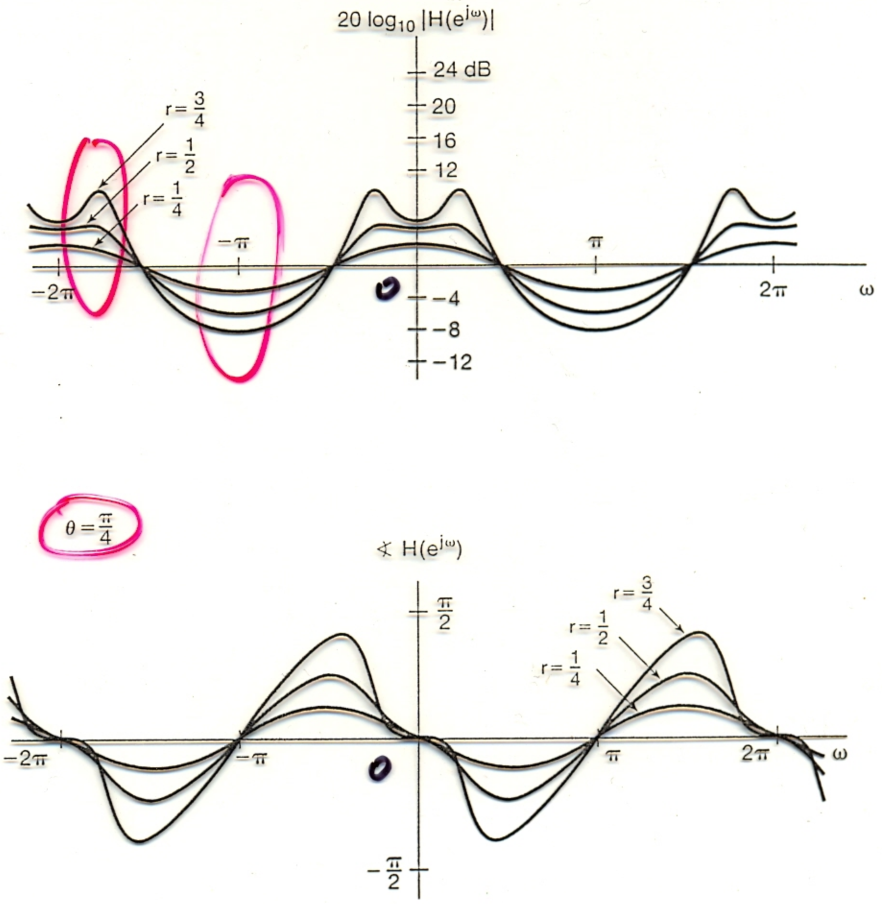
z-plane

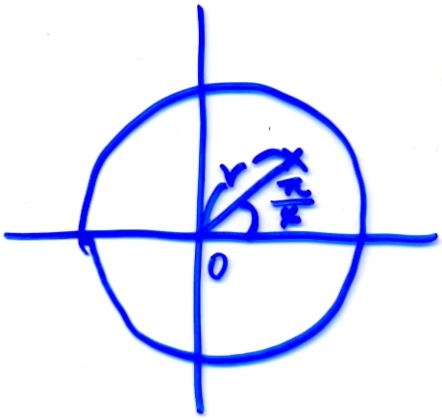


(cf. Fig 6.30: $\theta=0$)

(a)

Figure 6.31 Magnitude and phase of the frequency response of the second-order system of eq. (6.57): (a) $\theta = 0$; (b) $\theta = \pi/4$; (c) $\theta = \pi/2$; (d) $\theta = 3\pi/4$; (e) $\theta = \pi$. Each plot contains curves corresponding to $r = 1/4$, $1/2$, and $3/4$.





(cf. Fig. 6.30 : θ = π/4)

(b)

Figure 6.31 *Continued*

A special case of (6.57)

⇒ skip.

$$y[n] - (d_1 + d_2)y[n-1] + d_1 d_2 y[n-2] = x[n]. \tag{6.71}$$

In this case,

$$H(e^{j\omega}) = \frac{A}{1 - d_1 e^{-j\omega}} + \frac{B}{1 - d_2 e^{-j\omega}}, \tag{6.72}$$

