Naval Architecture & Ocean Engineering

[2009] [12]

# Innovative ship design

## -Elasticity –

## 2D Problems

May 2009

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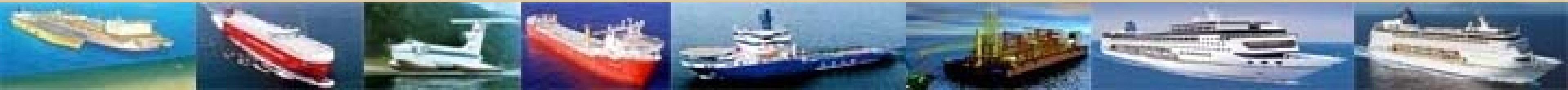

# Contents

## ① Elasticity

$\sum \mathbf{F} = m\mathbf{a}$ , $\mathbf{a} = 0$ in equilibrium

Stress —— Force Equilibrium

Strain —— Generalized Hooke' Law

Displacement —— Compatibility

Strain Energy

Virtual Displacement

Virtual Work

Differential Approach

Energy Method

**Governing Equation**

Analytic Solution

Approximation

Deflection Curve of the Beam

Scantling

② weak form : 'weighted average'*

| |
|---|
| Rayleigh-Ritz Method |
| Galerkin Method |
| FEM |


①

Wang,C.T.,
Applied Elasticity , McGRAW-HILL, 1953
응용탄성학, 이원 역, 숭실대학교 출판부, 1998

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Gere,J.M.,
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②

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"Methods of Applied Mathematics", 2nd edition, Dover, 1965

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"Finite Elements, An Introduction", Vol.1, Prentice-Hall, 1981

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"Computational Galerkin Methods", Springer, 1984



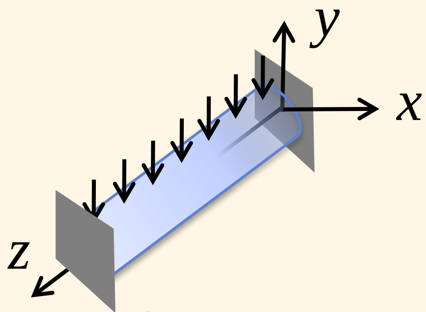

* Becker,E.B., "Finite Elements, An Introduction", Vol.1, Prentice-Hall, 1981, chapter1

# 2D Problems



# Summary : State of 2D Problems

**'A State of Plain Strain'**




If the ends of the cylinder are prevented from moving in the z coordinates, we can assume $w = 0$ at every cross section of the cylinder

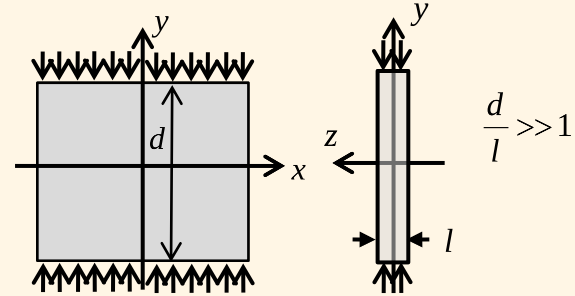
$$\varepsilon_x = \frac{\partial u}{\partial x},\ \varepsilon_y = \frac{\partial v}{\partial y},\ \varepsilon_z = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\ ,\gamma_{yz} = 0,\ \gamma_{zx} = 0$$

Where $u = u(x, y),\ v = v(x, y)$

$$\varepsilon_x = \varepsilon_x(x, y),\ \varepsilon_y = \varepsilon_y(x, y),\ \gamma_{xy} = \gamma_{xy}(x, y)$$

**'A State of Plain Stress'**




A thin plate under the lateral load uniformly distributed over its thickness

$$\sigma_z = 0,\ \tau_{zx} = 0,\ \tau_{zy} = 0$$

Where $\sigma_x = \sigma_x(x, y),\ \sigma_y = \sigma_y(x, y)$

$$\tau_{xy} = \tau_{xy}(u, v)$$

**Given : body force**

$$Z = 0,\ X = X(x, y),\ Y = Y(x, y)$$

**Find : Stress**

$$\sigma_x,\ \sigma_y,\ \tau_{xy}$$



# Summary

recall,

18 Variables → 15 Variables

- 9 Stress $\sigma_x, \tau_{yx}, \tau_{zx}, \tau_{xy}, \sigma_y, \tau_{zy}, \tau_{xz}, \tau_{yz}, \sigma_z$
- 6 Strain $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- 3 Displacement $u, v, w$

18 Equations → 15 Equations

**If we are interested in finding only the stress components in a body. we may reduce the system of equations to <u>six equations with six unknown stress components</u>**

| | |
|---|---|
| **Given : Body force** | $X, Y, Z$ |
| **Find : Stress** | $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{zx}$ |

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial X}{\partial x}+\nabla^2\sigma_x+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2}=0$$

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial Y}{\partial y}+\nabla^2\sigma_y+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2}=0$$

$$\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)+2\frac{\partial Z}{\partial z}+\nabla^2\sigma_z+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2}=0$$

$$\left(\frac{\partial Y}{\partial x}+\frac{\partial X}{\partial y}\right)+\nabla^2\tau_{xy}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial x\partial y}=0$$

$$\left(\frac{\partial Z}{\partial y}+\frac{\partial Y}{\partial z}\right)+\nabla^2\tau_{yz}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial y\partial z}=0$$

$$\left(\frac{\partial X}{\partial z}+\frac{\partial Z}{\partial x}\right)+\nabla^2\tau_{zx}+\frac{1}{\nu+1}\frac{\partial^2\Theta}{\partial z\partial x}=0$$

**6 Variables**
**6 Equations**

2

*6 Equations of force equilibrium*

$$\sum F_x=\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+X=0 \qquad \sum M_x=\tau_{yz}-\tau_{zy}=0$$

$$\sum F_y=\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+Y=0 \qquad \sum M_y=\tau_{xz}-\tau_{zx}=0$$

$$\sum F_z=\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z=0 \qquad \sum M_z=\tau_{xy}-\tau_{yx}=0$$

*6 Relations btw. Strain and Displacement*

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},$$

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y},\quad \gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}$$

3

**Compatibility equations** *3 independent Equations*

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y} \quad ,2\frac{\partial^2\varepsilon_x}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\partial z} \quad ,2\frac{\partial^2\varepsilon_y}{\partial z\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{zx}}{\partial z\partial x} \quad ,2\frac{\partial^2\varepsilon_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)$$

1

*6 Relations btw. 6 Strain and 6 Stress*

$$\sigma_x=\frac{\nu E}{(1+\nu)(1-2\nu)}e+\frac{E}{(1+\nu)}\varepsilon_x \quad ,\tau_{xy}=\frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\sigma_y=\frac{\nu E}{(1+\nu)(1-2\nu)}e+\frac{E}{(1+\nu)}\varepsilon_y \quad ,\tau_{yz}=\frac{E}{2(\nu+1)}\gamma_{yz}$$

$$\sigma_z=\frac{\nu E}{(1+\nu)(1-2\nu)}e+\frac{E}{(1+\nu)}\varepsilon_z \quad ,\tau_{zx}=\frac{E}{2(\nu+1)}\gamma_{zx}$$

$$,e=\varepsilon_x+\varepsilon_y+\varepsilon_z$$

- $X, Y, Z$ : bodyforce in x,y, and z direction repectively
- $e=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}$
- $\Theta=\sigma_x+\sigma_y+\sigma_z$
- $\mu, \lambda$ : Lame Elastic constant
- $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial y^2}$
- $G$ : Shear Moldulus
- $\nu$ : Poisson's Ratio
- $E$ : Young's Modulus



# Summary

18 Variables → 15 Variables

- 9 Stress $\sigma_x, \tau_{yx}, \tau_{zx}, \tau_{xy}, \sigma_y, \tau_{zy}, \tau_{xz}, \tau_{yz}, \sigma_z$
- 6 Strain $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- 3 Displacement $u, v, w$

**'A State of Plain Strain'**

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad , \gamma_{yz} = 0, \ \gamma_{zx} = 0$$

**(without gravitational force)**

**18 Equations → 15 Equations**

6 Equations of force equilibrium

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0 \qquad \sum M_x = \tau_{yz} - \tau_{zy} = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0 \qquad \sum M_y = \tau_{xz} - \tau_{zx} = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0 \qquad \sum M_z = \tau_{xy} - \tau_{yx} = 0$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

**Define stress function** $\psi(x, y)$ **such that**

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

**- reduce no. of variables satisfying force equilibrium eqn.**

6 Relations btw. Strain and Displacement

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

**Compatibility equations** 3 independent Equations

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad , 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \quad , 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \quad , 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

2

$$\boxed{\nabla^4 \psi = 0}$$

**Biharmonic Equation**

$\nabla^4 = (\nabla^2)^2$

1

6 Relations btw. 6 Strain and 6 Stress

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_x \quad , \tau_{xy} = \frac{E}{2(\nu+1)} \gamma_{xy}$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_y \quad , \tau_{yz} = \frac{E}{2(\nu+1)} \gamma_{yz}$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_z \quad , \tau_{zx} = \frac{E}{2(\nu+1)} \gamma_{zx}$$

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y]$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x + \varepsilon_y]$$

$$\tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y]$$

$$\varepsilon_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$



# Summary

**'A State of Plain Strain'**

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = 0,\ \gamma_{zx} = 0$$

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y},\quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x},\quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\boxed{\nabla^4 \psi = 0} \xrightarrow{\text{find}} \boxed{\psi} \xrightarrow{\text{find}} \boxed{\sigma_x,\ \sigma_y,\ \tau_{yx}} \xrightarrow{\text{find}} \boxed{\varepsilon_x,\ \varepsilon_y,\ \gamma_{yx}} \xrightarrow{\text{find}} \boxed{u,\ v,\ w}$$

**Biharmonic Equation**

$\nabla^4 = (\nabla^2)^2$

$$\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y]$$

$$\varepsilon_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$



# Summary

18 Variables → 15 Variables

- 9 Stress $\sigma_x, \tau_{yx}, \tau_{zx}, \tau_{xy}, \sigma_y, \tau_{zy}, \tau_{xz}, \tau_{yz}, \sigma_z$
- 6 Strain $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- 3 Displacement $u, v, w$

**'A State of Plain Stress'**

$$\sigma_z = 0,\ \tau_{zx} = 0,\ \tau_{zy} = 0$$

**Where**

$$\sigma_x = \sigma_x(x,y),\ \sigma_y = \sigma_y(x,y)$$

$$\tau_{xy} = \tau_{xy}(u,v)$$

**(without gravitational force)**

**18 Equations → 15 Equations**

6 Equations of force equilibrium

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0 \qquad \sum M_x = \tau_{yz} - \tau_{zy} = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0 \qquad \sum M_y = \tau_{xz} - \tau_{zx} = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0 \qquad \sum M_z = \tau_{xy} - \tau_{yx} = 0$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

**Define stress function $\psi(x,y)$ such that**

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y},\quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x},\quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

**- reduce no. of variables satisfying force equilibrium eqn.**

6 Relations btw. Strain and Displacement

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

**Compatibility equations** 3 independent Equations

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}, \quad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}, \quad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}, \quad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\nabla^4 \psi = 0$$

2

**Biharmonic Equation**

$\nabla^4 = (\nabla^2)^2$

1

6 Relations btw. 6 Strain and 6 Stress

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_x, \quad \tau_{xy} = \frac{E}{2(\nu+1)} \gamma_{xy}$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_y, \quad \tau_{yz} = \frac{E}{2(\nu+1)} \gamma_{yz}$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{(1+\nu)} \varepsilon_z, \quad \tau_{zx} = \frac{E}{2(\nu+1)} \gamma_{zx}$$

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu \varepsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu \varepsilon_x)$$

$$\tau_{xy} = G\gamma_{xy}$$

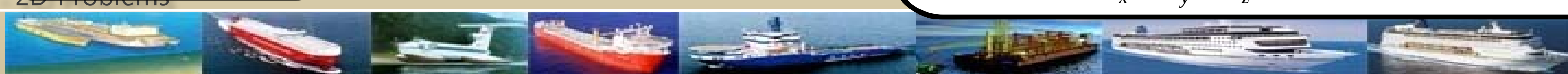
$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$





# Summary

**'A State of Plain Stress'**

$$\varepsilon_x = \frac{\partial u}{\partial x},\ \varepsilon_y = \frac{\partial v}{\partial y},\ \varepsilon_z = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\ ,\gamma_{yz} = 0,\ \gamma_{zx} = 0$$

$$\boxed{\nabla^4 \psi = 0} \xrightarrow{\text{find}} \boxed{\psi} \xrightarrow{\text{find}} \boxed{\sigma_x,\ \sigma_y,\ \tau_{yx}} \xrightarrow{\text{find}} \boxed{\varepsilon_x,\ \varepsilon_y,\ \varepsilon_z,\ \gamma_{yx}} \xrightarrow{\text{find}} \boxed{u,\ v,\ w}$$

**Biharmonic Equation**

$\nabla^4 = (\nabla^2)^2$

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y},\quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x},\quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

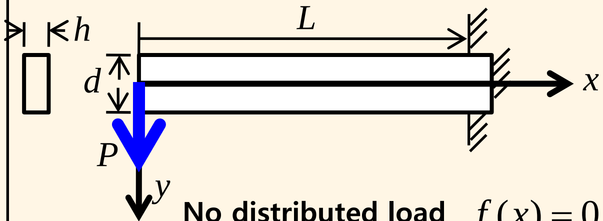
$$\gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\varepsilon_x = \frac{\partial u}{\partial x},\ \varepsilon_y = \frac{\partial v}{\partial y},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



# Summary - Comparison

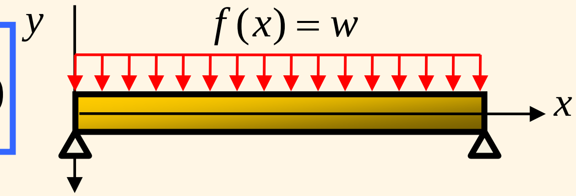



No distributed load $f(x)=0$

$v \leftarrow y$

**recall, differential equations of deflection curves***

$$\boxed{EI\frac{d^4y}{dx^4}=f(x)}$$




**displacement v**

$$v=\frac{\nu P}{2EI}xy^2+\frac{P}{6EI}x^3-\frac{PL^2}{2EI}x+\frac{PL^3}{3EI}$$

$$v=\frac{P}{6EI}x^3-\frac{PL^2}{2EI}x+\frac{PL^3}{3EI}\quad at\ \ y=0$$

**displacement v**

$$v=\frac{P}{6EI}x^3-\frac{P}{2EI}L^2x+\frac{P}{3EI}L^3$$

**displacement u**

$$u=-\frac{P}{2EI}x^2y+\frac{P}{3EI}(1+\frac{\nu}{2})y^3+\frac{P}{2EI}[L^3-(1+\nu)\frac{d^2}{2}]y$$

$$u=0\qquad at\ \ y=0$$

**displacement u**

**No displace u component from this solution**

**what is the difference between two solution and why?**

$$u=u(x,y)$$
$$v=v(x,y)$$

**The deflection of the beam at any point along its axis is the displacement of that point from its original position, measured in the y direction****

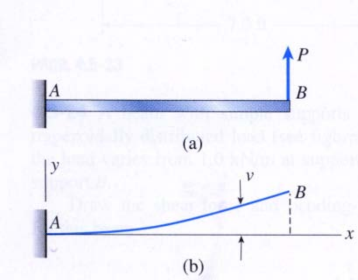



**FIG. 5-1** Bending of a cantilever beam: (a) beam with load, and (b) deflection curve



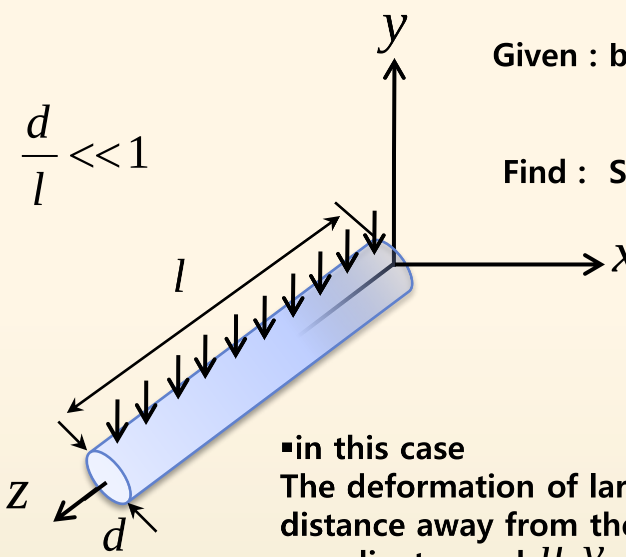
*Gere J. M., Mechanics of Materials (International Edition), Sixth Edition, Thomson, 2006, P906 (coordinates modified)
**Gere J. M., Mechanics of Materials (International Edition), Sixth Edition, Thomson, 2006, P301

## State of Plane Strain

# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**

$$\frac{d}{l} << 1$$




**Given : body force** $Z=0,\ X=X(x,y),\ Y=Y(x,y)$

**Find : Stress** $\sigma_x,\ \sigma_y,\ \tau_{xy}$

- **in this case**

**The deformation of large potion of the body at some distance away from the ends is independent of the** $z$ **coordinates and** $u, v$ **are function of** $x, y$ **only**

**If the ends of the cylinder are prevented from moving in the** $z$ **coordinates, we can assume** $w=0$ **at every cross section of the cylinder**

**In such a case, the strain components are function of** $x, y$

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$$

**'A State of Plain Strain'**

*Given : 3 Equations of force equilibrium :*

$$\sum F_x=\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\cancel{\frac{\partial\tau_{zx}}{\partial z}}+X=0$$

$$\sum F_y=\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\cancel{\frac{\partial\tau_{zy}}{\partial z}}+Y=0$$

$$\sum F_z=\cancel{\frac{\partial\tau_{xz}}{\partial x}}+\cancel{\frac{\partial\tau_{yz}}{\partial y}}+\cancel{\frac{\partial\sigma_z}{\partial z}}+\cancel{Z}=0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\boxed{\varepsilon_x=\frac{\partial u}{\partial x},}\quad \boxed{\varepsilon_y=\frac{\partial v}{\partial y},}\quad \varepsilon_z=\cancel{\frac{\partial w}{\partial z}},$$

$$\boxed{\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},}\quad \gamma_{yz}=\cancel{\frac{\partial v}{\partial z}}+\frac{\partial w}{\partial y},\quad \gamma_{zx}=\cancel{\frac{\partial w}{\partial x}}+\frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\qquad 2\frac{\partial^2\varepsilon_x}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\partial z}\qquad 2\frac{\partial^2\varepsilon_y}{\partial z\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{zx}}{\partial z\partial x}\qquad 2\frac{\partial^2\varepsilon_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z=\gamma_{xz}=\gamma_{yz}=0$

$$\sigma_x=\frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x+\nu\varepsilon_y]\qquad \tau_{xy}=\frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\sigma_x=\frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x+(1-\nu)\varepsilon_y]$$

$$\sigma_z=\frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x+\varepsilon_y]$$

**Generalized Hooke's Law for "Plane Strain State"**

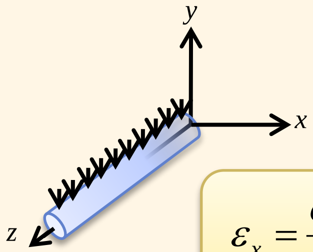
**Strain-stress relation for "Plane Stress State"**

$$\varepsilon_x=\frac{1+\nu}{E}[(1-\nu)\sigma_x-\nu\sigma_y]\qquad \gamma_{xy}=\frac{\tau_{xy}}{G}$$

$$\varepsilon_y=\frac{1+\nu}{E}[(1-\nu)\sigma_y-\nu\sigma_x]$$

# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**




**'A State of Plain Strain'**

$$X = X(x,y),\quad Y = Y(x,y),\quad Z = 0$$

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

**Where** $u = u(x,y),\ v = v(x,y)$

$$\varepsilon_x = \varepsilon_x(x,y),\ \varepsilon_y = \varepsilon_y(x,y),\ \gamma_{xy} = \gamma_{xy}(x,y)$$

**Generalized Hooke's Law for Plane Strain State**

$$\varepsilon_x,\ \varepsilon_y,\ \gamma_{xy} \quad \rightleftarrows \quad \sigma_x, \sigma_x, \sigma_z, \tau_{xy}$$

**Force equilibrium equation**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + Y = 0$$

$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \varepsilon_z = \cancel{\frac{\partial w}{\partial z}},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \gamma_{yz} = \cancel{\frac{\partial v}{\partial z}} + \frac{\partial w}{\partial y},\quad \gamma_{zx} = \cancel{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y] \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x + \varepsilon_y]$$

