Chapter 14 Analog Filters

Outline of the Chapter

General Considerations

- Filter Characteristics
- Classification of Filters
- Transfer Function of Filters
- Problem of Sensitivity

→

Second-Order Filters

- Special Cases
- RLC Realizations

→

Active Filters

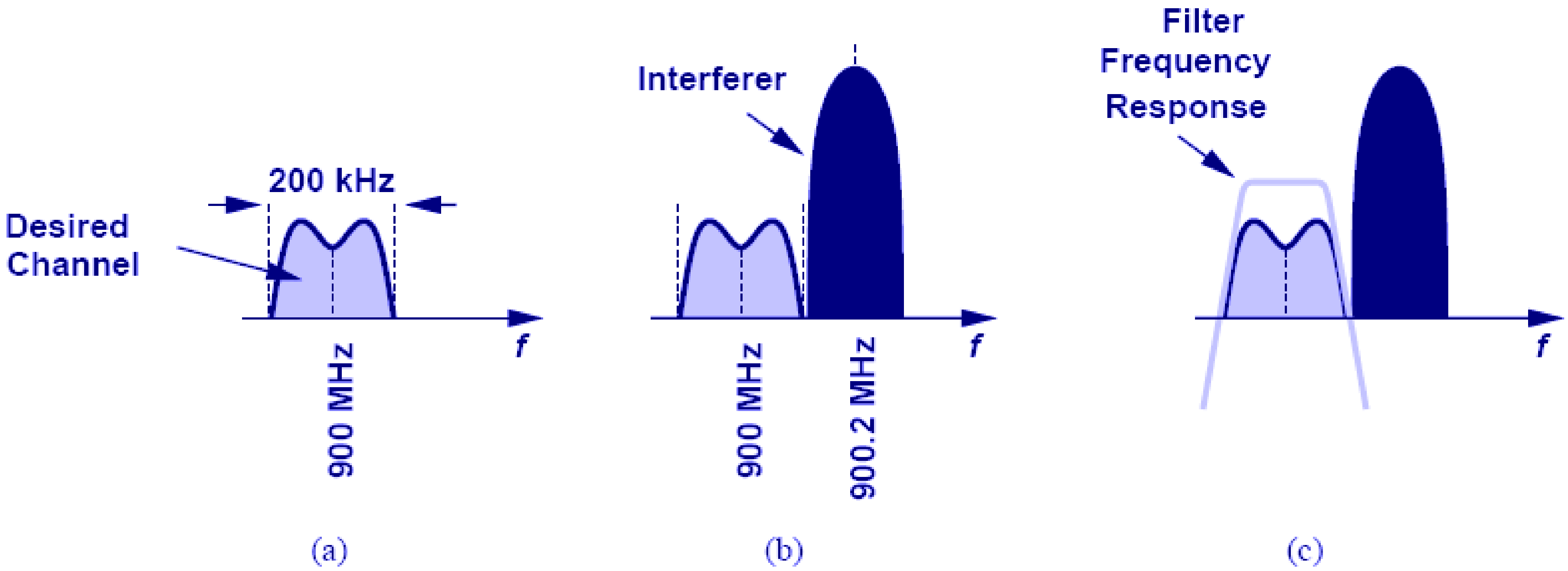
- Sallen and Key Filter
- KHN Biquad
- Tow-Thomas Biquad
- Biquads Based on Simulated Inductors

→

Approximation of Filter Response

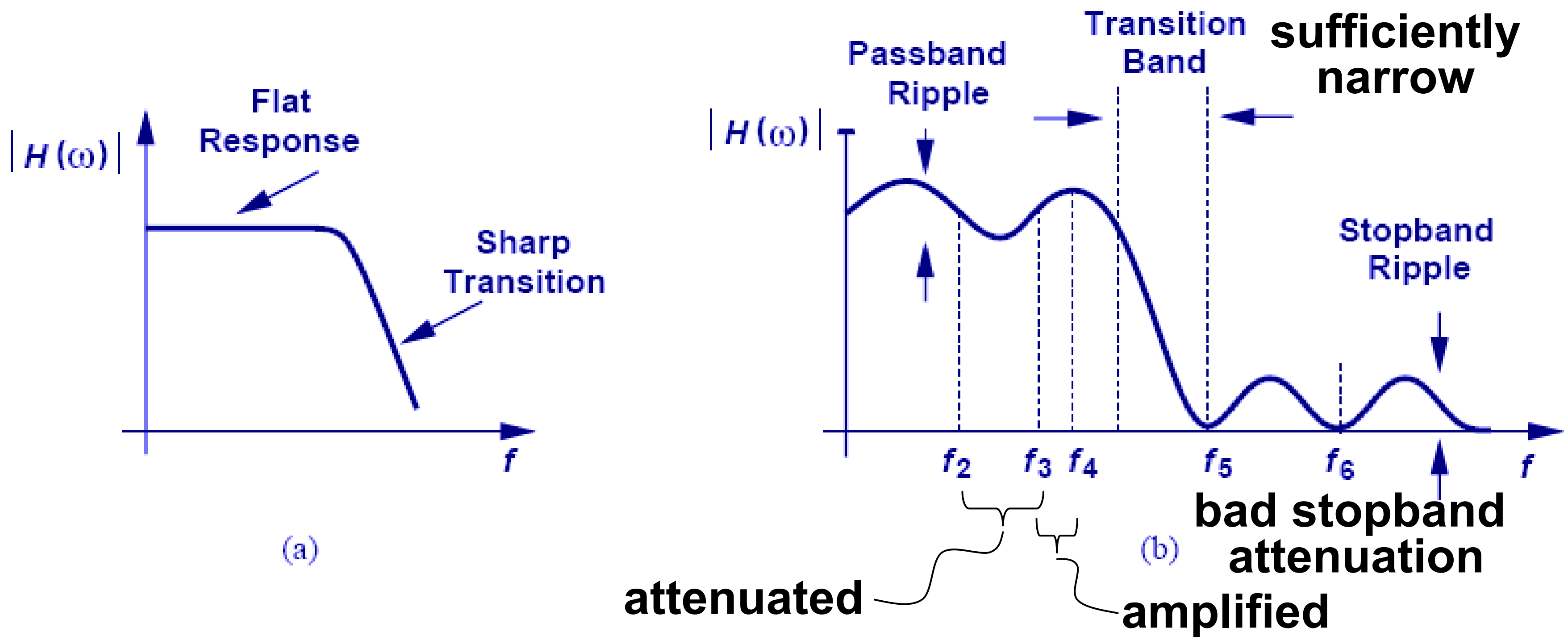
- Butterworth Response
- Chebyshev Response

Why We Need Filters



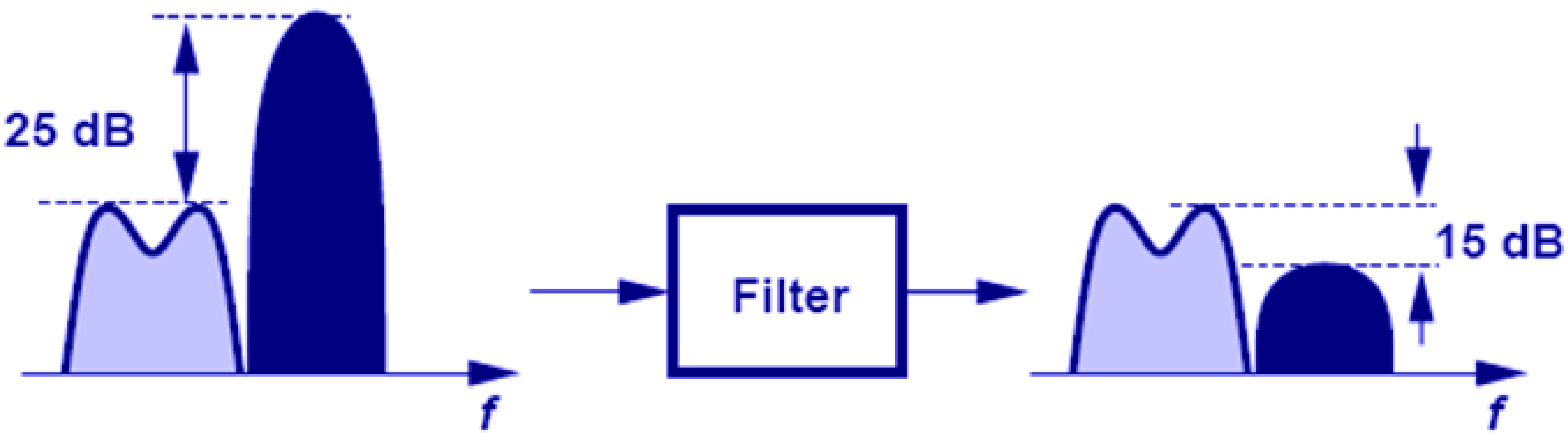
- In order to eliminate the unwanted interference that accompanies a signal, a filter is needed.

Filter Characteristics



- Ideally, a filter needs to have a flat pass band and a sharp roll-off in its transition band.
- Realistically, it has a rippling pass/stop band and a transition band.

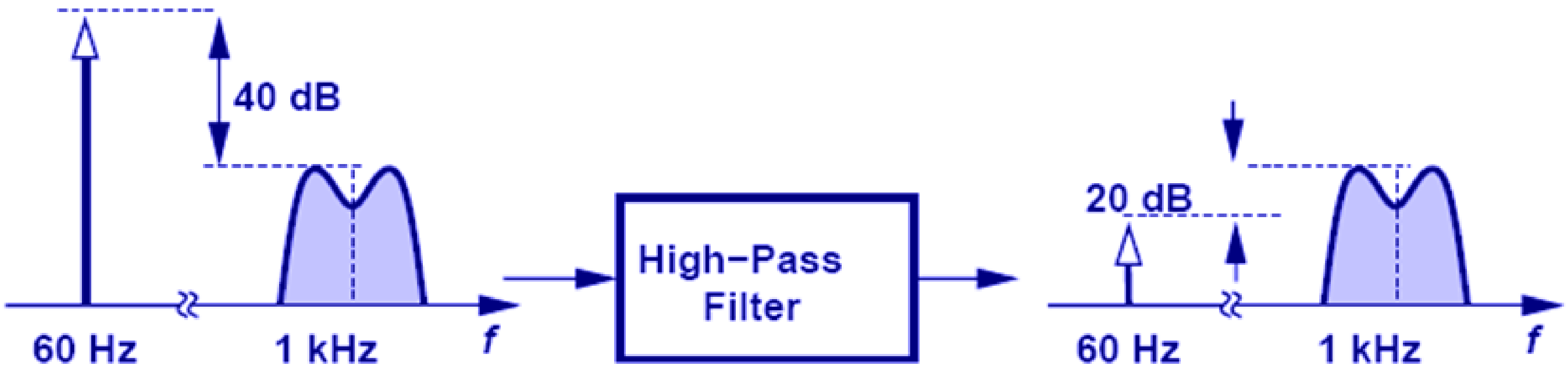
Example 14.1: Filter I



Problem : Adjacent channel interference is 25 dB above the signal. Determine the required stopband attenuation if Signal to Interference ratio must exceed 15 dB.

Solution: A filter with stopband attenuation of 40 dB

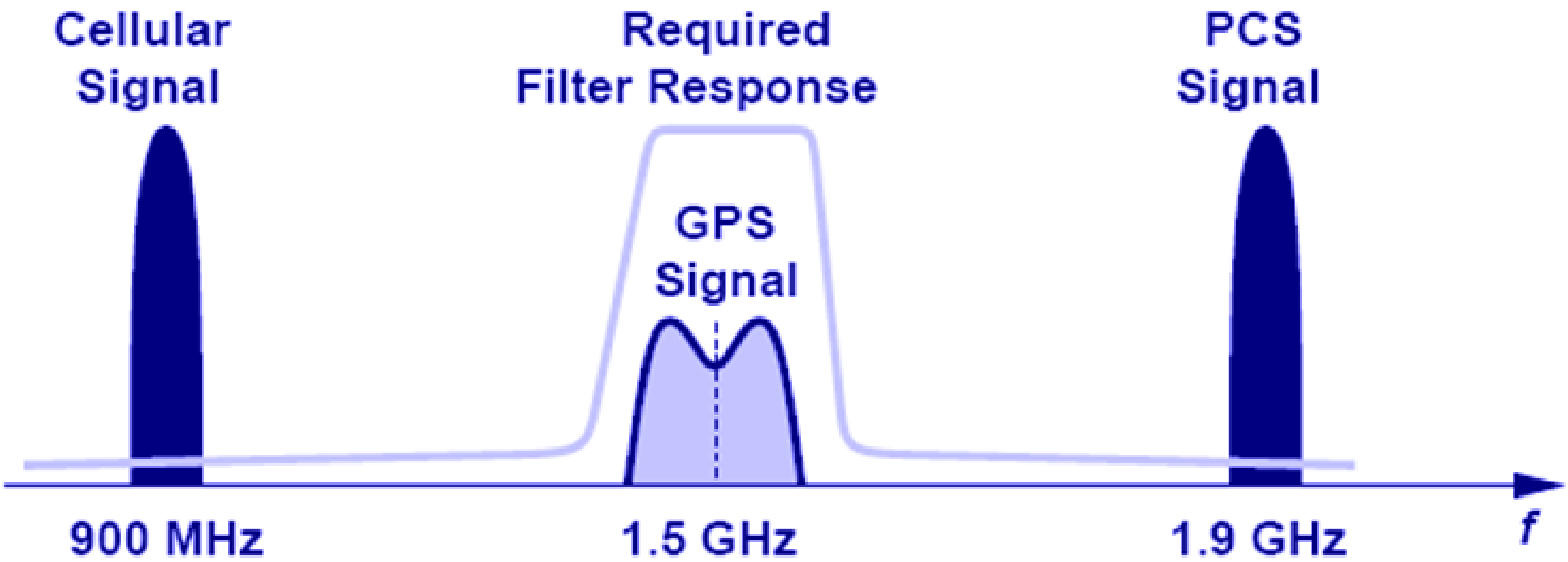
Example 14.2: Filter II



Problem: Adjacent 60-Hz channel interference is 40 dB above the signal. Determine the required stopband attenuation to ensure that the signal level remains 20dB above the interferer level.