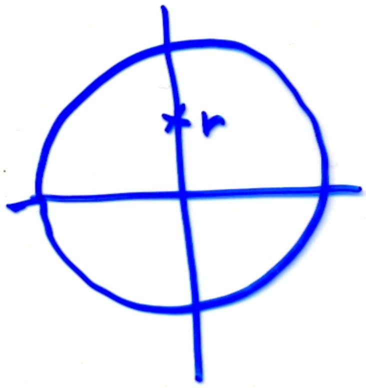
where

$$A = \frac{d_1}{d_1 - d_2}, \quad B = \frac{d_2}{d_2 - d_1}. \tag{6.73}$$

Thus,

(5.55), p.385 →

$$h[n] = [Ad_1^n + Bd_2^n]u[n], \tag{6.74}$$



(cf. Fig. 6.30 : $\theta = \frac{\pi}{2}$) $\theta = \frac{\pi}{2}$

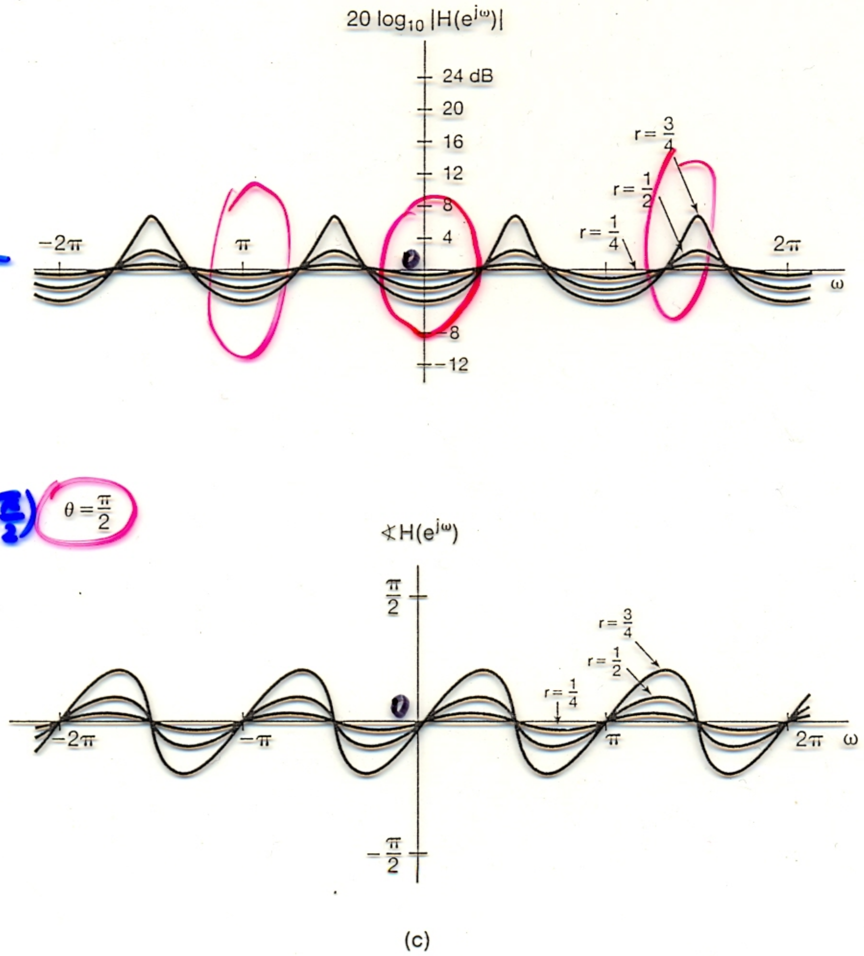
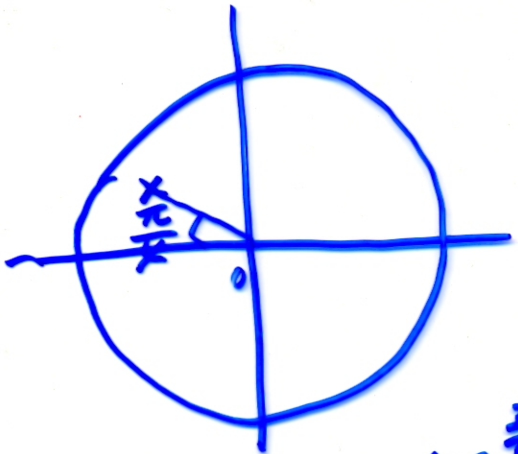


Figure 6.31 *Continued*

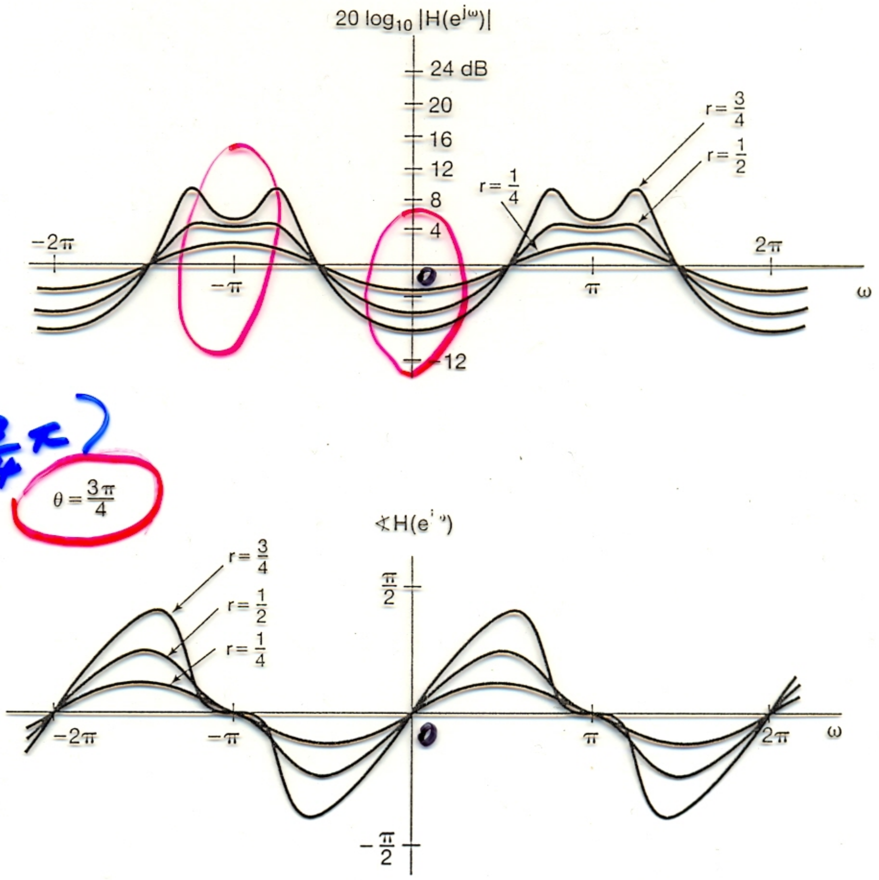
which is the sum of two decaying real exponentials. Also,

