frequency response is

$$H_d(e^{j\Omega}) = e^{-j\Omega\Delta/T}, \quad |\Omega| < \pi, \tag{7.35}$$

and is shown in Figure 7.29(b).

For appropriately band-limited inputs, the output of the system of Figure 7.24 with $H_d(e^{j\Omega})$ as in eq. (7.35) is a delayed replica of the input. For Δ/T an integer, the sequence $y_d[n]$ is a delayed replica of $x_d[n]$; that is,

$$Y_d(e^{j\Omega}) = e^{-j\Omega\Delta/T} X_d(e^{j\Omega}) \Rightarrow \quad y_d[n] = x_d\left[n - \frac{\Delta}{T}\right]. \quad \Rightarrow h[n] = \delta[n - \tfrac{\Delta}{T}]$$

For Δ/T not an integer, eq. (7.36), as written, has no meaning, since sequences are defined only at integer values of the index. However, we can interpret the relationship between $x_d[n]$ and $y_d[n]$ in these cases in terms of band-limited interpolation. The signals $x_c(t)$ and $x_d[n]$ are related through sampling and band-limited interpolation, as are $y_c(t)$ and $y_d[n]$. With $H_d(e^{j\Omega})$ in eq. (7.35), $y_d[n]$ is equal to samples of a shifted version of the band-limited interpolation of the sequence $x_d[n]$. This is illustrated in Figure 7.30 with $\Delta/T = 1/2$, which is sometimes referred to as a half-sample delay.

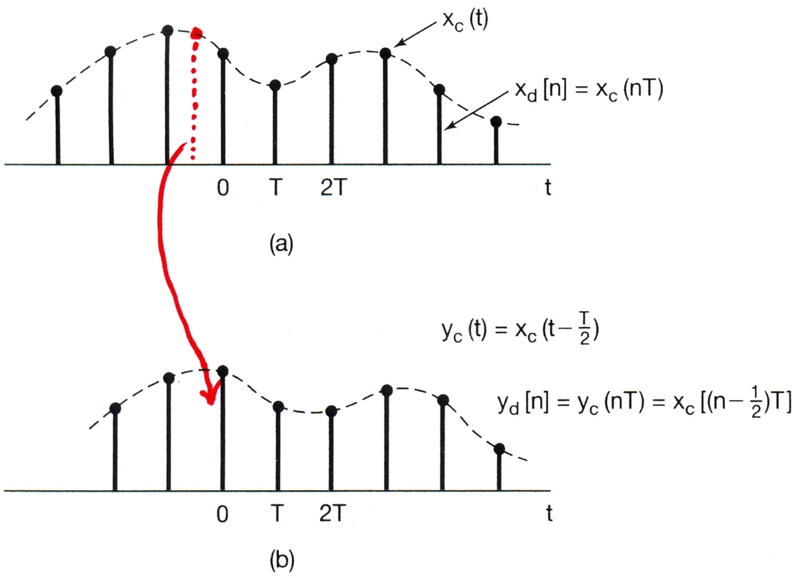


Figure 7.30 (a) Sequence of samples of a continuous-time signal $x_c(t)$; (b) sequence in (a) with a half-sample delay.

Example 7.3 $\Delta = \frac{1}{2}T$

The approach in Example 7.2 is also applicable to determining the impulse response $h_d[n]$ of the discrete-time filter in the half-sample delay system. With reference to Figure 7.24, let

$$x_c(t) = \frac{\sin(\pi t/T)}{\pi t}. \tag{7.37}$$

It follows from Example 7.2 that

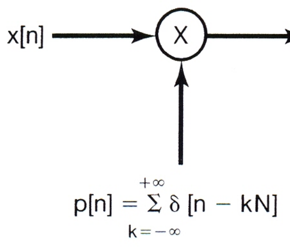
$$y_c(t) = x_c(t - \tfrac{T}{2}) = \frac{\sin(\pi(t - T/2)/T)}{\pi(t - T/2)} \qquad x_d[n] = x_c(nT) = \frac{1}{T}\delta[n].$$

$$\Rightarrow y_d[n] = y_c(nT) = \frac{\sin(\pi(n - 1/2))}{T\pi(n - 1/2)} \qquad = \delta[n-\alpha] \text{ if } \alpha \text{ is an integer}$$

$$\Rightarrow h[n] = \frac{\sin(\pi(n - 1/2))}{\pi(n - 1/2)} \qquad \left(\Rightarrow h[n] = \frac{\sin(\pi(n-\alpha))}{\pi(n-\alpha)} \text{ if } \Delta = \alpha T\right)$$