**Generalized Hooke's Law for "Plane Strain State"**

**Strain-stress relation for "Plane Stress State"**

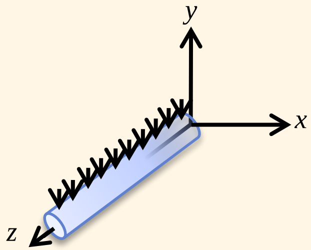
$$\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x]$$



# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**




**'A State of Plain Strain'**

$$X = X(x,y),\quad Y = Y(x,y),\quad Z = 0$$

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_x,\ \varepsilon_y,\ \gamma_{xy} \rightleftarrows \sigma_x, \sigma_x, \sigma_z, \tau_{xy}$$

$$u = u(x,y)$$
$$v = v(x,y)$$
$$\varepsilon_x = \varepsilon_x(x,y)$$
$$\varepsilon_y = \varepsilon_y(x,y)$$
$$\gamma_{xy} = \gamma_{xy}(x,y)$$
$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

**Force equilibrium equation**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0$$

$$\boxed{\begin{aligned}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \rho g_x = 0\\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho g_y = 0\end{aligned}}$$

**This equation is satisfied <u>if we introduce</u> a stress function** $\psi(x,y)$ **such that,**

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y},\quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x},\quad \tau_{yx} = \frac{\partial^2 \psi}{\partial x \partial y} - \rho g_x y - \rho g_y x$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + X = 0$$
$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + Y = 0$$
$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations* between *6 Strain and 3 Displacement* :

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x}},\quad \boxed{\varepsilon_y = \frac{\partial v}{\partial y}},\quad \cancel{\varepsilon_z = \frac{\partial w}{\partial z}},$$
$$\boxed{\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}},\quad \cancel{\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}},\quad \cancel{\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$
$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$
$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

*6 Relations* between *6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y] \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$
$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y]$$
$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x + \varepsilon_y]$$

**Generalized Hooke's Law for "Plane Strain State"**

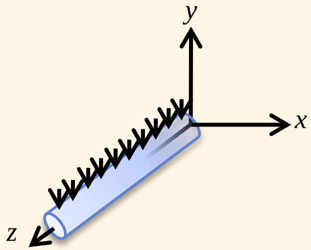
**Strain-stress relation for "Plane Stress State"**

$$\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\varepsilon_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x]$$



# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**




**'A State of Plain Strain'**

$$X = X(x,y),\quad Y = Y(x,y),\quad Z = 0$$

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_x,\ \varepsilon_y,\ \gamma_{xy} \rightleftarrows \sigma_x, \sigma_x, \sigma_z, \tau_{xy}$$

$$u = u(x,y)$$
$$v = v(u,v)$$
$$\varepsilon_x = \varepsilon_x(x,y)$$
$$\varepsilon_y = \varepsilon_y(x,y)$$
$$\gamma_{xy} = \gamma_{xy}(u,v)$$
$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

**Force equilibrium equation**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0$$

$$\Rightarrow \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \rho g_x = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho g_y = 0$$

$$\Rightarrow \boxed{\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0}$$

**This equation is satisfied if we introduce a *stress function* $\psi(x,y)$ such that,**

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y},\quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x},\quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y} - \rho g_x y - \rho g_y x$$

**If the gravitational force is neglected** ? **why?**

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y},\quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x},\quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + \cancel{X} = 0$$
$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + \cancel{Y} = 0$$
$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x}},\ \boxed{\varepsilon_y = \frac{\partial v}{\partial y}},\ \varepsilon_z = \cancel{\frac{\partial w}{\partial z}},$$
$$\boxed{\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}},\ \gamma_{yz} = \cancel{\frac{\partial v}{\partial z}} + \frac{\partial w}{\partial y},\ \gamma_{zx} = \cancel{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$
$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$
$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y] \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$
$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y]$$
$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x + \varepsilon_y]$$

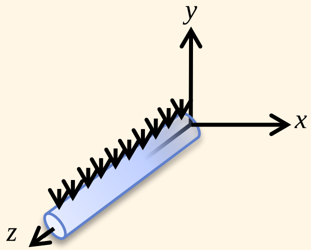
**Generalized Hooke's Law for "Plane Strain State"**

**Strain-stress relation for "Plane Stress State"**

$$\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\varepsilon_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x]$$

# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**




**'A State of Plain Strain'**

**Force equilibrium equation without gravitational force**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

$$u = u(x, y)$$
$$v = v(u, v)$$
$$\varepsilon_x = \varepsilon_x(x, y)$$
$$\varepsilon_y = \varepsilon_y(x, y)$$
$$\gamma_{xy} = \gamma_{xy}(u, v)$$
$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

**introduce a *stress function*** $\psi(x, y)$ **satisfying the equilibrium equation**

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

**the problem thus reduces to the determination of the *stress function*** $\psi$ **with appropriate boundary conditions.**

**find** $\psi \rightarrow$ **then find** $\sigma_x, \sigma_y, \tau_{xy}$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + \cancel{X} = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + \cancel{Y} = 0$$

$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x},} \quad \boxed{\varepsilon_y = \frac{\partial v}{\partial y},} \quad \cancel{\varepsilon_z = \frac{\partial w}{\partial z}},$$

$$\boxed{\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},} \quad \gamma_{yz} = \cancel{\frac{\partial v}{\partial z}} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \cancel{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y] \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x + \varepsilon_y]$$

**Generalized Hooke's Law for "Plane Strain State"**

**Strain-stress relation for "Plane Stress State"**

$$\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x]$$

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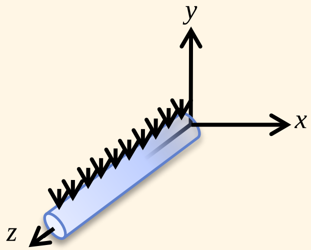



**'A State of Plain Strain'**

**Force equilibrium equation without gravitational force**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

$$u = u(x, y)$$
$$v = v(u, v)$$
$$\varepsilon_x = \varepsilon_x(x, y)$$
$$\varepsilon_y = \varepsilon_y(x, y)$$
$$\gamma_{xy} = \gamma_{xy}(u, v)$$
$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

*stress function* $\psi(x, y)$

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

**When solving for the stresses the compatibility equations must be used**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \qquad \because \frac{\partial}{\partial z} = 0$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + \cancel{X} = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + \cancel{Y} = 0$$

$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x},} \quad \boxed{\varepsilon_y = \frac{\partial v}{\partial y},} \quad \varepsilon_z = \cancel{\frac{\partial w}{\partial z}},$$

$$\boxed{\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},} \quad \gamma_{yz} = \cancel{\frac{\partial v}{\partial z}} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \cancel{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y] \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y]$$

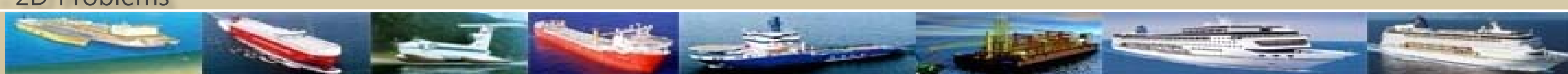
$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x + \varepsilon_y]$$

**Generalized Hooke's Law for "Plane Strain State"**

**Strain-stress relation for "Plane Stress State"**

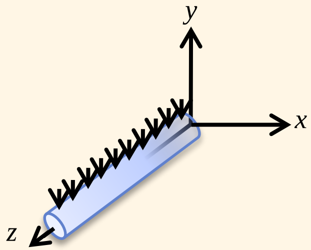
$$\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x]$$



# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**




**'A State of Plain Strain'**

**Force equilibrium equation without gravitational force**

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}=0$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0$$

$$u=u(x,y)$$
$$v=v(u,v)$$
$$\varepsilon_x=\varepsilon_x(x,y)$$
$$\varepsilon_y=\varepsilon_y(x,y)$$
$$\gamma_{xy}=\gamma_{xy}(u,v)$$
$$\varepsilon_z=\gamma_{xz}=\gamma_{yz}=0$$

***stress function*** $\psi(x,y)$

$$\sigma_x=\frac{\partial^2\psi}{\partial^2 y},\quad \sigma_y=\frac{\partial^2\psi}{\partial^2 x},\quad \tau_{yx}=-\frac{\partial^2\psi}{\partial x\partial y}$$

**When solving for the stresses the compatibility equations must be used**

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\qquad 2\cancel{\frac{\partial^2\varepsilon_x}{\partial y\partial z}}=\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\cancel{\frac{\partial\gamma_{xy}}{\partial z}}\right)$$

$$\cancel{\frac{\partial^2\varepsilon_y}{\partial z^2}}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\partial z}\qquad 2\cancel{\frac{\partial^2\varepsilon_y}{\partial z\partial x}}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\cancel{\frac{\partial\gamma_{xy}}{\partial z}}\right)\qquad \because \frac{\partial}{\partial z}=0$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2}+\cancel{\frac{\partial^2\varepsilon_x}{\partial z^2}}=\cancel{\frac{\partial^2\gamma_{zx}}{\partial z\partial x}}\qquad 2\frac{\partial^2\varepsilon_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\cancel{\frac{\partial\gamma_{xy}}{\partial z}}\right)$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x=\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\cancel{\frac{\partial\tau_{zx}}{\partial z}}+\cancel{X}=0$$
$$\sum F_y=\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\cancel{\frac{\partial\tau_{zy}}{\partial z}}+\cancel{Y}=0$$
$$\sum F_z=\cancel{\frac{\partial\tau_{xz}}{\partial x}}+\cancel{\frac{\partial\tau_{yz}}{\partial y}}+\cancel{\frac{\partial\sigma_z}{\partial z}}+\cancel{Z}=0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \cancel{\varepsilon_z=\frac{\partial w}{\partial z}},$$
$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \cancel{\gamma_{yz}=\frac{\partial v}{\partial z}}+\frac{\partial w}{\partial y},\quad \cancel{\gamma_{zx}=\frac{\partial w}{\partial x}}+\frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\qquad 2\frac{\partial^2\varepsilon_x}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$
$$\frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\partial z}\qquad 2\frac{\partial^2\varepsilon_y}{\partial z\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$
$$\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{zx}}{\partial z\partial x}\qquad 2\frac{\partial^2\varepsilon_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z=\gamma_{xz}=\gamma_{yz}=0$

$$\sigma_x=\frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x+\nu\varepsilon_y]\qquad \tau_{xy}=\frac{E}{2(\nu+1)}\gamma_{xy}$$
$$\sigma_x=\frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x+(1-\nu)\varepsilon_y]$$
$$\sigma_z=\frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x+\varepsilon_y]$$

**Generalized Hooke's Law for "Plane Strain State"**

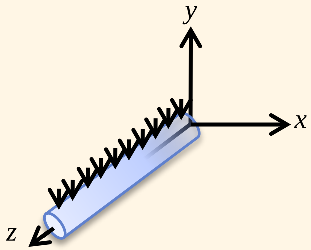
**Strain-stress relation for "Plane Stress State"**

$$\varepsilon_x=\frac{1+\nu}{E}[(1-\nu)\sigma_x-\nu\sigma_y]\qquad \gamma_{xy}=\frac{\tau_{xy}}{G}$$
$$\varepsilon_y=\frac{1+\nu}{E}[(1-\nu)\sigma_y-\nu\sigma_x]$$



# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**




**'A State of Plain Strain'**

**Force equilibrium equation without gravitational force**

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}=0$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0$$

$$u=u(x,y)$$
$$v=v(u,v)$$
$$\varepsilon_x=\varepsilon_x(x,y)$$
$$\varepsilon_y=\varepsilon_y(x,y)$$
$$\gamma_{xy}=\gamma_{xy}(u,v)$$
$$\varepsilon_z=\gamma_{xz}=\gamma_{yz}=0$$

*stress function* $\psi(x,y)$

$$\sigma_x=\frac{\partial^2\psi}{\partial^2 y},\quad \sigma_y=\frac{\partial^2\psi}{\partial^2 x},\quad \tau_{yx}=-\frac{\partial^2\psi}{\partial x\partial y}$$

**When solving for the stresses the compatibility equations must be used**

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\qquad \cancel{2\frac{\partial^2\varepsilon_x}{\partial y\partial z}}=\frac{\partial}{\partial x}\left(-\cancel{\frac{\partial\gamma_{yz}}{\partial x}}+\cancel{\frac{\partial\gamma_{zx}}{\partial y}}+\cancel{\frac{\partial\gamma_{xy}}{\partial z}}\right)$$

$$\cancel{\frac{\partial^2\varepsilon_y}{\partial z^2}}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\cancel{\frac{\partial^2\gamma_{yz}}{\partial y\partial z}}\qquad \cancel{2\frac{\partial^2\varepsilon_y}{\partial z\partial x}}=\frac{\partial}{\partial y}\left(\cancel{\frac{\partial\gamma_{yz}}{\partial x}}-\cancel{\frac{\partial\gamma_{zx}}{\partial y}}+\cancel{\frac{\partial\gamma_{xy}}{\partial z}}\right)\quad \because \gamma_{xz}=\gamma_{yz}=0$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2}+\cancel{\frac{\partial^2\varepsilon_x}{\partial z^2}}=\cancel{\frac{\partial^2\gamma_{zx}}{\partial z\partial x}}\qquad 2\frac{\partial^2\varepsilon_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\cancel{\frac{\partial\gamma_{yz}}{\partial x}}+\cancel{\frac{\partial\gamma_{zx}}{\partial y}}-\cancel{\frac{\partial\gamma_{xy}}{\partial z}}\right)$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x=\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\cancel{\frac{\partial\tau_{zx}}{\partial z}}+\cancel{X}=0$$

$$\sum F_y=\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\cancel{\frac{\partial\tau_{zy}}{\partial z}}+\cancel{Y}=0$$

$$\sum F_z=\cancel{\frac{\partial\tau_{xz}}{\partial x}}+\cancel{\frac{\partial\tau_{yz}}{\partial y}}+\cancel{\frac{\partial\sigma_z}{\partial z}}+\cancel{Z}=0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\boxed{\varepsilon_x=\frac{\partial u}{\partial x}},\quad \boxed{\varepsilon_y=\frac{\partial v}{\partial y}},\quad \varepsilon_z=\cancel{\frac{\partial w}{\partial z}},$$

$$\boxed{\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}},\quad \gamma_{yz}=\cancel{\frac{\partial v}{\partial z}}+\frac{\partial w}{\partial y},\quad \gamma_{zx}=\cancel{\frac{\partial w}{\partial x}}+\frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\qquad 2\frac{\partial^2\varepsilon_x}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\partial z}\qquad 2\frac{\partial^2\varepsilon_y}{\partial z\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{zx}}{\partial z\partial x}\qquad 2\frac{\partial^2\varepsilon_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z=\gamma_{xz}=\gamma_{yz}=0$

$$\sigma_x=\frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x+\nu\varepsilon_y]\qquad \tau_{xy}=\frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\sigma_x=\frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x+(1-\nu)\varepsilon_y]$$

$$\sigma_z=\frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x+\varepsilon_y]$$

**Generalized Hooke's Law for "Plane Strain State"**

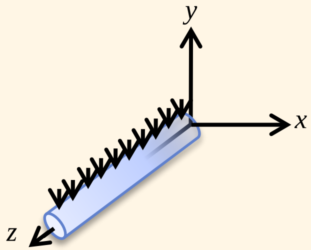
**Strain-stress relation for "Plane Stress State"**

$$\varepsilon_x=\frac{1+\nu}{E}[(1-\nu)\sigma_x-\nu\sigma_y]\qquad \gamma_{xy}=\frac{\tau_{xy}}{G}$$

$$\varepsilon_y=\frac{1+\nu}{E}[(1-\nu)\sigma_y-\nu\sigma_x]$$

# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**




**'A State of Plain Strain'**

**Force equilibrium equation without gravitational force**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

$$u = u(x, y)$$
$$v = v(u, v)$$
$$\varepsilon_x = \varepsilon_x(x, y)$$
$$\varepsilon_y = \varepsilon_y(x, y)$$
$$\gamma_{xy} = \gamma_{xy}(u, v)$$
$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

*stress function* $\psi(x, y)$

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

**When solving for the stresses the compatibility equations must be used**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\cancel{\frac{\partial^2 \varepsilon_x}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

$$\cancel{\frac{\partial^2 \varepsilon_y}{\partial z^2}} + \cancel{\frac{\partial^2 \varepsilon_z}{\partial y^2}} = \cancel{\frac{\partial^2 \gamma_{yz}}{\partial y \partial z}} \qquad 2\cancel{\frac{\partial^2 \varepsilon_y}{\partial z \partial x}} = \frac{\partial}{\partial y}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} - \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right) \qquad \because \varepsilon_z = 0$$

$$\cancel{\frac{\partial^2 \varepsilon_z}{\partial x^2}} + \cancel{\frac{\partial^2 \varepsilon_x}{\partial z^2}} = \cancel{\frac{\partial^2 \gamma_{zx}}{\partial z \partial x}} \qquad 2\cancel{\frac{\partial^2 \varepsilon_z}{\partial x \partial y}} = \frac{\partial}{\partial z}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} - \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + \cancel{X} = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + \cancel{Y} = 0$$

$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \cancel{\frac{\partial w}{\partial z}},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \cancel{\frac{\partial v}{\partial z}} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \cancel{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y] \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x + \varepsilon_y]$$

**Generalized Hooke's Law for "Plane Strain State"**

**Strain-stress relation for "Plane Stress State"**

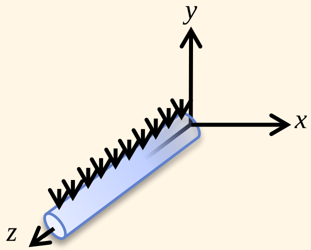
$$\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x]$$



# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**




**'A State of Plain Strain'**

**Force equilibrium equation without gravitational force**

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}=0$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0$$

$$u=u(x,y)$$
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$$\varepsilon_x=\varepsilon_x(x,y)$$
$$\varepsilon_y=\varepsilon_y(x,y)$$
$$\gamma_{xy}=\gamma_{xy}(u,v)$$
$$\varepsilon_z=\gamma_{xz}=\gamma_{yz}=0$$

*stress function* $\psi(x,y)$

$$\sigma_x=\frac{\partial^2\psi}{\partial^2 y},\quad \sigma_y=\frac{\partial^2\psi}{\partial^2 x},\quad \tau_{yx}=-\frac{\partial^2\psi}{\partial x\partial y}$$

**When solving for the stresses the compatibility equations must be used**

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\qquad \cancel{2\frac{\partial^2\varepsilon_x}{\partial y\partial z}}=\frac{\partial}{\partial x}\left(-\cancel{\frac{\partial\gamma_{yz}}{\partial x}}+\cancel{\frac{\partial\gamma_{zx}}{\partial y}}+\cancel{\frac{\partial\gamma_{xy}}{\partial z}}\right)$$

$$\cancel{\frac{\partial^2\varepsilon_y}{\partial z^2}}+\cancel{\frac{\partial^2\varepsilon_z}{\partial y^2}}=\cancel{\frac{\partial^2\gamma_{yz}}{\partial y\partial z}}\qquad \cancel{2\frac{\partial^2\varepsilon_y}{\partial z\partial x}}=\frac{\partial}{\partial y}\left(\cancel{\frac{\partial\gamma_{yz}}{\partial x}}-\cancel{\frac{\partial\gamma_{zx}}{\partial y}}+\cancel{\frac{\partial\gamma_{xy}}{\partial z}}\right)$$

$$\cancel{\frac{\partial^2\varepsilon_z}{\partial x^2}}+\cancel{\frac{\partial^2\varepsilon_x}{\partial z^2}}=\cancel{\frac{\partial^2\gamma_{zx}}{\partial z\partial x}}\qquad \cancel{2\frac{\partial^2\varepsilon_z}{\partial x\partial y}}=\frac{\partial}{\partial z}\left(\cancel{\frac{\partial\gamma_{yz}}{\partial x}}+\cancel{\frac{\partial\gamma_{zx}}{\partial y}}-\cancel{\frac{\partial\gamma_{xy}}{\partial z}}\right)$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x=\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\cancel{\frac{\partial\tau_{zx}}{\partial z}}+\cancel{X}=0$$
$$\sum F_y=\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\cancel{\frac{\partial\tau_{zy}}{\partial z}}+\cancel{Y}=0$$
$$\sum F_z=\cancel{\frac{\partial\tau_{xz}}{\partial x}}+\cancel{\frac{\partial\tau_{yz}}{\partial y}}+\cancel{\frac{\partial\sigma_z}{\partial z}}+\cancel{Z}=0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\boxed{\varepsilon_x=\frac{\partial u}{\partial x},}\quad \boxed{\varepsilon_y=\frac{\partial v}{\partial y},}\quad \varepsilon_z=\cancel{\frac{\partial w}{\partial z}},$$
$$\boxed{\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},}\quad \gamma_{yz}=\cancel{\frac{\partial v}{\partial z}}+\frac{\partial w}{\partial y},\quad \gamma_{zx}=\cancel{\frac{\partial w}{\partial x}}+\frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\qquad \cancel{2\frac{\partial^2\varepsilon_x}{\partial y\partial z}}=\frac{\partial}{\partial x}\left(-\cancel{\frac{\partial\gamma_{yz}}{\partial x}}+\cancel{\frac{\partial\gamma_{zx}}{\partial y}}+\cancel{\frac{\partial\gamma_{xy}}{\partial z}}\right)$$
$$\cancel{\frac{\partial^2\varepsilon_y}{\partial z^2}}+\cancel{\frac{\partial^2\varepsilon_z}{\partial y^2}}=\cancel{\frac{\partial^2\gamma_{yz}}{\partial y\partial z}}\qquad \cancel{2\frac{\partial^2\varepsilon_y}{\partial z\partial x}}=\frac{\partial}{\partial y}\left(\cancel{\frac{\partial\gamma_{yz}}{\partial x}}-\cancel{\frac{\partial\gamma_{zx}}{\partial y}}+\cancel{\frac{\partial\gamma_{xy}}{\partial z}}\right)$$
$$\cancel{\frac{\partial^2\varepsilon_z}{\partial x^2}}+\cancel{\frac{\partial^2\varepsilon_x}{\partial z^2}}=\cancel{\frac{\partial^2\gamma_{zx}}{\partial z\partial x}}\qquad \cancel{2\frac{\partial^2\varepsilon_z}{\partial x\partial y}}=\frac{\partial}{\partial z}\left(\cancel{\frac{\partial\gamma_{yz}}{\partial x}}+\cancel{\frac{\partial\gamma_{zx}}{\partial y}}-\cancel{\frac{\partial\gamma_{xy}}{\partial z}}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z=\gamma_{xz}=\gamma_{yz}=0$

$$\sigma_x=\frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x+\nu\varepsilon_y]\qquad \tau_{xy}=\frac{E}{2(\nu+1)}\gamma_{xy}$$
$$\sigma_x=\frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x+(1-\nu)\varepsilon_y]$$
$$\sigma_z=\frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x+\varepsilon_y]$$

