7.4 DISCRETE-TIME PROCESSING OF CONTINUOUS-TIME SIGNALS

In many applications, there is a significant advantage offered in processing a continuous-time signal by first converting it to a discrete-time signal and, after discrete-time processing, converting back to a continuous-time signal. The discrete-time signal processing can be implemented with a general- or special-purpose computer, with microprocessors, or with any of the variety of devices that are specifically oriented toward discrete-time signal processing.

In broad terms, this approach to continuous-time signal processing can be viewed as the cascade of three operations, as indicated in Figure 7.19, where $x_c(t)$ and $y_c(t)$ are continuous-time signals and $x_d[n]$ and $y_d[n]$ are the discrete-time signals corresponding to $x_c(t)$ and $y_c(t)$. The overall system is, of course, a continuous-time system in the sense that its input and output are both continuous-time signals. The theoretical basis for converting a continuous-time signal to a discrete-time signal and reconstructing a continuous-time signal from its discrete-time representation lies in the sampling theorem, as discussed in Section 7.1. Through the process of periodic sampling with the sampling frequency consistent with the conditions of the sampling theorem, the continuous-time signal $x_c(t)$ is exactly represented by a sequence of instantaneous sample values $x_c(nT)$; that is, the discrete-time sequence $x_d[n]$ is related to $x_c(t)$ by

$$x_d[n] = x_c(nT). \tag{7.16}$$

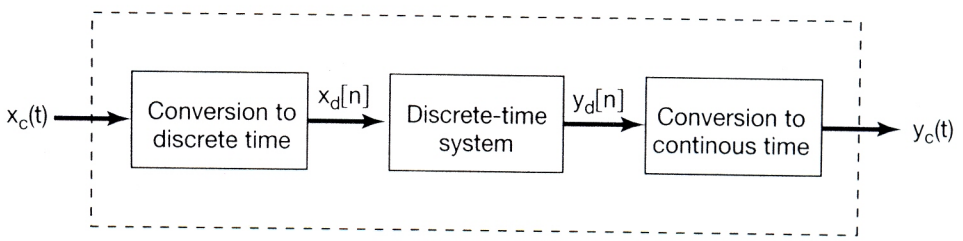


Figure 7.19 Discrete-time processing of continuous-time signals.

The transformation of $x_c(t)$ to $x_d[n]$ corresponding to the first system in Figure 7.19 will be referred to as *continuous-to-discrete-time conversion* and will be abbreviated C/D. The reverse operation corresponding to the third system in Figure 7.19 will be abbreviated D/C, representing *discrete-time to continuous-time conversion*. The D/C operation performs an interpolation between the sample values provided to it as input. That is, the D/C operation produces a continuous-time signal $y_c(t)$ which is related to the discrete-time signal $y_d[n]$ by

$$y_d[n] = y_c(nT).$$

This notation is made explicit in Figure 7.20. In systems such as digital computers and digital systems for which the discrete-time signal is represented in digital form, the device commonly used to implement the C/D conversion is referred to as an *analog-to-digital* (A-to-D) converter, and the device used to implement the D/C conversion is referred to as a *digital-to-analog* (D-to-A) converter.