§7.5 Sampling of Discrete-Time Signals

§7.5.1 Impulse-Train Sampling



$$= \begin{cases} x[n] & \text{if } n = kN \\ 0 & \text{, otherwise} \end{cases}$$

$$= \sum_{k=-\infty}^{\infty} x[kN]\,\delta[n-kN]$$

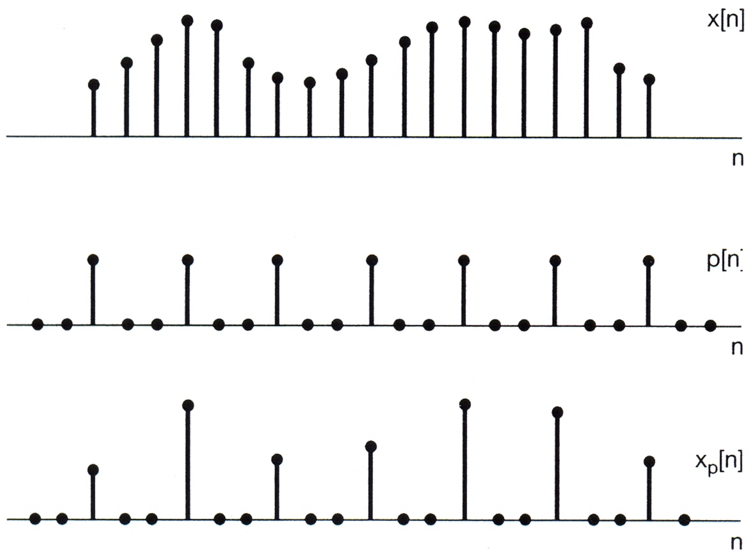


Figure 7.31 Discrete-time sampling.

$a_k = \frac{1}{N}\sum_{n\in\langle N\rangle} x[n]\,e^{-jk\omega_s n}$

$x[n] = \sum_{k\in\langle N\rangle} a_k\,e^{jk\omega_s n}$

$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k\,\delta(\omega - k\omega_s)$

$x[n] = p[n] \Rightarrow a_k = \frac{1}{N}$

$\omega_s = 2\pi/N$

As in Example 5.6, the Fourier transform of the sampling sequence $p[n]$ is

$$\Rightarrow P(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s), \tag{7.41}$$

where ω_s, the sampling frequency, equals $2\pi/N$. Combining eqs. (7.40) and (7.41), we have

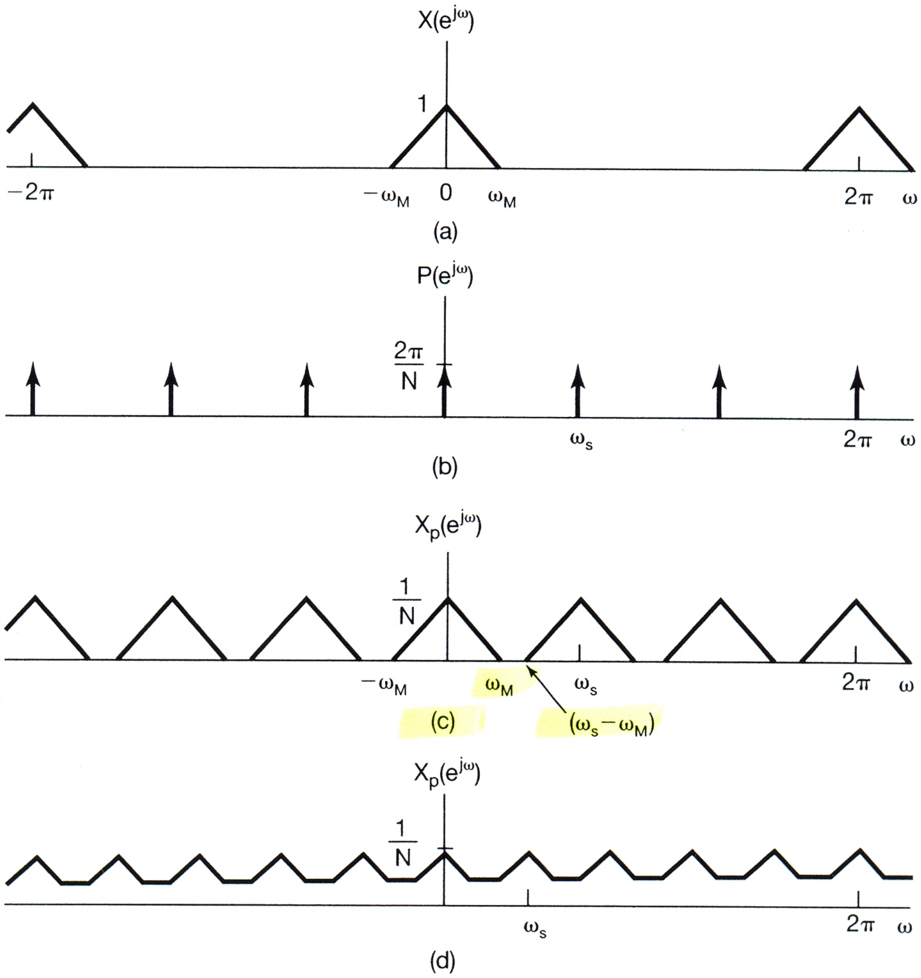
$$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(e^{j(\omega-k\omega_s)}\right). \tag{7.42}$$