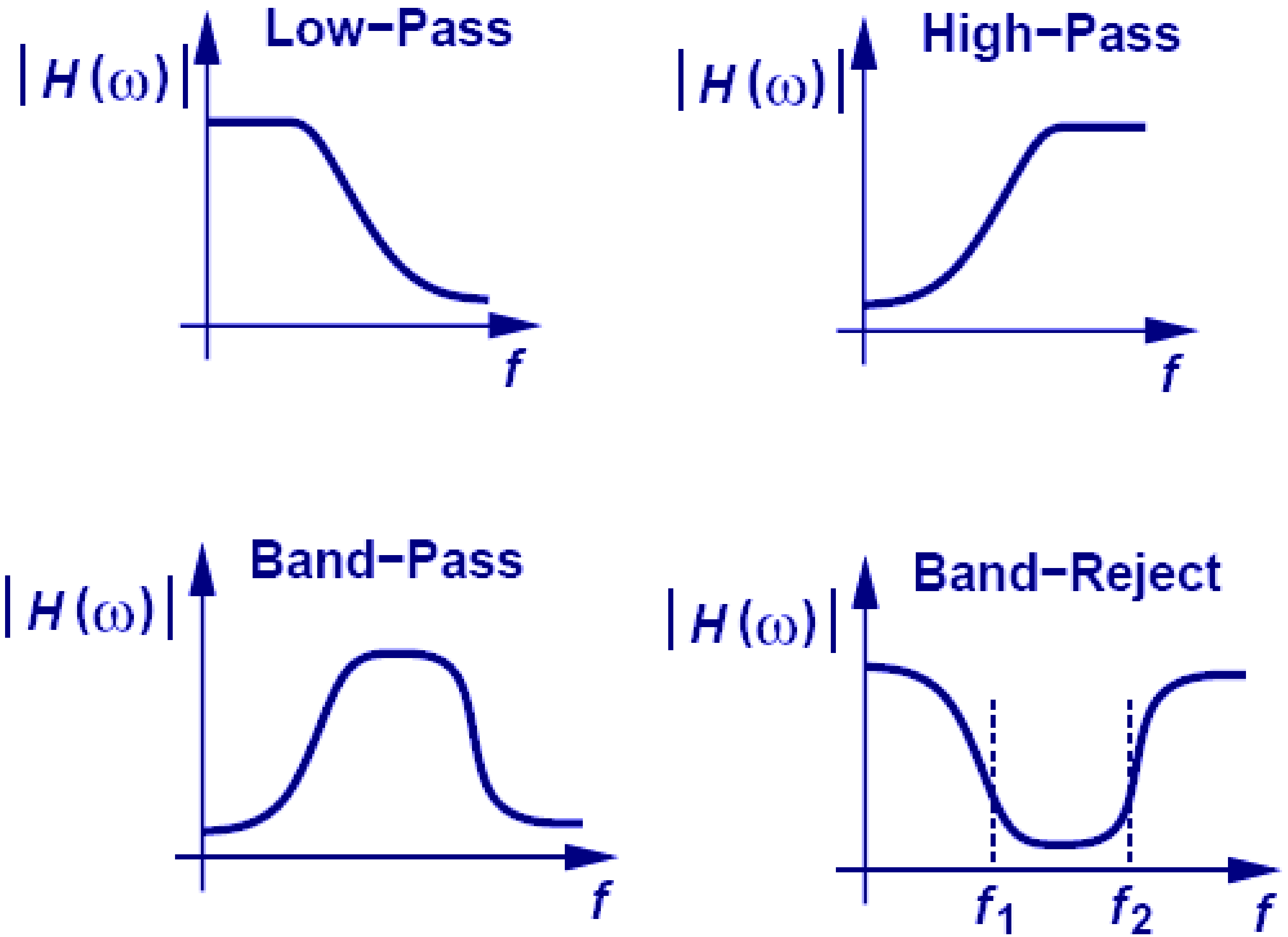
Solution: A high-pass filter with stopband attenuation of 60 dB at 60Hz.

Example 14.3: Filter III

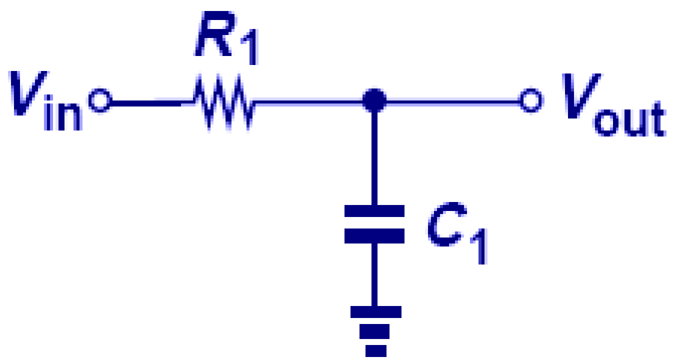


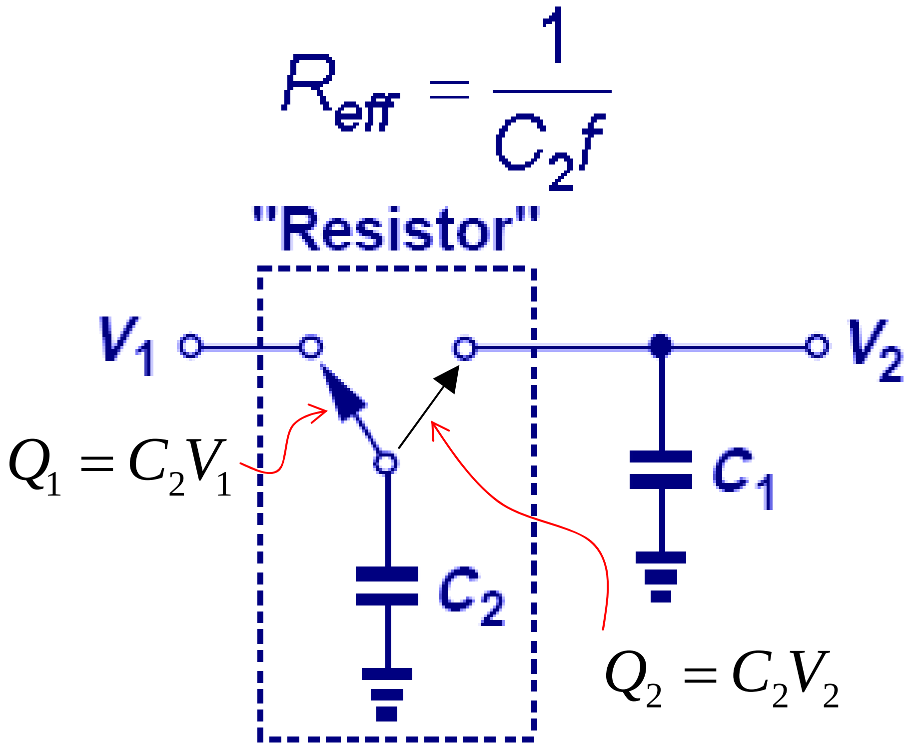
- A bandpass filter around 1.5 GHz is required to reject the adjacent Cellular and PCS signals.

Classification of Filters I



Classification of Filters II



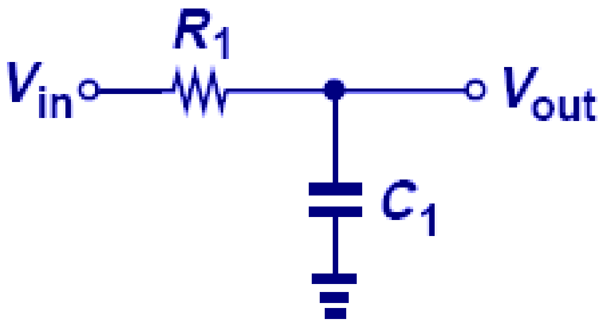


Continuous-time

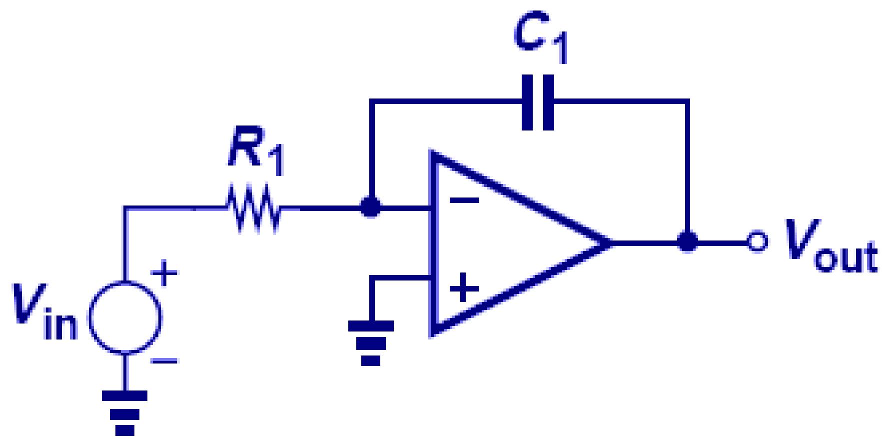
Discrete-time

if $V_1 > V_2$, C_2 absorbs charge from V_1 and delivers it to $V_2 \Rightarrow \approx$ a resistor

Classification of Filters III

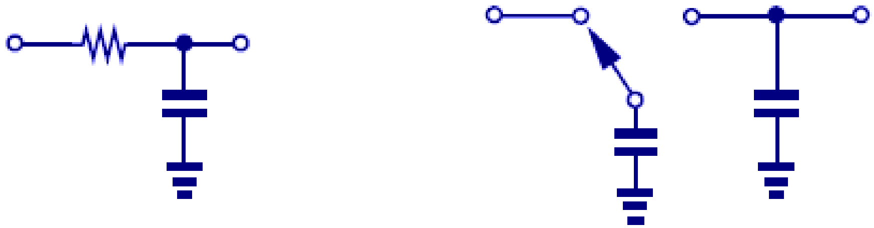
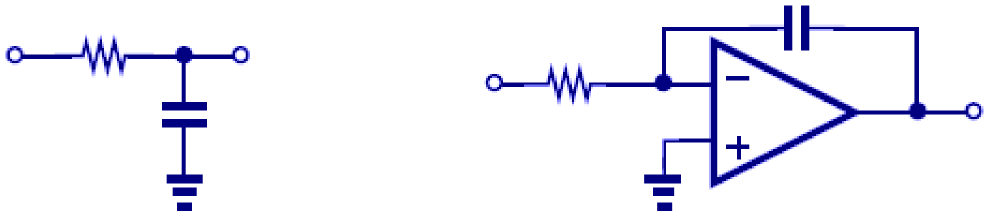


Passive

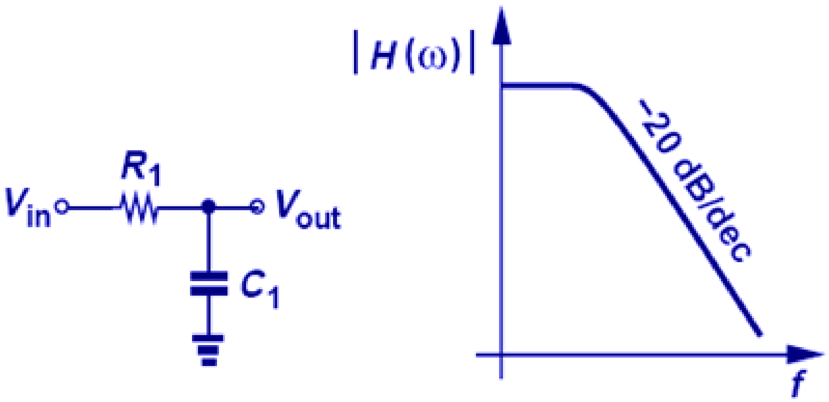


Active

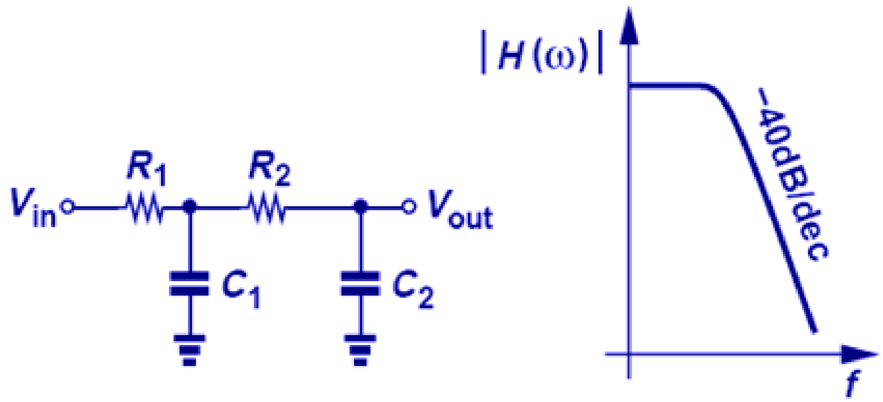
Summary of Filter Classifications

	Low-Pass	High-Pass	Band-Pass	Band-Reject
Frequency Response				
Continuous-Time and Discrete-Time				
Passive and Active				

Filter Transfer Function



(a)



(b)

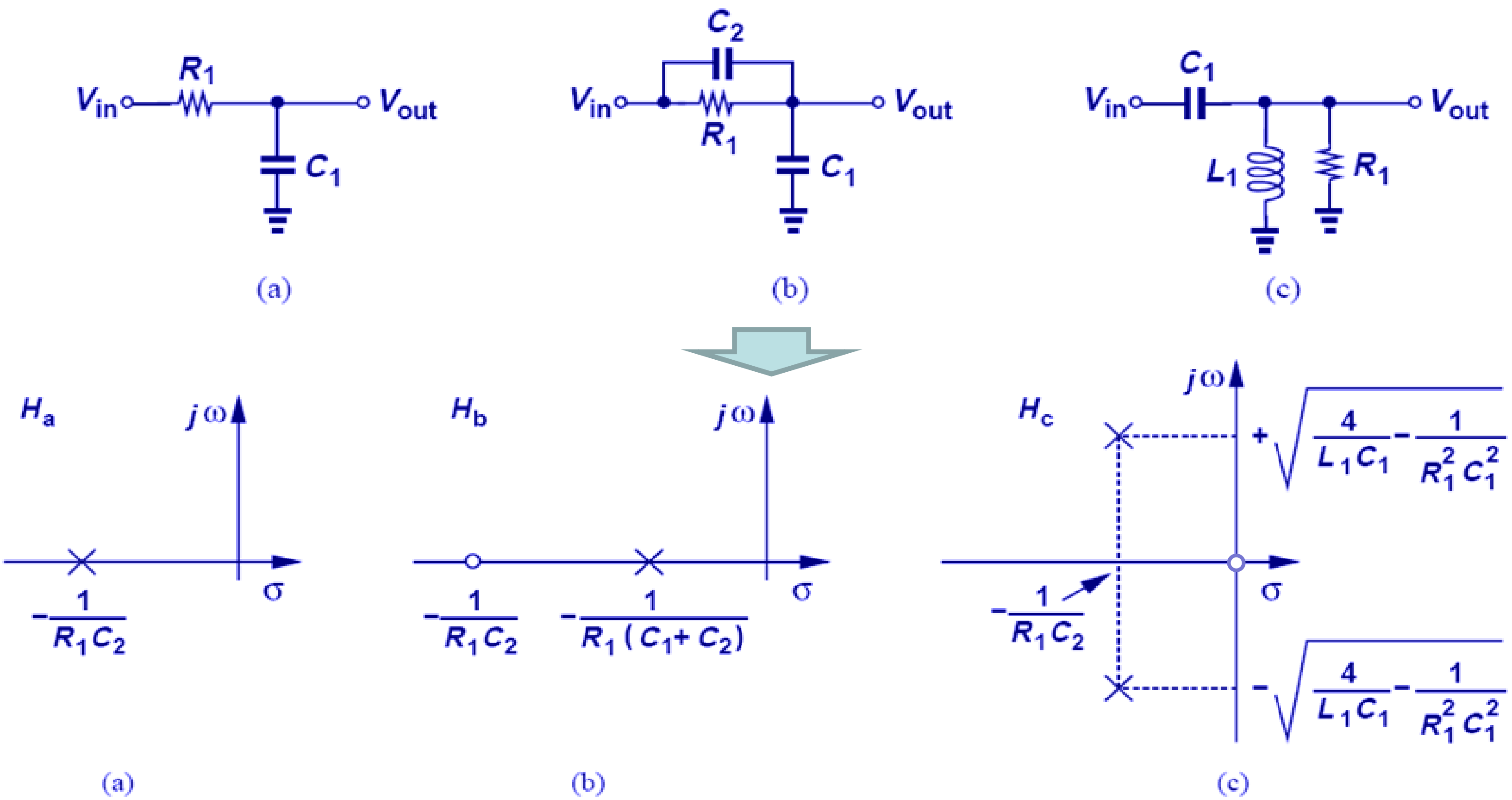
- Filter (a) has a transfer function with -20dB/dec roll-off.
- Filter (b) has a transfer function with -40dB/dec roll-off and provides a higher selectivity.