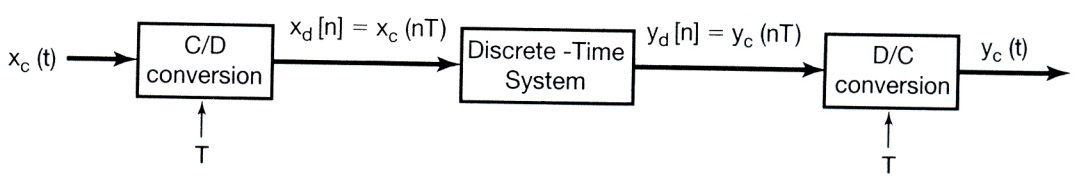
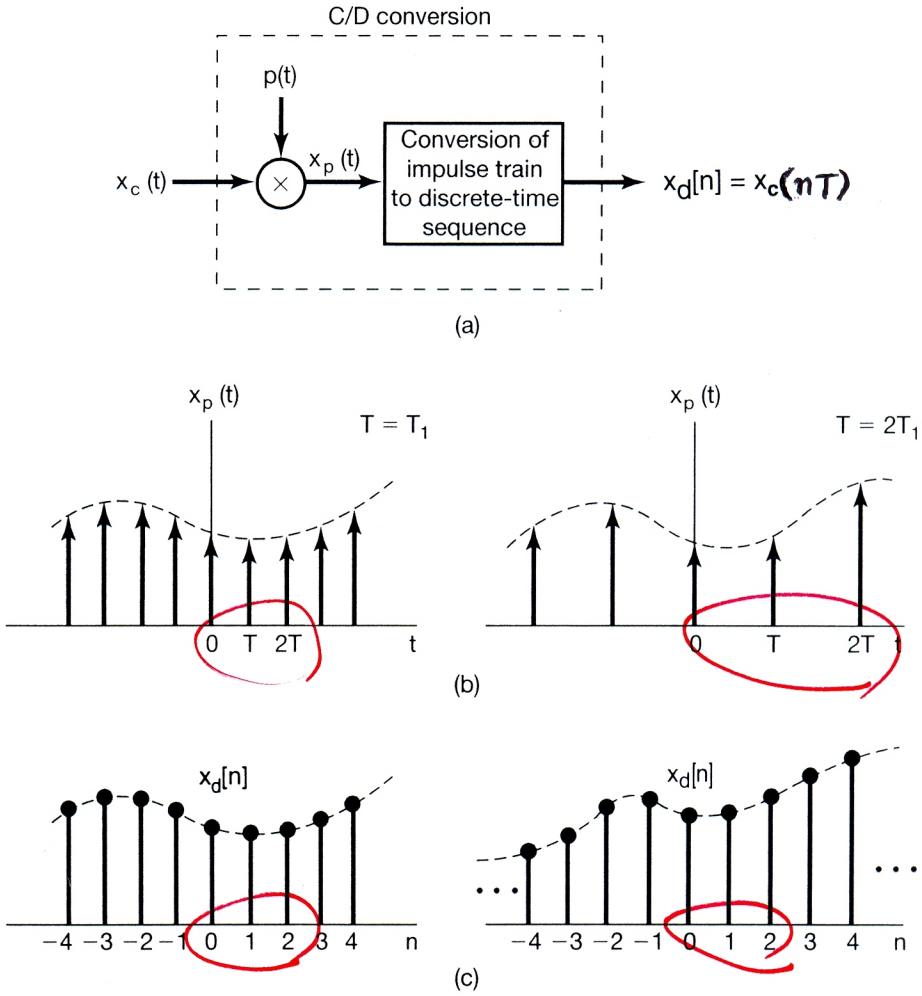


Figure 7.20 Notation for continuous-to-discrete-time conversion and discrete-to-continuous-time conversion. T represents the sampling period.



The time axis is scaled by T/1

Figure 7.21 Sampling with a periodic impulse train followed by conversion to a discrete-time sequence: (a) overall system; (b) $x_p(t)$ for two sampling rates. The dashed envelope represents $x_c(t)$; (c) the output sequence for the two different sampling rates.

$$\boxed{X_d(e^{j\Omega}) \triangleq \mathcal{F}\{x_d[n]\} = X_p(j\Omega/T) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2\pi k)/T)} \quad (7.23)$$

To begin let us express $X_p(j\omega)$, the continuous-time Fourier transform of $x_p(t)$, in terms of the sample values of $x_c(t)$ by applying the Fourier transform to eq. (7.3). Since

<proof>

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t - nT), \tag{7.17}$$

and since the transform of $\delta(t - nT)$ is $e^{-j\omega nT}$, it follows that

$$X_p(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\omega nT} \tag{7.18}$$

Now consider the discrete-time Fourier transform of $x_d[n]$, that is,

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_d[n]e^{-j\Omega n}, \tag{7.19}$$

or, using eq. (7.16),

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\Omega n}. \tag{7.20}$$

Comparing eqs. (7.18) and (7.20), we see that $X_d(e^{j\Omega})$ and $X_p(j\omega)$ are related through

$$X_d(e^{j\Omega}) = X_p(j\Omega/T). \tag{7.21}$$

Also, recall that, as developed in eq. (7.6) and illustrated in Figure 7.3,

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s)). \tag{7.22}$$

Consequently,

$$X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\Omega - 2\pi k)/T). \tag{7.23}$$

The relationship among $X_c(j\omega)$, $X_p(j\omega)$, and $X_d(e^{j\Omega})$ is illustrated in Figure 7.22 for two different sampling rates. Here, $X_d(e^{j\Omega})$ is a frequency-scaled version of $X_p(j\omega)$