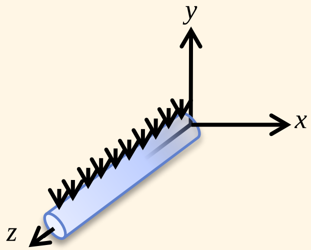
**Generalized Hooke's Law for "Plane Strain State"**

**Strain-stress relation for "Plane Stress State"**

$$\varepsilon_x=\frac{1+\nu}{E}[(1-\nu)\sigma_x-\nu\sigma_y]\qquad \gamma_{xy}=\frac{\tau_{xy}}{G}$$
$$\varepsilon_y=\frac{1+\nu}{E}[(1-\nu)\sigma_y-\nu\sigma_x]$$

# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**




**'A State of Plain Strain'**

**Force equilibrium equation without gravitational force**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

$$u = u(x, y)$$
$$v = v(u, v)$$
$$\varepsilon_x = \varepsilon_x(x, y)$$
$$\varepsilon_y = \varepsilon_y(x, y)$$
$$\gamma_{xy} = \gamma_{xy}(u, v)$$
$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + \cancel{X} = 0$$
$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + \cancel{Y} = 0$$
$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x},} \quad \boxed{\varepsilon_y = \frac{\partial v}{\partial y},} \quad \varepsilon_z = \cancel{\frac{\partial w}{\partial z}},$$
$$\boxed{\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},} \quad \gamma_{yz} = \cancel{\frac{\partial v}{\partial z}} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \cancel{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\cancel{\frac{\partial^2 \varepsilon_x}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$
$$\cancel{\frac{\partial^2 \varepsilon_y}{\partial z^2}} + \cancel{\frac{\partial^2 \varepsilon_z}{\partial y^2}} = \cancel{\frac{\partial^2 \gamma_{yz}}{\partial y \partial z}} \qquad 2\cancel{\frac{\partial^2 \varepsilon_y}{\partial z \partial x}} = \frac{\partial}{\partial y}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} - \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$
$$\cancel{\frac{\partial^2 \varepsilon_z}{\partial x^2}} + \cancel{\frac{\partial^2 \varepsilon_x}{\partial z^2}} = \cancel{\frac{\partial^2 \gamma_{zx}}{\partial z \partial x}} \qquad 2\cancel{\frac{\partial^2 \varepsilon_z}{\partial x \partial y}} = \frac{\partial}{\partial z}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} - \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

*stress function* $\psi(x, y)$

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y] \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$
$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y]$$
$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x + \varepsilon_y]$$

**Generalized Hooke's Law for "Plane Strain State"**

**When solving for the stresses the compatibility equations must be used**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

**Strain-stress relation for "Plane Stress State"**

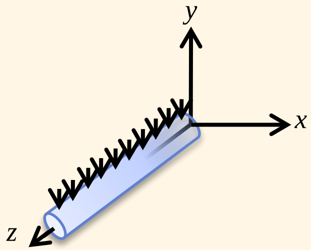
$$\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\varepsilon_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x]$$

$$\frac{\partial^2}{\partial y^2}\frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y] + \frac{\partial^2}{\partial x^2}\frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x] = \frac{\partial^2}{\partial x \partial y}\left(\frac{\tau_{xy}}{G}\right)$$



# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**




**'A State of Plain Strain'**

**Force equilibrium equation without gravitational force**

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$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

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$$v = v(u, v)$$

$$\varepsilon_x = \varepsilon_x(x, y)$$

$$\varepsilon_y = \varepsilon_y(x, y)$$

$$\gamma_{xy} = \gamma_{xy}(u, v)$$

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

*stress function* $\psi(x, y)$

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

**recall,** $G = \dfrac{E}{2(1+\nu)}$

**When solving for the stresses the compatibility equations must be used**

$$\frac{\partial^2}{\partial y^2}\frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y] + \frac{\partial^2}{\partial x^2}\frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x] = \frac{\partial^2}{\partial x \partial y}\left(\frac{\tau_{xy}}{G}\right)$$

$$\Downarrow$$

$$\frac{1+\nu}{E}\left\{\frac{\partial^2}{\partial y^2}[(1-\nu)\sigma_x - \nu\sigma_y] + \frac{\partial^2}{\partial x^2}[(1-\nu)\sigma_y - \nu\sigma_x]\right\} = \frac{2(1+\nu)}{E}\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

$$\Downarrow$$

$$\frac{\partial^2}{\partial y^2}[(1-\nu)\sigma_x - \nu\sigma_y] + \frac{\partial^2}{\partial x^2}[(1-\nu)\sigma_y - \nu\sigma_x] = 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + \cancel{X} = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + \cancel{Y} = 0$$

$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x}}, \quad \boxed{\varepsilon_y = \frac{\partial v}{\partial y}}, \quad \varepsilon_z = \cancel{\frac{\partial w}{\partial z}},$$

$$\boxed{\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}}, \quad \gamma_{yz} = \cancel{\frac{\partial v}{\partial z}} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \cancel{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\cancel{\frac{\partial^2 \varepsilon_x}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

$$\cancel{\frac{\partial^2 \varepsilon_y}{\partial z^2}} + \cancel{\frac{\partial^2 \varepsilon_z}{\partial y^2}} = \cancel{\frac{\partial^2 \gamma_{yz}}{\partial y \partial z}} \qquad 2\cancel{\frac{\partial^2 \varepsilon_y}{\partial z \partial x}} = \frac{\partial}{\partial y}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} - \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

$$\cancel{\frac{\partial^2 \varepsilon_z}{\partial x^2}} + \cancel{\frac{\partial^2 \varepsilon_x}{\partial z^2}} = \cancel{\frac{\partial^2 \gamma_{zx}}{\partial z \partial x}} \qquad 2\cancel{\frac{\partial^2 \varepsilon_z}{\partial x \partial y}} = \frac{\partial}{\partial z}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} - \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y] \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x + \varepsilon_y]$$

**Generalized Hooke's Law for "Plane Strain State"**

**Strain-stress relation for "Plane Stress State"**

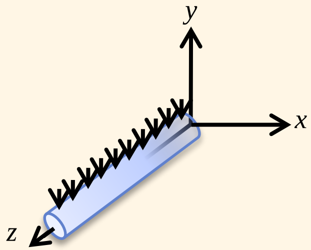
$$\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x]$$



# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**




**'A State of Plain Strain'**

$$u = u(x, y)$$
$$v = v(u, v)$$
$$\varepsilon_x = \varepsilon_x(x, y)$$
$$\varepsilon_y = \varepsilon_y(x, y)$$
$$\gamma_{xy} = \gamma_{xy}(u, v)$$
$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

**Force equilibrium equation without gravitational force**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

*__stress function__* $\psi(x, y)$

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

**When solving for the stresses the compatibility equations must be used**

$$\frac{\partial^2}{\partial y^2}[(1-\nu)\sigma_x - \nu\sigma_y] + \frac{\partial^2}{\partial x^2}[(1-\nu)\sigma_y - \nu\sigma_x] = 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

$$(1-\nu)\frac{\partial^2 \sigma_x}{\partial y^2} - \nu\frac{\partial^2 \sigma_y}{\partial y^2} + (1-\nu)\frac{\partial^2 \sigma_y}{\partial x^2} - \nu\frac{\partial^2 \sigma_x}{\partial x^2} = 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

$$(1-\nu)\left(\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2}\right) - \nu\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right) = 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + \cancel{X} = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + \cancel{Y} = 0$$

$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x},} \quad \boxed{\varepsilon_y = \frac{\partial v}{\partial y},} \quad \cancel{\varepsilon_z = \frac{\partial w}{\partial z}},$$

$$\boxed{\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},} \quad \cancel{\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}}, \quad \cancel{\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\cancel{\frac{\partial^2 \varepsilon_y}{\partial z^2}} + \cancel{\frac{\partial^2 \varepsilon_z}{\partial y^2}} = \cancel{\frac{\partial^2 \gamma_{yz}}{\partial y \partial z}}$$

$$\cancel{\frac{\partial^2 \varepsilon_z}{\partial x^2}} + \cancel{\frac{\partial^2 \varepsilon_x}{\partial z^2}} = \cancel{\frac{\partial^2 \gamma_{zx}}{\partial z \partial x}}$$

$$2\cancel{\frac{\partial^2 \varepsilon_x}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

$$2\cancel{\frac{\partial^2 \varepsilon_y}{\partial z \partial x}} = \frac{\partial}{\partial y}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} - \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

$$2\cancel{\frac{\partial^2 \varepsilon_z}{\partial x \partial y}} = \frac{\partial}{\partial z}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} - \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y] \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x + \varepsilon_y]$$

**Generalized Hooke's Law for "Plane Strain State"**

**Strain-stress relation for "Plane Stress State"**

$$\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

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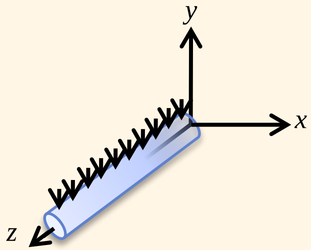
# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**




**'A State of Plain Strain'**

**Force equilibrium equation without gravitational force**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

$$u = u(x,y)$$
$$v = v(u,v)$$
$$\varepsilon_x = \varepsilon_x(x,y)$$
$$\varepsilon_y = \varepsilon_y(x,y)$$
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*stress function* $\psi(x,y)$

$$\boxed{\sigma_x = \frac{\partial^2 \psi}{\partial^2 y}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}}$$

**When solving for the stresses the compatibility equations must be used**

$$(1-\nu)\left(\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2}\right) - \nu\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right) = 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

$$(1-\nu)\left(\frac{\partial^4 \psi}{\partial y^4} + \frac{\partial^4 \psi}{\partial x^4}\right) - \nu\left(\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^2 \partial x^2}\right) = -2\frac{\partial^4 \psi}{\partial x^2 \partial y^2}$$

$$(1-\nu)\left(\frac{\partial^4 \psi}{\partial y^4} + \frac{\partial^4 \psi}{\partial x^4}\right) - \nu\left(2\frac{\partial^4 \psi}{\partial x^2 \partial y^2}\right) + \left(2\frac{\partial^4 \psi}{\partial x^2 \partial y^2}\right) = 0$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + \cancel{X} = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + \cancel{Y} = 0$$

$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \cancel{\frac{\partial w}{\partial z}},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \cancel{\frac{\partial v}{\partial z}} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \cancel{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\cancel{\frac{\partial^2 \varepsilon_x}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

$$\cancel{\frac{\partial^2 \varepsilon_y}{\partial z^2}} + \cancel{\frac{\partial^2 \varepsilon_z}{\partial y^2}} = \cancel{\frac{\partial^2 \gamma_{yz}}{\partial y \partial z}} \qquad 2\cancel{\frac{\partial^2 \varepsilon_y}{\partial z \partial x}} = \frac{\partial}{\partial y}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} - \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

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*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

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$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x + \varepsilon_y]$$

**Generalized Hooke's Law for "Plane Strain State"**

**Strain-stress relation for "Plane Stress State"**

$$\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

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# 2D Problem - The Governing Differential Equation

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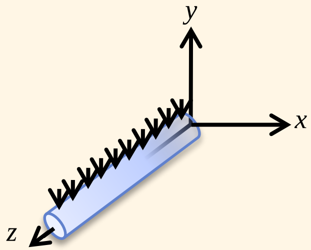


**'A State of Plain Strain'**

**Force equilibrium equation without gravitational force**

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$$\varepsilon_x = \varepsilon_x(x,y)$$
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$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + \cancel{X} = 0$$
$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + \cancel{Y} = 0$$
$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x},} \quad \boxed{\varepsilon_y = \frac{\partial v}{\partial y},} \quad \varepsilon_z = \cancel{\frac{\partial w}{\partial z}},$$
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**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\cancel{\frac{\partial^2 \varepsilon_x}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$
$$\cancel{\frac{\partial^2 \varepsilon_y}{\partial z^2}} + \cancel{\frac{\partial^2 \varepsilon_z}{\partial y^2}} = \cancel{\frac{\partial^2 \gamma_{yz}}{\partial y \partial z}} \qquad 2\cancel{\frac{\partial^2 \varepsilon_y}{\partial z \partial x}} = \frac{\partial}{\partial y}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} - \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$
$$\cancel{\frac{\partial^2 \varepsilon_z}{\partial x^2}} + \cancel{\frac{\partial^2 \varepsilon_x}{\partial z^2}} = \cancel{\frac{\partial^2 \gamma_{zx}}{\partial z \partial x}} \qquad 2\cancel{\frac{\partial^2 \varepsilon_z}{\partial x \partial y}} = \frac{\partial}{\partial z}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} - \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

*stress function* $\psi(x,y)$

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

**When solving for the stresses the compatibility equations must be used**

$$(1-\nu)\left(\frac{\partial^4 \psi}{\partial y^4} + \frac{\partial^4 \psi}{\partial x^4}\right) - \nu\left(2\frac{\partial^4 \psi}{\partial x^2 \partial y^2}\right) + \left(2\frac{\partial^4 \psi}{\partial x^2 \partial y^2}\right) = 0$$

$$(1-\nu)\left(\frac{\partial^4 \psi}{\partial y^4} + \frac{\partial^4 \psi}{\partial x^4}\right) + (1-\nu)\left(2\frac{\partial^4 \psi}{\partial x^2 \partial y^2}\right) = 0$$

$$(1-\nu)\left(\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4}\right) = 0$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y] \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$
$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y]$$
$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x + \varepsilon_y]$$

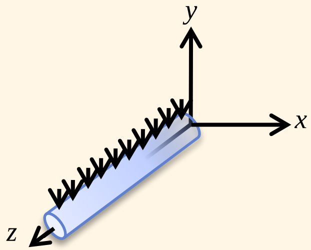
**Generalized Hooke's Law for "Plane Strain State"**

**Strain-stress relation for "Plane Stress State"**

$$\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\varepsilon_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x]$$

# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**




**'A State of Plain Strain'**

**Force equilibrium equation without gravitational force**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

$$u = u(x,y)$$
$$v = v(u,v)$$
$$\varepsilon_x = \varepsilon_x(x,y)$$
$$\varepsilon_y = \varepsilon_y(x,y)$$
$$\gamma_{xy} = \gamma_{xy}(u,v)$$
$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + \cancel{X} = 0$$
$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + \cancel{Y} = 0$$
$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x},} \quad \boxed{\varepsilon_y = \frac{\partial v}{\partial y},} \quad \cancel{\varepsilon_z = \frac{\partial w}{\partial z}},$$
$$\boxed{\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},} \quad \cancel{\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}}, \quad \cancel{\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad \cancel{2\frac{\partial^2 \varepsilon_x}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$
$$\cancel{\frac{\partial^2 \varepsilon_y}{\partial z^2}} + \cancel{\frac{\partial^2 \varepsilon_z}{\partial y^2}} = \cancel{\frac{\partial^2 \gamma_{yz}}{\partial y \partial z}} \qquad \cancel{2\frac{\partial^2 \varepsilon_y}{\partial z \partial x}} = \frac{\partial}{\partial y}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} - \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$
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*stress function* $\psi(x,y)$

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

**When solving for the stresses the compatibility equations must be used**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \Rightarrow (1-\nu)\left(\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4}\right) = 0$$

$\because 0 < \nu < 1$

$i.g.$ 0.27 ~ 0.30 for steel*

$$\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0$$

✓***Poisson's ratio 'v'***

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y] \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$
$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y]$$
$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x + \varepsilon_y]$$

**Generalized Hooke's Law for "Plane Strain State"**

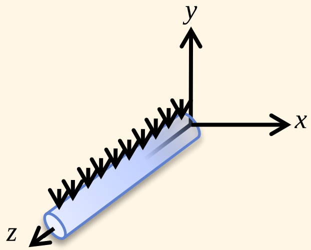
**Strain-stress relation for "Plane Stress State"**

$$\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\varepsilon_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x]$$

*Gere J.M., Mechanics of Materials, International Student Edition, Thomson, 2006, p913

# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**




**'A State of Plain Strain'**

**Force equilibrium equation without gravitational force**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

$$u = u(x, y)$$
$$v = v(u, v)$$
$$\varepsilon_x = \varepsilon_x(x, y)$$
$$\varepsilon_y = \varepsilon_y(x, y)$$
$$\gamma_{xy} = \gamma_{xy}(u, v)$$
$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + \cancel{X} = 0$$
$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + \cancel{Y} = 0$$
$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \cancel{\frac{\partial w}{\partial z}},$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \cancel{\frac{\partial v}{\partial z}} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \cancel{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\cancel{\frac{\partial^2 \varepsilon_x}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$
$$\cancel{\frac{\partial^2 \varepsilon_y}{\partial z^2}} + \cancel{\frac{\partial^2 \varepsilon_z}{\partial y^2}} = \cancel{\frac{\partial^2 \gamma_{yz}}{\partial y \partial z}} \qquad 2\cancel{\frac{\partial^2 \varepsilon_y}{\partial z \partial x}} = \frac{\partial}{\partial y}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} - \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$
$$\cancel{\frac{\partial^2 \varepsilon_z}{\partial x^2}} + \cancel{\frac{\partial^2 \varepsilon_x}{\partial z^2}} = \cancel{\frac{\partial^2 \gamma_{zx}}{\partial z \partial x}} \qquad 2\cancel{\frac{\partial^2 \varepsilon_z}{\partial x \partial y}} = \frac{\partial}{\partial z}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} - \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

*stress function* $\psi(x, y)$

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

**When solving for the stresses the compatibility equations must be used**

$$\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 = (\nabla^2)^2$$

$$\boxed{\nabla^4 \psi = 0}$$

**Biharmonic Equation**

$\nabla^2$ **Laplace or Harmonic Operator**

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y] \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$
$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y]$$
$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x + \varepsilon_y]$$

**Generalized Hooke's Law for "Plane Strain State"**

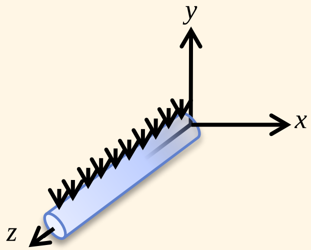
**Strain-stress relation for "Plane Stress State"**

$$\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\varepsilon_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x]$$



# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**




**'A State of Plain Strain'**

$$u = u(x, y)$$
$$v = v(u, v)$$
$$\varepsilon_x = \varepsilon_x(x, y)$$
$$\varepsilon_y = \varepsilon_y(x, y)$$
$$\gamma_{xy} = \gamma_{xy}(u, v)$$
$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

**solve**

$$\boxed{\nabla^4 \psi = 0}$$

**Stress determined**

$$\boxed{\sigma_x = \frac{\partial^2 \psi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial x^2}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}}$$