General Transfer Function

$$H(s)=\alpha\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$$\left.\begin{array}{l}\mathbf{z_k} = \textbf{zero frequencies} \\ \mathbf{p_k} = \textbf{pole frequencies}\end{array}\right\} = \sigma + j\omega$$

Example 14.4 : Pole-Zero Diagram

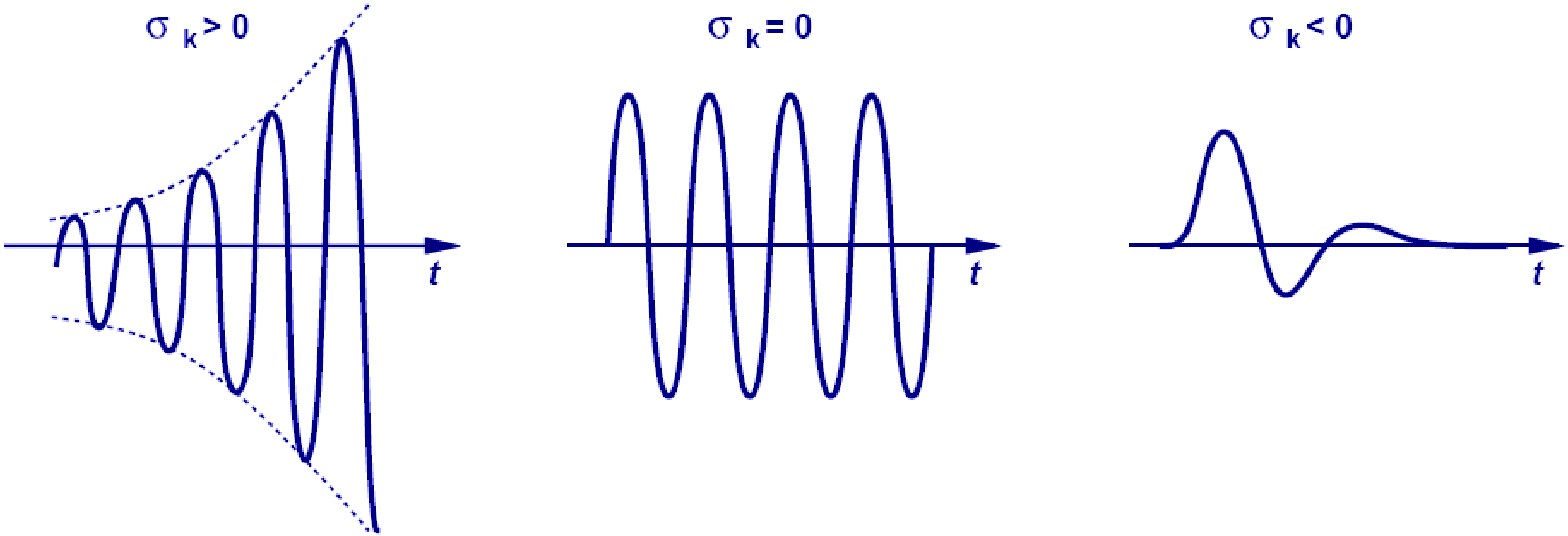


$$H_a(s) = \frac{1}{R_1C_1s+1} \qquad H_b(s) = \frac{R_1C_2s+1}{R_1(C_1+C_2)s+1} \qquad H_c(s) = \frac{C_1s}{R_1L_1C_1s^2+L_1s+R_1}$$

Example 14.5: Position of the poles

Impulse response contains

$$\exp(p_k t) = \exp(\sigma_k t)\exp(j\omega_k t)$$



Poles on the RHP
Unstable
(no good)

Poles on the jω axis
Oscillatory
(no good)

Poles on the LHP
Decaying
(good)

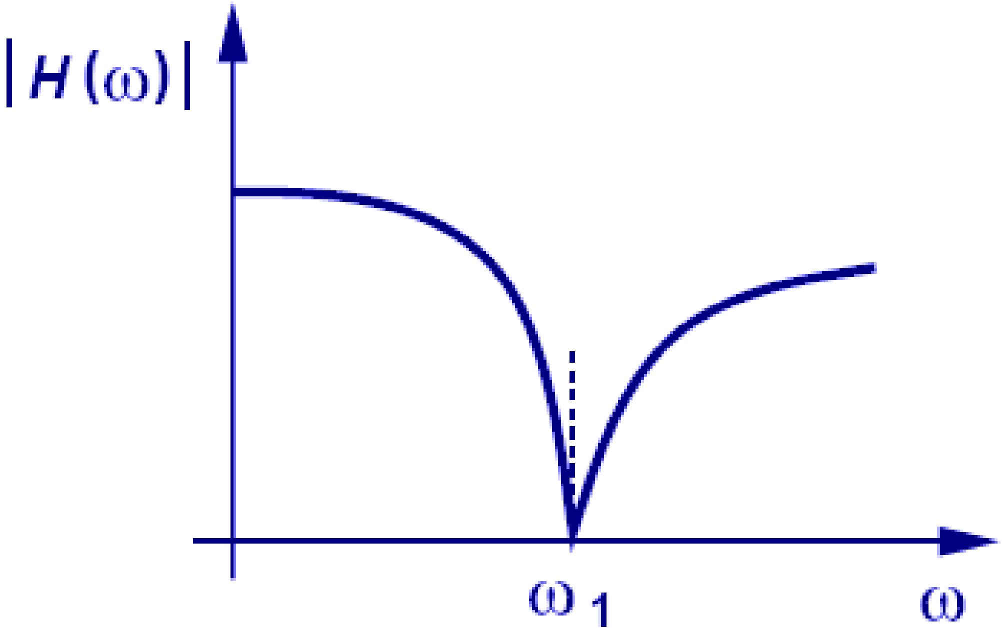
Transfer Function

$$H(s) = \alpha \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

- **The order of the numerator *m* ≤ The order of the denominator *n***
  Otherwise, H(s)→∞ as s→∞.
- **For a physically-realizable transfer function, complex zeros or poles occur in conjugate pairs.** $z_1 = \sigma_1 + j\omega_1 \quad z_2 = \sigma_1 - j\omega_1$
- **If a zero is located on the jω axis, $z_{1,2} = \pm j\omega_1$, H(s) drops to zero at ω_1.**

The numerator contains a product such as $(s - j\omega_1)(s + j\omega_1) = s^2 + \omega_1^2$, which vanishes at $s = j\omega_1$

Imaginary Zeros



- **Imaginary zero is used to create a null at certain frequency. For this reason, imaginary zeros are placed only in the stop band.**

Sensitivity

$$S_C^P = \frac{dP}{P} \Big/ \frac{dC}{C}$$

P=Filter Parameter

C=Component Value

Example:

In simple RC filter, the -3dB corner frequency is given by $1/(R_1C_1)$

Errors in the cut-off frequency:

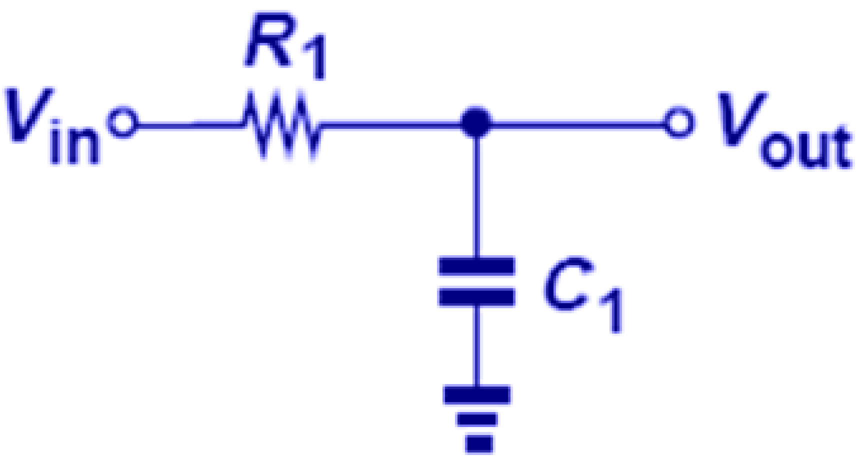
(a) the value of components varies with process and temperature in ICs

(b) The available values of components deviate from those required by the design

- **Sensitivity indicates the variation of a filter parameter due to variation of a component value.**

Example 14.6: Sensitivity

Problem: Determine the sensitivity of ω_0 with respect to R_1.



$$\omega_0 = 1/(R_1 C_1)$$

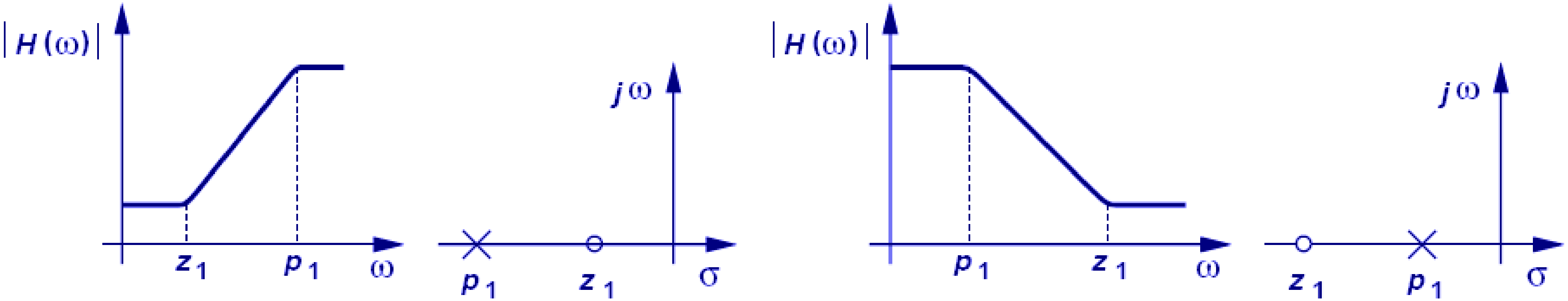
$$\frac{d\omega_0}{dR_1} = \frac{-1}{R_1^2 C_1}$$

$$\frac{d\omega_0}{\omega_0} = -\frac{dR_1}{R_1}$$

$$S_{R_1}^{\omega_0} = -1$$

- **For example, a +5% change in R_1 translates to a -5% error in ω_0.**

First-Order Filters

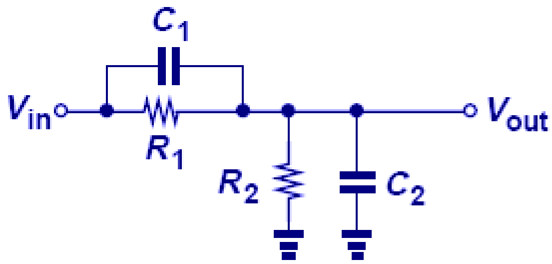


(a) (b)

$$H(s) = \alpha \frac{s + z_1}{s + p_1}$$

- First-order filters are represented by the transfer function shown above.
- Low/high pass filters can be realized by changing the relative positions of poles and zeros.

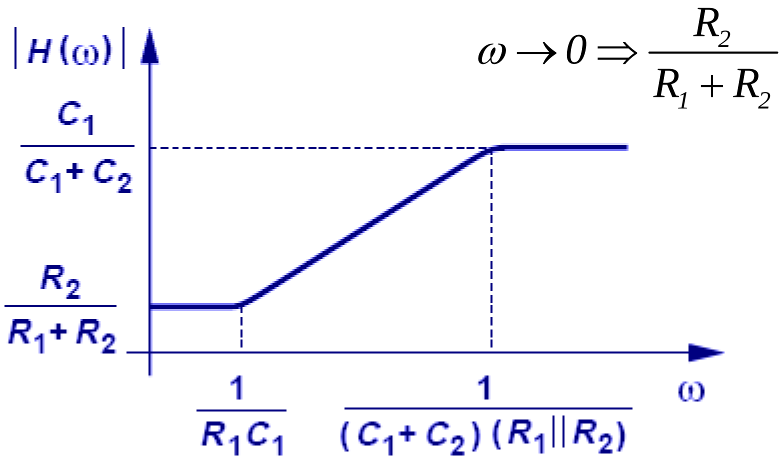
Example 14.8: First-Order Filter I



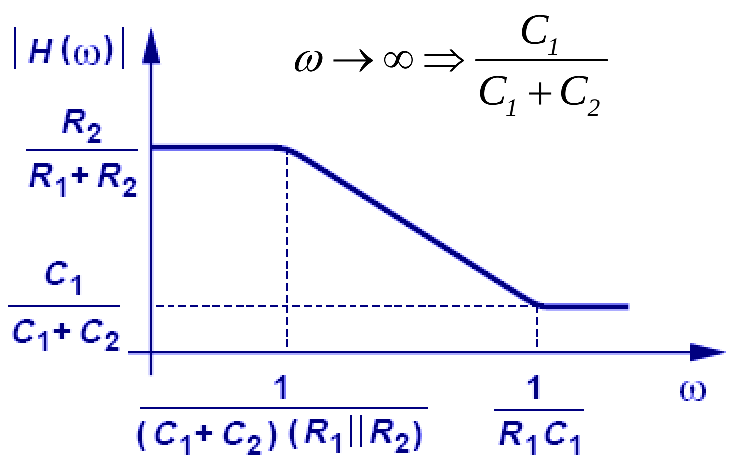
(a)

$$\frac{V_{out}}{V_{in}}(s)=\frac{R_2(R_1C_1s+1)}{R_1R_2(C_1+C_2)s+R_1+R_2}$$

$$z_1=-1/(R_1C_1),\ p_1=-[(C_1+C_2)R_1\parallel R_2]^{-1}$$



(b)