Equation (7.42) is the counterpart for discrete-time sampling of eq. (7.6) for continuous-time sampling and is illustrated in Figure 7.32. In Figure 7.32(c), with $\omega_s - \omega_M > \omega_M$, or equivalently, $\omega_s > 2\omega_M$, there is no aliasing [i.e., the nonzero portions of the replicas of $X(e^{j\omega})$ do not overlap], whereas with $\omega_s < 2\omega_M$, as in Figure 7.32(d), frequency-domain aliasing results. In the absence of aliasing, $X(e^{j\omega})$ is faithfully reproduced around $\omega = 0$ and integer multiples of 2π. Consequently, $x[n]$ can be recovered from $x_p[n]$ by means of a lowpass filter with gain N and a cutoff frequency greater than

(proof of 7.42)

$$x_p[n] = x[n]\,p[n] \Rightarrow X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{0}^{2\pi} P(e^{j\theta})\,X(e^{j(\omega-\theta)})\,d\theta$$

$$= (7.42)$$



$\omega_M < \omega_s - \omega_M$

$\Rightarrow 2\omega_M < \omega_s$

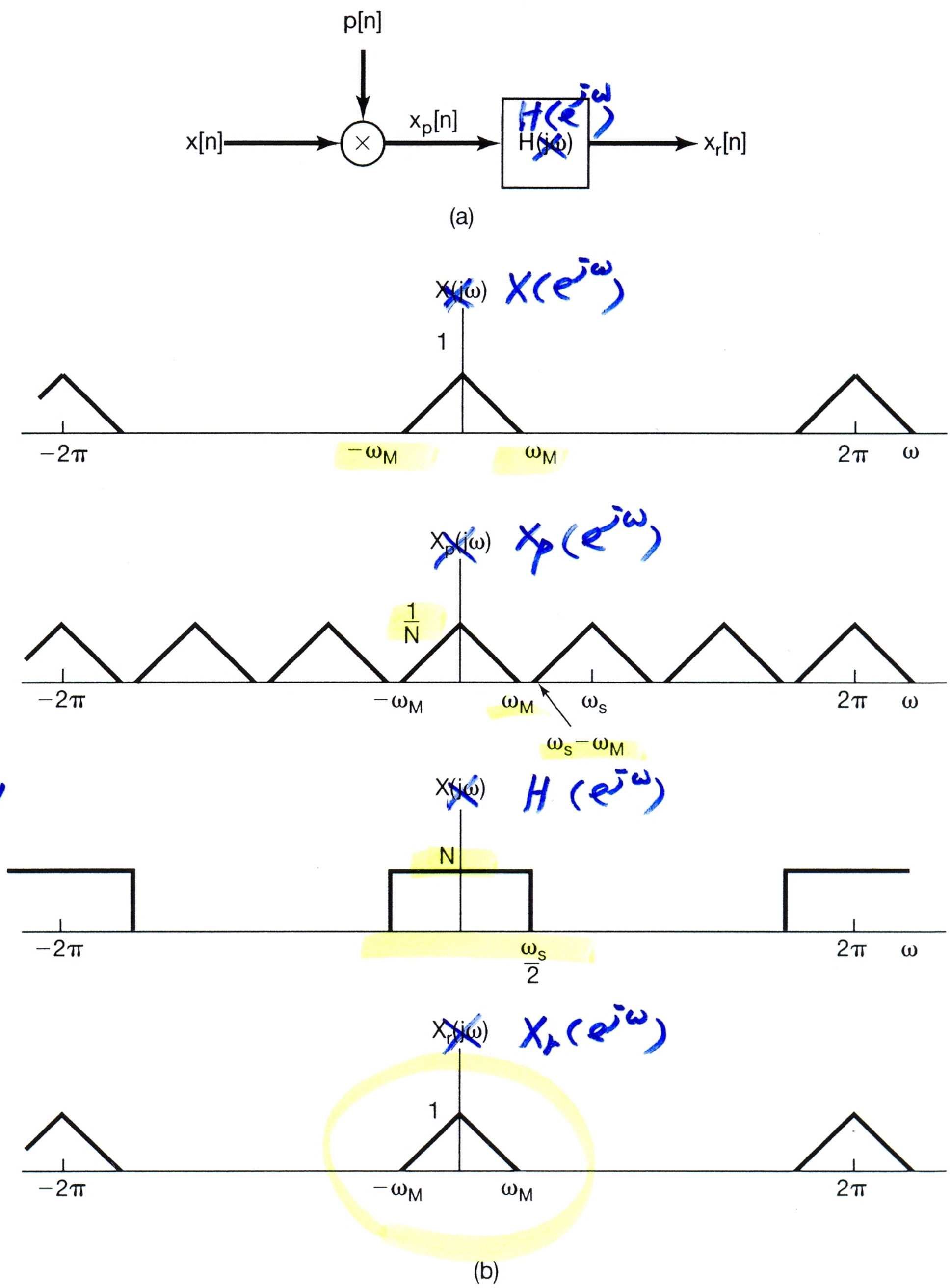
for no aliasing

Figure 7.32 Effect in the frequency domain of impulse-train sampling of a discrete-time signal: (a) spectrum of original signal; (b) spectrum of sampling sequence; (c) spectrum of sampled signal with $\omega_s > 2\omega_M$; (d) spectrum of sampled signal with $\omega_s < 2\omega_M$. Note that aliasing occurs.