The frequency axis is scaled by T

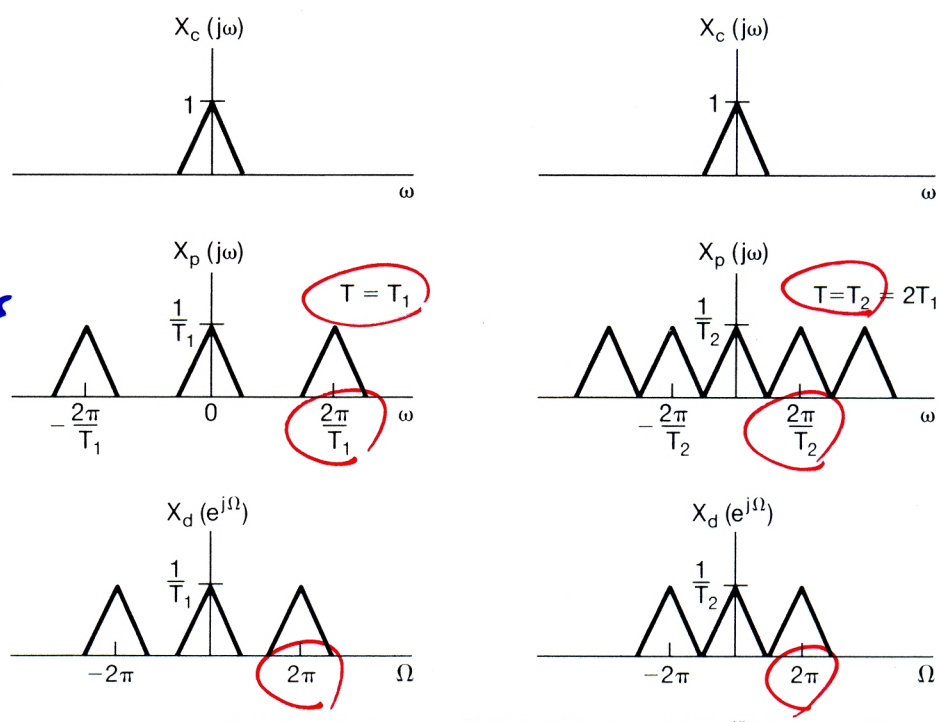


Figure 7.22 Relationship between $X_c(j\omega)$, $X_p(j\omega)$, and $X_d(e^{j\Omega})$ for two different sampling rates.

and, in particular, is periodic in Ω with period 2π. This periodicity is, of course, characteristic of any discrete-time Fourier transform. The spectrum of $x_d[n]$ is related to that of $x_c(t)$ through periodic replication, represented by eq. (7.22), followed by linear frequency scaling, represented by eq. (7.21). The periodic replication is a consequence of the first step in the conversion process in Figure 7.21, namely, the impulse-train sampling. The linear frequency scaling in eq. (7.21) can be thought of informally as a consequence of the normalization in time introduced by converting from the impulse train $x_p(t)$ to the discrete-time sequence $x_d[n]$. From the time-scaling property of the Fourier transform in Section 4.3.5, scaling of the time axis by $1/T$ will introduce a scaling of the frequency axis by T. Thus, the relationship $\Omega = \omega T$ is consistent with the notion that, in converting from $x_p(t)$ to $x_d[n]$, the time axis is scaled by $1/T$.

In the overall system of Figure 7.19, after processing with a discrete-time system, the resulting sequence is converted back to a continuous-time signal. This process is the reverse of the steps in Figure 7.21. Specifically, from the sequence $y_d[n]$, a continuous-time impulse train $y_p(t)$ can be generated. Recovery of the continuous-time signal $y_c(t)$ from this impulse train is then accomplished by means of lowpass filtering, as illustrated in Figure 7.23.