(c)

$$\frac{1}{R_1C_1}<\frac{1}{(C_1+C_2)(R_1\parallel R_2)}$$

$$1+\frac{C_2}{C_1}<1+\frac{R_1}{R_2}$$

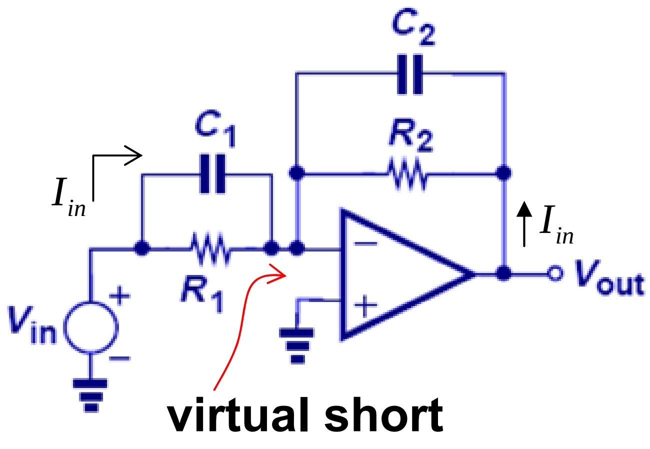
$R_2C_2 < R_1C_1$

$$\frac{1}{R_1C_1}>\frac{1}{(C_1+C_2)(R_1\parallel R_2)}$$

$$1+\frac{C_2}{C_1}>1+\frac{R_1}{R_2}$$

$R_2C_2 > R_1C_1$

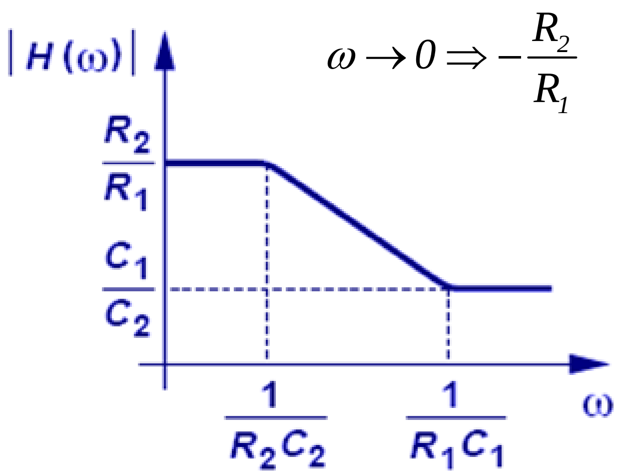
Example 14.9: First-Order Filter II



(a)

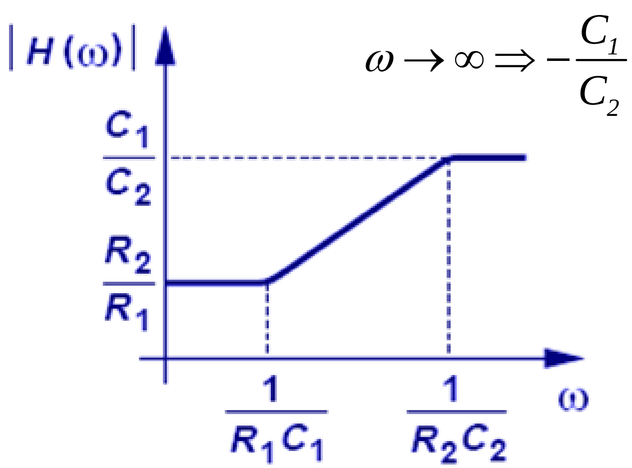
$$\frac{V_{out}}{V_{in}}(s) = \frac{-(R_2 \parallel \frac{1}{C_2 s})}{R_1 \parallel \frac{1}{C_1 s}}$$

$$= -\frac{R_2}{R_1} \cdot \frac{R_1 C_1 s + 1}{R_2 C_2 s + 1}$$



(b)

$R_2C_2 < R_1C_1$



(c)

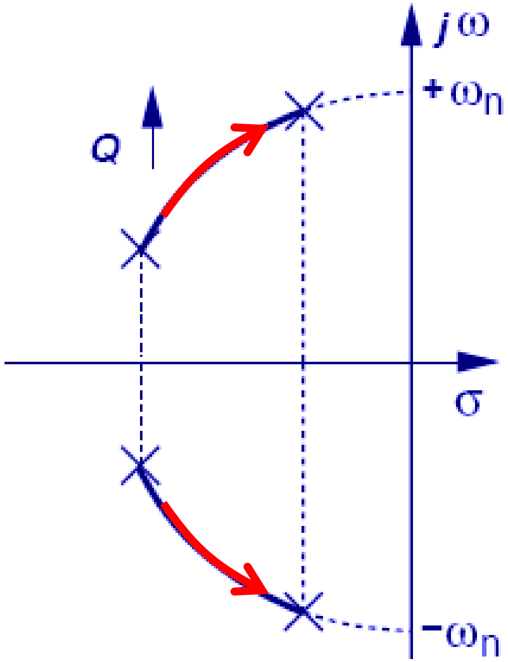
$R_2C_2 > R_1C_1$

Second-Order Filters

General transfer function

$$H(s) = \frac{\alpha s^2 + \beta s + \gamma}{s^2 + \frac{\omega_n}{Q} s + \omega_n^2}$$

$$p_{1,2} = -\frac{\omega_n}{2Q} \pm j\omega_n \sqrt{1 - \frac{1}{4Q^2}}$$



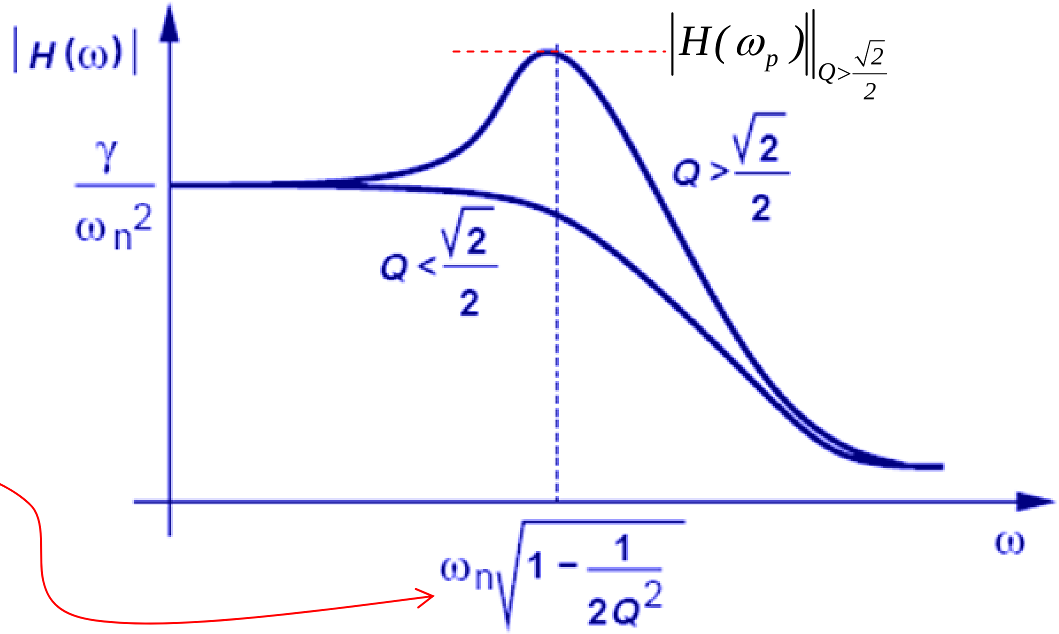
- **Second-order filters are characterized by the "biquadratic" equation with two complex poles shown above.**
- **When Q increases, the real part decreases while the imaginary part approaches $\pm\omega_n$.**

  => the poles look very imaginary thereby bringing the circuit closer to instability.

Second-Order Low-Pass Filter

$\alpha = \beta = 0$

$$|H(j\omega)|^2 = \frac{\gamma^2}{\left(\omega_n^2 - \omega^2\right)^2 + \left(\frac{\omega_n}{Q}\omega\right)^2}$$



$$|H(\omega_p)|\Big|_{Q>\frac{\sqrt{2}}{2}}$$

$$\frac{\partial |H(j\omega)|^2}{\partial \omega} = 0$$

for $Q > \frac{\sqrt{2}}{2}$

$$\frac{|H(\omega_p)|\Big|_{Q>\frac{\sqrt{2}}{2}}}{\gamma / \omega_n^2}$$

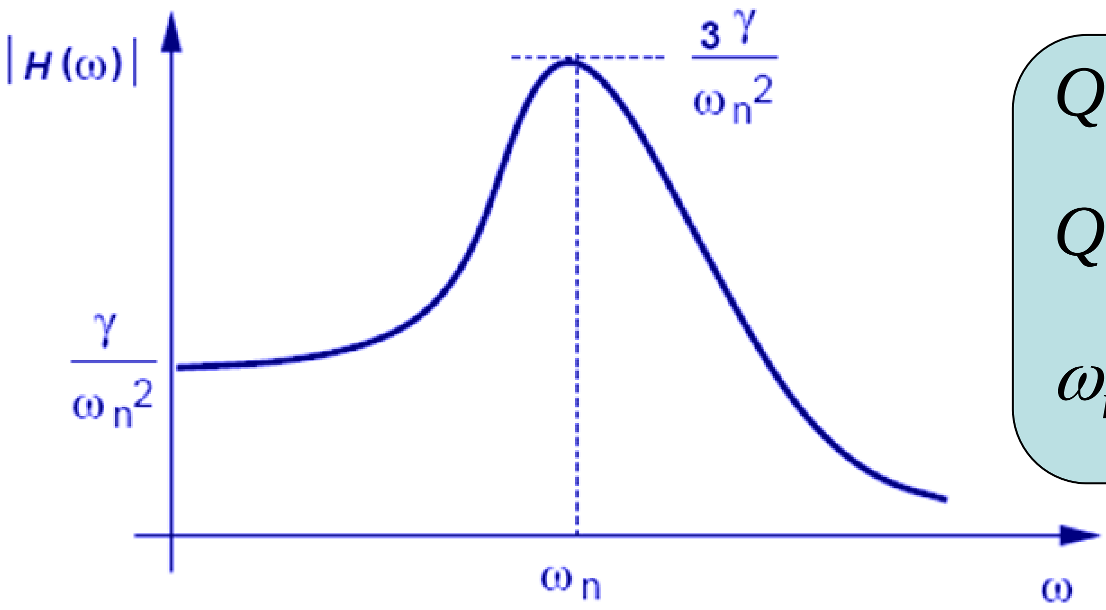
Peak magnitude normalized to the passband magnitude: $Q/\sqrt{1-(4Q^2)^{-1}}$

Example 14.10: Second-Order LPF

Problem: Q of a second-order LPF = 3.

Estimate the magnitude and frequency of the peak in the frequency response.

$$|H(j\omega)|^2 = \frac{\gamma^2}{\left(\omega_n^2 - \omega_n^2\right)^2 + \left(\dfrac{\omega_n^2}{3}\right)^2}$$



$$Q = 3$$

$$Q / \sqrt{1 - 1/(4Q^2)} \approx 3$$

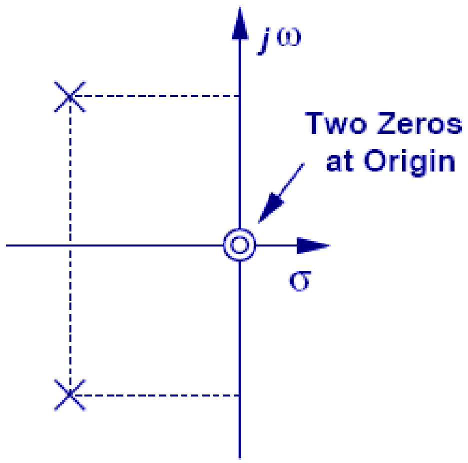
$$\omega_n \sqrt{1 - 1/(2Q^2)} \approx \omega_n$$

Second-Order High-Pass Filter

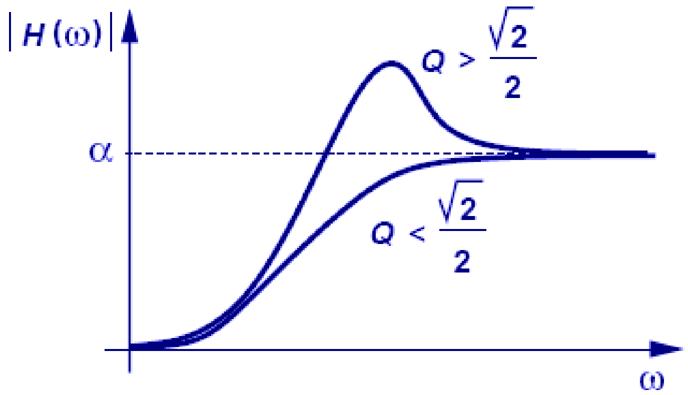
The zero(s) must fall below the poles

$$H(s) = \frac{\alpha s^2}{s^2 + \frac{\omega_n}{Q} s + \omega_n^2}$$

$$\beta = \gamma = 0$$



(a)



(b)

Frequency of the peak: $\omega_n / \sqrt{1 - 1/(2Q^2)}$ for $Q > \frac{\sqrt{2}}{2}$

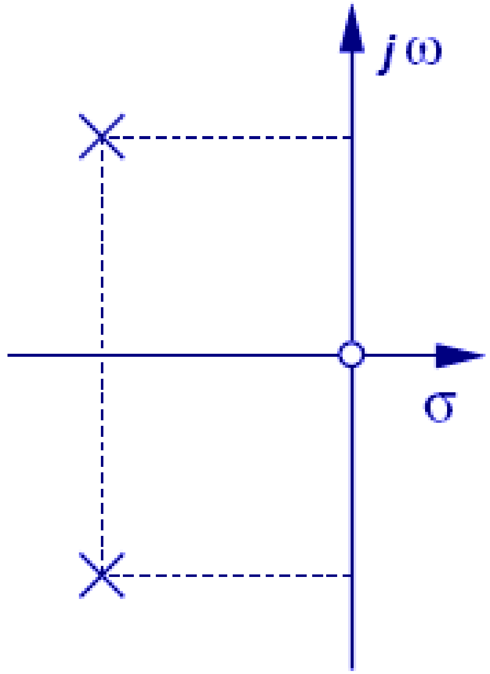
Peak magnitude normalized to the passband magnitude: $Q / \sqrt{1 - (4Q^2)^{-1}}$

Second-Order Band-Pass Filter

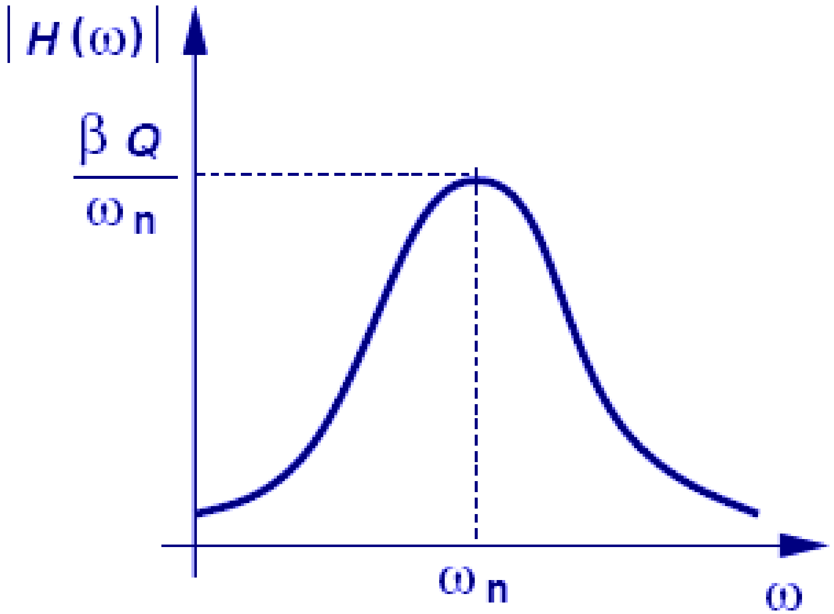
$$H(s) = \frac{\beta s}{s^2 + \frac{\omega_n}{Q} s + \omega_n^2}$$

$$\alpha = \gamma = 0$$

The magnitude approaches zero for both s→0 and s→ ∞, reaching a maximum in between