ω_M and less than $\omega_s - \omega_M$, as illustrated in Figure 7.33, where we have specified the cutoff frequency of the lowpass filter as $\omega_s/2$. If the overall system of Figure 7.33(a) is applied to a sequence for which $\omega_s < 2\omega_M$, so that aliasing results, $x_r[n]$ will no longer be equal to $x[n]$. However, as with continuous-time sampling, the two sequences *will* be equal at multiples of the sampling period; that is, corresponding to eq. (7.13), we have

$$x_r[kN] = x[kN], \quad k = 0, \pm 1, \pm 2, \ldots, \tag{7.43}$$

independently of whether aliasing occurs. (See Problem 7.46.)



$\omega_M < \omega_s - \omega_M$

$\Rightarrow 2\omega_M < \omega_s$

for no aliasing

$\Rightarrow$ Exact reconstruction is possible if ω_M is finite.

Figure 7.33 Exact recovery of a discrete-time signal from its samples using an ideal lowpass filter: (a) block diagram for sampling and reconstruction of a band-limited signal from its samples; (b) spectrum of the signal $x[n]$; (c) spectrum of $x_p[n]$; (d) frequency response of an ideal lowpass filter with cutoff frequency $\omega_s/2$; (e) spectrum of the reconstructed signal $x_r[n]$. For the example depicted here $\omega_s > 2\omega_M$ so that no aliasing occurs and consequently $x_r[n] = x[n]$.

Example 7.4

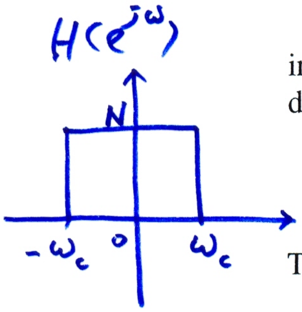
Consider a sequence $x[n]$ whose Fourier transform $X(e^{j\omega})$ has the property that

$$X(e^{j\omega}) = 0 \qquad \text{for} \quad 2\pi/9 \le |\omega| \le \pi.$$

To determine the lowest rate at which $x[n]$ may be sampled without the possibility of aliasing, we must find the largest N such that

$$\frac{2\pi}{N} \geq 2\left(\frac{2\pi}{9}\right) \Longrightarrow N \leq 9/2.$$

We conclude that $N_{\max} = 4$, and the corresponding sampling frequency is $2\pi/4 = \pi/2$.



The reconstruction of $x[n]$ through the use of a lowpass filter applied to $x_p[n]$ can be interpreted in the time domain as an interpolation formula similar to eq. (7.11). With $h[n]$ denoting the impulse response of the lowpass filter, we have

$$h[n] = \frac{N\omega_c}{\pi}\frac{\sin \omega_c n}{\omega_c n}. \tag{7.44}$$

The reconstructed sequence is then

$$x_r[n] = x_p[n] * h[n], \quad = \sum_{k=-\infty}^{\infty} x_p[n]\, h[n-k] \tag{7.45}$$

or equivalently,

$$= \sum_{k=-\infty}^{\infty} x[kN]\, h[n-kN]$$

$$x_r[n] = \sum_{k=-\infty}^{+\infty} x[kN]\frac{N\omega_c}{\pi}\frac{\sin \omega_c(n-kN)}{\omega_c(n-kN)}. \tag{7.46}$$

If $\omega_c = \omega_s/2 = \frac{2\pi}{2N}$, then for any choice of N, $x_p[kN] = x[kN]$, $k = 0, \pm 1, \cdots$ (7.43) even if $\omega_s > 2\omega_M$ is not satisfied.

Equation (7.46) represents ideal band-limited interpolation and requires the implementation of an ideal lowpass filter. In typical applications a suitable approximation for the lowpass filter in Figure 7.33 is used, in which case the equivalent interpolation formula is of the form

$$x_r[n] = \sum_{k=-\infty}^{+\infty} x[kN]h_r[n-kN], \tag{7.47}$$

where $h_r[n]$ is the impulse response of the interpolating filter. Some specific examples, including the discrete-time counterparts of the zero-order hold and first-order hold discussed in Section 7.2 for continuous-time interpolation, are considered in Problem 7.50.

7.5.2 Discrete-Time Decimation and Interpolation

There are a variety of important applications of the principles of discrete-time sampling, such as in filter design and implementation or in communication applications. In many of these applications it is inefficient to represent, transmit, or store the sampled sequence $x_p[n]$ directly in the form depicted in Figure 7.31, since, in between the sampling instants, $x_p[n]$ is *known* to be zero. Thus, the sampled sequence is typically replaced by a new sequence $x_b[n]$, which is simply every Nth value of $x_p[n]$; that is,

$$x_b[n] = x_p[nN]. \tag{7.48}$$

Also, equivalently,

$$x_b[n] = x[nN], \tag{7.49}$$

since $x_p[n]$ and $x[n]$ are equal at integer multiples of N. The operation of extracting every Nth sample is commonly referred to as *decimation*.[3] The relationship between $x[n]$, $x_p[n]$, and $x_b[n]$ is illustrated in Figure 7.34.