$$s[n] = \left[A\left(\frac{1-d_1^{n+1}}{1-d_1}\right) + B\left(\frac{1-d_2^{n+1}}{1-d_2}\right)\right]u[n]. \tag{6.75}$$

The system with frequency response given by eq. (6.70) corresponds to the cascade of two first-order systems. Therefore, we can deduce most of its properties from our understanding of the first-order case. For example, the log-magnitude and phase plots for eq. (6.70) can be obtained by adding together the plots for each of the two first-order terms. Also, as we saw for first-order systems, the response of the system is fast if $|d_1|$ and $|d_2|$ are small, but the system has a long settling time if either of these magnitudes is near 1. Furthermore, if d_1 and d_2 are negative, the response is oscillatory. The case when both d_1 and d_2 are positive is the counterpart of the overdamped case in continuous time, with the impulse and step responses settling without oscillation.



(cf. Fig. 6.30: $\theta = \frac{3}{4}\pi$)



(d)

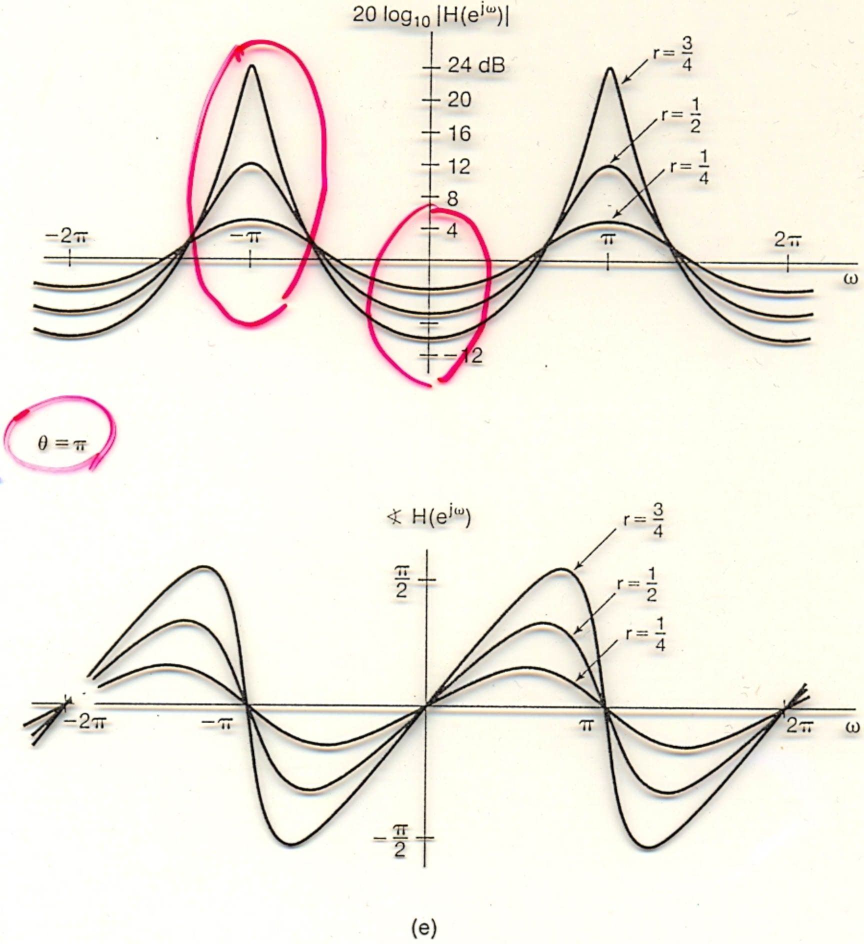
Figure 6.31 *Continued*

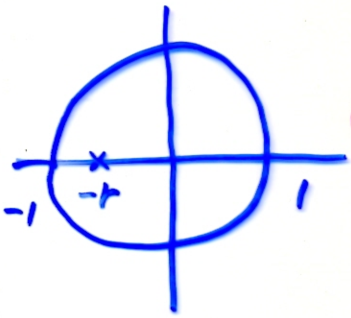
In this section, we have restricted attention to those causal first- and second-order systems that are stable and for which the frequency response can be defined. In particular, the causal system described by eq. (6.51) is unstable for $|a| \geq 1$. Also, the causal system described by eq. (6.57) is unstable if $r \geq 1$, and that described by eq. (6.71) is unstable if either $|d_1|$ or $|d_2|$ exceeds 1.