(a)



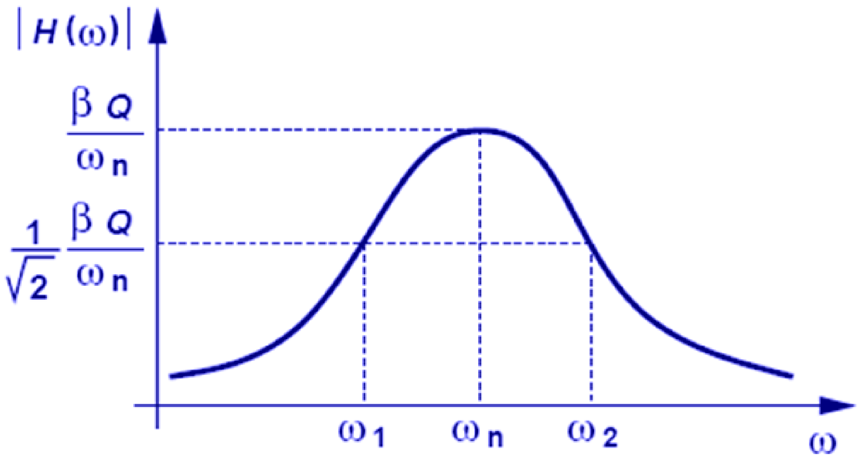
(b)

Example 14.2: -3-dB Bandwidth

Problem: Determine the -3dB bandwidth of a band-pass response.

$$H(s) = \frac{\beta s}{(s^2 + \frac{\omega_n}{Q} s + \omega_n^2)}$$

The response reaches $1/\sqrt{2}$ times its peak value at -3dB frequency

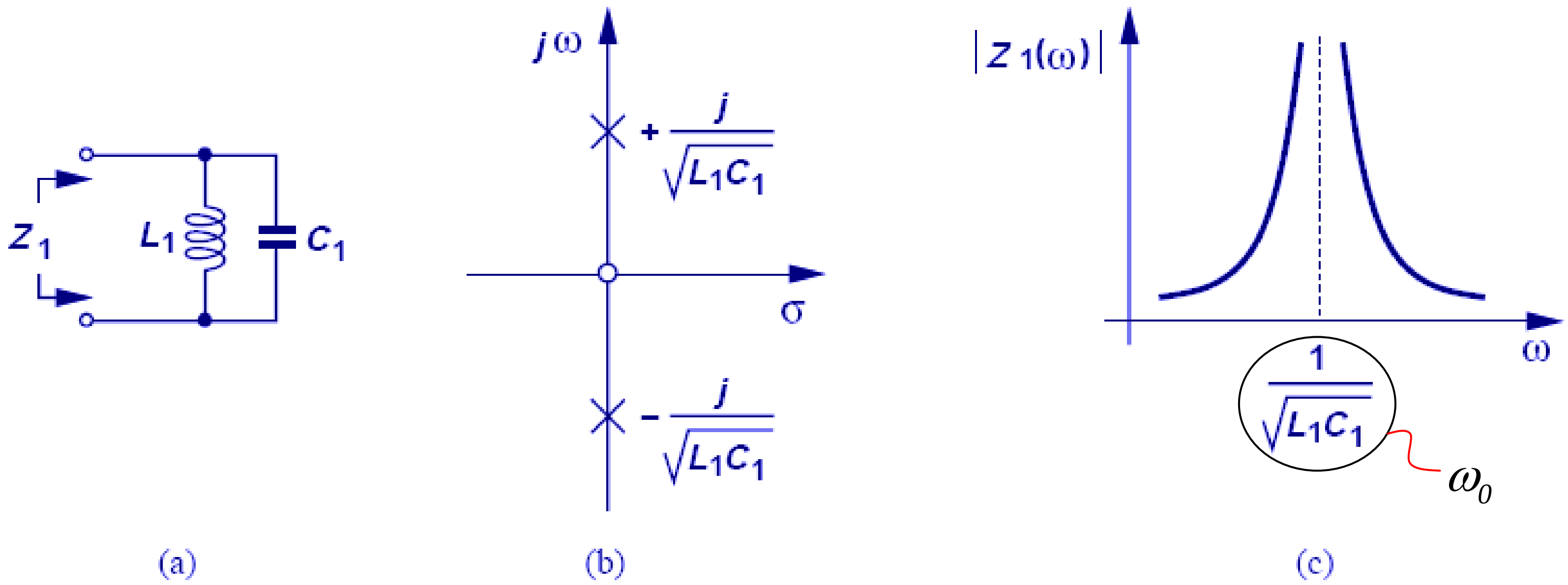


$$\frac{\beta^2 \omega^2}{(\omega_n{}^2 - \omega^2)^2 + (\frac{\omega_n}{Q}\omega)^2} = \frac{\beta^2 Q^2}{2\omega_n{}^2}$$

$$\omega_{1,2} = \omega_0 \left[\sqrt{1 + \frac{1}{4Q^2}} \pm \frac{1}{2Q} \right]$$

$$BW = \frac{\omega_0}{Q}$$

LC Realization of Second-Order Filters



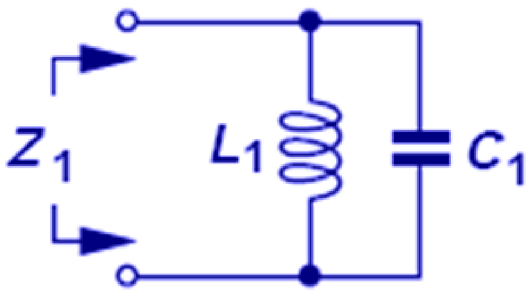
$$Z_1 = (L_1 s) \,||\, \frac{1}{C_1 s} = \frac{L_1 s}{L_1 C_1 s^2 + 1}$$

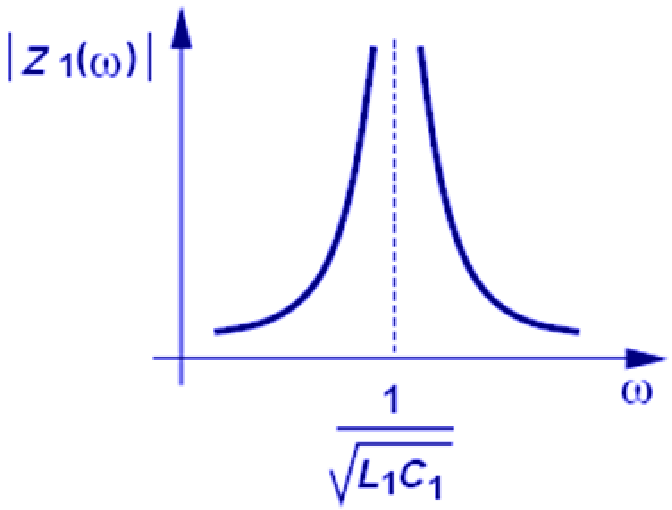
$|Z_1| = 0$ at $\omega = 0$ and ∞

$|Z_1| = \infty$ at $\omega = \omega_0$

- **An LC tank realizes two imaginary poles at $\pm j/(L_1C_1)^{1/2}$, which implies infinite impedance at $\omega=1/(L_1C_1)^{1/2}$.**

Example 14.13: LC Tank





$$Z_1 = (L_1 s) \| \frac{1}{C_1 s} = \frac{L_1 s}{L_1 C_1 s^2 + 1}$$

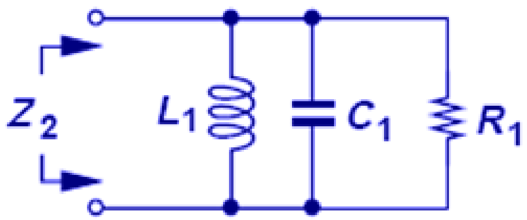
- **At ω=0, the inductor acts as a short.**
- **At ω=∞, the capacitor acts as a short.**

$|Z_1| = 0$ at $\omega = 0$ and ∞

RLC Realization of Second-Order Filters

$$Z_2 = R_1 \| \frac{L_1 s}{L_1 C_1 s^2 + 1} = \frac{R_1 L_1 s}{R_1 L_1 C_1 s^2 + L_1 s + R_1}$$

$$= \frac{R_1 L_1 s}{R_1 L_1 C_1 (s^2 + \frac{1}{R_1 C_1} s + \frac{1}{L_1 C_1})} = \frac{R_1 L_1 s}{R_1 L_1 C_1 (s^2 + \frac{\omega_n}{Q} s + \omega_n^{\ 2})}$$



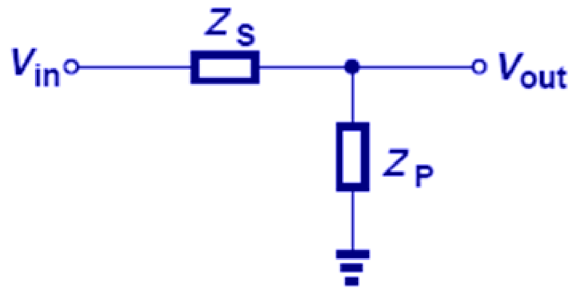
$$\omega_n = \frac{1}{\sqrt{L_1 C_1}}, \quad Q = R_1 \sqrt{\frac{C_1}{L_1}}$$

$$p_{1,2} = -\frac{\omega_n}{2Q} \pm j\omega_n \sqrt{1 - \frac{1}{4Q^2}}$$

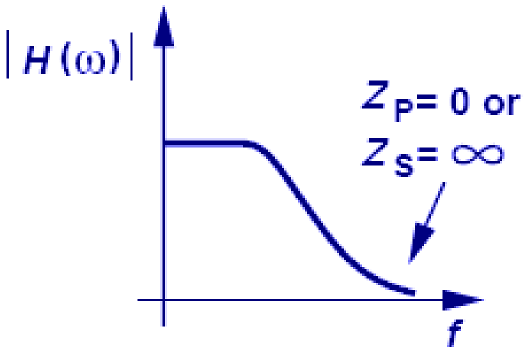
$$= -\frac{1}{2R_1 C_1} \pm j\frac{1}{\sqrt{L_1 C_1}} \sqrt{1 - \frac{L_1}{4R_1^2 C_1}}$$

- **With a resistor, the poles are no longer pure imaginary which implies there will be no infinite impedance at any ω.**

Voltage Divider Using General Impedances

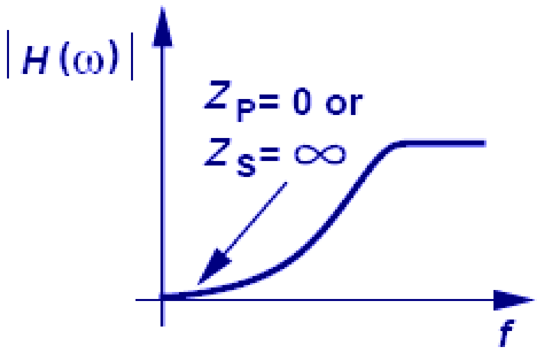


$$\frac{V_{out}}{V_{in}}(s) = \frac{Z_P}{Z_S + Z_P}$$



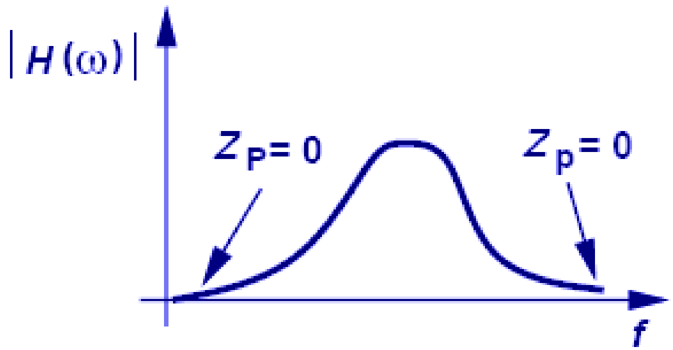
(a)

Low-pass



(b)

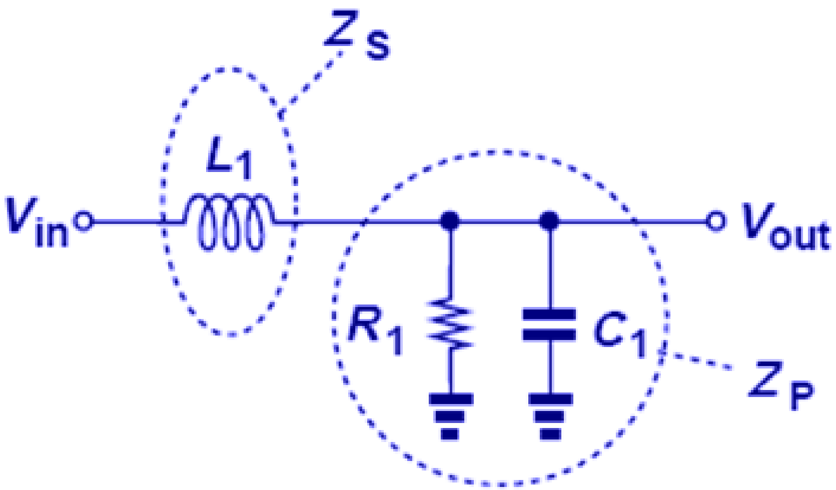
High-pass



(c)