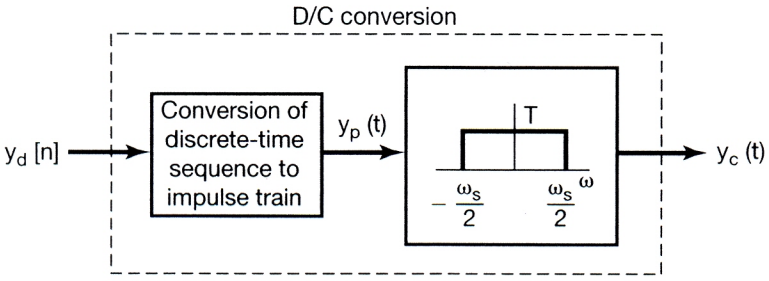
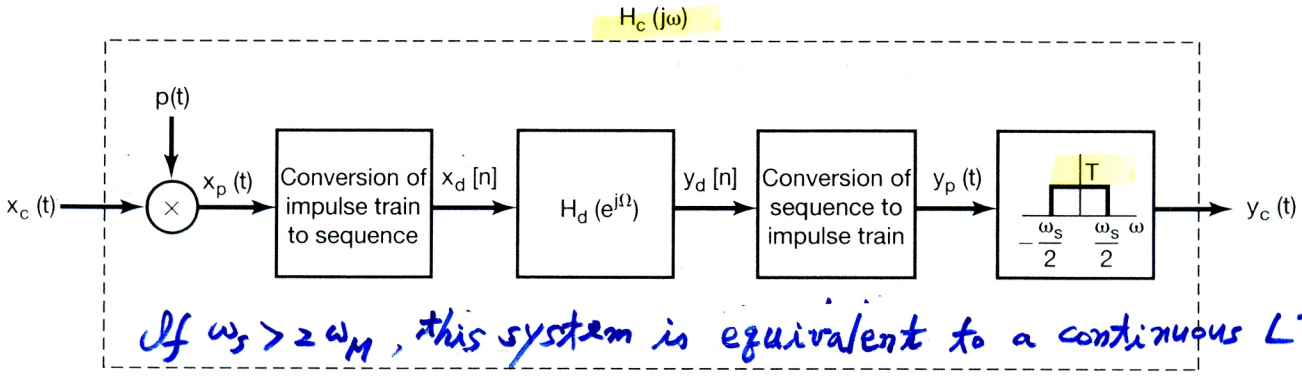


Figure 7.23 Conversion of a discrete-time sequence to a continuous-time signal.

Now let us consider the overall system of Figure 7.19, represented as shown in Figure 7.24. Clearly, if the discrete-time system is an identity system (i.e., $x_d[n] = y_d[n]$), then, assuming that the conditions of the sampling theorem are met, the overall system will be an identity system. The characteristics of the overall system with a more general frequency response $H_d(e^{j\Omega})$ are perhaps best understood by examining the representative example depicted in Figure 7.25. On the left-hand side of the figure are the representative



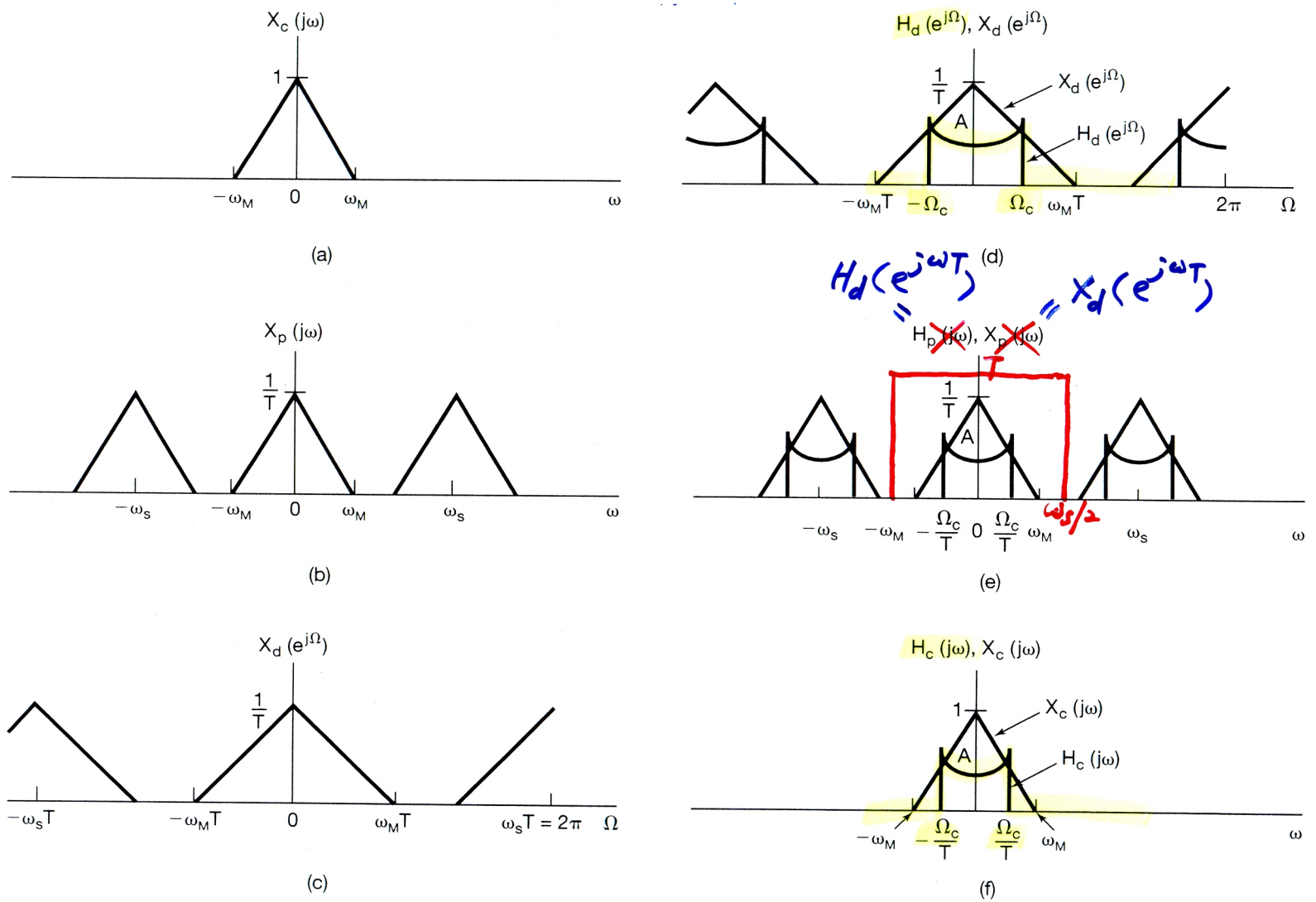
If $\omega_s > 2\omega_M$, this system is equivalent to a continuous LTI system.