To determine the effect in the frequency domain of decimation, we wish to determine the relationship between $X_b(e^{j\omega})$—the Fourier transform of $x_b[n]$—and $X(e^{j\omega})$. To this end, we note that

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_b[k]e^{-j\omega k}, \tag{7.50}$$

or, using eq. (7.48),

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_p[kN]e^{-j\omega k}. \tag{7.51}$$

If we let $n = kN$, or equivalently $k = n/N$, we can write

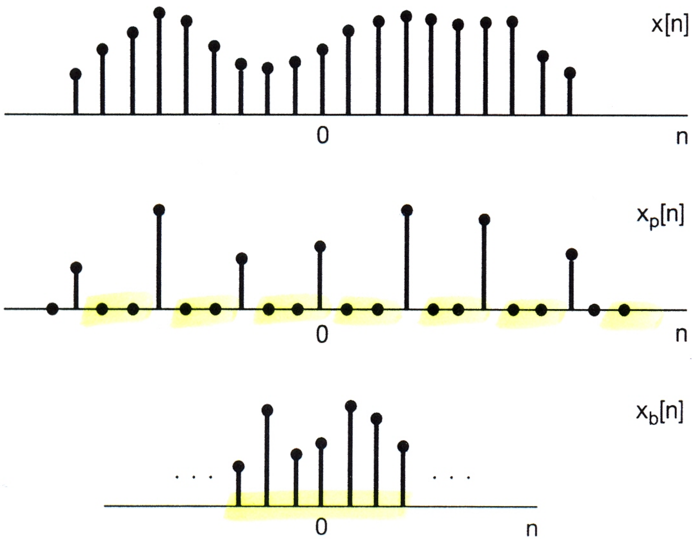
$$X_b(e^{j\omega}) = \sum_{\substack{n=\text{integer}\\ \text{multiple of } N}} x_p[n]e^{-j\omega n/N},$$

and since $x_p[n] = 0$ when n is not an integer multiple of N, we can also write

$$X_b(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_p[n]e^{-j\omega n/N}. \tag{7.52}$$

Downsampling or Decimation

The time axis is scaled by $\frac{1}{N}$



$= x_p[nN] = x[nN]$

Figure 7.34 Relationship between $x_p[n]$ corresponding to sampling and $x_b[n]$ corresponding to decimation.

[3] Technically, decimation would correspond to extracting every *tenth* sample. However, it has become common terminology to refer to the operation as decimation even when N is not equal to 10.

Furthermore, we recognize the right-hand side of eq. (7.52) as the Fourier transform of $x_p[n]$; that is,

$$\sum_{n=-\infty}^{+\infty} x_p[n]e^{-j\omega n/N} = X_p(e^{j\omega/N}). \tag{7.53}$$

Thus, $\frac{2\pi}{N}$ eqs. (7.52) and (7.53), we conclude that

$$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega - k\frac{2\pi}{N})}) \Longrightarrow X_b(e^{j\omega}) = X_p(e^{j\omega/N}). = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N})$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega - k\omega_s)}) \qquad = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega - N\omega_s)/N})$$

illustrated in Figure 7.35, and from ...ce and the decimated sequence diff... normalization. If the original spectrum $X(e^{j\omega})$ is appropriately band limited, so that there is no aliasing present in $X_p(e^{j\omega})$, then, as shown in the figure, the effect of decimation is to spread the spectrum of the original sequence over a larger portion of the frequency band.

Choose N so that $\omega_s > 2\omega_M$ ($\Leftrightarrow \pi > N\omega_M$)

$\omega_s \triangleq \frac{2\pi}{N}$

The freq. axis is scaled by N

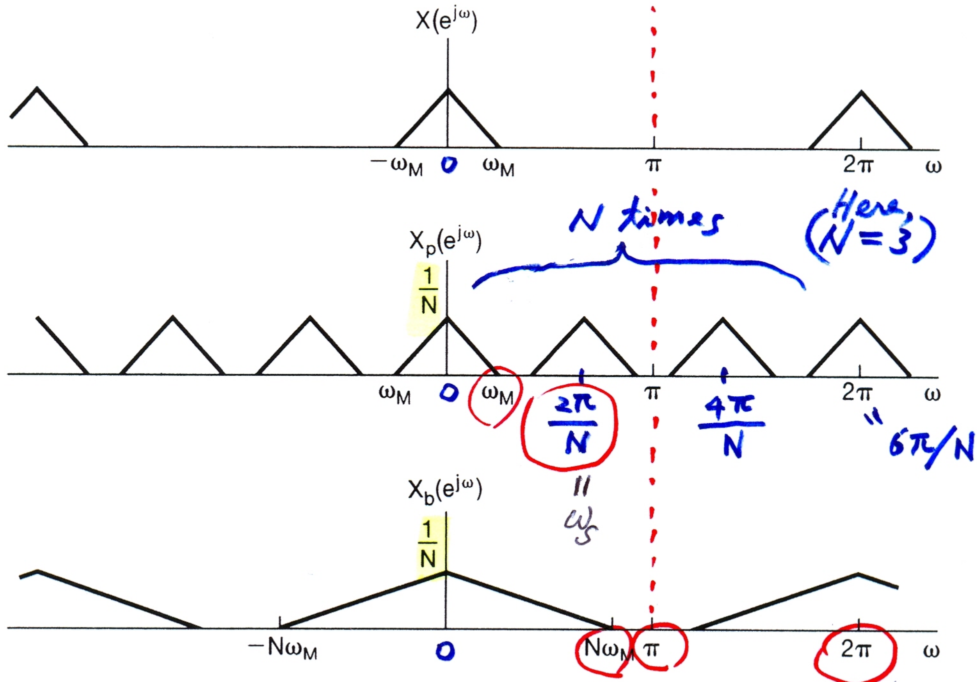


Figure 7.35 Frequency-domain illustration of the relationship between sampling and decimation.