**Strain determined**

$$\boxed{\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y], \quad \varepsilon_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x], \quad \gamma_{xy} = \frac{\tau_{xy}}{G}}$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + \cancel{X} = 0$$
$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + \cancel{Y} = 0$$
$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \cancel{\frac{\partial w}{\partial z}},$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \cancel{\frac{\partial v}{\partial z}} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \cancel{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\cancel{\frac{\partial^2 \varepsilon_x}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$
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$$\cancel{\frac{\partial^2 \varepsilon_z}{\partial x^2}} + \cancel{\frac{\partial^2 \varepsilon_x}{\partial z^2}} = \cancel{\frac{\partial^2 \gamma_{zx}}{\partial z \partial x}} \qquad 2\cancel{\frac{\partial^2 \varepsilon_z}{\partial x \partial y}} = \frac{\partial}{\partial z}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} - \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Strain State",** $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y] \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$
$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y]$$
$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x + \varepsilon_y]$$

**Generalized Hooke's Law for "Plane Strain State"**

**Strain-stress relation for "Plane Stress State"**

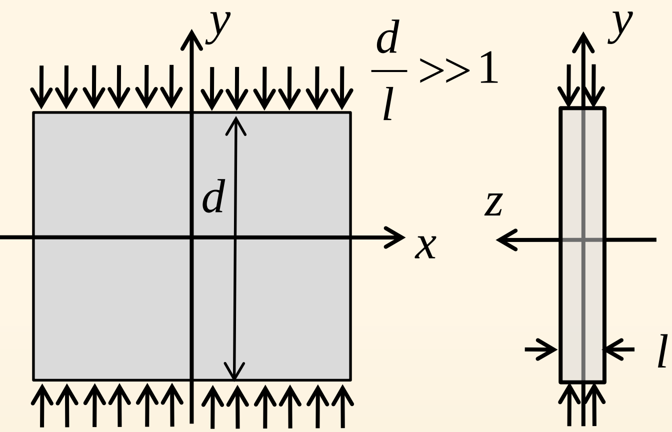
$$\varepsilon_x = \frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\varepsilon_y = \frac{1+\nu}{E}[(1-\nu)\sigma_y - \nu\sigma_x]$$



## State of Plain Stress

# 2D Problem - The Governing Differential Equation

**A thin plate under the lateral load uniformly distributed over its thickness**




$$\frac{d}{l} >> 1$$

▪**body force**

$$Z = 0$$

$$X = X(x, y)$$

$$Y = Y(x, y)$$

▪**in this case**

**The plate is under the action of forces applied at the boundary, parallel to the plane of the plate and distributed uniformly over its thickness**

**We see that the surfaces of the plate** $z = \pm\frac{h}{2}$ **will be free of external forces and the stress components** $\sigma_z, \tau_{zx}, \tau_{zy}$ **are zero there**

**If the plate is thin we can assume that these components are zero throughout the thickness of the plate and the other three stress components** $\sigma_x, \sigma_y, \tau_{xy}$ **remain practically constant over the thickness of the plate**

**'A State of Plain Stress'**

$$\sigma_x = \sigma_x(x, y)$$

$$\sigma_y = \sigma_y(x, y)$$

$$\tau_{xy} = \tau_{xy}(u, v) \qquad \sigma_z = \tau_{zx} = \tau_{zy} = 0$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + Y = 0$$

$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

**compatibility equations**

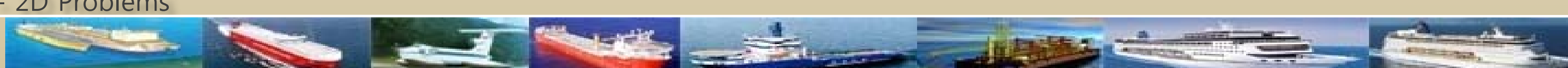
$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

*6 Relations between 6 Strain and 6 Stress*





# 2D Problem - The Governing Differential Equation

## Generalized Hooke's Law for Plane Stress

*6 Relations between 6 Strain and 6 Stress*

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \qquad \gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] \qquad \gamma_{yz} = \frac{2(\nu+1)}{E}\tau_{yz}$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \qquad \gamma_{zx} = \frac{2(\nu+1)}{E}\tau_{zx}$$

=

*6 Relations between 6 Strain and 6 Stress*

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x \qquad \tau_{xy} = \frac{E}{2(\nu+1)}\gamma_{xy}$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_y \qquad \tau_{yz} = \frac{E}{2(\nu+1)}\gamma_{yz}$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_z \qquad \tau_{zx} = \frac{E}{2(\nu+1)}\gamma_{zx}$$

$$, e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

**In case of "Plane Stress State",** $\sigma_z = \tau_{yz} = \tau_{zx} = 0$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

**Generalized Hooke's Law for "Plane Stress State"**

**Strain-stress relation for "Plane Stress State"**

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

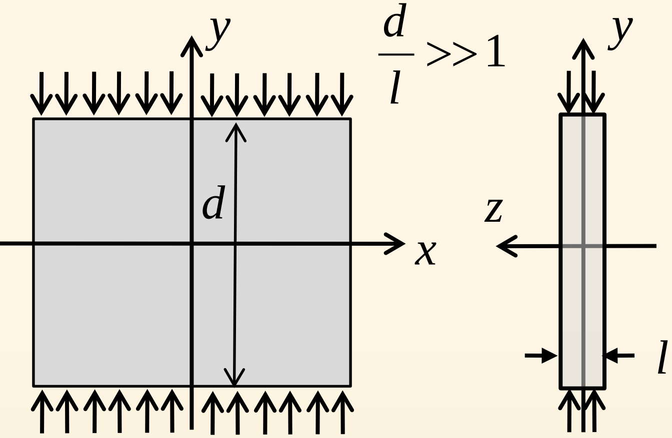
$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)$$

$$\tau_{xy} = G\gamma_{xy}$$



# 2D Problem - The Governing Differential Equation

**A thin plate under the lateral load uniformly distributed over its thickness**




$$\frac{d}{l} >> 1$$

**'A State of Plain Stress'**

▪**body force**

$Z = 0$

$X = X(x, y)$

$Y = Y(x, y)$

$\sigma_x = \sigma_x(x, y)$

$\sigma_y = \sigma_y(x, y)$

$\tau_{xy} = \tau_{xy}(u, v)$

$\sigma_z = \tau_{zx} = \tau_{zy} = 0$

**Generalized Hooke's Law for Plane Strain State**

$$\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy} \Leftrightarrow \sigma_x, \sigma_y, \tau_{xy}$$

**Force equilibrium equation without gravitational force**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

**and introduce stress function $\psi(x, y)$ satisfying the force equilibrium equations**

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + Y = 0$$

$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \cancel{\varepsilon_z = \frac{\partial w}{\partial z}},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \cancel{\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}}, \quad \cancel{\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

*6 Relations between 6 Strain and 6 Stress*

In case of "Plane Stress State", $\sigma_z = \tau_{yz} = \tau_{zx} = 0$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

Generalized Hooke's Law for "Plane Stress State"

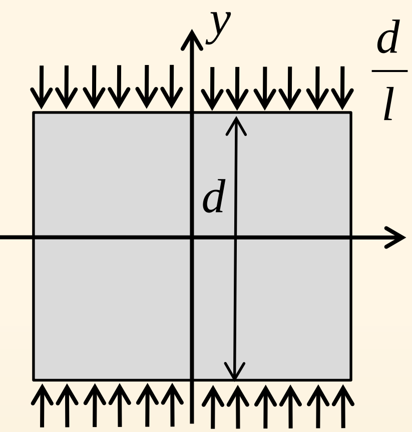
**Strain-stress relation for "Plane Stress State"**

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

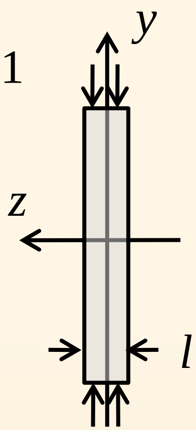
$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x), \quad \tau_{xy} = G\gamma_{xy}$$

# 2D Problem - The Governing Differential Equation

**A thin plate under the lateral load uniformly distributed over its thickness**




$$\frac{d}{l} >> 1$$




**'A State of Plain Stress'**

▪**body force**

$Z = 0$

$X = X(x, y)$

$Y = Y(x, y)$

$\sigma_x = \sigma_x(x, y)$

$\sigma_y = \sigma_y(x, y)$

$\tau_{xy} = \tau_{xy}(x, y)$

$\sigma_z = \tau_{zx} = \tau_{zy} = 0$

**When solving for the stresses the compatibility equations must be used**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\cancel{\frac{\partial^2 \varepsilon_x}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\cancel{\frac{\partial^2 \varepsilon_y}{\partial z^2}} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad 2\cancel{\frac{\partial^2 \varepsilon_y}{\partial z \partial x}} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \cancel{\frac{\partial^2 \varepsilon_x}{\partial z^2}} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial \varepsilon_x}{\partial z} = \frac{1}{E}\left(\frac{\partial \sigma_x}{\partial z} - \nu \frac{\partial \sigma_y}{\partial z}\right) = 0, \quad \because \frac{\partial \sigma_x}{\partial z} = \frac{\partial \sigma_y}{\partial z} = 0$$

**in same way** $\dfrac{\partial \varepsilon_y}{\partial z} = 0, \dfrac{\partial \varepsilon_z}{\partial z} = 0$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + Y = 0$$

$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations* between *6 Strain and 3 Displacement :*

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \cancel{\frac{\partial w}{\partial z}},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \cancel{\frac{\partial v}{\partial z}} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \cancel{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

*6 Relations* **between** *6 Strain and 6 Stress*

**In case of "Plane Stress State",** $\sigma_z = \tau_{yz} = \tau_{zx} = 0$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

**Generalized Hooke's Law for "Plane Stress State"**

**Strain-stress relation for "Plane Stress State"**

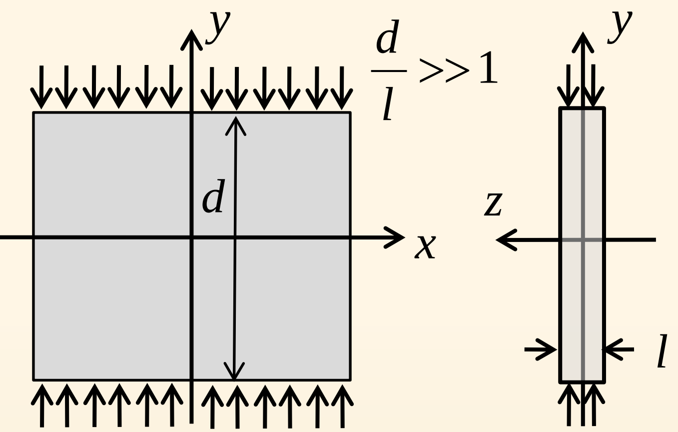
$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x), \qquad \tau_{xy} = G\gamma_{xy}$$



# 2D Problem - The Governing Differential Equation

**A thin plate under the lateral load uniformly distributed over its thickness**




$$\frac{d}{l} >> 1$$

**'A State of Plain Stress'**

▪**body force**

$$Z = 0$$
$$X = X(x, y)$$
$$Y = Y(x, y)$$

$$\sigma_x = \sigma_x(x, y)$$
$$\sigma_y = \sigma_y(x, y)$$
$$\tau_{xy} = \tau_{xy}(u, v)$$
$$\sigma_z = \tau_{zx} = \tau_{zy} = 0$$

**When solving for the stresses the compatibility equations must be used**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\cancel{\frac{\partial^2 \varepsilon_y}{\partial z^2}} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \cancel{\frac{\partial^2 \gamma_{yz}}{\partial y \partial z}}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \cancel{\frac{\partial^2 \varepsilon_x}{\partial z^2}} = \cancel{\frac{\partial^2 \gamma_{zx}}{\partial z \partial x}}$$

$$2\cancel{\frac{\partial^2 \varepsilon_x}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\cancel{\frac{\partial^2 \varepsilon_y}{\partial z \partial x}} = \frac{\partial}{\partial y}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} - \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\gamma_{zx} = \gamma_{zy} = 0 \quad \because \tau_{zx} = \tau_{zy} = 0$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + Y = 0$$

$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \cancel{\frac{\partial w}{\partial z}},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \cancel{\frac{\partial v}{\partial z}} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \cancel{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Stress State",** $\sigma_z = \tau_{yz} = \tau_{zx} = 0$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

**Generalized Hooke's Law for "Plane Stress State"**

**Strain-stress relation for "Plane Stress State"**

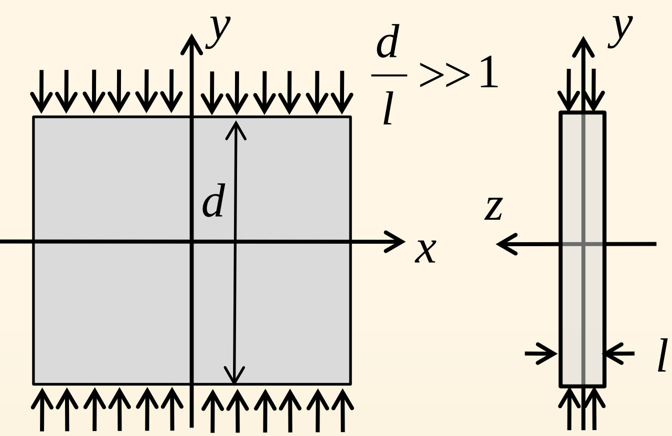
$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x), \qquad \tau_{xy} = G\gamma_{xy}$$



# 2D Problem - The Governing Differential Equation

**A thin plate under the lateral load uniformly distributed over its thickness**




$$\frac{d}{l} >> 1$$

**'A State of Plain Stress'**

▪**body force**

$$Z = 0 \qquad X = X(x,y) \qquad Y = Y(x,y)$$

$$\sigma_x = \sigma_x(x,y) \qquad \sigma_y = \sigma_y(x,y) \qquad \tau_{xy} = \tau_{xy}(u,v)$$

$$\sigma_z = \tau_{zx} = \tau_{zy} = 0$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + Y = 0$$

$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \cancel{\varepsilon_z = \frac{\partial w}{\partial z}},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \cancel{\gamma_{yz} = \frac{\partial v}{\partial z}} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \cancel{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

**When solving for the stresses the compatibility equations must be used**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\cancel{\frac{\partial^2 \varepsilon_x}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

$$\cancel{\frac{\partial^2 \varepsilon_y}{\partial z^2}} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \cancel{\frac{\partial^2 \gamma_{yz}}{\partial y \partial z}} \qquad 2\cancel{\frac{\partial^2 \varepsilon_y}{\partial z \partial x}} = \frac{\partial}{\partial y}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} - \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \cancel{\frac{\partial^2 \varepsilon_x}{\partial z^2}} = \cancel{\frac{\partial^2 \gamma_{zx}}{\partial z \partial x}} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} - \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

$$\frac{\partial \gamma_{xy}}{\partial z} = \frac{2(\nu+1)}{E}\frac{\partial \tau_{xy}}{\partial z} = 0 \qquad \because \frac{\partial \tau_{xy}}{\partial z} = 0$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Stress State",** $\sigma_z = \tau_{yz} = \tau_{zx} = 0$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

**Generalized Hooke's Law for "Plane Stress State"**

**Strain-stress relation for "Plane Stress State"**

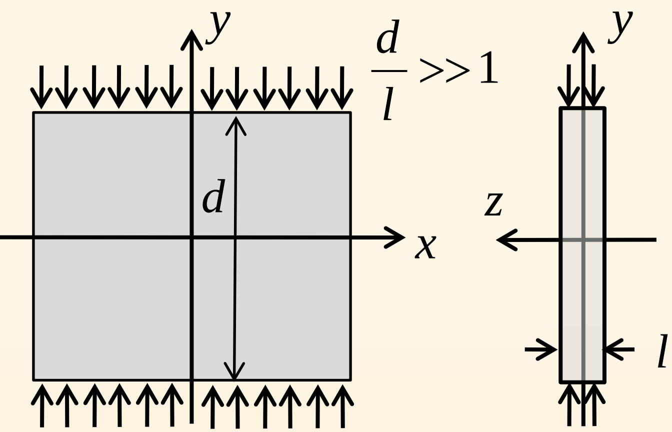
$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x), \qquad \tau_{xy} = G\gamma_{xy}$$



# 2D Problem - The Governing Differential Equation

**A thin plate under the lateral load uniformly distributed over its thickness**




$$\frac{d}{l} >> 1$$

**'A State of Plain Stress'**

▪**body force**

$Z = 0$

$X = X(x, y)$

$Y = Y(x, y)$

$\sigma_x = \sigma_x(x, y)$

$\sigma_y = \sigma_y(x, y)$

$\tau_{xy} = \tau_{xy}(u, v)$

$\sigma_z = \tau_{zx} = \tau_{zy} = 0$

**Generalized Hooke's Law for Plane Strain State**

$$\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy} \Leftrightarrow \sigma_x, \sigma_y, \tau_{xy}$$

***stress function*** $\psi(x, y)$

$$\sigma_x = \frac{\partial^2 \psi}{\partial^2 y}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial^2 x}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

**When solving for the stresses the compatibility equations must be used**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \longrightarrow \boxed{\nabla^4 \psi = 0}$$

**biharmonic Equation**

**In same way with the case of Plane Strain**

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + Y = 0$$

$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad \cancel{2\frac{\partial^2 \varepsilon_x}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

$$\cancel{\frac{\partial^2 \varepsilon_y}{\partial z^2}} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \cancel{\frac{\partial^2 \gamma_{yz}}{\partial y \partial z}} \qquad \cancel{2\frac{\partial^2 \varepsilon_y}{\partial z \partial x}} = \frac{\partial}{\partial y}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} - \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \cancel{\frac{\partial^2 \varepsilon_x}{\partial z^2}} = \cancel{\frac{\partial^2 \gamma_{zx}}{\partial z \partial x}} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} - \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Stress State",** $\sigma_z = \tau_{yz} = \tau_{zx} = 0$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu \sigma_y) \qquad \gamma_{xy} = \frac{2(\nu + 1)}{E}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu \sigma_x)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

**Generalized Hooke's Law for "Plane Stress State"**

**Strain-stress relation for "Plane Stress State"**

$$\sigma_x = \frac{E}{1 - \nu^2}(\varepsilon_x + \nu \varepsilon_y)$$

$$\sigma_y = \frac{E}{1 - \nu^2}(\varepsilon_y + \nu \varepsilon_x), \quad \tau_{xy} = G \gamma_{xy}$$

# 2D Problem - The Governing Differential Equation

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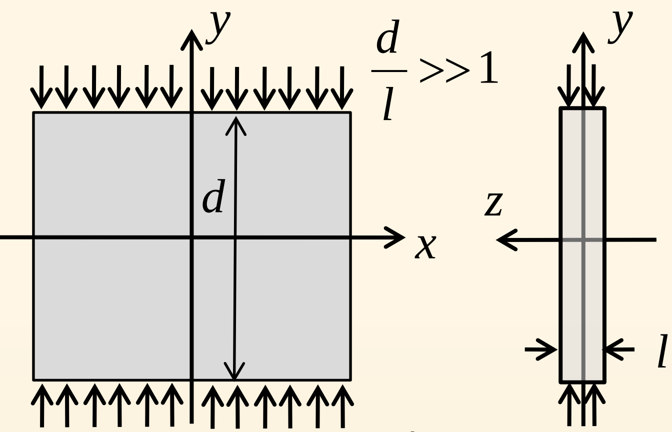



**'A State of Plain Stress'**

▪**body force**

$Z = 0$

$X = X(x, y)$

$Y = Y(x, y)$

$\sigma_x = \sigma_x(x, y)$

$\sigma_y = \sigma_y(x, y)$

$\tau_{xy} = \tau_{xy}(x, y)$

$\sigma_z = \tau_{zx} = \tau_{zy} = 0$

solve

$$\boxed{\nabla^4 \psi = 0}$$

**Stress determined**

$$\sigma_x = \frac{\partial^2 \psi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial x^2}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

**Strain determined**

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y], \quad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x], \quad \varepsilon_z = -\frac{\nu}{E}[\sigma_x + \sigma_y], \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + Y = 0$$

$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\cancel{\frac{\partial^2 \varepsilon_x}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

$$\cancel{\frac{\partial^2 \varepsilon_y}{\partial z^2}} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \cancel{\frac{\partial^2 \gamma_{yz}}{\partial y \partial z}} \qquad 2\cancel{\frac{\partial^2 \varepsilon_y}{\partial z \partial x}} = \frac{\partial}{\partial y}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} - \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \cancel{\frac{\partial^2 \varepsilon_x}{\partial z^2}} = \cancel{\frac{\partial^2 \gamma_{zx}}{\partial z \partial x}} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} - \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Stress State",** $\sigma_z = \tau_{yz} = \tau_{zx} = 0$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