Band-pass

Low-pass Filter Implementation with Voltage Divider

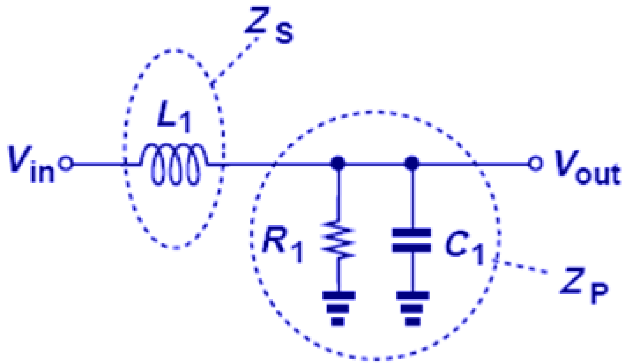


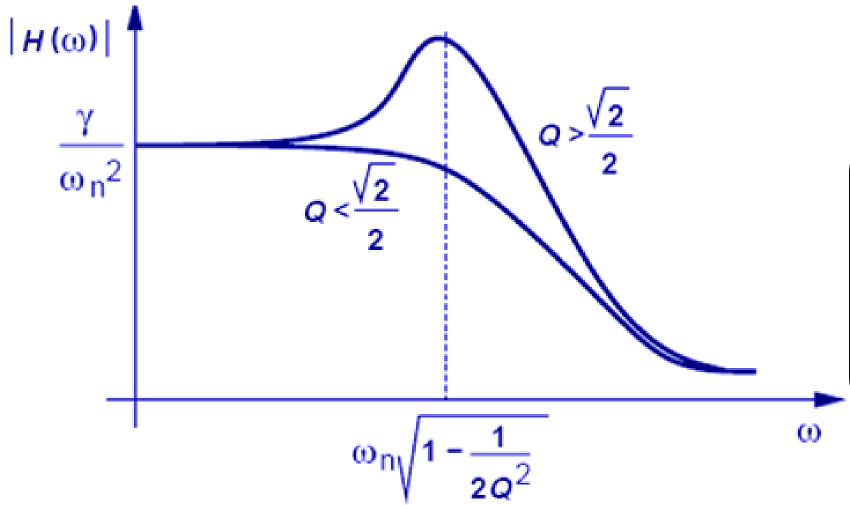
$$Z_S = L_1 s \rightarrow \infty \text{ as } s \rightarrow \infty$$

$$Z_P = \frac{1}{C_1 s} \parallel R_1 \rightarrow 0 \text{ as } s \rightarrow \infty$$

$$\frac{V_{out}}{V_{in}}(s) = \frac{R_1}{R_1 C_1 L_1 s^2 + L_1 s + R_1}$$

Example 14.14: Frequency Peaking





$$\frac{V_{out}}{V_{in}}(s) = \frac{R_1}{R_1C_1L_1s^2 + L_1s + R_1}$$

$$\text{Let } |D|^2 = \left(R_1 - R_1C_1L_1\omega^2\right)^2 + L_1^2\omega^2$$

Voltage gain greater than unity (peaking) occurs when a solution exists for

$$\frac{d|D|^2}{d(\omega^2)} = 2(-R_1C_1L_1)(R_1 - R_1C_1L_1\omega^2) + L_1^2$$

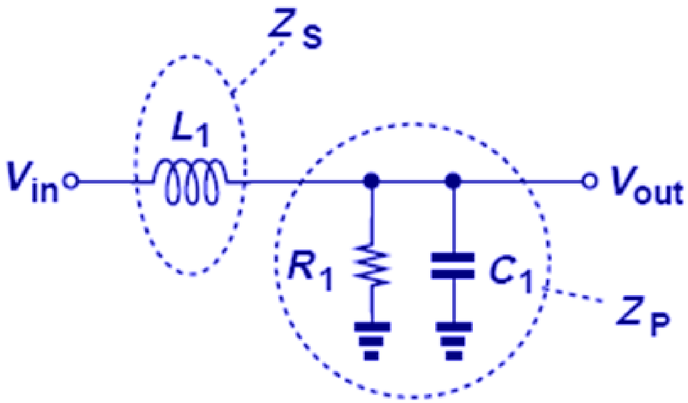
$$= 0 \qquad \Rightarrow \omega = \frac{1}{L_1C_1} - \frac{1}{2R_1^2C_1^2} > 0$$

$$2R_1^2\frac{C_1}{L_1} > 1 \qquad \because Q = R_1\sqrt{\frac{C_1}{L_1}}$$

Thus, when $Q = R_1 \cdot \sqrt{\frac{C_1}{L_1}} > \frac{1}{\sqrt{2}}$,

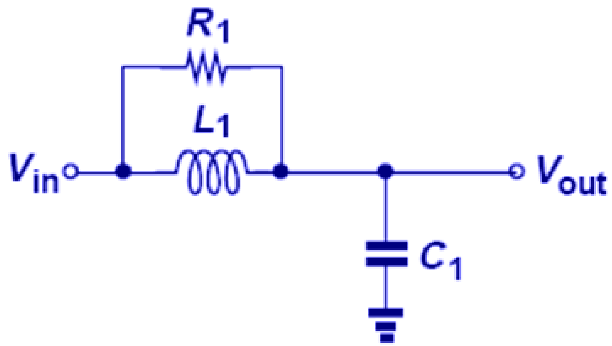
peaking occurs.

Example 14.15: Low-pass Circuit Comparison



(a)

Good



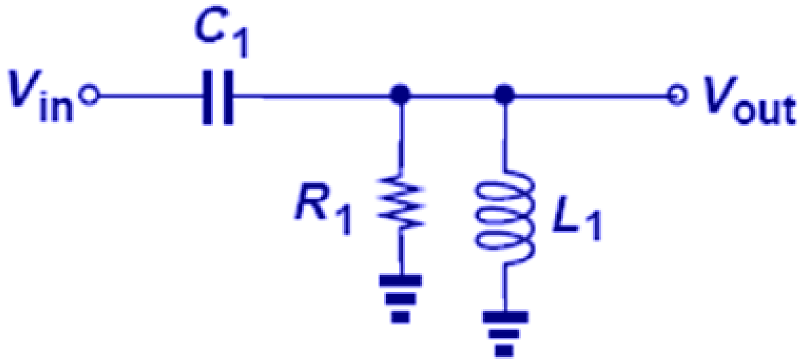
(b)

Bad

- **The circuit (a) has a -40dB/dec roll-off at high frequency.**
- **However, the circuit (b) exhibits only a -20dB/dec roll-off since the parallel combination of L_1 and R_1 is dominated by R_1 because $L_1\omega \to \infty$, thereby reduces the circuit to R_1 and C_1.**

High-pass Filter Implementation with Voltage Divider

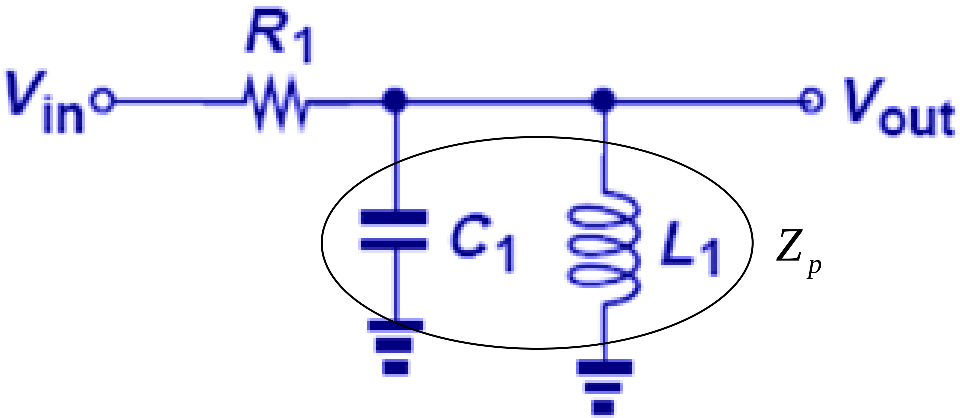
$$\because H(s) = \frac{\alpha s^2}{s^2 + \frac{\omega_n}{Q}s + \omega_n^2} \text{ for high pass filter}$$



$$\frac{V_{out}}{V_{in}}(s) = \frac{(L_1 s) \,||\, R_1}{(L_1 s) \,||\, R_1 + \frac{1}{C_1 s}} = \frac{L_1 C_1 R_1 s^2}{R_1 C_1 L_1 s^2 + L_1 s + R_1}$$

Band-pass Filter Implementation with Voltage Divider

Z_p must contain both a capacitor and an inductor so that it approaches zero as s → 0 or s → ∞



$$\because H(s) = \frac{\beta s}{s^2 + \frac{\omega_n}{Q}s + \omega_n^2}$$

$$\frac{V_{out}}{V_{in}}(s) = \frac{(L_1 s) \| \frac{1}{C_1 s}}{(L_1 s) \| \frac{1}{C_1 s} + R_1} = \frac{L_1 s}{R_1 C_1 L_1 s^2 + L_1 s + R_1}$$

Summary

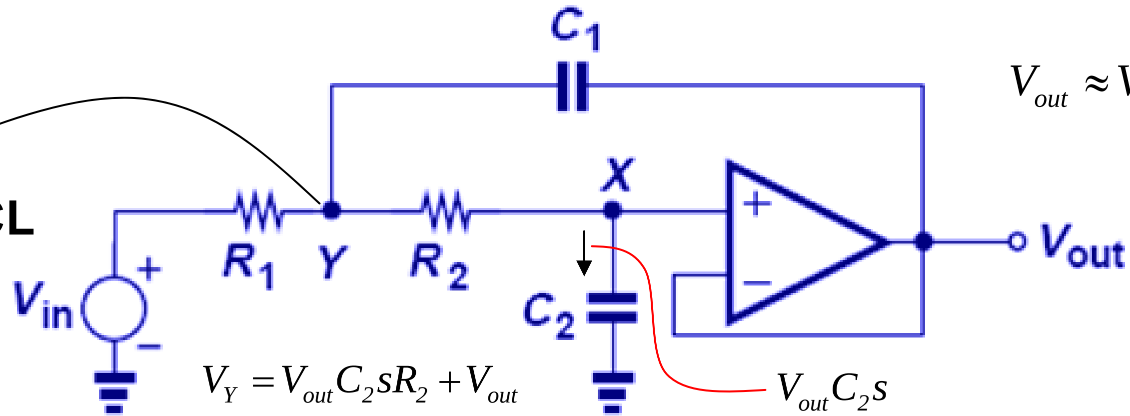
- Analog filters prove essential in removing unwanted frequency components that may accompany a desired signal.
- The frequency response of a filter consists of a passband, stopband, and a transition band between the two. The passband and stopband may exhibit some ripple.
- Filters can be classified as LP, HP, BP, or BR topologies. They can be realized as continuous-time or discrete-time configurations, and as passive or active circuits.
- The frequency response of filters has dependences on various component values and, therefore, suffers from sensitivity to component variations.
- First-order passive or active filters can readily provide a LP or HP response, but their transition band is quite wide and stopband attenuation only moderate.
- Second-order filters have a greater stopband attenuation and are widely used. For a well-behaved frequency and time response, the Q of these filters is typically maintained below $\sqrt{2}/2$

Why Active Filter?

- Passive filters constrain the type of transfer function.
- They may require bulky inductors.

Sallen and Key (SK) Filter: Low-Pass

By applying KCL



$$V_{out} \approx V_X$$

$$V_Y = V_{out} C_2 s R_2 + V_{out}$$

$$V_{out} C_2 s$$

$$\frac{V_{out}}{V_{in}}(s) = \frac{1}{R_1 R_2 C_1 C_2 s^2 + (R_1 + R_2) C_2 s + 1}$$

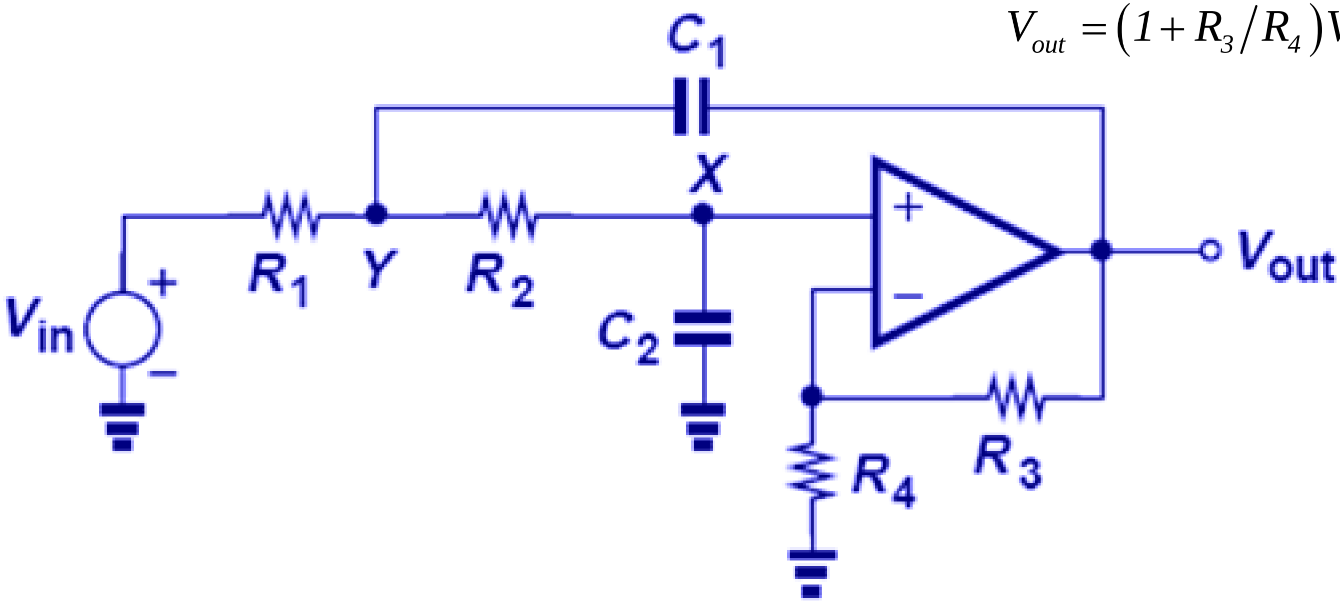
$$H(s) = \frac{\alpha s^2 + \beta s + \gamma}{s^2 + \frac{\omega_n}{Q} s + \omega_n^2}$$

$$Q = \frac{1}{R_1 + R_2} \sqrt{R_1 R_2 \frac{C_1}{C_2}}$$

$$\omega_n = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}$$

- **Sallen and Key filters are examples of active filters. This particular filter implements a low-pass, second-order transfer function.**

Example 14.16: SK Filter with Voltage Gain

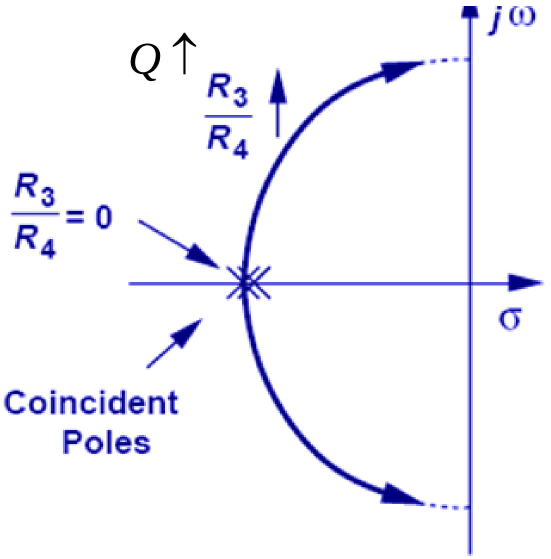


$$V_{out} = (1 + R_3/R_4)V_X$$

$$\frac{V_{out}}{V_{in}}(s) = \frac{1 + \frac{R_3}{R_4}}{R_1R_2C_1C_2s^2 + \left(R_1C_2 + R_2C_2 - R_1\frac{R_3}{R_4}C_1\right)s + 1}$$

Example 14.17: SK Filter Poles

Problem: Assuming $R_1=R_2$, $C_1=C_2$, Does such a filter contain complex poles?