Figure 7.24 Overall system for filtering a continuous-time signal using a discrete-time filter.

Note : $y_c(nT) = y_d[n]$

$x_c(nT) = x_d[n]$

Assume : $\omega_s > 2\omega_M$.



The freq. axis is scaled by T

That is, $\Omega = \omega T$

The freq. axis is scaled by $1/T$

That is, $\Omega = \omega T$ $(\omega = \Omega/T)$

Figure 7.25 Frequency-domain illustration of the system of Figure 7.24: (a) continuous-time spectrum $X_c(j\omega)$; (b) spectrum after impulse-train sampling; (c) spectrum of discrete-time sequence $x_d[n]$; (d) $H_d(e^{j\Omega})$ and $X_d(e^{j\Omega})$ that are multiplied to form $Y_d(e^{j\Omega})$; (e) spectra that are multiplied to form $Y_p(j\omega)$; (f) spectra that are multiplied to form $Y_c(j\omega)$.

$$\Rightarrow H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}) & \text{if } |\omega| < \omega_s/2 \\ 0, & \text{otherwise} \end{cases} \iff H_d(e^{j\Omega}) = H_c(j\Omega/T) \text{ for } |\Omega| \le \pi$$

spectra $X_c(j\omega)$, $X_p(j\omega)$, and $X_d(e^{j\Omega})$, where we assume that $\omega_M < \omega_s/2$, so that there is no aliasing. The spectrum $Y_d(e^{j\Omega})$ corresponding to the output of the discrete-time filter is the product of $X_d(e^{j\Omega})$ and $H_d(e^{j\Omega})$, and this is depicted in Figure 7.25(d) by overlaying $H_d(e^{j\Omega})$ and $X_d(e^{j\Omega})$. The transformation to $Y_c(j\omega)$ then corresponds to applying a frequency scaling and lowpass filtering, resulting in the spectra indicated in Figure 7.25(e) and (f). Since $Y_d(e^{j\Omega})$ is the product of the two overlaid spectra in Figure 7.25(d), the frequency scaling and lowpass filtering are applied to both. In comparing Figures 7.25(a) and (f), we see that

$$Y_c(j\omega) = X_c(j\omega)H_d(e^{j\omega T}). \tag{7.24}$$

Consequently, for inputs that are sufficiently band limited, so that the sampling theorem is satisfied, the overall system of Figure 7.24 is, in fact, equivalent to a continuous-time LTI system with frequency response $H_c(j\omega)$ which is related to the discrete-time frequency response $H_d(e^{j\Omega})$ through

$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}), & |\omega| < \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}. \tag{7.25}$$

The equivalent frequency response for this continuous-time filter is one period of the frequency response of the discrete-time filter with a linear scale change applied to the frequency axis. This relationship between the discrete-time frequency response and the equivalent continuous-time frequency response is illustrated in Figure 7.26.