**Generalized Hooke's Law for "Plane Stress State"**

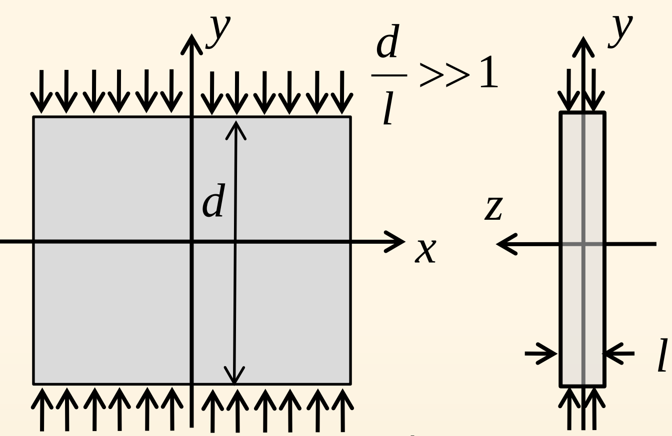
**Strain-stress relation for "Plane Stress State"**

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x), \quad \tau_{xy} = G\gamma_{xy}$$

# 2D Problem - The Governing Differential Equation

**A thin plate under the lateral load uniformly distributed over its thickness**




$$\frac{d}{l} >> 1$$

**'A State of Plain Stress'**

▪**body force**

$$Z = 0$$
$$X = X(x, y)$$
$$Y = Y(x, y)$$

$$\sigma_x = \sigma_x(x, y)$$
$$\sigma_y = \sigma_y(x, y)$$
$$\tau_{xy} = \tau_{xy}(x, y)$$
$$\sigma_z = \tau_{zx} = \tau_{zy} = 0$$

**solve**

$$\boxed{\nabla^4 \psi = 0}$$

**Stress determined**

$$\boxed{\sigma_x = \frac{\partial^2 \psi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial x^2}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}}$$

**Strain determined**

$$\boxed{\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y], \quad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x], \quad \varepsilon_y = -\frac{1}{E}[\sigma_x + \sigma_y], \quad \gamma_{xy} = \frac{\tau_{xy}}{G}}$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \cancel{\frac{\partial \tau_{zx}}{\partial z}} + X = 0$$
$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + Y = 0$$
$$\sum F_z = \cancel{\frac{\partial \tau_{xz}}{\partial x}} + \cancel{\frac{\partial \tau_{yz}}{\partial y}} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\cancel{\frac{\partial^2 \varepsilon_x}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$
$$\cancel{\frac{\partial^2 \varepsilon_y}{\partial z^2}} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \cancel{\frac{\partial^2 \gamma_{yz}}{\partial y \partial z}} \qquad 2\cancel{\frac{\partial^2 \varepsilon_y}{\partial z \partial x}} = \frac{\partial}{\partial y}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} - \cancel{\frac{\partial \gamma_{zx}}{\partial y}} + \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$
$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \cancel{\frac{\partial^2 \varepsilon_x}{\partial z^2}} = \cancel{\frac{\partial^2 \gamma_{zx}}{\partial z \partial x}} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\cancel{\frac{\partial \gamma_{yz}}{\partial x}} + \cancel{\frac{\partial \gamma_{zx}}{\partial y}} - \cancel{\frac{\partial \gamma_{xy}}{\partial z}}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Stress State",** $\sigma_z = \tau_{yz} = \tau_{zx} = 0$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$
$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$
$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

**Generalized Hooke's Law for "Plane Stress State"**

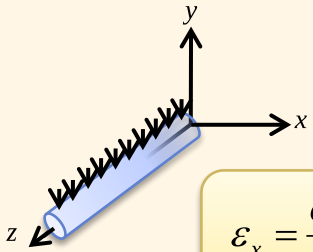
**Strain-stress relation for "Plane Stress State"**

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$
$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x), \quad \tau_{xy} = G\gamma_{xy}$$



# 2D Problem - The Governing Differential Equation

**A prismatic cylinder under the lateral load uniformly distributed along the axis**




**'A State of Plain Strain'**

$$X = X(x,y), \quad Y = Y(x,y), \quad Z = 0$$

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

**Where** $u = u(x,y), \ v = v(x,y)$

$$\varepsilon_x = \varepsilon_x(x,y), \quad \varepsilon_y = \varepsilon_y(x,y), \quad \gamma_{xy} = \gamma_{xy}(x,y)$$

**Generalized Hooke's Law for Plane Strain State**

$$\varepsilon_x, \varepsilon_y, \gamma_{xy} \rightleftarrows \sigma_x, \sigma_x, \sigma_z, \tau_{xy}$$

**Force equilibrium equation**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0$$

**Compatibility equation**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

*Given : 3 Equations of force equilibrium :*

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

$$\sum F_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

*6 Relations between 6 Strain and 3 Displacement :*

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

**compatibility equations**

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

*6 Relations between 6 Strain and 6 Stress*

**In case of "Plane Stress State",** $\sigma_z = \tau_{yz} = \tau_{zx} = 0$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \gamma_{xy} = \frac{2(\nu+1)}{E}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

**Generalized Hooke's Law for "Plane Stress State"**

**Strain-stress relation for "Plane Stress State"**

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

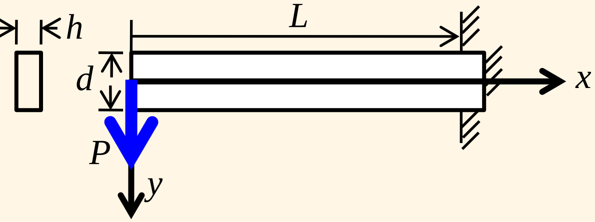
$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x), \quad \tau_{xy} = G\gamma_{xy}$$



# Example

# Example - Cantilever

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**




A cantilever beam of narrow rectangular cross section under an end load $P$ .
With its width $h$ small compared with the depth $d$ , the loaded beam may be regarded with as an <u>example in plane stress</u>

Boundary condition : shearing force at $x = 0$ is equal to $P$

If $P$ is large compared with $\rho g$ , the gravitational force can be neglected

From the static considerations,
Bending moment at any section will be proportional to $x$ and
the stress component $\sigma_x$ at any point of the section will be proportional to $y$

$$\sigma_x \propto xy$$

Accordingly we shall <u>assume</u> for a trial

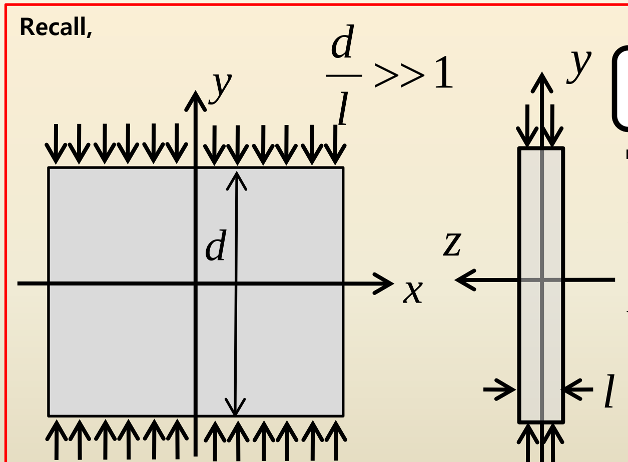
$$\frac{\partial^2 \psi}{\partial y^2} = c_1 xy \quad , c_1 : \text{constant} \quad \because \sigma_x = \frac{\partial^2 \psi}{\partial y^2},$$

$$\int \frac{\partial^2 \psi}{\partial y^2} dy = \int c_1 xy \, dy \Rightarrow \frac{\partial \psi}{\partial y} = \frac{c_1}{2} xy^2 + f_1(x)$$

$$\int \frac{\partial \psi}{\partial y} dy = \int \left[ \frac{c_1}{2} xy^2 + f_1(x) \right] dy$$

$$\psi = \frac{c_1}{6} xy^3 + y f_1(x) + f_2(x)$$

Recall,




'A State of Plain Stress'

- body force

$Z = 0$

$X = X(x, y)$

$Y = Y(x, y)$

$\sigma_x = \sigma_x(x, y)$

$\sigma_y = \sigma_y(x, y)$

$\tau_{xy} = \tau_{xy}(x, y)$

$\sigma_z = \tau_{zx} = \tau_{zy} = 0$

solve

$$\nabla^4 \psi = 0$$

Stress determined

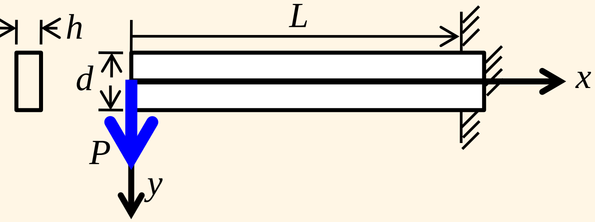
$$\sigma_x = \frac{\partial^2 \psi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial x^2}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

Strain determined

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu \sigma_y], \quad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu \sigma_x], \quad \varepsilon_z = -\frac{1}{E}[\sigma_x + \sigma_y], \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

# Example - Cantilever

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**

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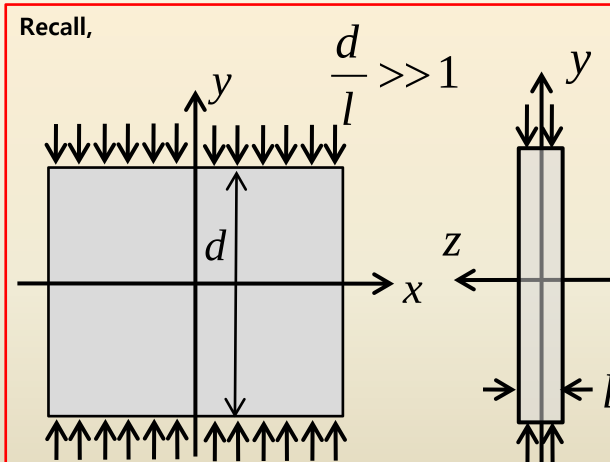
$$\int \frac{\partial^2\psi}{\partial y^2} dy = \int c_1 xy \, dy \;\Rightarrow\; \frac{\partial\psi}{\partial y} = \frac{c_1}{2} xy^2 + f_1(x)$$

$$\int \frac{\partial\psi}{\partial y} dy = \int \left[ \frac{c_1}{2} xy^2 + f_1(x) \right] dy$$

$$\psi = \frac{c_1}{6} xy^3 + y f_1(x) + f_2(x)$$

$, f_1(x), f_2(x) : funciton\ of\ x$

Recall,




'A State of Plain Stress'

- body force

$Z = 0$

$X = X(x,y)$

$Y = Y(x,y)$

$\sigma_x = \sigma_x(x,y)$

$\sigma_y = \sigma_y(x,y)$

$\tau_{xy} = \tau_{xy}(x,y)$

$\sigma_z = \tau_{zx} = \tau_{zy} = 0$

solve

$$\nabla^4\psi = 0$$

Stress determined

$$\boxed{\sigma_x = \frac{\partial^2\psi}{\partial y^2},} \quad \sigma_y = \frac{\partial^2\psi}{\partial x^2}, \quad \tau_{yx} = -\frac{\partial^2\psi}{\partial x \partial y}$$

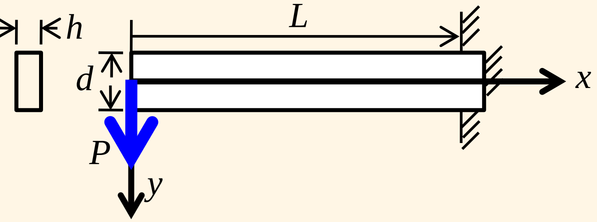
Strain determined

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y], \quad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x], \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$



# Example - Cantilever

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**




A cantilever beam of narrow rectangular cross section under an end load $P$ . With its width $h$ small compared with the depth $d$ , the loaded beam may be regarded with as an <u>example in plane stress</u>

Boundary condition : shearing force at $x=0$ is equal to $P$

If $P$ is large compared with $\rho g$ , the gravitational force can be neglected

From the static considerations,
Bending moment at any section will be proportional to $x$ and
the stress component $\sigma_x$ at any point of the section will be proportional to $y$

$$\sigma_x \propto xy$$

Accordingly <u>assume</u> for a trial

$$\boxed{\sigma_x = \frac{\partial^2 \psi}{\partial y^2} = c_1 xy}$$

$$\psi = \frac{c_1}{6} xy^3 + y f_1(x) + f_2(x)$$

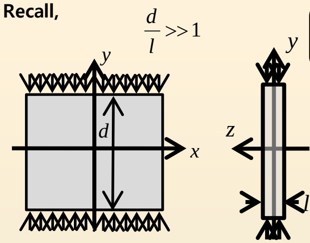
Substitution of $\psi$ into governing equation

$$\boxed{\nabla^4 \psi = 0} \quad , \nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

$$\frac{\partial^4}{\partial x^4}\left(\frac{c_1}{6} xy^3 + y f_1(x) + f_2(x)\right) + 2\frac{\partial^4}{\partial x^2 \partial y^2}\left(\frac{c_1}{6} xy^3 + y f_1(x) + f_2(x)\right) + \frac{\partial^4}{\partial y^4}\left(\frac{c_1}{6} xy^3 + y f_1(x) + f_2(x)\right) = 0$$

$$\frac{\partial^4}{\partial x^4}\left(y f_1(x) + f_2(x)\right) = 0$$

Recall,




**'A State of Plain Stress'**

▪**body force**

$Z = 0$ $\quad \sigma_x = \sigma_x(x,y)$

$X = X(x,y)$ $\quad \sigma_y = \sigma_y(x,y)$

$Y = Y(x,y)$ $\quad \tau_{xy} = \tau_{xy}(x,y)$

$\sigma_z = \tau_{zx} = \tau_{zy} = 0$

solve $\boxed{\nabla^4 \psi = 0}$

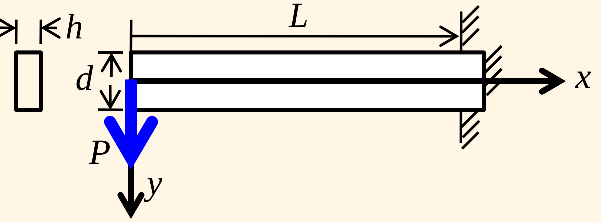
Stress determined $\boxed{\sigma_x = \frac{\partial^2 \psi}{\partial y^2}}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial x^2}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$

Strain determined

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y], \quad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x], \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

# Example - Cantilever

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**




A cantilever beam of narrow rectangular cross section under an end load $P$ .
With its width $h$ small compared with the depth $d$ , the loaded beam may be regarded with as an <u>example in plane stress</u>

Boundary condition : shearing force at $x=0$ is equal to $P$

If $P$ is large compared with $\rho g$ , the gravitational force can be neglected

From the static considerations,
Bending moment at any section will be proportional to $x$ and
the stress component $\sigma_x$ at any point of the section will be proportional to $y$

$$\sigma_x \propto xy$$

Accordingly <u>assume</u> for a trial $\boxed{\sigma_x = \dfrac{\partial^2 \psi}{\partial y^2} = c_1 xy}$

$$\psi = \frac{c_1}{6} xy^3 + y f_1(x) + f_2(x)$$

then, from the governing equation $\boxed{\nabla^4 \psi = 0}$

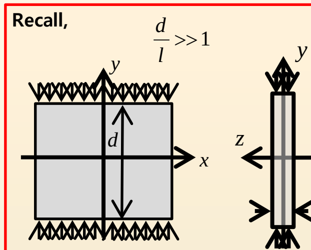
$$\frac{\partial^4}{\partial x^4}\left(y f_1(x) + f_2(x)\right) = 0$$

$$y\frac{\partial^4 f_1(x)}{\partial x^4} + \frac{\partial^4 f_2(x)}{\partial x^4} = 0 \quad \xrightarrow{\because \text{x,y : independent}} \quad \frac{\partial^4 f_1(x)}{\partial x^4} = 0 \ \ and \ \ \frac{\partial^4 f_2(x)}{\partial x^4} = 0$$

integrate four times

$$\therefore \begin{cases} f_1(x) = c_2 x^3 + c_3 x^2 + c_4 x + c_5 \\ f_2(x) = c_6 x^3 + c_7 x^2 + c_8 x + c_9 \end{cases}$$

Recall,




**'A State of Plain Stress'**

▪body force

$Z = 0$ $\quad \sigma_x = \sigma_x(x,y)$

$X = X(x,y)$ $\quad \sigma_y = \sigma_y(x,y)$

$Y = Y(x,y)$ $\quad \tau_{xy} = \tau_{xy}(x,y)$

$\sigma_z = \tau_{zx} = \tau_{zy} = 0$

solve $\boxed{\nabla^4 \psi = 0}$ ⇒ Stress determined $\boxed{\sigma_x = \dfrac{\partial^2 \psi}{\partial y^2}},\ \sigma_y = \dfrac{\partial^2 \psi}{\partial x^2},\ \tau_{yx} = -\dfrac{\partial^2 \psi}{\partial x \partial y}$

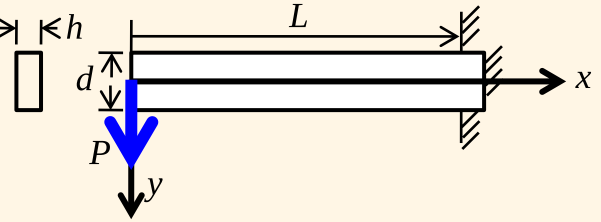
Strain determined

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y],\ \ \varepsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x],\ \ \gamma_{xy} = \frac{\tau_{xy}}{G}$$



# Example - Cantilever

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**

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From the static considerations,
Bending moment at any section will be proportional to $x$ and
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$$\sigma_x \propto xy$$

Accordingly <u>assume</u> for a trial $\boxed{\sigma_x = \dfrac{\partial^2 \psi}{\partial y^2} = c_1 xy}$

$$\psi = \frac{c_1}{6} xy^3 + y f_1(x) + f_2(x)$$

then, from the governing equation $\boxed{\nabla^4 \psi = 0}$

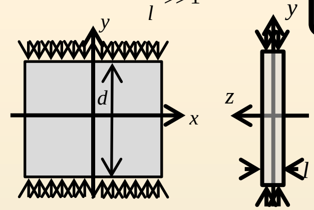
$$\begin{cases} f_1(x) = c_2 x^3 + c_3 x^2 + c_4 x + c_5 \\ f_2(x) = c_6 x^3 + c_7 x^2 + c_8 x + c_9 \end{cases}$$

$$\psi = \frac{c_1}{6} xy^3 + y\left[c_2 x^3 + c_3 x^2 + c_4 x + c_5\right] + \left[c_6 x^3 + c_7 x^2 + c_8 x + c_9\right]$$

$$\boxed{\sigma_y = \frac{\partial^2 \psi}{\partial^2 x}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}}$$

Recall,

$\frac{d}{l} >> 1$




**'A State of Plain Stress'**

▪**body force**

$Z = 0$ $\qquad \sigma_x = \sigma_x(x,y)$

$X = X(x,y)$ $\qquad \sigma_y = \sigma_y(x,y)$

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solve

$\boxed{\nabla^4 \psi = 0}$ ⇒ Stress determined $\boxed{\sigma_x = \dfrac{\partial^2 \psi}{\partial y^2},}$ $\boxed{\sigma_y = \dfrac{\partial^2 \psi}{\partial x^2}, \quad \tau_{yx} = -\dfrac{\partial^2 \psi}{\partial x \partial y}}$

Strain determined

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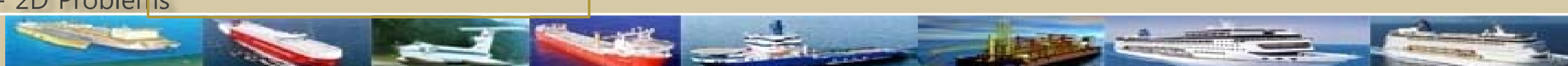
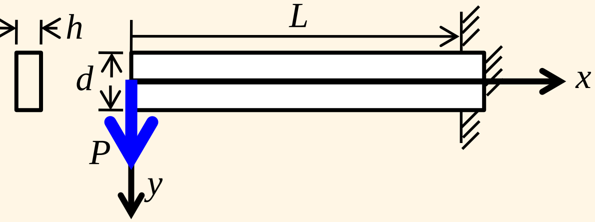


# Example - Cantilever

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**




A cantilever beam of narrow rectangular cross section under an end load $P$ . With its width $h$ small compared with the depth $d$ , the loaded beam may be regarded with as an example in plane stress

Boundary condition : shearing force at $x=0$ is equal to $P$

If $P$ is large compared with $\rho g$ , the gravitational force can be neglected