$\theta = 0$ or π

6.7 EXAMPLES OF TIME- AND FREQUENCY-DOMAIN ANALYSIS OF SYSTEMS

Throughout this chapter, we have illustrated the importance of viewing systems in both the time domain and the frequency domain and the importance of being aware of trade-offs in the behavior between the two domains. In this section, we illustrate some of these issues further. In Section 6.7.1, we discuss these trade-offs for continuous time in the context of an automobile suspension system. In Section 6.7.2, we discuss an important class of discrete-time filters referred to as moving-average or nonrecursive systems.





(e)

Figure 6.31 *Continued*

6.7.1 Analysis of an Automobile Suspension System

A number of the points that we have made concerning the characteristics and trade-offs in continuous-time systems can be illustrated in the interpretation of an automobile suspension system as a lowpass filter. Figure 6.32 shows a diagrammatic representation of a simple suspension system comprised of a spring and dashpot (shock absorber). The road surface can be thought of as a superposition of rapid small-amplitude changes in elevation (high frequencies), representing the roughness of the road surface, and gradual changes in elevation (low frequencies) due to the general topography. The automobile suspension system is generally intended to filter out rapid variations in the ride caused by the road surface (i.e., the system acts as a lowpass filter).

The basic purpose of the suspension system is to provide a smooth ride, and there is no sharp, natural division between the frequencies to be passed and those to be rejected. Thus, it is reasonable to accept and, in fact, prefer a lowpass filter that has a gradual

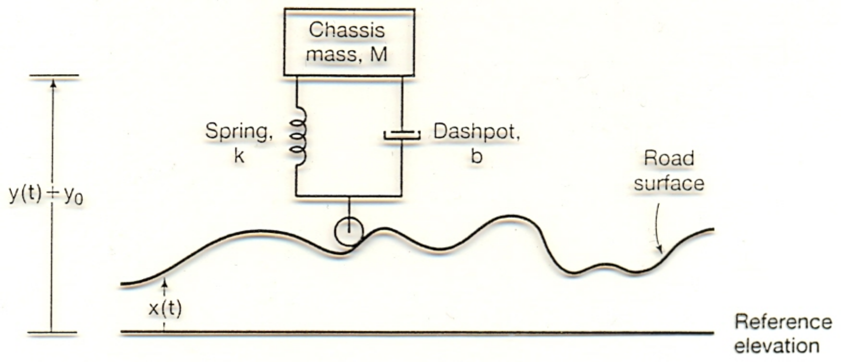


Figure 6.32 Diagrammatic representation of an automotive suspension system. Here, y_0 represents the distance between the chassis and the road surface when the automobile is at rest, $y(t) + y_0$ the position of the chassis above the reference elevation, and $x(t)$ the elevation of the road above the reference elevation.

transition from passband to stopband. Furthermore, the time-domain characteristics of the system are important. If the impulse response or step response of the suspension system exhibits ringing, then a large bump in the road (modeled as an impulse input) or a curb (modeled as a step input) will result in an uncomfortable oscillatory response. In fact, a common test for a suspension system is to introduce an excitation by depressing and then releasing the chassis. If the response exhibits ringing, it is an indication that the shock absorbers need to be replaced.

Cost and ease of implementation also play an important role in the design of automobile suspension systems. Many studies have been carried out to determine the most desirable frequency-response characteristics for suspension systems from the point of view of passenger comfort. In situations where the cost may be warranted, such as for passenger railway cars, intricate and costly suspension systems are used. For the automotive industry, cost is an important factor, and simple, less costly suspension systems are generally used. A typical automotive suspension system consists simply of the chassis connected to the wheels through a spring and a dashpot.

In the diagrammatic representation in Figure 6.32, y_0 represents the distance between the chassis and the road surface when the automobile is at rest, $y(t) + y_0$ the position of the chassis above the reference elevation, and $x(t)$ the elevation of the road above the reference elevation. The differential equation governing the motion of the chassis is then

$$M\frac{d^2y(t)}{dt} + b\frac{dy(t)}{dt} + ky(t) = kx(t) + b\frac{dx(t)}{dt}, \tag{6.76}$$

where M is the mass of the chassis and k and b are the spring and shock absorber constants, respectively. The frequency response of the system is

$$H(j\omega) = \frac{k + bj\omega}{(j\omega)^2M + b(j\omega) + k},$$

or

$$H(j\omega) = \frac{\omega_n^2 + 2\zeta\omega_n(j\omega)}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}, \tag{6.77}$$

where $\omega_n \triangleq \sqrt{\frac{k}{M}}$, $2\zeta\omega_n \triangleq \frac{b}{M}$

* Trade off between the time and frequency domains (?)
– quick recovery and drive comfort