The equivalence of the overall system of Figure 7.24 to an LTI system is somewhat surprising in view of the fact that multiplication by an impulse train is *not* a time-invariant operation. In fact, the overall system of Figure 7.24 is not time invariant for arbitrary inputs. For example, if $x_c(t)$ was a narrow rectangular pulse of duration less than T, then a time shift of $x_c(t)$ could generate a sequence $x[n]$ that either had all zero values or had one nonzero value, depending on the alignment of the rectangular pulse relative to the

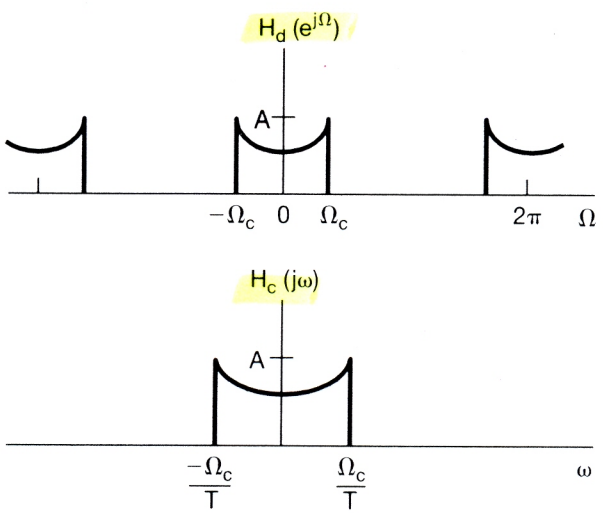


Figure 7.26 Discrete-time frequency response and the equivalent continuous-time frequency response for the system of Figure 7.24.

sampling impulse train. However, as suggested by the spectra of Figure 7.25, for *band-limited input signals* with a sampling rate sufficiently high so as to avoid aliasing, the system of Figure 7.24 *is* equivalent to a continuous-time LTI system. For such inputs, Figure 7.24 and eq. (7.25) provide the conceptual basis for continuous-time processing using discrete-time filters. This is now explored further in the context of some examples.

7.4.1 Digital Differentiator

Consider the discrete-time implementation of a continuous-time band-limited differentiating filter. As discussed in Section 3.9.1, the frequency response of a continuous-time differentiating filter is

$$H_c(j\omega) = j\omega, \tag{7.26}$$

and that of a band-limited differentiator with cutoff frequency ω_c is

$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}), & |\omega| < \omega_s/2 \\ 0, & \text{otherwise} \end{cases}$$

$$H_c(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}, \tag{7.27}$$

$\Updownarrow$

as sketched in Figure 7.27. Using eq. (7.25) with a sampling frequency $\omega_s = 2\omega_c$, we see that the corresponding discrete-time transfer function is

$$H_d(e^{j\Omega}) = H_c\left(j\frac{\Omega}{T}\right) \quad \text{for } |\Omega| < \pi \quad \Longrightarrow \quad H_d(e^{j\Omega}) = j\left(\frac{\Omega}{T}\right), \quad |\Omega| < \pi, \tag{7.28}$$

as sketched in Figure 7.28. With this discrete-time transfer function, $y_c(t)$ in Figure 7.24 will be the derivative of $x_c(t)$, assuming that there is no aliasing in sampling $x_c(t)$.

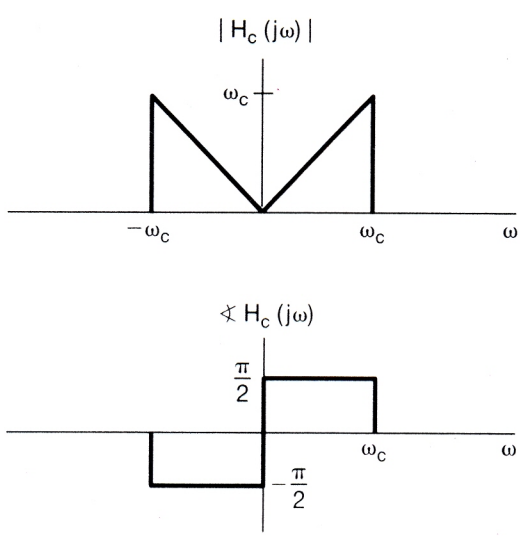


Figure 7.27 Frequency response of a continuous-time ideal band-limited differentiator $H_c(j\omega) = j\omega$, $|\omega| < \omega_c$.