(cf.) Sampling of continuous-time signals in Fig. 7.22

If the original sequence $x[n]$ is obtained by sampling a continuous-time signal, the process of decimation can be viewed as reducing the sampling rate on the signal by a factor of N. To avoid aliasing, $X(e^{j\omega})$ cannot occupy the full frequency band. In other words, if the signal can be decimated without introducing aliasing, then the original continuous-time signal was oversampled, and thus, the sampling rate can be reduced without aliasing. With the interpretation of the sequence $x[n]$ as samples of a continuous-time signal, the process of decimation is often referred to as *downsampling*.

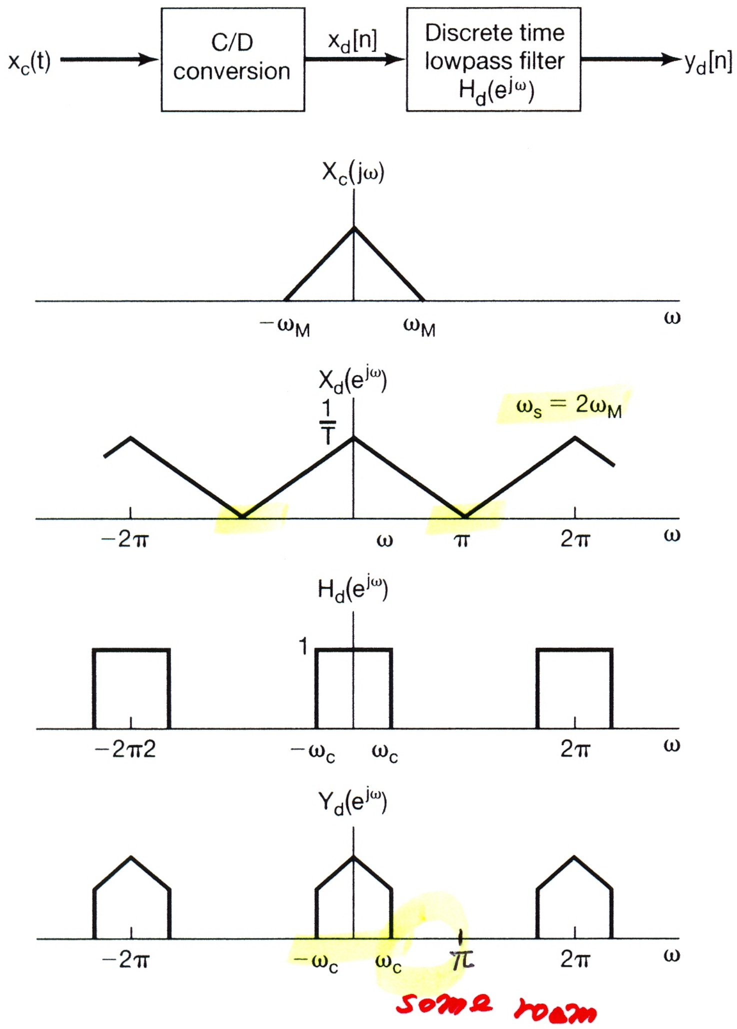


Figure 7.36 Continuous-time signal that was originally sampled at the Nyquist rate. After discrete-time filtering, the resulting sequence can be further downsampled. Here $X_c(j\omega)$ is the continuous-time Fourier transform of $x_c(t)$, $X_d(e^{j\omega})$ and $Y_d(e^{j\omega})$ are the discrete-time Fourier transforms of $x_d[n]$ and $y_d[n]$ respectively, and $H_d(e^{j\omega})$ is the frequency response of the discrete-time lowpass filter depicted in the block diagram.

$\omega_s \simeq 2\omega_M$

In some applications in which a sequence is obtained by sampling a continuous-time signal, the original sampling rate may be as low as possible without introducing aliasing, but after additional processing and filtering, the bandwidth of the sequence may be reduced. An example of such a situation is shown in Figure 7.36. Since the output of the discrete-time filter is band limited, downsampling or decimation can be applied.

Just as in some applications it is useful to downsample, there are situations in which it is useful to convert a sequence to a *higher* equivalent sampling rate, a process referred to as *upsampling or interpolation.* Upsampling is basically the reverse of decimation or downsampling. As illustrated in Figures 7.34 and 7.35, in decimation we first sample and then retain only the sequence values at the sampling instants. To upsample, we reverse the process. For example, referring to Figure 7.34, we consider upsampling the sequence $x_b[n]$ to obtain $x[n]$. From $x_b[n]$, we form the sequence $x_p[n]$ by inserting $N - 1$ points with zero amplitude between each of the values in $x_b[n]$. The interpolated sequence $x[n]$ is then obtained from $x_p[n]$ by lowpass filtering. The overall procedure is summarized in Figure 7.37.