From the static considerations,
Bending moment at any section will be proportional to $x$ and
the stress component $\sigma_x$ at any point of the section will be proportional to $y$

$$\sigma_x \propto xy$$

Accordingly assume for a trial

$$\boxed{\sigma_x = \frac{\partial^2 \psi}{\partial y^2} = c_1 xy}$$

$$\psi = \frac{c_1}{6} xy^3 + y f_1(x) + f_2(x)$$

then, from the governing equation $\boxed{\nabla^4 \psi = 0}$

$$\psi = \frac{c_1}{6} xy^3 + y\left[c_2 x^3 + c_3 x^2 + c_4 x + c_5\right] + \left[c_6 x^3 + c_7 x^2 + c_8 x + c_9\right]$$

$$\boxed{\sigma_y = \frac{\partial^2 \psi}{\partial x^2}} = \frac{\partial^2}{\partial x^2}\left\{\frac{c_1}{6} xy^3 + y\left[c_2 x^3 + c_3 x^2 + c_4 x + c_5\right] + \left[c_6 x^3 + c_7 x^2 + c_8 x + c_9\right]\right\} \Rightarrow \therefore \sigma_y = 6(c_2 y + c_6)x + 2(c_3 y + c_7)$$

$$\boxed{\tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}} = -\frac{\partial^2}{\partial x \partial y}\left\{\frac{c_1}{6} xy^3 + y\left[c_2 x^3 + c_3 x^2 + c_4 x + c_5\right] + \left[c_6 x^3 + c_7 x^2 + c_8 x + c_9\right]\right\} \Rightarrow \therefore \tau_{xy} = -\frac{c_1}{2} y^2 - 3c_2 x^2 - 2c_3 x - c_4$$

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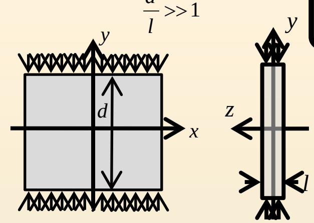


**'A State of Plain Stress'**

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solve

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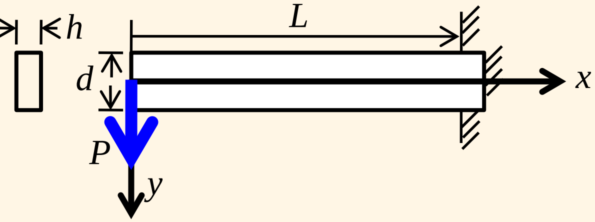


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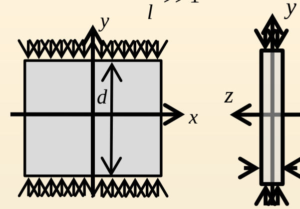


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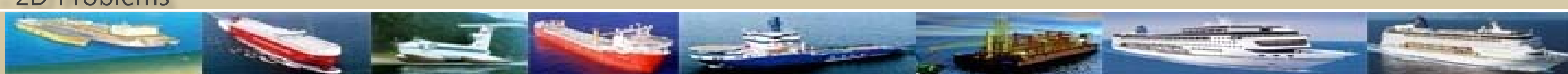

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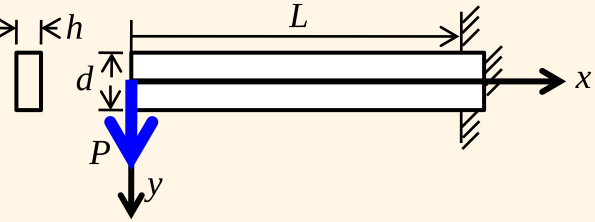


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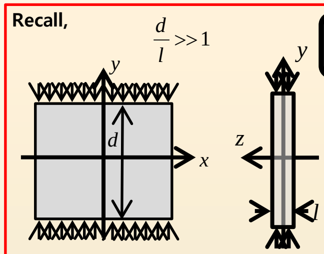


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---

from boundary condition $\sigma_y = 0$ at $y = \pm\frac{d}{2}$

$$\sigma_y\Big|_{y=d/2} = 6(c_2\frac{d}{2} + c_6)x + 2(c_3\frac{d}{2} + c_7) = 0$$

$$\sigma_y\Big|_{y=-d/2} = 6(-c_2\frac{d}{2} + c_6)x + 2(-c_3\frac{d}{2} + c_7) = 0$$

Since this must be valid for all $x$

$$(c_2\frac{d}{2} + c_6) = 0, \quad (c_3\frac{d}{2} + c_7) = 0$$

$$(-c_2\frac{d}{2} + c_6) = 0, \quad (-c_3\frac{d}{2} + c_7) = 0$$

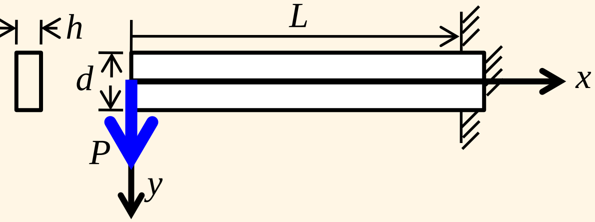
$$c_2 = 0, \; c_6 = 0, \; c_3 = 0, \; c_7 = 0$$

$$\therefore \begin{cases} \sigma_y = 0 \\ \tau_{xy} = -\frac{c_1}{2}y^2 - c_4 \end{cases}$$



# Example - Cantilever

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**

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from boundary condition $\tau_{xy} = 0 \text{ at } y = \pm\dfrac{d}{2}$

$$\tau_{xy}\Big|_{y=\pm d/2} = -\frac{c_1}{2}\frac{d^2}{4} - c_4 \Rightarrow c_4 = -\frac{c_1}{8}d^2 \Rightarrow \tau_{xy} = -\frac{c_1}{2}y^2 + \frac{c_1}{8}d^2$$

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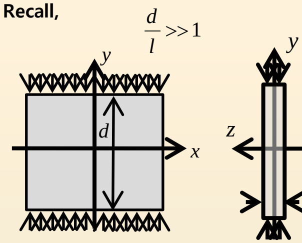


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solve $\boxed{\nabla^4\psi = 0}$ ⇨ Stress determined $\boxed{\sigma_x = \dfrac{\partial^2\psi}{\partial y^2}}$, $\boxed{\sigma_y = \dfrac{\partial^2\psi}{\partial x^2}, \quad \tau_{yx} = -\dfrac{\partial^2\psi}{\partial x\partial y}}$

Strain determined

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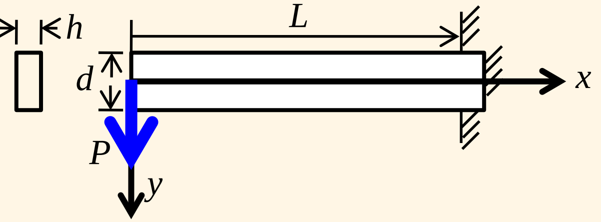


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from boundary condition the sum of the distributed shearing forces must be equal to P on the loaded end of the beam ,

$$-\int_{-d/2}^{+d/2} \tau_{xy} h dy = P$$

$\tau_{xy}$ :negative

$$-\int_{-d/2}^{+d/2} \tau_{xy} h dy = -\int_{-d/2}^{+d/2} \left(-\frac{c_1}{2} y^2 + \frac{c_1}{8} d^2\right) h dy = \int_{-d/2}^{+d/2} \frac{c_1}{8} h(4y^2 - d^2) dy = P$$

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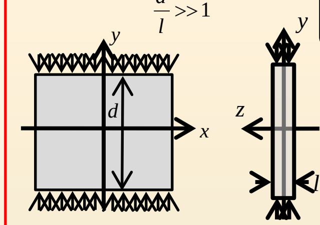


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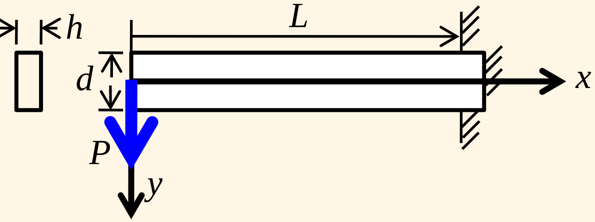


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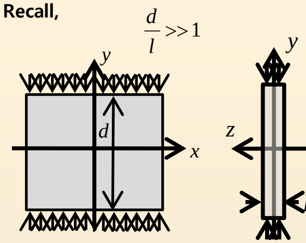


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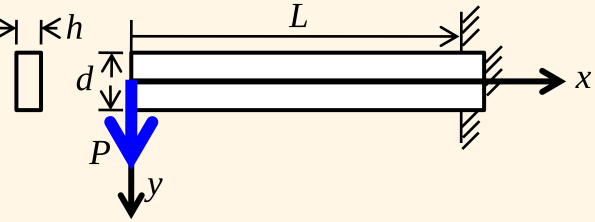


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then, from the governing equation $\boxed{\nabla^4\psi = 0}$

$$\psi = \frac{c_1}{6}xy^3 + y\left[c_2x^3 + c_3x^2 + c_4x + c_5\right] + \left[c_6x^3 + c_7x^2 + c_8x + c_9\right]$$

from $\sigma_x = \frac{\partial^2\psi}{\partial y^2}$, $\boxed{\sigma_y = \frac{\partial^2\psi}{\partial x^2},\quad \tau_{yx} = -\frac{\partial^2\psi}{\partial x\partial y}}$ $\Rightarrow$

$$\sigma_y = 6(c_2y + c_6)x + 2(c_3y + c_7)$$

$$\tau_{xy} = -\frac{c_1}{2}y^2 - 3c_2x^2 - 2c_3x - c_4$$

from boundary condition $\sigma_y = 0$ at $y = \pm\frac{d}{2}$ $\Rightarrow$ $\sigma_y = 0,\ \tau_{xy} = -\frac{c_1}{2}y^2 - c_4$

from boundary condition $\tau_{xy} = 0$ at $y = \pm\frac{d}{2}$ $\Rightarrow$ $\tau_{xy} = -\frac{c_1}{2}y^2 + \frac{c_1}{8}d^2$

---

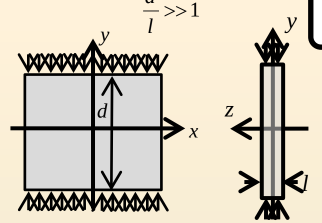
from boundary condition the sum of the distributed shearing forces must be equal to P on the loaded end of the beam ,

$$P = -\frac{c_1}{12}hd^3 \qquad \therefore c_1 = -\frac{12}{hd^3}P \Rightarrow \tau_{xy} = \frac{12}{hd^3}P\left(\frac{1}{2}y^2 - \frac{1}{8}d^2\right)$$

and $\boxed{\sigma_x = -\frac{12}{hd^3}Pxy}$

Recall,

$\frac{d}{l} >> 1$




**'A State of Plain Stress'**

▪body force

$Z = 0$ $\quad \sigma_x = \sigma_x(x,y)$

$X = X(x,y)$ $\quad \sigma_y = \sigma_y(x,y)$

$Y = Y(x,y)$ $\quad \tau_{xy} = \tau_{xy}(x,y)$

solve $\boxed{\nabla^4\psi = 0}$ $\Rightarrow$ Stress determined $\sigma_z = \tau_{zx} = \tau_{zy} = 0$

$\boxed{\sigma_x = \frac{\partial^2\psi}{\partial y^2},}$ $\boxed{\sigma_y = \frac{\partial^2\psi}{\partial x^2},\quad \tau_{yx} = -\frac{\partial^2\psi}{\partial x\partial y}}$

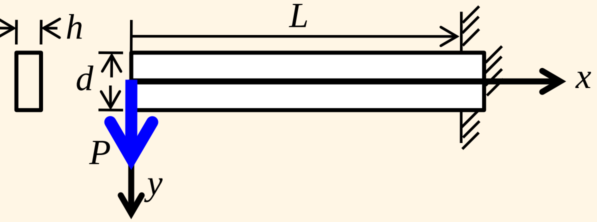
Strain determined

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y],\quad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x],\ \gamma_{xy} = \frac{\tau_{xy}}{G}$$



# Example - Cantilever

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**




A cantilever beam of narrow rectangular cross section under an end load $P$ .
With its width $h$ small compared with the depth $d$ , the loaded beam may be regarded with as an <u>example in plane stress</u>

Boundary condition : shearing force at $x=0$ is equal to $P$

If $P$ is large compared with $\rho g$ , the gravitational force can be neglected

From the static considerations,
Bending moment at any section will be proportional to $x$ and
the stress component $\sigma_x$ at any point of the section will be proportional to $y$

$$\sigma_x \propto xy$$

Accordingly <u>assume</u> for a trial $\boxed{\sigma_x = \dfrac{\partial^2 \psi}{\partial y^2} = c_1 xy}$

$$\psi = \frac{c_1}{6} xy^3 + y f_1(x) + f_2(x)$$

then, from the governing equation $\boxed{\nabla^4 \psi = 0}$

$$\psi = \frac{c_1}{6} xy^3 + y\left[c_2 x^3 + c_3 x^2 + c_4 x + c_5\right] + \left[c_6 x^3 + c_7 x^2 + c_8 x + c_9\right]$$

from $\boxed{\sigma_x = \dfrac{\partial^2 \psi}{\partial y^2},\quad \sigma_y = \dfrac{\partial^2 \psi}{\partial x^2},\quad \tau_{yx} = -\dfrac{\partial^2 \psi}{\partial x \partial y}}$

and boundary conditions $\sigma_y = 0$ at $y = \pm \dfrac{d}{2}$ , $\tau_{xy} = 0$ at $y = \pm \dfrac{d}{2}$ , $-\displaystyle\int_{-d/2}^{+d/2} \tau_{xy} h\, dy = P$

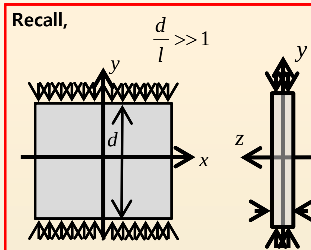
$$\sigma_x = -\frac{12}{hd^3} Pxy\ ,\quad \sigma_y = 0,\quad \tau_{xy} = \frac{12}{hd^3} P\left(\frac{1}{2} y^2 - \frac{1}{8} d^2\right)$$

$$\boxed{\sigma_x = -\frac{Pxy}{I}\ ,\quad \sigma_y = 0,\quad \tau_{xy} = \frac{I}{2} P\left(y^2 - \frac{1}{4} d^2\right)}$$

moment of inertia of the cross section

$$I = \frac{hd^3}{12}$$

Recall,




**'A State of Plain Stress'**

▪body force

$Z = 0$ $\quad \sigma_x = \sigma_x(x,y)$

$X = X(x,y)$ $\quad \sigma_y = \sigma_y(x,y)$

$Y = Y(x,y)$ $\quad \tau_{xy} = \tau_{xy}(x,y)$

$\sigma_z = \tau_{zx} = \tau_{zy} = 0$

solve $\boxed{\nabla^4 \psi = 0}$

Stress determined $\boxed{\sigma_x = \dfrac{\partial^2 \psi}{\partial y^2},}$ $\boxed{\sigma_y = \dfrac{\partial^2 \psi}{\partial x^2},\quad \tau_{yx} = -\dfrac{\partial^2 \psi}{\partial x \partial y}}$

Strain determined

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu \sigma_y],\quad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu \sigma_x],\quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$



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# Example - Cantilever

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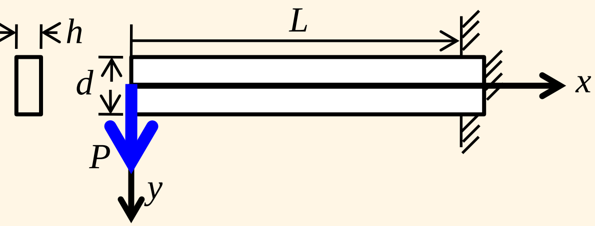


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$$\sigma_x = -\frac{Pxy}{I}, \quad \sigma_y = 0, \quad \tau_{xy} = \frac{I}{2}P\left(y^2 - \frac{1}{4}d^2\right)$$

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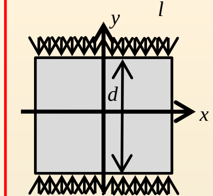
$$\varepsilon_x = \frac{\sigma_x}{E} = -\frac{Pxy}{EI}$$

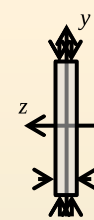
$$\varepsilon_y = -\frac{\nu\sigma_x}{E} = \frac{\nu Pxy}{EI}$$

$$\gamma_{xy} = \frac{2(1+\nu)\tau_{xy}}{E} = \frac{P(1+\nu)}{EI}\left(y^2 - \frac{d^2}{4}\right)$$

Recall,

$\frac{d}{l} >> 1$




**'A State of Plain Stress'**

▪**body force**

$Z = 0$ $\quad \sigma_x = \sigma_x(x, y)$

$X = X(x, y)$ $\quad \sigma_y = \sigma_y(x, y)$

$Y = Y(x, y)$ $\quad \tau_{xy} = \tau_{xy}(x, y)$

solve

$$\nabla^4 \psi = 0$$

Stress determined $\quad \sigma_z = \tau_{zx} = \tau_{zy} = 0$

$$\sigma_x = \frac{\partial^2\psi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\psi}{\partial x^2}, \quad \tau_{yx} = -\frac{\partial^2\psi}{\partial x \partial y}$$

Strain determined

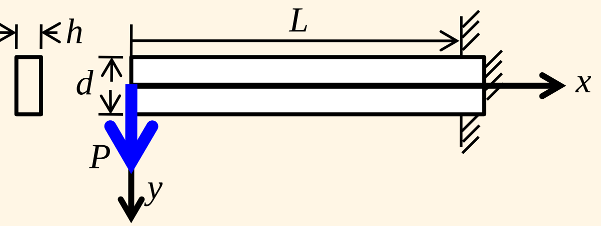
$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y], \quad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x], \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$



# Example - Cantilever

$$I = \frac{hd^3}{12}$$

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**




A cantilever beam of narrow rectangular cross section under an end load $P$ . With its width $h$ small compared with the depth $d$ , the loaded beam may be regarded with as an <u>example in plane stress</u>

Boundary condition : shearing force at $x = 0$ is equal to $P$

If $P$ is large compared with $\rho g$ , the gravitational force can be neglected

$$\sigma_x = -\frac{Pxy}{I}, \quad \sigma_y = 0, \quad \tau_{xy} = \frac{I}{2} P\left(y^2 - \frac{1}{4}d^2\right)$$

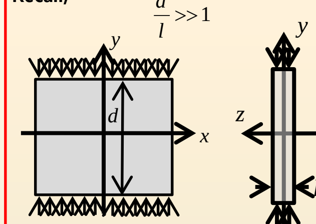
$$\varepsilon_x = -\frac{Pxy}{EI}, \ \varepsilon_y = \frac{\nu Pxy}{EI}, \ \gamma_{xy} = \frac{P(1+\nu)}{EI}(y^2 - \frac{d^2}{4})$$

$$\frac{\partial u}{\partial x} = -\frac{Pxy}{EI}$$

$$\frac{\partial v}{\partial y} = \frac{\nu Pxy}{EI}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{P(1+\nu)}{EI}(y^2 - \frac{d^2}{4})$$

Recall, $\frac{d}{l} \gg 1$




**'A State of Plain Stress'**

▪body force

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$X = X(x, y)$ $\quad \sigma_y = \sigma_y(x, y)$

$Y = Y(x, y)$ $\quad \tau_{xy} = \tau_{xy}(x, y)$

solve

Stress determined $\quad \sigma_z = \tau_{zx} = \tau_{zy} = 0$

$$\nabla^4 \psi = 0 \qquad \sigma_x = \frac{\partial^2 \psi}{\partial y^2}, \ \sigma_y = \frac{\partial^2 \psi}{\partial x^2}, \ \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

Strain determined

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y], \ \varepsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x], \ \gamma_{xy} = \frac{\tau_{xy}}{G}$$

<u>Strain and Displacement</u> :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \cancel{\varepsilon_z = \frac{\partial w}{\partial z}},$$

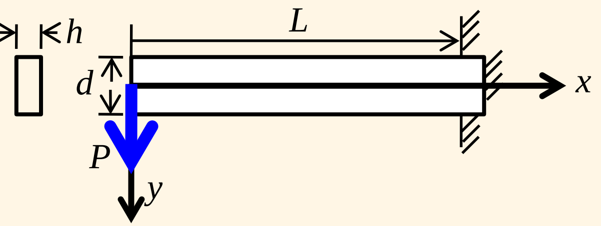
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \cancel{\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}}, \quad \gamma_{zx} = \cancel{\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}}$$

moment of inertia of the cross section

$$I = \frac{hd^3}{12}$$

# Example - Cantilever

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**




**A cantilever beam of narrow rectangular cross section under an end load $P$ . With its width $h$ small compared with the depth $d$ , the loaded beam may be regarded with as an <u>example in plane stress</u>**

**Boundary condition : shearing force at $x = 0$ is equal to $P$**

**If $P$ is large compared with $\rho g$ , the gravitational force can be neglected**

$$\sigma_x = -\frac{Pxy}{I}, \quad \sigma_y = 0, \quad \tau_{xy} = \frac{I}{2}P\left(y^2 - \frac{1}{4}d^2\right)$$