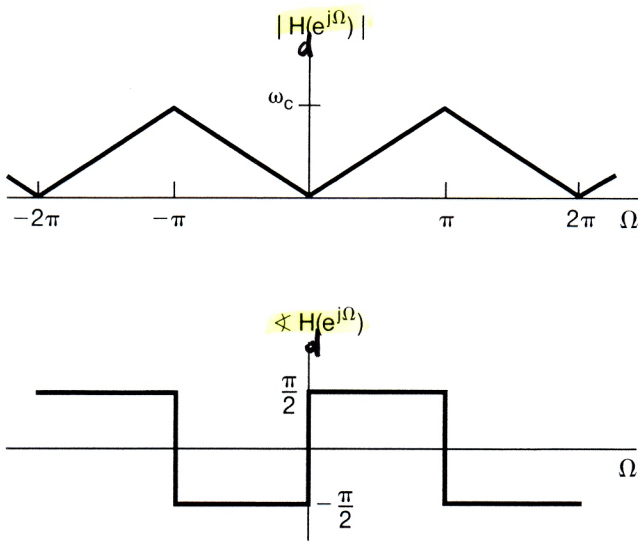


Figure 7.28 Frequency response of discrete-time filter used to implement a continuous-time band-limited differentiator.

Example 7.2

By considering the output of the digital differentiator for a continuous-time sinc input, we may conveniently determine the impulse response $h_d[n]$ of the discrete-time filter in the implementation of the digital differentiator. With reference to Figure 7.24, let

$$x_c(t) = \frac{\sin(\pi t/T)}{\pi t}, \tag{7.29}$$

where T is the sampling period. Then

$$X_c(j\omega) = \begin{cases} 1, & |\omega| < \pi/T \\ 0, & \text{otherwise} \end{cases},$$

Confirm that ω_c (BW of x_c) $< \omega_s/2$

which is sufficiently band limited to ensure that sampling $x_c(t)$ at frequency $\omega_s = 2\pi/T$ does not give rise to any aliasing. It follows that the output of the digital differentiator is

$$y_c(t) = \frac{d}{dt}x_c(t) = \frac{\cos(\pi t/T)}{Tt} - \frac{\sin(\pi t/T)}{\pi t^2}. \tag{7.30}$$

For $x_c(t)$ as given by eq. (7.29), the corresponding signal $x_d[n]$ in Figure 7.24 may be expressed as

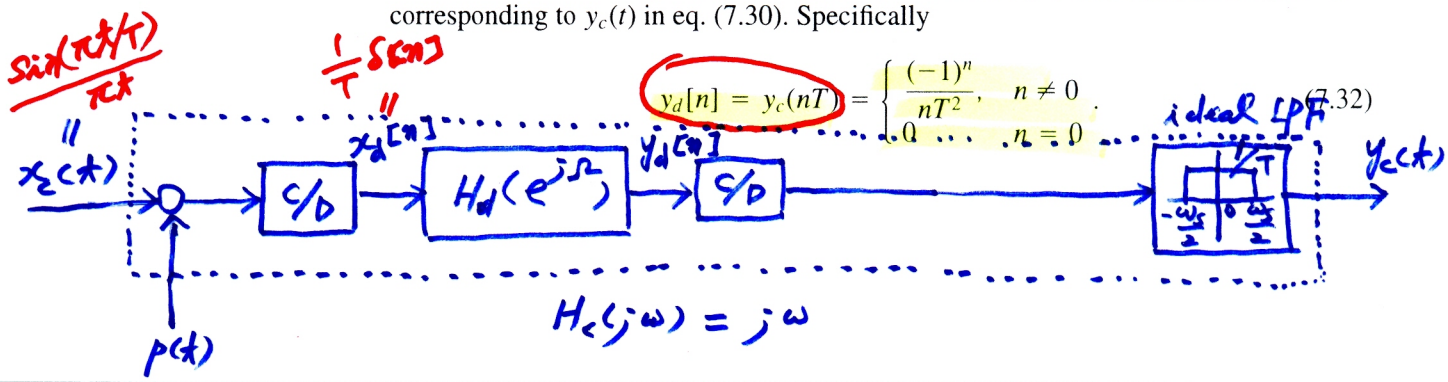
$$x_d[n] = x_c(nT) = \frac{\sin \pi n}{\pi n T} \Longrightarrow \quad x_d[n] = x_c(nT) = \frac{1}{T}\delta[n]. \Rightarrow \text{plays the role of impulse input} \tag{7.31}$$