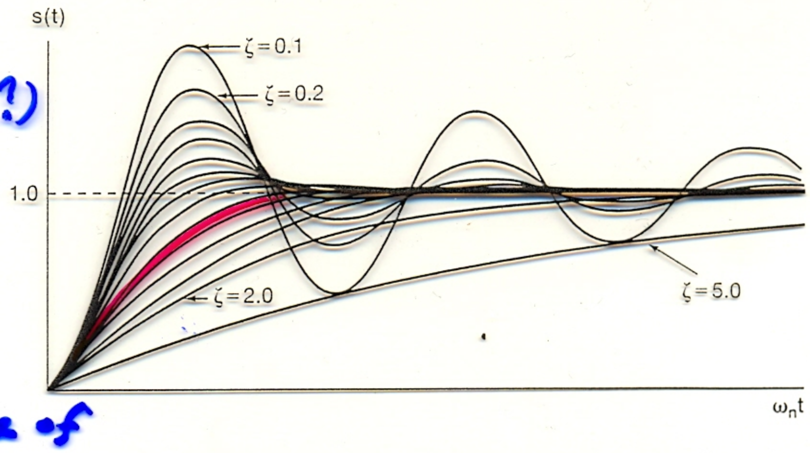


Figure 6.34 Step response of the automotive suspension system for various values of the damping ratio (ζ = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.2, 1.5, 2.0, 5.0).

By an appropriate choice of ω_n or k, the filter cutoff frequency is controlled, ⇒ smooth ride

domains. Generally, the shock absorber damping is chosen to have a rapid rise time and yet avoid overshoot and ringing. This choice corresponds to the critically damped case, with $\zeta = 1.0$, considered in Section 6.5.2.

6.7.2 Examples of Discrete-Time Nonrecursive Filters

<Fact 1> For a discrete-time filter to be causal and to have exactly linear phase, it must be a FIR filter.

<proof> Let
$$\begin{cases} \mathcal{F}\{h[n]\} = H(e^{j\omega}) = H_r(e^{j\omega})\, e^{-jM\omega}, \\ \mathcal{F}\{h_r[n]\} = H_r(e^{j\omega}), \end{cases}$$
where $H_r(e^{j\omega})$ is real and even.

Then,
$$\begin{cases} h_r[n] = h_r[-n] \quad (\text{since } \mathcal{F}\{h_r[-n]\} = H_r(e^{-j\omega})) \\ h[n] = h_r[n-M] \end{cases}$$

These imply that

$$h[M+n] = h[M-n] \Rightarrow h[n] = h[2M-n]$$

$$\Rightarrow \quad h[n] = 0 \text{ if } n<0 \text{ and } n>2M$$

$\square$

The moving-average filter

$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k]$$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{N+M+1}\, e^{j\omega[(N-M)/2]} \times \underbrace{\frac{\sin[\omega(M+N+1)/2]}{\sin(\omega/2)}}_{= H_r(e^{j\omega})}$$