Upsampling or Interpolation – Reverse of decimation.

For example, consider upsampling the seq. $x_b[n]$ to obtain $x[n]$ by referring to Fig. 7.36 and Fig. 7.35

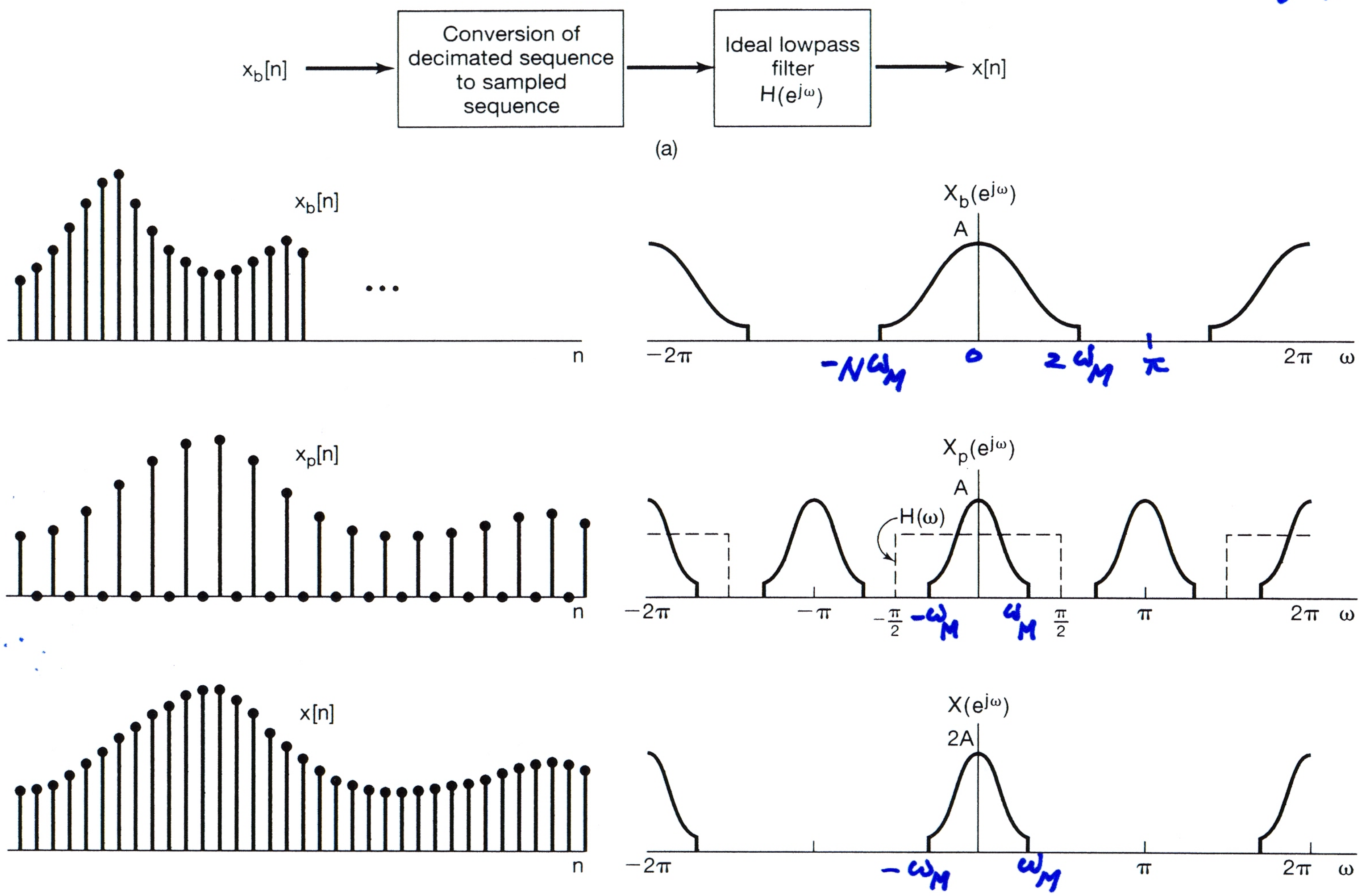


Figure 7.37 Upsampling: (a) overall system; (b) associated sequences and spectra for upsampling by a factor of 2.

The time-axis is scaled by N (Here, $N=2$)

upsampling or Interpolation

The freq. axis is scaled by $1/N$ (Here $N=2$)

$\omega_s \triangleq \frac{2\pi}{N} > 2\omega_M$

(Here, $\frac{\pi}{2} > \omega_M$)

$\left\{ \begin{array}{l} \omega_s = \frac{2\pi}{N} > 2\omega_M \\ \omega_M = \frac{2\pi}{9} \end{array} \right.$

$\Downarrow$

$N \le 4.5$

choose $N = 4$

Example 7.5

In this example, we illustrate how a combination of interpolation and decimation may be used to further downsample a sequence without incurring aliasing. It should be noted that maximum possible downsampling is achieved once the non-zero portion of one period of the discrete-time spectrum has expanded to fill the entire band from $-\pi$ to π.