That is, for $n \neq 0$, $x_c(nT) = 0$, while

$$x_d[0] = x_c(0) = \frac{1}{T}$$

which can be verified by l'Hôpital's rule. We can similarly evaluate $y_d[n]$ in Figure 7.24 corresponding to $y_c(t)$ in eq. (7.30). Specifically

$$y_d[n] = y_c(nT) = \begin{cases} \dfrac{(-1)^n}{nT^2}, & n \neq 0 \\ 0, & n = 0 \end{cases}. \tag{7.32}$$



which can be verified for $n \neq 0$ by direct substitution into eq. (7.30) and for $n = 0$ by application of l'Hôpital's rule.

Thus when the input to the discrete-time filter given by eq. (7.28) is the scaled unit impulse in eq. (7.31), the resulting output is given by eq. (7.32). We then conclude that the impulse response of this filter is given by

$$\left.\begin{matrix}(7.32)\\(7.31)\end{matrix}\right\} \Longrightarrow \quad h_d[n] = \begin{cases} \dfrac{(-1)^n}{nT}, & n \neq 0 \\ 0, & n = 0 \end{cases}$$

• noncausal
• not BIBO stable

7.4.2 Half-Sample Delay

In this section, we consider the implementation of a time shift (delay) of a continuous-time signal through the use of a system in the form of Figure 7.19. Thus, we require that the input and output of the overall system be related by

$$y_c(t) = x_c(t - \Delta) \tag{7.33}$$

when the input $x_c(t)$ is band limited and the sampling rate is high enough to avoid aliasing and where Δ represents the delay time. From the time-shifting property derived in Section 4.3.2

$$Y_c(j\omega) = e^{-j\omega\Delta} X_c(j\omega).$$

From eq. (7.25), the equivalent continuous-time system to be implemented must be band limited. Therefore, we take

$H_d(e^{j\Omega}) = e^{-j\Omega\Delta/T}$ for $|\Omega| < \pi$ (9.35)

$$\Longleftarrow \quad H_c(j\omega) = \begin{cases} e^{-j\omega\Delta}, & |\omega| < \omega_c \\ 0, & \text{otherwise} \end{cases}, \tag{7.34}$$

where ω_c is the cutoff frequency of the continuous-time filter. That is, $H_c(j\omega)$ corresponds to a time shift as in eq. (7.33) for band-limited signals and rejects all frequencies greater than ω_c. The magnitude and phase of the frequency response are shown in Figure 7.29(a). With the sampling frequency ω_s taken as $\omega_s = 2\omega_c$, the corresponding discrete-time

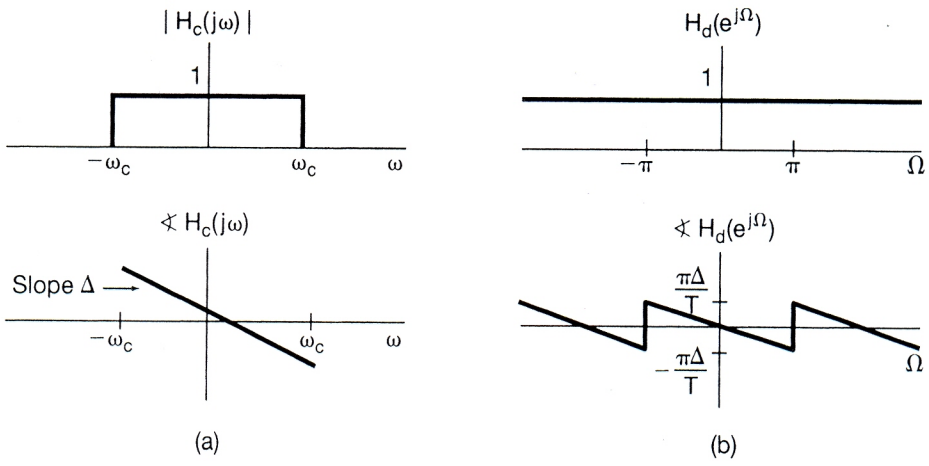


Figure 7.29 (a) Magnitude and phase of the frequency response for a continuous-time delay; (b) magnitude and phase of the frequency response for the corresponding discrete-time delay.