linear phase

narrower passband ⇒ smoother
more complex

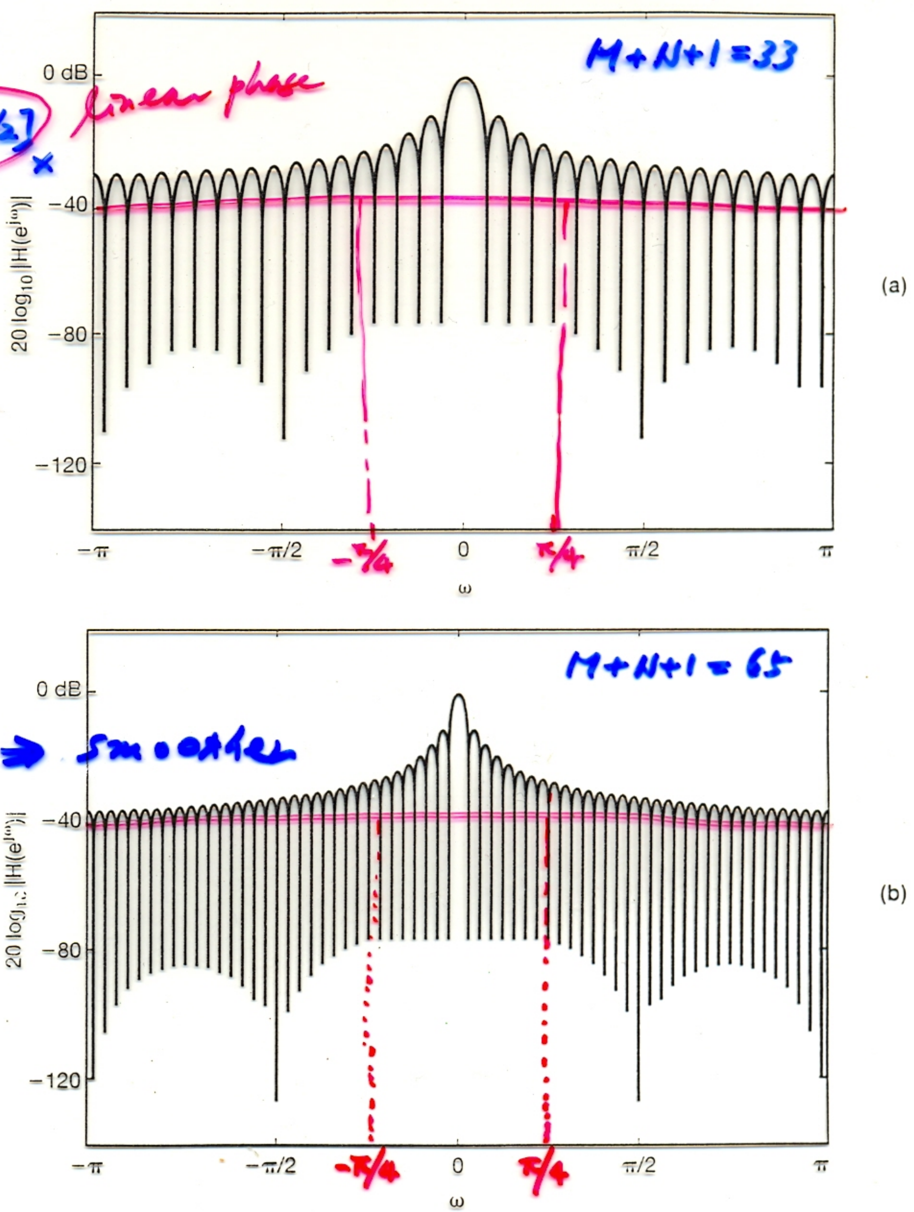
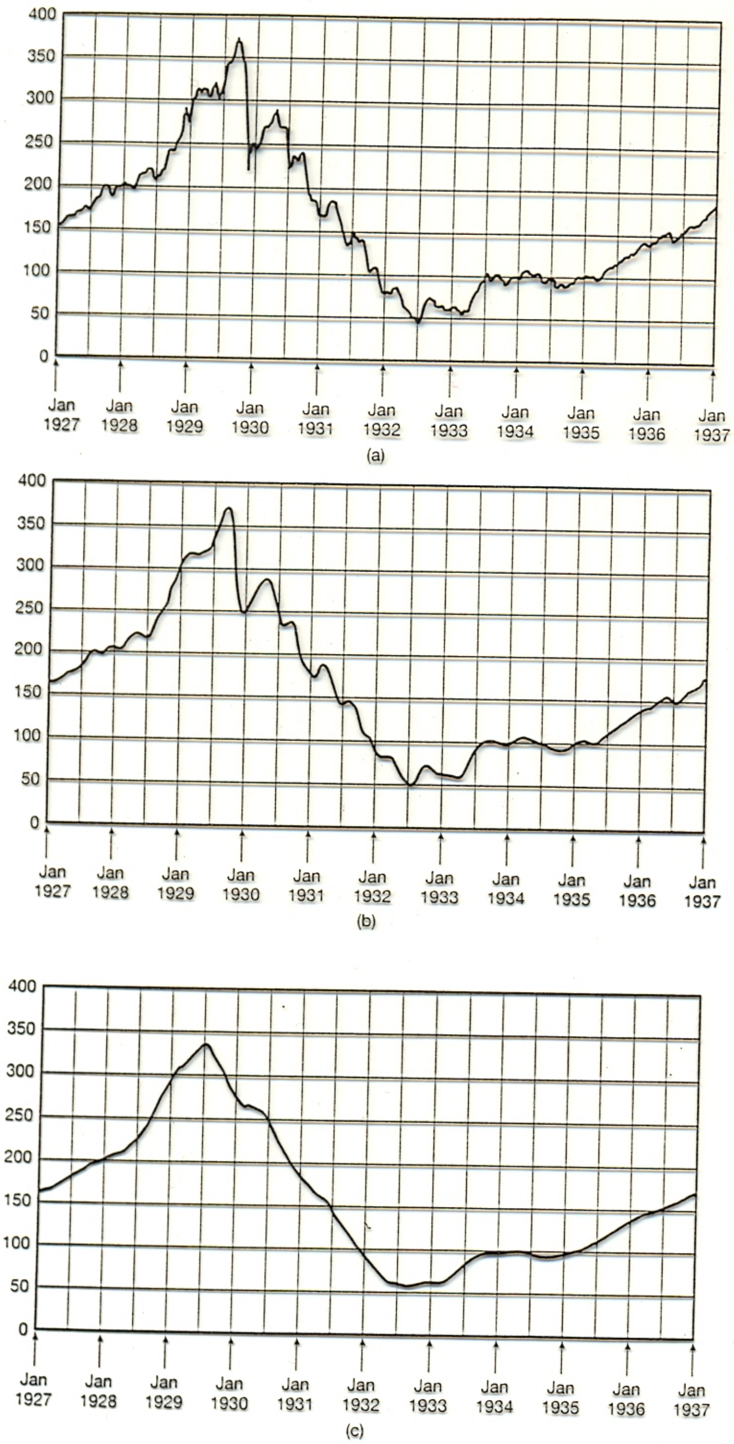


Figure 6.35 Log-magnitude plots for the moving-average filter of eqs. (6.78) and (6.79) for (a) $M + N + 1 = 33$ and (b) $M + N + 1 = 65$.

the response of the filter, let us consider a filter of the form of eq. (6.80), with $N = M = 16$ and the filter coefficients chosen to be

$$(6.81)$$



actual data

51-day moving average

201-day moving average

Figure 6.36 Effect of lowpass filtering on the Dow Jones weekly stock market index over a 10-year period using moving-average filters: (a) weekly index; (b) 51-day moving average applied to (a); (c) 201-day moving average applied to (a). The weekly stock market index and the two moving averages are discrete-time sequences. For clarity in the graphical display, the three sequences are shown here with their individual values connected by straight lines to form a continuous curve.