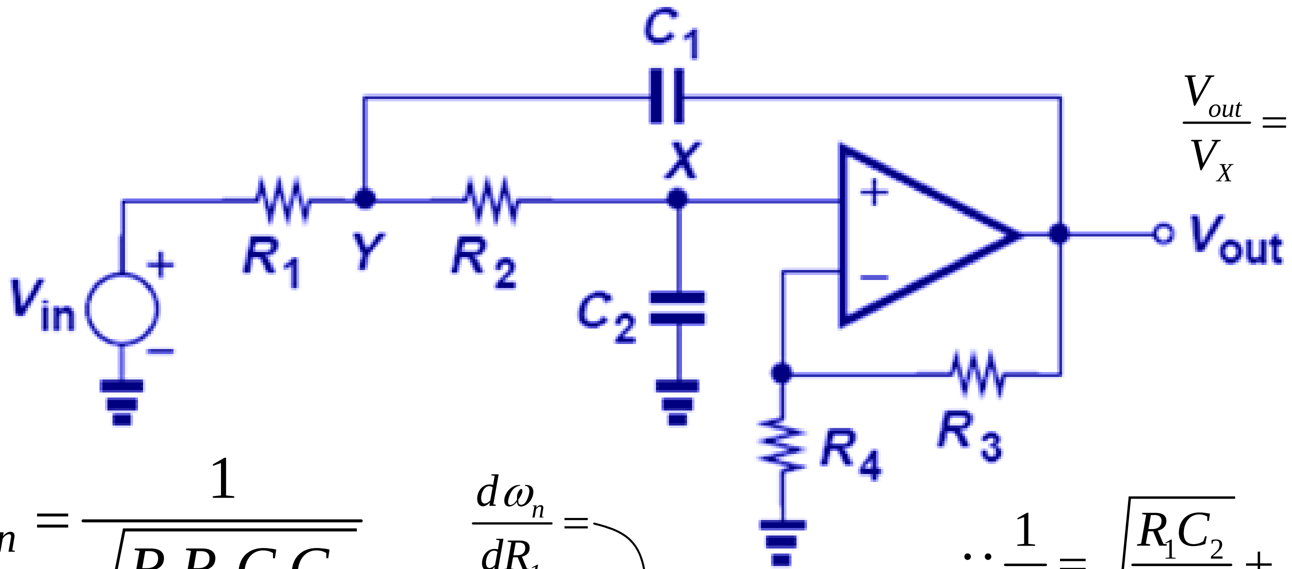
$$\frac{V_{out}}{V_{in}}(s)=\frac{1+\frac{R_3}{R_4}}{R_1R_2C_1C_2s^2+(R_1C_2+R_2C_2-R_1\frac{R_3}{R_4}C_1)s+1}$$

$$\frac{1}{Q}=\sqrt{\frac{R_1C_2}{R_2C_1}}+\sqrt{\frac{R_2C_2}{R_1C_1}}-\sqrt{\frac{R_1C_1}{R_2C_2}}\frac{R_3}{R_4}$$

$$Q=\frac{1}{2-\frac{R_3}{R_4}}$$

- **The poles begin with real, equal values for $R_3/R_4=0$ and become complex for $R_3/R_4>0$.**

Sensitivity in Low-Pass SK Filter



$$\frac{V_{out}}{V_X} = K = 1 + R_3/R_4$$

the gain of filter

$$\because \omega_n = \frac{1}{\sqrt{R_1R_2C_1C_2}} \qquad \frac{d\omega_n}{dR_1} =$$

$$\because \frac{1}{Q} = \sqrt{\frac{R_1C_2}{R_2C_1}} + \sqrt{\frac{R_2C_2}{R_1C_1}} - \sqrt{\frac{R_1C_1}{R_2C_2}}\frac{R_3}{R_4}$$

$$S_{R_1}^{\omega_n} = S_{R_2}^{\omega_n} = S_{C_1}^{\omega_n} = S_{C_2}^{\omega_n} = -\frac{1}{2}$$

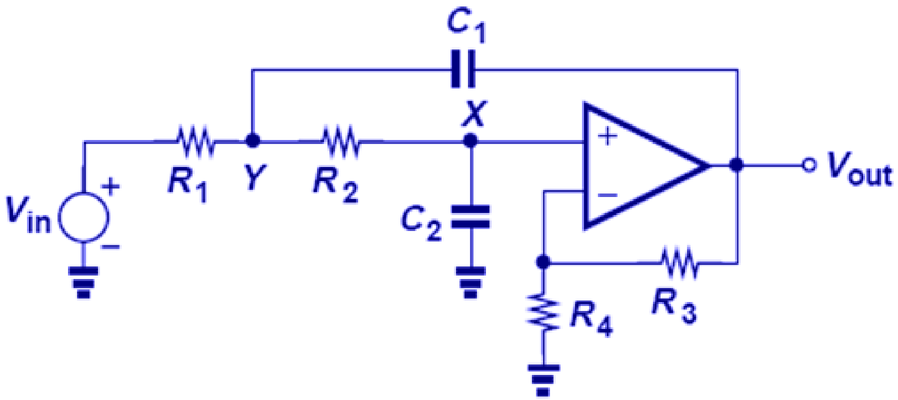
$$S_{C_1}^{Q} = -S_{C_2}^{Q} = -\frac{1}{2} + Q\left(\sqrt{\frac{R_2C_2}{R_1C_1}} + \sqrt{\frac{R_1C_2}{R_2C_1}}\right)$$

$$S_{R_1}^{Q} = -S_{R_2}^{Q} = -\frac{1}{2} + Q\sqrt{\frac{R_2C_2}{R_1C_1}}$$

$$S_{K}^{Q} = QK\sqrt{\frac{R_1C_1}{R_2C_2}}$$

Example 14.18: SK Filter Sensitivity I

Problem: Determine the Q sensitivities of the SK filter for the common choice $R_1=R_2=R$, $C_1=C_2=C$.



$$S_{R_1}^{Q} = -S_{R_2}^{Q} = -\frac{1}{2} + \frac{1}{3-K}$$

$$S_{C_1}^{Q} = -S_{C_2}^{Q} = -\frac{1}{2} + \frac{2}{3-K}$$

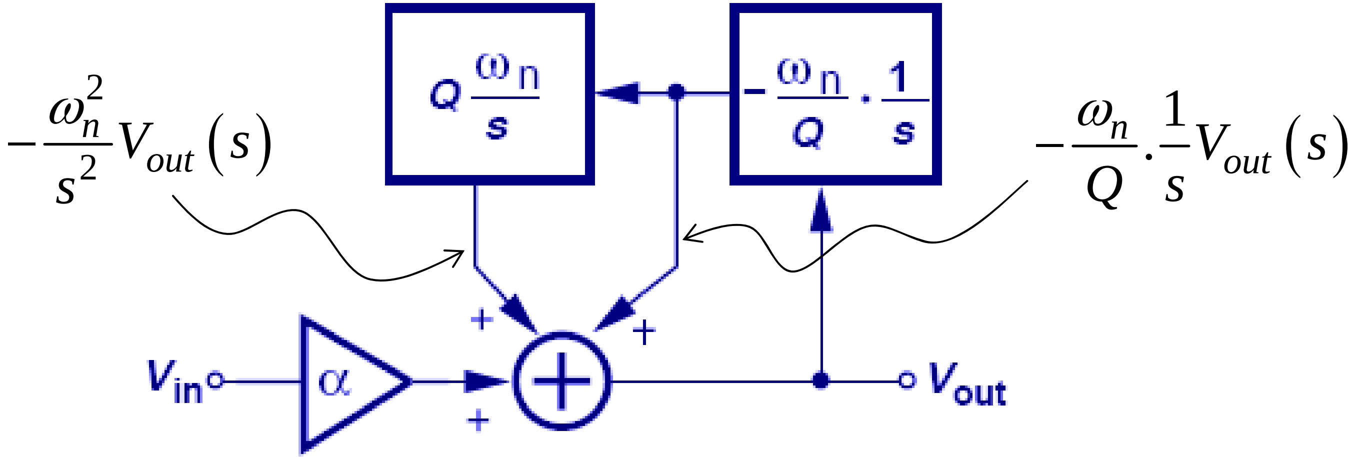
$$S_{K}^{Q} = \frac{K}{3-K}$$

With $K=1$,

$$\left|S_{R_1}^{Q}\right| = \left|S_{R_2}^{Q}\right| = 0$$

$$\left|S_{C_1}^{Q}\right| = \left|S_{C_2}^{Q}\right| = \left|S_{K}^{Q}\right| = \frac{1}{2}$$

Integrator-Based Biquads

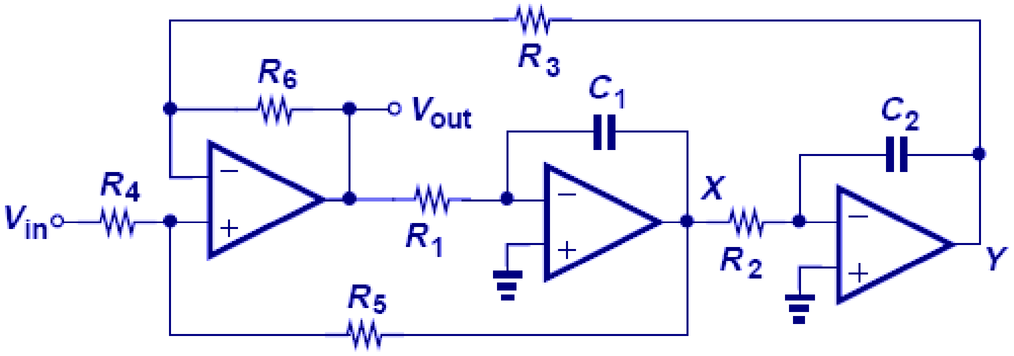


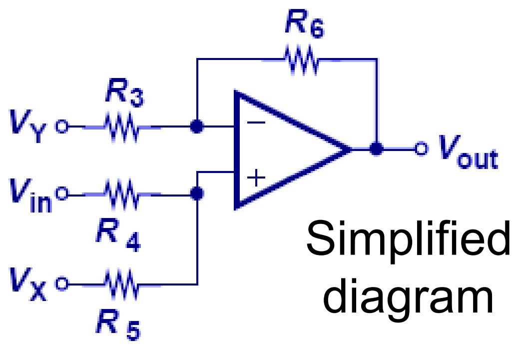
$$\frac{V_{out}}{V_{in}}(s)=\frac{\alpha s^2}{s^2+\frac{\omega_n}{Q}s+\omega_n^2}$$

$$V_{out}(s)=\alpha V_{in}(s)-\frac{\omega_n}{Q}\cdot\frac{1}{s}V_{out}(s)-\frac{\omega_n^2}{s^2}V_{out}(s)$$

- **It is possible to use integrators to implement biquadratic transfer functions.**

KHN (Kerwin, Huelsman, and Newcomb) Biquads





Simplified diagram

$$V_X = -\frac{1}{R_1 C_1 s} V_{out},\ V_Y = -\frac{1}{R_2 C_2 s} V_X = \frac{1}{R_1 R_2 C_1 C_2 s^2} V_{out}$$

$$V_{out} = \frac{V_{in} R_5 + V_X R_4}{R_4 + R_5}\left(1 + \frac{R_6}{R_3}\right) - V_Y \frac{R_6}{R_3}$$

Comparing with $V_{out}(s) = \alpha V_{in}(s) - \frac{\omega_n}{Q} . \frac{1}{s} V_{out}(s) - \frac{\omega_n^2}{s^2} V_{out}(s)$

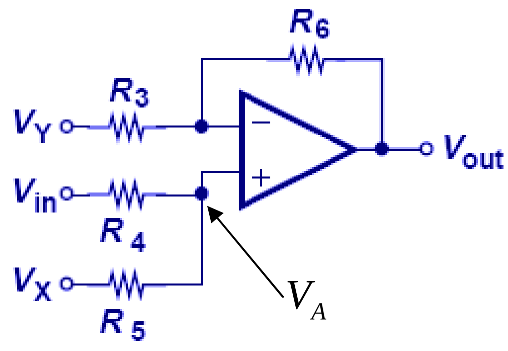
$$\alpha = \frac{R_5}{R_4 + R_5}\left(1 + \frac{R_6}{R_3}\right)$$

$$\frac{\omega_n}{Q} = \frac{R_4}{R_4 + R_5} . \frac{1}{R_1 C_1} . \left(1 + \frac{R_6}{R_3}\right)$$

$$\omega_n^2 = \frac{R_6}{R_3} . \frac{1}{R_1 R_2 C_1 C_2}$$

Calculation of V_{out} with Simplified Circuit

To obtain V_A,



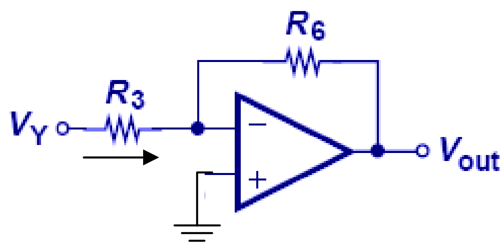
$$V_A\Big|_{V_{in}=0} = \frac{R_4}{R_4+R_5}V_X \quad V_A\Big|_{V_X=0} = \frac{R_5}{R_4+R_5}V_{in}$$

$$V_A = V_A\Big|_{V_{in}=0} + V_A\Big|_{V_X=0} = \frac{R_4V_X + R_5V_{in}}{R_4+R_5}$$

To obtain V_{out} for given V_A, we ground V_Y

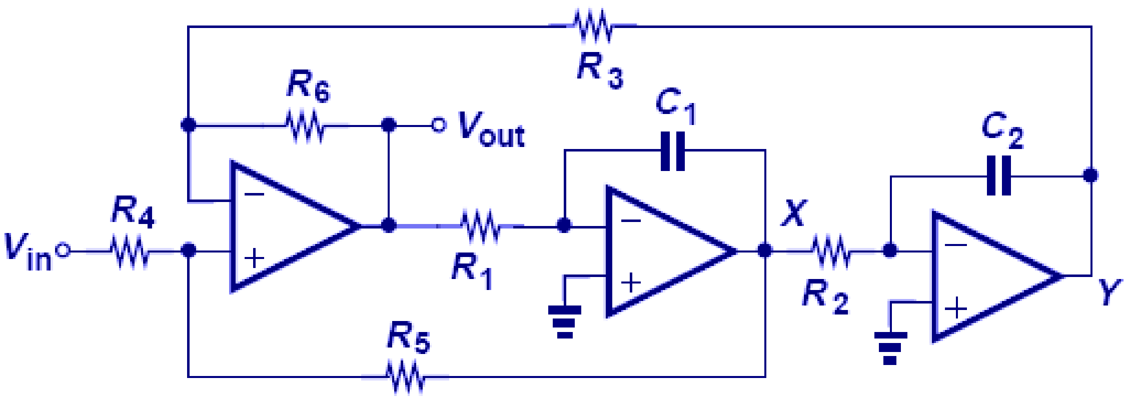
$$V_{out} = \frac{V_A}{R_3}(R_3+R_6) = V_A\left(1+\frac{R_6}{R_3}\right)$$