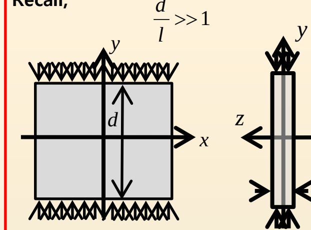
$$\varepsilon_x = -\frac{Pxy}{EI}, \quad \varepsilon_y = \frac{\nu Pxy}{EI}, \quad \gamma_{xy} = \frac{P(1+\nu)}{EI}\left(y^2 - \frac{d^2}{4}\right)$$

$$\frac{\partial u}{\partial x} = -\frac{Pxy}{EI} \quad \text{Integrate} \quad \int \frac{\partial u}{\partial x} dx = \int \left(-\frac{Pxy}{EI}\right) dx \quad u = -\frac{P}{2EI}x^2 y + g_1(y)$$

$$\frac{\partial v}{\partial y} = \frac{\nu Pxy}{EI} \quad \int \frac{\partial v}{\partial y} dy = \int \frac{\nu Pxy}{EI} dy \quad v = \frac{\nu P}{2EI}xy^2 + g_2(x)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{P(1+\nu)}{EI}\left(y^2 - \frac{d^2}{4}\right)$$

Recall,




**'A State of Plain Stress'**

- **body force**

$Z = 0$ $\quad \sigma_x = \sigma_x(x, y)$

$X = X(x, y)$ $\quad \sigma_y = \sigma_y(x, y)$

$Y = Y(x, y)$ $\quad \tau_{xy} = \tau_{xy}(x, y)$

solve

$\nabla^4 \psi = 0$

Stress determined $\quad \sigma_z = \tau_{zx} = \tau_{zy} = 0$

$$\sigma_x = \frac{\partial^2 \psi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial x^2}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

Strain determined

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y], \quad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x], \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

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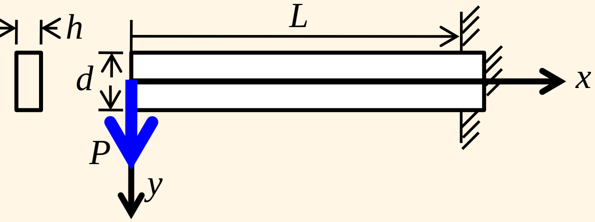
moment of inertia of the cross section

$$I=\frac{hd^3}{12}$$

# Example - Cantilever

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**

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$$\sigma_x=-\frac{Pxy}{I}\ ,\ \sigma_y=0,\quad \tau_{xy}=\frac{I}{2}P\left(y^2-\frac{1}{4}d^2\right)$$

$$\varepsilon_x=-\frac{Pxy}{EI},\ \varepsilon_y=\frac{\nu Pxy}{EI},\ \gamma_{xy}=\frac{P(1+\nu)}{EI}(y^2-\frac{d^2}{4})$$

$$\frac{\partial u}{\partial x}=-\frac{Pxy}{EI}$$

$$\frac{\partial v}{\partial y}=\frac{\nu Pxy}{EI}$$

**Integrate**

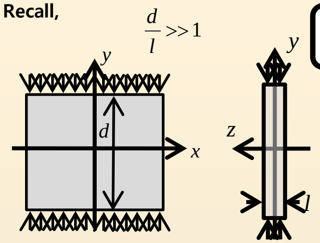
$$u=-\frac{P}{2EI}x^2y+g_1(y)$$

$$v=\frac{\nu P}{2EI}xy^2+g_2(x)$$

$$\boxed{\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}}=\frac{P(1+\nu)}{EI}(y^2-\frac{d^2}{4})$$

$$\frac{\partial}{\partial y}\left(-\frac{P}{2EI}x^2y+g_1(y)\right)+\frac{\partial}{\partial x}\left(\frac{\nu P}{2EI}xy^2+g_2(x)\right)=\frac{P(1+\nu)}{EI}(y^2-\frac{d^2}{4})$$

**Recall,**




**'A State of Plain Stress'**

**▪body force**

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$X=X(x,y)$ $\quad \sigma_y=\sigma_y(x,y)$

$Y=Y(x,y)$ $\quad \tau_{xy}=\tau_{xy}(x,y)$

solve

$$\nabla^4\psi=0$$

Stress determined $\quad \sigma_z=\tau_{zx}=\tau_{zy}=0$

$$\sigma_x=\frac{\partial^2\psi}{\partial y^2},\ \sigma_y=\frac{\partial^2\psi}{\partial x^2},\ \tau_{yx}=-\frac{\partial^2\psi}{\partial x\partial y}$$

Strain determined

$$\varepsilon_x=\frac{1}{E}[\sigma_x-\nu\sigma_y],\ \varepsilon_y=\frac{1}{E}[\sigma_y-\nu\sigma_x],\ \gamma_{xy}=\frac{\tau_{xy}}{G}$$

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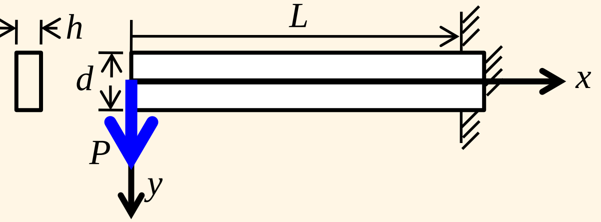
# Example - Cantilever

$$I = \frac{hd^3}{12}$$

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**

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If $P$ is large compared with $\rho g$ , the gravitational force can be neglected

$$\boxed{\sigma_x = -\frac{Pxy}{I}, \quad \sigma_y = 0, \quad \tau_{xy} = \frac{I}{2}P\left(y^2 - \frac{1}{4}d^2\right)}$$

$$\frac{\partial}{\partial y}\left(-\frac{P}{2EI}x^2 y + g_1(y)\right) + \frac{\partial}{\partial x}\left(\frac{\nu P}{2EI}xy^2 + g_2(x)\right) = \frac{P(1+\nu)}{EI}(y^2 - \frac{d^2}{4})$$

$$-\frac{P}{2EI}x^2 + \frac{\partial g_1(y)}{\partial y} + \frac{\nu P}{2EI}y^2 + \frac{\partial g_2(x)}{\partial x} = \frac{P(1+\nu)}{EI}(y^2 - \frac{d^2}{4})$$

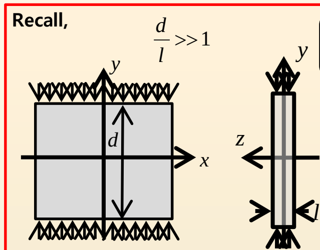
$$\frac{\partial g_1(y)}{\partial y} + \frac{\nu P}{2EI}y^2 - \frac{P(1+\nu)}{EI}y^2 = -\frac{\partial g_2(x)}{\partial x} + \frac{P}{2EI}x^2 - \frac{P(1+\nu)}{4EI}d^2$$

$$\underbrace{\frac{\partial g_1(y)}{\partial y} - \frac{P}{EI}(1+\frac{\nu}{2})y^2}_{\text{Function of y}} = \underbrace{-\frac{\partial g_2(x)}{\partial x} + \frac{P}{2EI}x^2 - \frac{P(1+\nu)}{4EI}d^2}_{\text{Function of x}}$$

$$\frac{\partial g_1(y)}{\partial y} - \frac{P}{EI}(1+\frac{\nu}{2})y^2 = -\frac{\partial g_2(x)}{\partial x} + \frac{P}{2EI}x^2 - \frac{P(1+\nu)}{4EI}d^2 = a_1$$

Recall,

$\frac{d}{l} >> 1$




**'A State of Plain Stress'**

▪body force

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$Y = Y(x,y)$ $\quad \tau_{xy} = \tau_{xy}(x,y)$

solve

Stress determined $\quad \sigma_z = \tau_{zx} = \tau_{zy} = 0$

$$\nabla^4 \psi = 0 \quad \Rightarrow \quad \sigma_x = \frac{\partial^2 \psi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial x^2}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

Strain determined

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y], \quad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x], \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

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moment of inertia of the cross section

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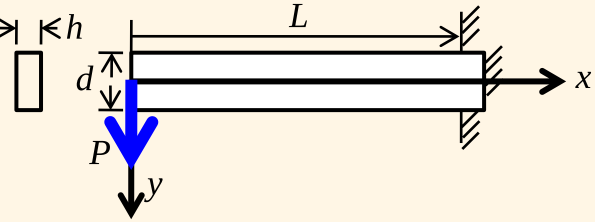
# Example - Cantilever

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**

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If $P$ is large compared with $\rho g$ , the gravitational force can be neglected

$$\sigma_x = -\frac{Pxy}{I}, \quad \sigma_y = 0, \quad \tau_{xy} = \frac{I}{2}P\left(y^2 - \frac{1}{4}d^2\right)$$

$$\frac{\partial g_1(y)}{\partial y} - \frac{P}{EI}(1+\frac{\nu}{2})y^2 = -\frac{\partial g_2(x)}{\partial x} + \frac{P}{2EI}x^2 - \frac{P(1+\nu)}{4EI}d^2 = a_1$$

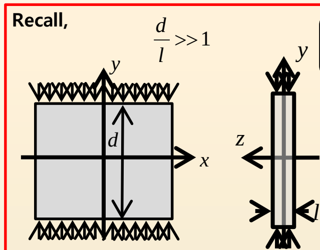
$$\therefore \frac{\partial g_1(y)}{\partial y} - \frac{P}{EI}(1+\frac{\nu}{2})y^2 = a_1, \quad -\frac{\partial g_2(x)}{\partial x} + \frac{P}{2EI}x^2 - \frac{P(1+\nu)}{4EI}d^2 = a_1$$

$$\frac{\partial g_1(y)}{\partial y} = \frac{P}{EI}(1+\frac{\nu}{2})y^2 + a_1, \quad \frac{\partial g_2(x)}{\partial x} = \frac{P}{2EI}x^2 - \frac{P(1+\nu)}{4EI}d^2 - a_1$$

$$\int \frac{\partial g_1(y)}{\partial y}dy = \int\left(\frac{P}{EI}(1+\frac{\nu}{2})y^2 + a_1\right)dy$$

$$g_1(y) = \frac{P}{3EI}(1+\frac{\nu}{2})y^3 + a_1 y + a_2$$

Recall,




'A State of Plain Stress'

- body force

$Z = 0 \qquad \sigma_x = \sigma_x(x,y)$

$X = X(x,y) \qquad \sigma_y = \sigma_y(x,y)$

$Y = Y(x,y) \qquad \tau_{xy} = \tau_{xy}(x,y)$

solve

Stress determined $\quad \sigma_z = \tau_{zx} = \tau_{zy} = 0$

$$\nabla^4\psi = 0 \quad \Rightarrow \quad \sigma_x = \frac{\partial^2\psi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\psi}{\partial x^2}, \quad \tau_{yx} = -\frac{\partial^2\psi}{\partial x\partial y}$$

Strain determined

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y], \quad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x], \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

Strain and Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

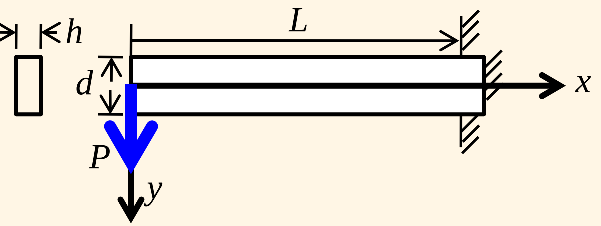
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$



# Example - Cantilever

$$I=\frac{hd^3}{12}$$

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**




A cantilever beam of narrow rectangular cross section under an end load $P$ . With its width $h$ small compared with the depth $d$ , the loaded beam may be regarded with as an example in plane stress

Boundary condition : shearing force at $x=0$ is equal to $P$

If $P$ is large compared with $\rho g$ , the gravitational force can be neglected

$$\sigma_x=-\frac{Pxy}{I}\ ,\ \sigma_y=0,\quad \tau_{xy}=\frac{I}{2}P\left(y^2-\frac{1}{4}d^2\right)$$

---

$$\frac{\partial g_1(y)}{\partial y}-\frac{P}{EI}(1+\frac{\nu}{2})y^2=-\frac{\partial g_2(x)}{\partial x}+\frac{P}{2EI}x^2-\frac{P(1+\nu)}{4EI}d^2=a_1$$

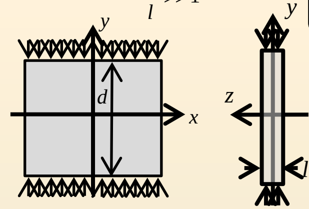
$$\therefore \frac{\partial g_1(y)}{\partial y}-\frac{P}{EI}(1+\frac{\nu}{2})y^2=a_1,\quad -\frac{\partial g_2(x)}{\partial x}+\frac{P}{2EI}x^2-\frac{P(1+\nu)}{4EI}d^2=a_1$$

$$\frac{\partial g_1(y)}{\partial y}=\frac{P}{EI}(1+\frac{\nu}{2})y^2+a_1,\quad \frac{\partial g_2(x)}{\partial x}=\frac{P}{2EI}x^2-\frac{P(1+\nu)}{4EI}d^2-a_1$$

$$\int\frac{\partial g_2(x)}{\partial x}dx=\int\left(\frac{P}{2EI}x^2-\frac{P(1+\nu)}{4EI}d^2-a_1\right)dx$$

$$g_2(x)=\frac{P}{6EI}x^3-\frac{(1+\nu)P}{4EI}d^2x-a_1x+a_3$$

Recall,




$\frac{d}{l}>>1$

**'A State of Plain Stress'**

▪body force

$Z=0$ $\quad \sigma_x=\sigma_x(x,y)$

$X=X(x,y)\quad \sigma_y=\sigma_y(x,y)$

$Y=Y(x,y)\quad \tau_{xy}=\tau_{xy}(x,y)$

solve

Stress determined $\quad \sigma_z=\tau_{zx}=\tau_{zy}=0$

$\nabla^4\psi=0$ $\quad \sigma_x=\frac{\partial^2\psi}{\partial y^2},\ \sigma_y=\frac{\partial^2\psi}{\partial x^2},\ \tau_{yx}=-\frac{\partial^2\psi}{\partial x\partial y}$

Strain determined

$\varepsilon_x=\frac{1}{E}[\sigma_x-\nu\sigma_y],\ \varepsilon_y=\frac{1}{E}[\sigma_y-\nu\sigma_x],\ \gamma_{xy}=\frac{\tau_{xy}}{G}$

Strain and Displacement :

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \cancel{\varepsilon_z=\frac{\partial w}{\partial z}},$$

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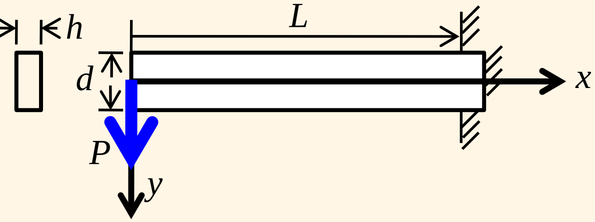
# Example - Cantilever

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**

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$$\frac{\partial g_1(y)}{\partial y}-\frac{P}{EI}(1+\frac{\nu}{2})y^2=-\frac{\partial g_2(x)}{\partial x}+\frac{P}{2EI}x^2-\frac{P(1+\nu)}{4EI}d^2=a_1$$

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$$\frac{\partial g_1(y)}{\partial y}=\frac{P}{EI}(1+\frac{\nu}{2})y^2+a_1,\quad \frac{\partial g_2(x)}{\partial x}=\frac{P}{2EI}x^2-\frac{P(1+\nu)}{4EI}d^2-a_1$$

$$g_1(y)=\frac{P}{3EI}(1+\frac{\nu}{2})y^3+a_1y+a_2$$

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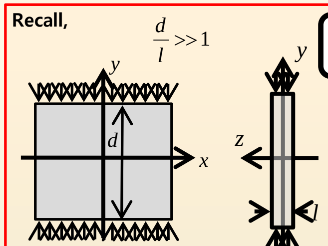


**'A State of Plain Stress'**

▪body force

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solve

Stress determined $\quad \sigma_z=\tau_{zx}=\tau_{zy}=0$

$$\nabla^4\psi=0 \quad \sigma_x=\frac{\partial^2\psi}{\partial y^2},\ \sigma_y=\frac{\partial^2\psi}{\partial x^2},\ \tau_{yx}=-\frac{\partial^2\psi}{\partial x\partial y}$$

Strain determined

$$\varepsilon_x=\frac{1}{E}[\sigma_x-\nu\sigma_y],\ \varepsilon_y=\frac{1}{E}[\sigma_y-\nu\sigma_x],\ \gamma_{xy}=\frac{\tau_{xy}}{G}$$

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$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},$$

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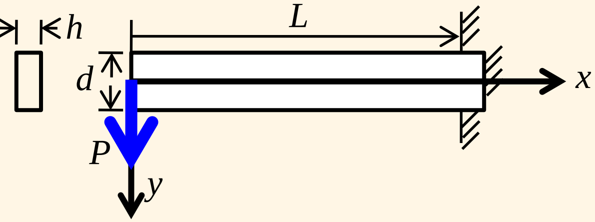
# Example - Cantilever

$$I = \frac{hd^3}{12}$$

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**

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$$\sigma_x = -\frac{Pxy}{I}, \quad \sigma_y = 0, \quad \tau_{xy} = \frac{I}{2}P\left(y^2 - \frac{1}{4}d^2\right)$$

$$\varepsilon_x = -\frac{Pxy}{EI}, \quad \varepsilon_y = \frac{\nu Pxy}{EI}, \quad \gamma_{xy} = \frac{P(1+\nu)}{EI}\left(y^2 - \frac{d^2}{4}\right)$$

$$\frac{\partial u}{\partial x} = -\frac{Pxy}{EI}, \quad \frac{\partial v}{\partial y} = \frac{\nu Pxy}{EI}$$

Integrate

$$u = -\frac{P}{2EI}x^2 y + g_1(y)$$

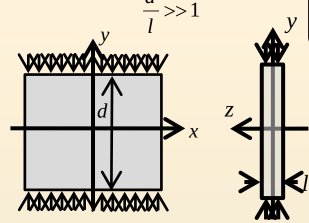
$$v = \frac{\nu P}{2EI}xy^2 + g_2(x)$$

$$\boxed{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}} = \frac{P(1+\nu)}{EI}\left(y^2 - \frac{d^2}{4}\right)$$

$$\begin{cases} g_1(y) = \dfrac{P}{3EI}\left(1 + \dfrac{\nu}{2}\right)y^3 + a_1 y + a_2 \\ g_2(x) = \dfrac{P}{6EI}x^3 - \dfrac{(1+\nu)P}{4EI}d^2 x - a_1 x + a_3 \end{cases}$$

Recall,

$$\frac{d}{l} >> 1$$




**'A State of Plain Stress'**

▪**body force**

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$$Y = Y(x, y) \qquad \tau_{xy} = \tau_{xy}(x, y)$$

solve

$$\nabla^4 \psi = 0$$

Stress determined

$$\sigma_z = \tau_{zx} = \tau_{zy} = 0$$

$$\sigma_x = \frac{\partial^2 \psi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial x^2}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

Strain determined

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y], \quad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x], \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

Strain and Displacement :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \cancel{\varepsilon_z = \frac{\partial w}{\partial z}},$$

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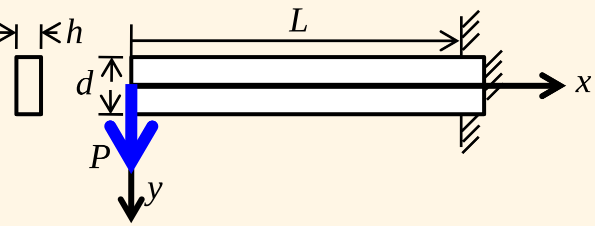
# Example - Cantilever

$$I = \frac{hd^3}{12}$$

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**

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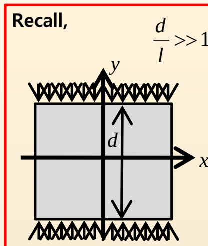
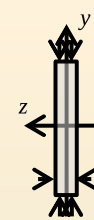
$$u = -\frac{P}{2EI}x^2 y + g_1(y), \quad v = \frac{\nu P}{2EI}xy^2 + g_2(x), \text{ where } \begin{cases} g_1(y) = \frac{P}{3EI}\left(1+\frac{\nu}{2}\right)y^3 + a_1 y + a_2 \\ g_2(x) = \frac{P}{6EI}x^3 - \frac{(1+\nu)P}{4EI}d^2 x - a_1 x + a_3 \end{cases}$$

$$\therefore \begin{cases} u = -\frac{P}{2EI}x^2 y + \frac{P}{3EI}\left(1+\frac{\nu}{2}\right)y^3 + a_1 y + a_2 \\ v = \frac{\nu P}{2EI}xy^2 + \frac{P}{6EI}x^3 - \frac{(1+\nu)P}{4EI}d^2 x - a_1 x + a_3 \end{cases}$$