Consider the sequence $x[n]$ whose Fourier transform $X(e^{j\omega})$ is illustrated in Figure 7.38(a). As discussed in Example 7.4, the lowest rate at which impulse-train sampling may be used on this sequence without incurring aliasing is $2\pi/4$. This corresponds to

The freq. axis is scaled by $N = 4$ (Downsampling)

The freq. axis is scaled by $N_1 = 1/2$ (Upsampling)

The freq. axis is scaled by $N_2 = 9$ (Downsampling)

$\Rightarrow N = N_1 N_2 = 4.5$

$\Rightarrow \omega_s = 2\omega_M$

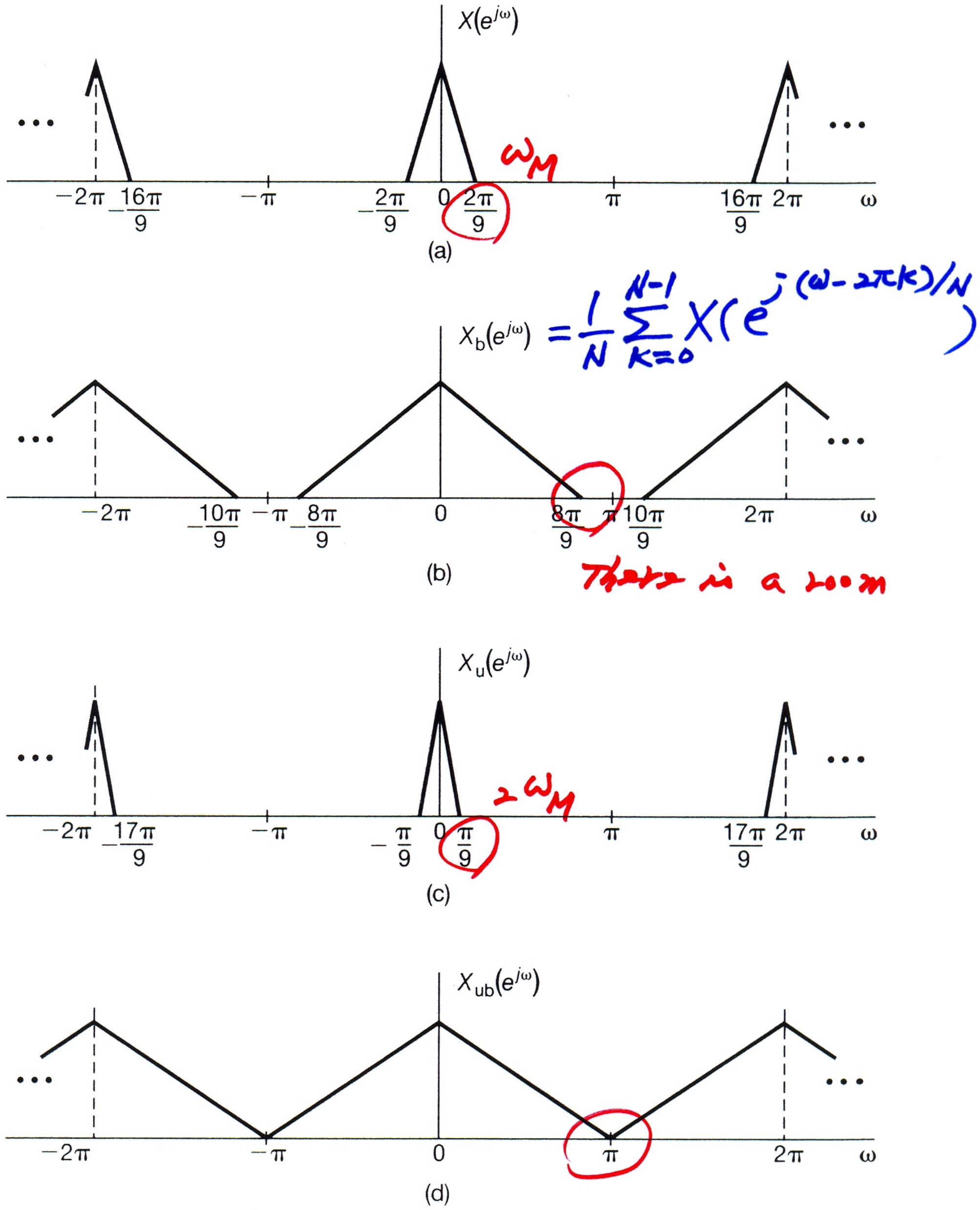


Figure 7.38 Spectra associated with Example 7.5. (a) Spectrum of $x[n]$; (b) spectrum after downsampling by 4; (c) spectrum after upsampling $x[n]$ by a factor of 2; (d) spectrum after upsampling $x[n]$ by 2 and then downsampling by 9.

HW#7

4, 6, 19, 26, 31, 37, 38, 43, 50, 52