The general form of a discrete-time nonrecursive filter

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k] \qquad (6.80)$$

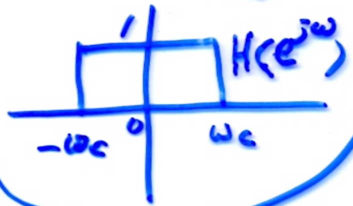
Choose $b_k \triangleq \begin{cases} \sin(2\pi k/33)/\pi k, & |k| \le 32 \\ 0, & |k| > 32 \end{cases}$

Then, the impulse response of the filter is

$$h[n] = \begin{cases} \dfrac{\sin(2\pi n/33)}{\pi n}, & |n| \le 32 \\ 0, & |n| > 32 \end{cases}. \qquad (6.82)$$

Recall that

$$h[n] = \frac{\sin \omega_c n}{\pi n} \quad (6.20)$$



Comparing this impulse response with eq. (6.20), we see that eq. (6.82) corresponds to truncating, for $|n| > 32$, the impulse response for the ideal lowpass filter with cutoff frequency $\omega_c = 2\pi/33$.

In general, the coefficients b_k can be adjusted so that the cutoff is at a desired frequency. For the example shown in Figure 6.37, the cutoff frequency was chosen to match approximately the cutoff frequency of Figure 6.35 for $N = M = 16$. Figure 6.37(a) shows the impulse response of the filter, and Figure 6.37(b) shows the log magnitude of the frequency response in dB. Comparing this frequency response to Figure 6.35, we observe that the passband of the filter has approximately the same width, but that the transition to the stopband is sharper.

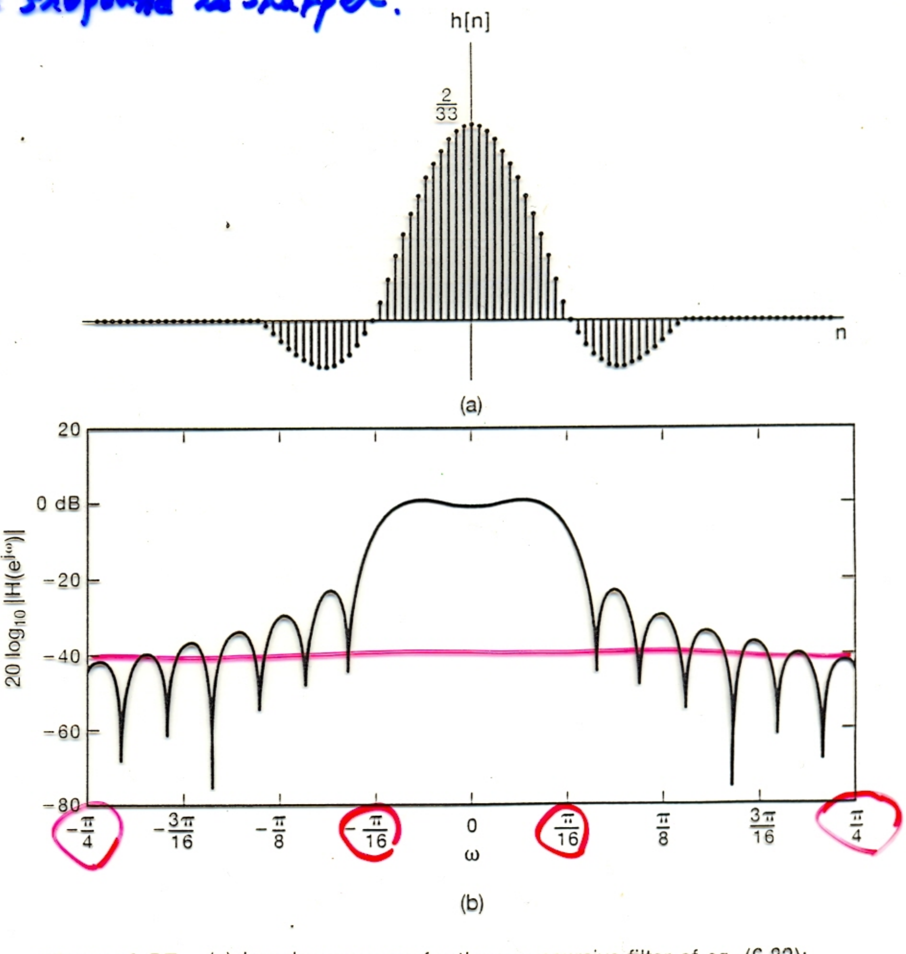


Figure 6.37 (a) Impulse response for the nonrecursive filter of eq. (6.82); (b) log magnitude of the frequency response of the filter.