Recall,




**'A State of Plain Stress'**

▪body force

$Z = 0$ $\quad \sigma_x = \sigma_x(x,y)$

$X = X(x,y)$ $\quad \sigma_y = \sigma_y(x,y)$

$Y = Y(x,y)$ $\quad \tau_{xy} = \tau_{xy}(x,y)$

solve

Stress determined $\quad \sigma_z = \tau_{zx} = \tau_{zy} = 0$

$$\nabla^4 \psi = 0 \quad \Rightarrow \quad \sigma_x = \frac{\partial^2\psi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\psi}{\partial x^2}, \quad \tau_{yx} = -\frac{\partial^2\psi}{\partial x \partial y}$$

Strain determined

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y], \quad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x], \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

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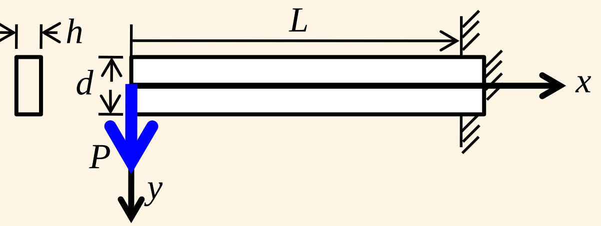
$$I = \frac{hd^3}{12}$$

# Example - Cantilever

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**

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If $P$ is large compared with $\rho g$ , the gravitational force can be neglected

$$\sigma_x = -\frac{Pxy}{I}, \quad \sigma_y = 0, \quad \tau_{xy} = \frac{I}{2}P\left(y^2 - \frac{1}{4}d^2\right)$$

$$\varepsilon_x = -\frac{Pxy}{EI}, \quad \varepsilon_y = \frac{\nu Pxy}{EI}, \quad \gamma_{xy} = \frac{P(1+\nu)}{EI}\left(y^2 - \frac{d^2}{4}\right)$$

$$\begin{cases} u = -\dfrac{P}{2EI}x^2 y + \dfrac{P}{3EI}\left(1+\dfrac{\nu}{2}\right)y^3 + a_1 y + a_2 \\ v = \dfrac{\nu P}{2EI}xy^2 + \dfrac{P}{6EI}x^3 - \dfrac{(1+\nu)P}{4EI}d^2 x - a_1 x + a_3 \end{cases}$$

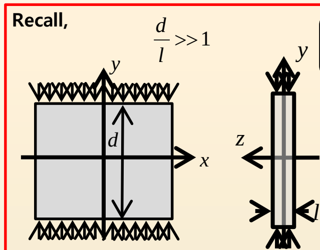
assume that the point $(x,y)=(L,0)$ is fixed. $u = v = \dfrac{\partial v}{\partial x} = 0$ at $(x=L, \ y=0)$

$$\left.\frac{\partial v}{\partial x}\right|_{x=L,y=0} = \left[\frac{\nu P}{2EI}y^2 + \frac{P}{2EI}x^2 - \frac{(1+\nu)P}{4EI}d^2 - a_1\right]_{x=L,y=0} = \frac{P}{2EI}L^2 - \frac{(1+\nu)P}{4EI}d^2 - a_1 = 0$$

$$v|_{x=L,y=0} = \frac{P}{6EI}L^3 - \frac{(1+\nu)P}{4EI}d^2 L - a_1 L + a_3 = 0, \qquad u|_{x=L,y=0} = a_2 = 0$$

Recall,

$\frac{d}{l} >> 1$




**'A State of Plain Stress'**

▪body force

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$X = X(x,y) \qquad \sigma_y = \sigma_y(x,y)$

$Y = Y(x,y) \qquad \tau_{xy} = \tau_{xy}(x,y)$

solve

Stress determined $\quad \sigma_z = \tau_{zx} = \tau_{zy} = 0$

$$\nabla^4 \psi = 0 \quad \Rightarrow \quad \sigma_x = \frac{\partial^2 \psi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial x^2}, \quad \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

Strain determined

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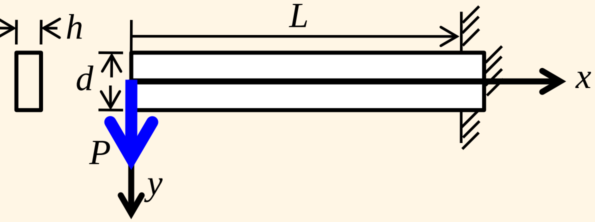
# Example - Cantilever

$$I = \frac{hd^3}{12}$$

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**

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If $P$ is large compared with $\rho g$ , the gravitational force can be neglected

$$\boxed{\sigma_x = -\frac{Pxy}{I}, \quad \sigma_y = 0, \quad \tau_{xy} = \frac{I}{2}P\left(y^2 - \frac{1}{4}d^2\right)}$$

---

assume that the point $(x, y) = (L, 0)$ is fixed. $u = v = \frac{\partial v}{\partial x} = 0$ at $(x = L, \ y = 0)$

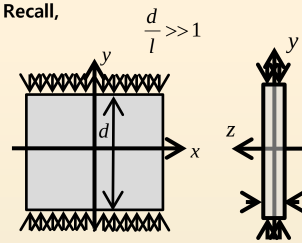
$$\begin{cases} \dfrac{P}{6EI}L^3 - \dfrac{(1+\nu)P}{4EI}d^2L - a_1L + a_3 = 0 \cdots (1) \\ \dfrac{P}{2EI}L^2 - \dfrac{(1+\nu)P}{4EI}d^2 - a_1 = 0 \Rightarrow a_1 = \dfrac{P}{2EI}L^2 - \dfrac{(1+\nu)P}{4EI}d^2 \cdots (2) \\ a_2 = 0 \end{cases}$$

$$(2) \to (1): \ \frac{P}{6EI}L^3 - \frac{(1+\nu)P}{4EI}d^2L - \left(\frac{P}{2EI}L^2 - \frac{(1+\nu)P}{4EI}d^2\right)L + a_3 = 0$$

$$\Downarrow$$

$$-\frac{P}{3EI}L^3 + a_3 = 0 \Rightarrow \therefore a_3 = \frac{P}{3EI}L^3 \cdots (3)$$

Recall,




**'A State of Plain Stress'**

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$Y = Y(x, y)$ $\quad \tau_{xy} = \tau_{xy}(x, y)$

solve $\quad$ Stress determined $\quad \sigma_z = \tau_{zx} = \tau_{zy} = 0$

$$\nabla^4 \psi = 0 \Rightarrow \sigma_x = \frac{\partial^2 \psi}{\partial y^2}, \ \sigma_y = \frac{\partial^2 \psi}{\partial x^2}, \ \tau_{yx} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

Strain determined

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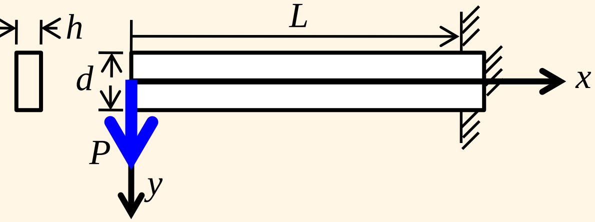
moment of inertia of the cross section

$$I = \frac{hd^3}{12}$$

# Example - Cantilever

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**

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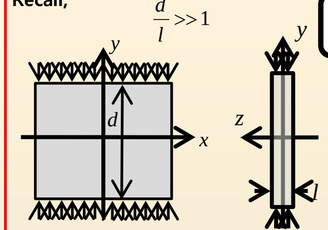


'A State of Plain Stress'

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solve $\quad$ Stress determined $\quad \sigma_z = \tau_{zx} = \tau_{zy} = 0$

$$\nabla^4 \psi = 0 \Rightarrow \sigma_x = \frac{\partial^2\psi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\psi}{\partial x^2}, \quad \tau_{yx} = -\frac{\partial^2\psi}{\partial x \partial y}$$

Strain determined

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y], \quad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x], \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

---

assume that the point $(x,y) = (L,0)$ is fixed. $u = v = \frac{\partial v}{\partial x} = 0$ at $(x = L, \ y = 0)$

$$\begin{cases} \frac{P}{6EI}L^3 - \frac{(1+\nu)P}{4EI}d^2L - a_1L + a_3 = 0 \cdots (1) \\ \frac{P}{2EI}L^2 - \frac{(1+\nu)P}{4EI}d^2 - a_1 = 0 \Rightarrow a_1 = \frac{P}{2EI}L^2 - \frac{(1+\nu)P}{4EI}d^2 \cdots (2) \\ a_2 = 0 \end{cases}$$

$$a_3 = \frac{P}{3EI}L^3 \cdots (3)$$

$$(3) \rightarrow (1): \ \frac{P}{6EI}L^3 - \frac{(1+\nu)P}{4EI}d^2L - a_1L + \frac{P}{3EI}L^3 = 0$$

$$\Downarrow$$

$$\frac{P}{2EI}L^3 - \frac{(1+\nu)P}{4EI}d^2L - a_1L = 0 \Rightarrow \therefore a_1 = \frac{P}{2EI}L^2 - \frac{(1+\nu)P}{4EI}d^2$$

<u>Strain and Displacement</u> :

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \cancel{\varepsilon_z = \frac{\partial w}{\partial z}},$$

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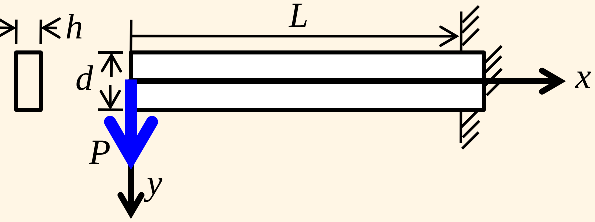
moment of inertia of the cross section

$$I = \frac{hd^3}{12}$$

# Example - Cantilever

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**

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Boundary condition : shearing force at $x = 0$ is equal to $P$

If $P$ is large compared with $\rho g$ , the gravitational force can be neglected

$$\sigma_x = -\frac{Pxy}{I}, \quad \sigma_y = 0, \quad \tau_{xy} = \frac{I}{2}P\left(y^2 - \frac{1}{4}d^2\right)$$

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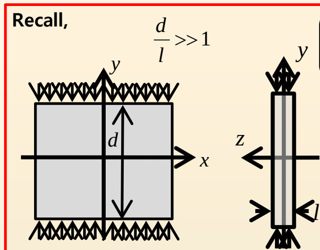
Recall,




**'A State of Plain Stress'**

▪**body force**

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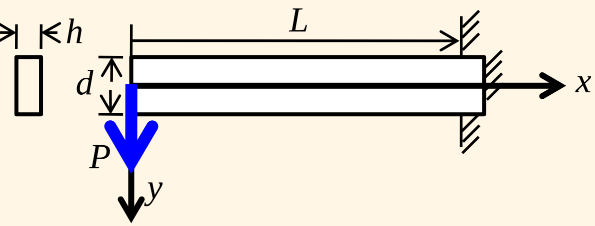


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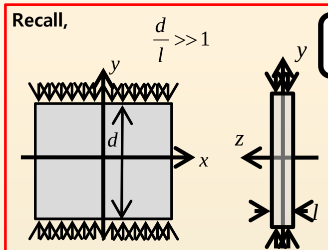


**'A State of Plain Stress'**

- **body force**

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moment of inertia of the cross section

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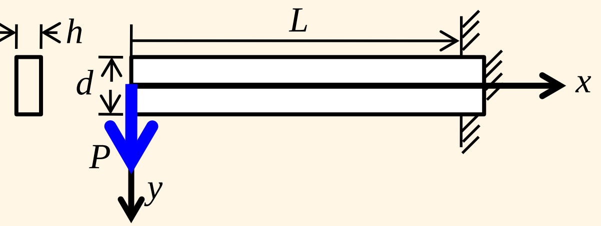


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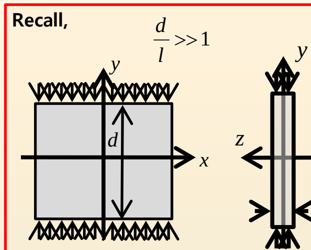


**'A State of Plain Stress'**

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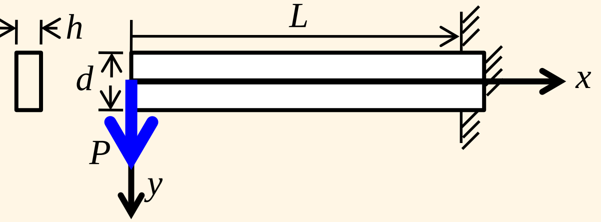


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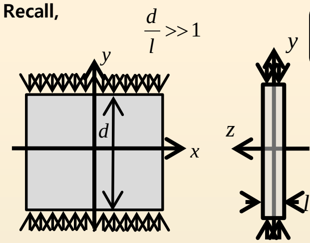


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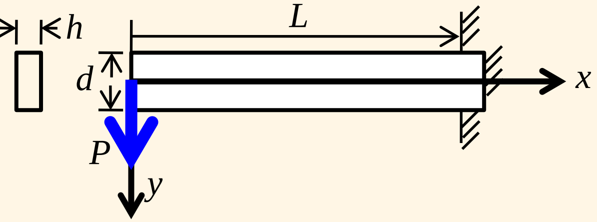
# Example - Cantilever

$$I=\frac{hd^3}{12}$$

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**

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$$\varepsilon_x=-\frac{Pxy}{EI},\ \varepsilon_y=\frac{\nu Pxy}{EI},\ \gamma_{xy}=\frac{P(1+\nu)}{EI}(y^2-\frac{d^2}{4})$$

$$u=-\frac{P}{2EI}x^2y+\frac{P}{3EI}(1+\frac{\nu}{2})y^3+\frac{P}{2EI}[L^3-(1+\nu)\frac{d^2}{2}]y$$

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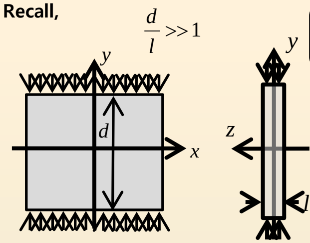
Recall,




'A State of Plain Stress'

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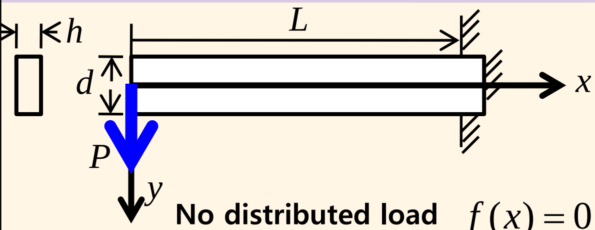
Displacement determined $u,v$



# Example - Cantilever

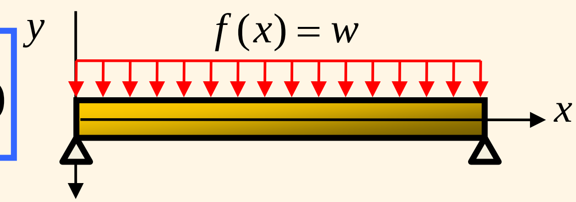



No distributed load $f(x)=0$

$v \leftarrow y$

**recall, differential equations of deflection curves***

$$\boxed{EI\frac{d^4 y}{dx^4} = f(x)}$$




| Boundary Condition | |
|---|---|
| No bending moment at x=0 | $\rightarrow EIv''(0)=0$ |
| Shear force P at x=0 | $\rightarrow EIv'''(0)=P$ |
| No displacement at x=0 | $\rightarrow v(L)=0$ |
| No slope at x=0 | $\rightarrow v'(L)=0,$ |

$$EI\frac{d^4 v(x)}{dx^4}=0$$

**Integrate four times**

$$v'''(x)=\frac{1}{EI}c_1$$

$$v''(x)=c_2+\frac{1}{EI}c_1 x$$

$$v'(x)=c_3+c_2 x+\frac{1}{2EI}c_1 x^2$$

$$v(x)=c_4+c_3 x+\frac{1}{2}c_2 x^2+\frac{1}{6EI}c_1 x^3$$

$$v''(0)=\frac{1}{EI}c_2 \qquad ,v'''=\frac{1}{EI}c_1$$

$$v'(L)=\frac{1}{EI}\left(c_3+c_2 L+\frac{1}{2}c_1 L^2\right) \qquad ,v(L)=\frac{1}{EI}\left(c_4+c_3 L+\frac{1}{2}c_2 L^2+\frac{1}{6}c_1 L^3\right)$$

**by the boundary condition**

$$0=\frac{1}{EI}c_2 \Rightarrow \therefore c_2=0$$

$$\frac{P}{EI}=\frac{1}{EI}c_1 \Rightarrow \therefore c_1=P$$

$$c_3+c_2 L+\frac{1}{2}c_1 L^2=0 \Rightarrow c_3=-\frac{1}{2}PL^2$$

$$c_4+c_3 L+\frac{1}{2}c_2 L^2+\frac{1}{6}c_1 L^3=0 \Rightarrow c_4=\frac{1}{2}PL^3-\frac{1}{6}PL^3=\frac{1}{3}PL^3$$

$$\therefore EIv(x)=\frac{1}{3}PL^3-\frac{1}{2}PL^2 x+\frac{1}{6}Px^3$$

$$\boxed{v(x)=\frac{1}{6EI}Px^3-\frac{1}{2EI}PL^2 x-\frac{1}{3EI}PL^3}$$



*Gere J. M., Mechanics of Materials (International Edition), Sixth Edition, Thomson, 2006, P906 (coordinates modified)

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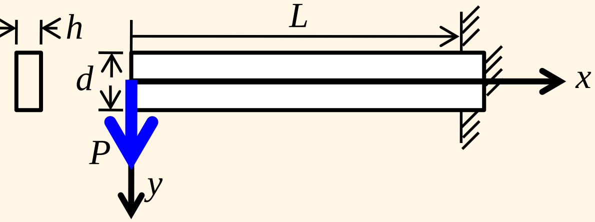
# Example - Comparison

**Bending of a Narrow Cantilever of Rectangular Cross Section under an End Load**

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$$u = -\frac{P}{2EI}x^2 y + \frac{P}{3EI}(1+\frac{\nu}{2})y^3 + \frac{P}{2EI}[L^3 - (1+\nu)\frac{d^2}{2}]y, \quad v = \frac{\nu P}{2EI}xy^2 + \frac{P}{6EI}x^3 - \frac{PL^2}{2EI}x + \frac{PL^3}{3EI}$$

**The equation of the deflection curve is given by the expression of $v$ at $y = 0$**

$$v = \frac{\nu P}{2EI}xy^2 + \frac{P}{6EI}x^3 - \frac{PL^2}{2EI}x + \frac{PL^3}{3EI}$$

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**recall, differential equations of deflection curves***

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same

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moment of inertia of the cross section

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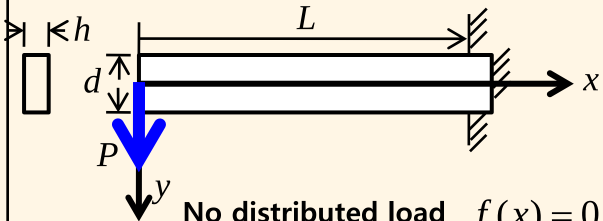


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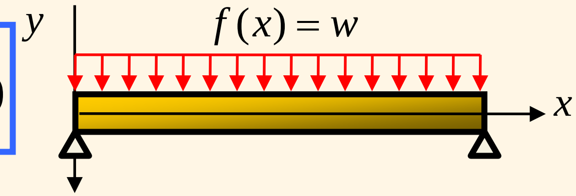


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$$v = \frac{P}{6EI}x^3 - \frac{PL^2}{2EI}x + \frac{PL^3}{3EI} \quad at \;\; y = 0$$

**same displacement v**

$$v = \frac{P}{6EI}x^3 - \frac{P}{2EI}L^2 x + \frac{P}{3EI}L^3$$

$$u = -\frac{P}{2EI}x^2 y + \frac{P}{3EI}(1+\frac{\nu}{2})y^3 + \frac{P}{2EI}[L^3 - (1+\nu)\frac{d^2}{2}]y$$

$$u = 0 \quad at \;\; y = 0$$

**No displace u component from this solution**

**what is the difference between two solution and why?**

$$u = u(x, y)$$
$$v = v(x, y)$$

**The deflection of the beam at any point along its axis is the displacement of that point from its original position, measured in the y direction****

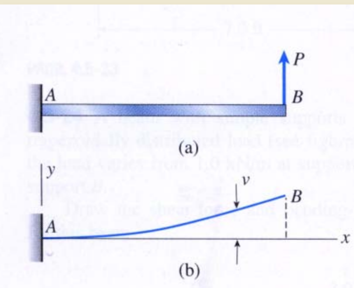


**FIG. 5-1** Bending of a cantilever beam: (a) beam with load, and (b) deflection curve



*Gere J. M., Mechanics of Materials (International Edition), Sixth Edition, Thomson, 2006, P906 (coordinates modified)
**Gere J. M., Mechanics of Materials (International Edition), Sixth Edition, Thomson, 2006, P301