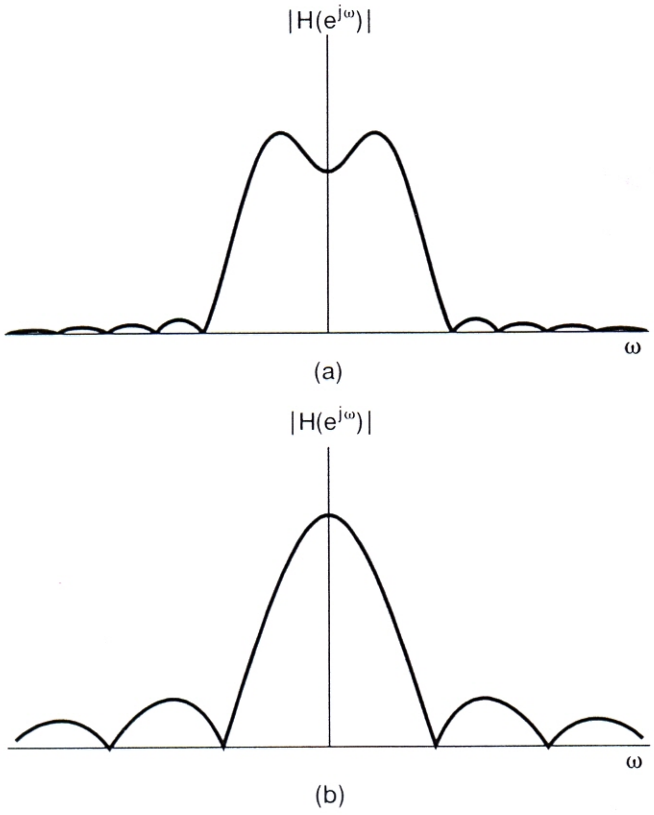


Figure 6.38 Comparison, on a linear amplitude scale, of the frequency responses of (a) Figure 6.37 and (b) Figure 6.35.

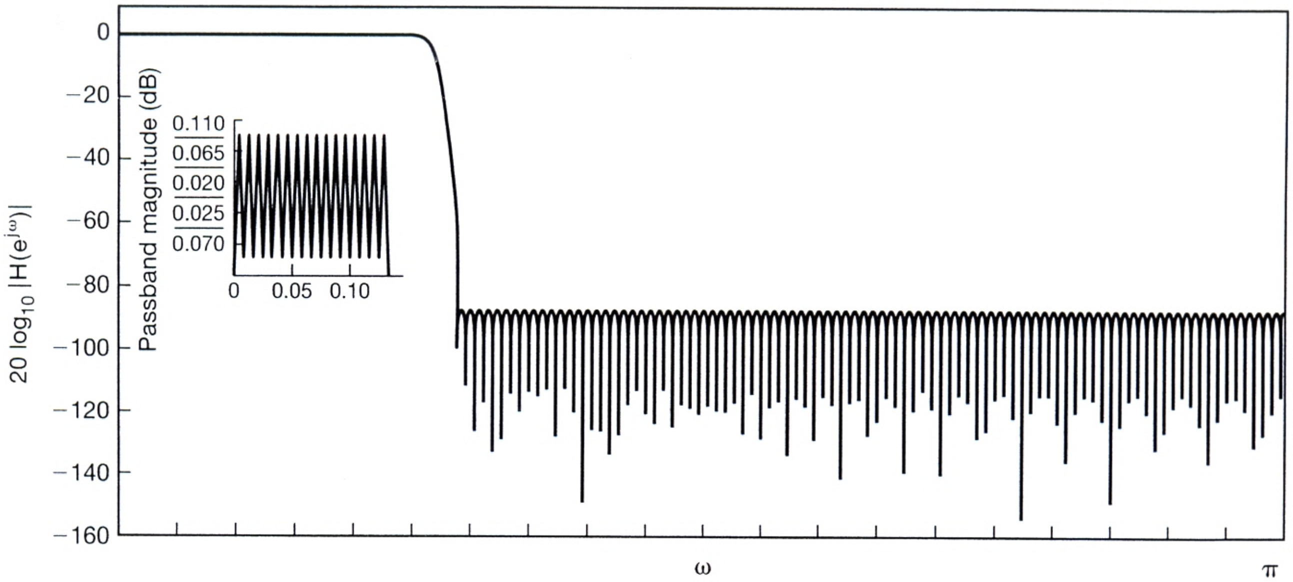


Figure 6.39 Lowpass nonrecursive filter with 251 coefficients designed to obtain the sharpest possible cutoff.

A noncausal FIR filter with an impulse response that is real and even can be converted to a causal filter with a linear-phase frequency response

Suppose that (i.e., $h[n] = 0$ for all $|n| > N$). If we now define the nonrecursive LTI system resulting from a simple N-step delay of $h[n]$, i.e.,

$$h_1[n] = h[n - N], \tag{6.83}$$

then $h_1[n] = 0$ for all $n < 0$, so that this LTI system is causal. Furthermore, from the time-shift property for discrete-time Fourier transforms, we see that the frequency response of the system is

$$H_1(e^{j\omega}) = H(e^{j\omega})e^{-j\omega N}. \tag{6.84}$$

Since $H(e^{j\omega})$ has zero phase, $H_1(e^{j\omega})$ does indeed have linear phase.

$\boxed{\text{HW \#6}}$

12, 16, 22.(c), 23, 28.(a)(iv), (vi), (b) 29.(a)
31. (a), (b), 41, 46, 56, 59