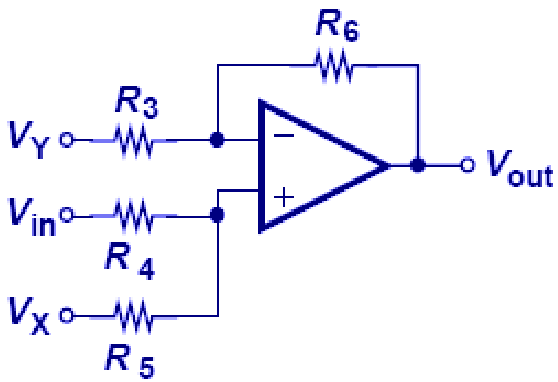
To obtain V_{out} for given V_Y, we ground V_A



$$V_{out} = -\frac{V_Y}{R_3}R_6 = V_Y\left(\frac{R_6}{R_3}\right)$$

Versatility of KHN Biquads



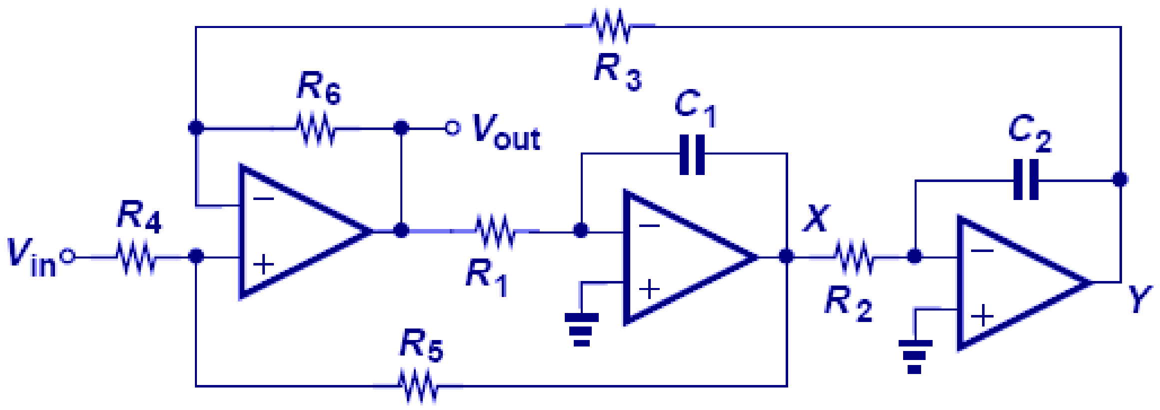


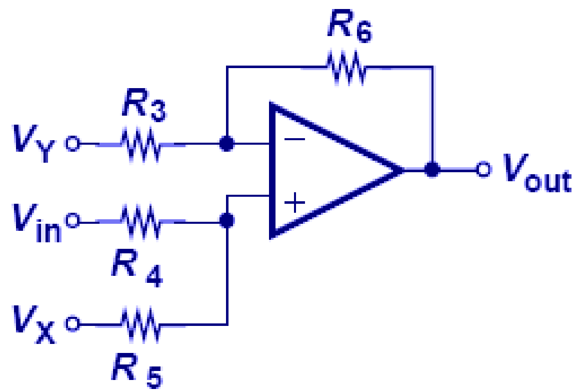
High-pass: $\dfrac{V_{out}}{V_{in}}(s) = \dfrac{\alpha s^2}{s^2 + \dfrac{\omega_n}{Q}s + \omega_n^2}$

Band-pass: $\dfrac{V_X}{V_{in}}(s) = \dfrac{\alpha s^2}{s^2 + \dfrac{\omega_n}{Q}s + \omega_n^2} \cdot \dfrac{-1}{R_1C_1s}$ $\quad \because V_X = -\dfrac{V_{out}}{R_1}\dfrac{1}{sC_1}$

Low-pass: $\dfrac{V_Y}{V_{in}}(s) = \dfrac{\alpha s^2}{s^2 + \dfrac{\omega_n}{Q}s + \omega_n^2} \cdot \dfrac{1}{R_1R_2C_1C_2s^2}$ $\quad \because V_Y = -\dfrac{V_X}{R_2}\dfrac{1}{sC_2}$

Sensitivity in KHN Biquads





$$\left| S^{\omega_n}_{R_1,R_2,C_1,C_2,R_4,R_5,R_3,R_6} \right| = 0.5$$

$$\left| S^{Q}_{R_1,R_2,C_1,C_2} \right| = 0.5, \quad \left| S^{Q}_{R_4,R_5} \right| = \frac{R_5}{R_4 + R_5} < 1,$$

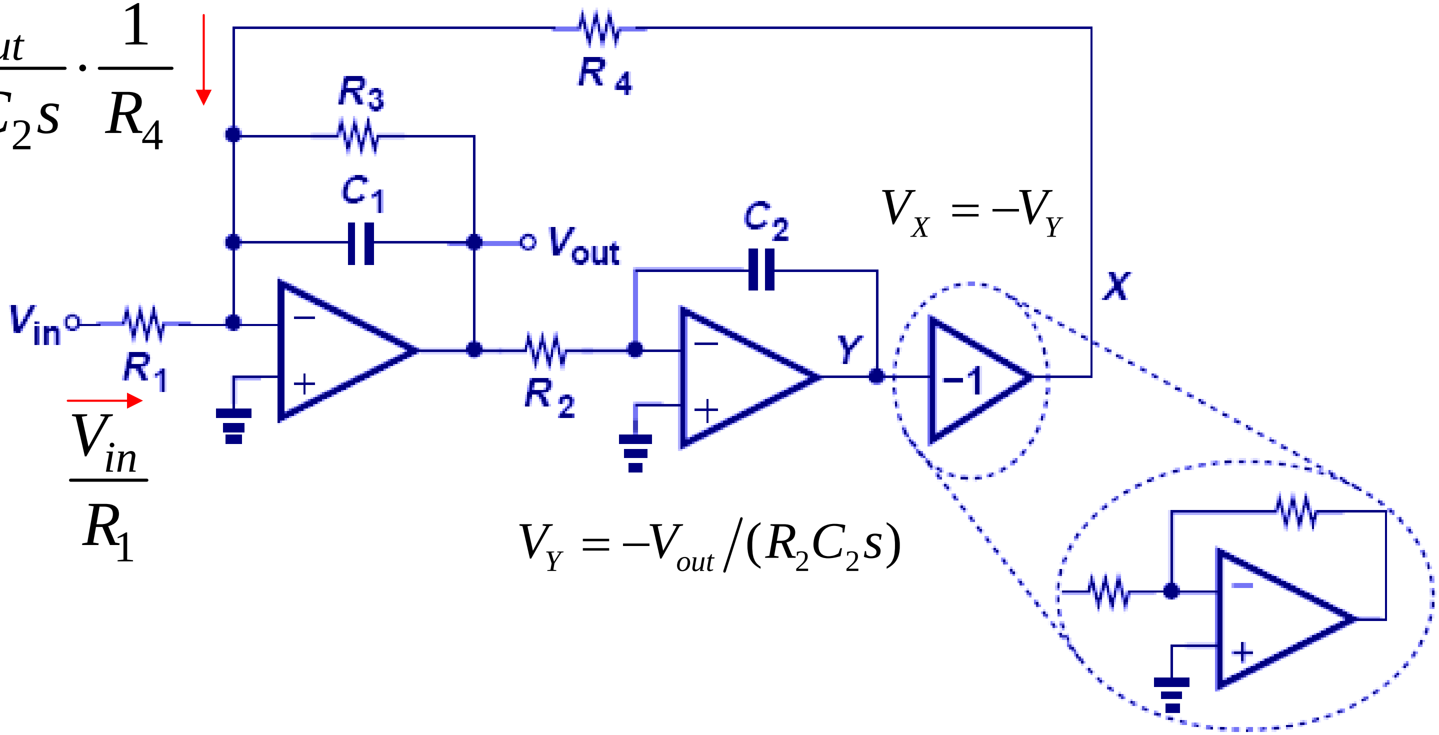
$$\left| S^{Q}_{R_3,R_6} \right| = \frac{Q}{2} \frac{\left| R_3 - R_6 \right|}{1 + \dfrac{R_5}{R_4}} \sqrt{\frac{R_2 C_2}{R_3 R_6 R_1 C_1}}$$

if $R_3 = R_6$,

then $S^{Q}_{R_3,R_6}$ vanishes

Tow-Thomas Biquad

$$\frac{V_{out}}{R_2C_2s}\cdot\frac{1}{R_4}$$



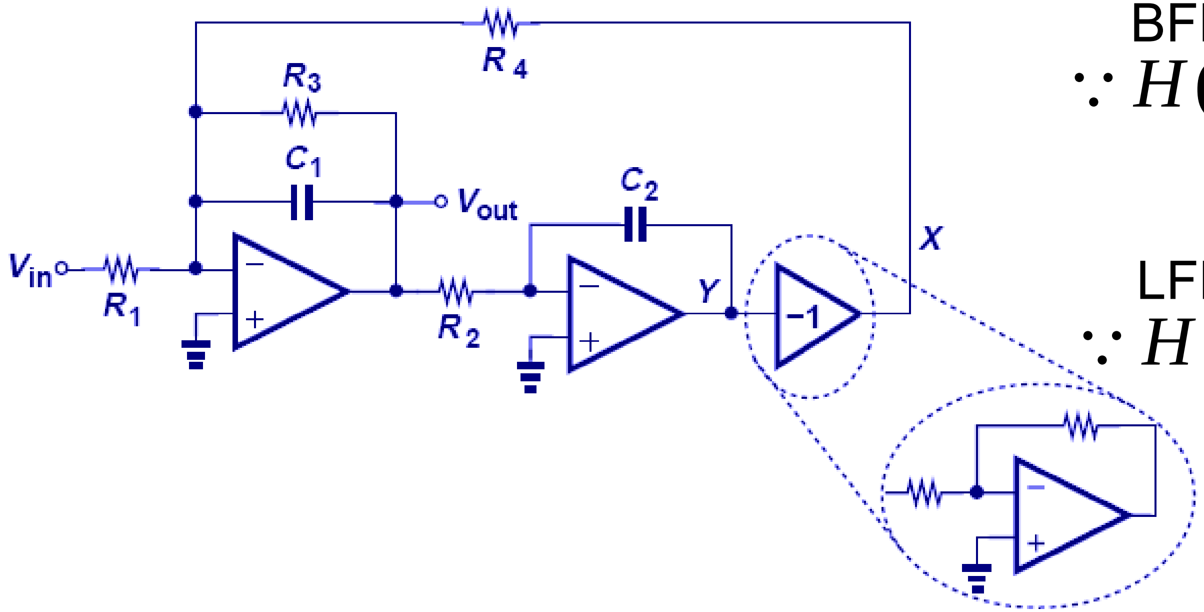
$$V_X = -V_Y$$

$$\frac{V_{in}}{R_1}$$

$$V_Y = -V_{out}/(R_2C_2s)$$

$$\left(\frac{V_{out}}{R_2C_2s}\cdot\frac{1}{R_4}+\frac{V_{in}}{R_1}\right)\left(R_3\,||\,\frac{1}{sC_1}\right) = -V_{out}$$

Tow-Thomas Biquad



BFP

$$\because H(s) = \frac{\beta s}{s^2 + \frac{\omega_n}{Q}s + \omega_n^2}$$

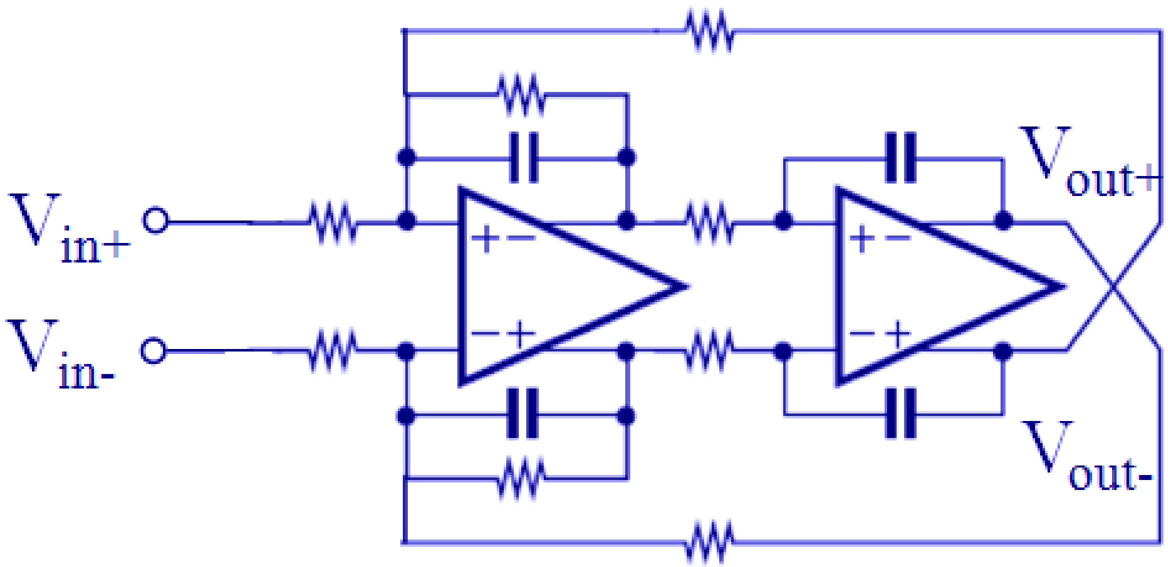
LFP

$$\because H(s) = \frac{\gamma}{s^2 + \frac{\omega_n}{Q}s + \omega_n^2}$$

Band-pass: $\frac{V_{out}}{V_{in}} = -\frac{R_2R_3R_4}{R_1} \cdot \frac{C_2 s}{R_2R_3R_4C_1C_2s^2 + R_2R_4C_2s + R_3}$

Low-pass: $\frac{V_Y}{V_{in}} = \frac{R_3R_4}{R_1} \cdot \frac{1}{R_2R_3R_4C_1C_2s^2 + R_2R_4C_2s + R_3}$

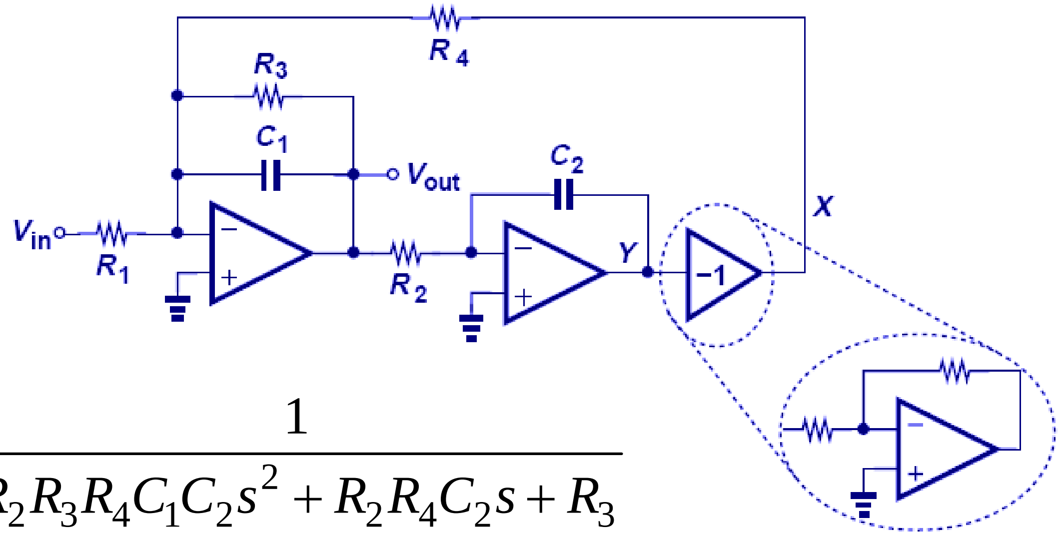
Differential Tow-Thomas Biquads



- **An important advantage of this topology over the KHN biquad is accrued in integrated circuit design, where differential integrators obviate the need for the inverting stage in the loop.**

Example 14.20: Tow-Thomas Biquad

$$H(s) = \frac{\cancel{\alpha} s^2 + \cancel{\beta} s + \gamma}{s^2 + \frac{\omega_n}{Q} s + \omega_n^2}$$



$$\text{Low-pass: } \frac{V_Y}{V_{in}} = \frac{R_3 R_4}{R_1} \cdot \frac{1}{R_2 R_3 R_4 C_1 C_2 s^2 + R_2 R_4 C_2 s + R_3}$$

Note that ω_n and Q of the Tow-Thomas filter can be adjusted (tuned) independently.

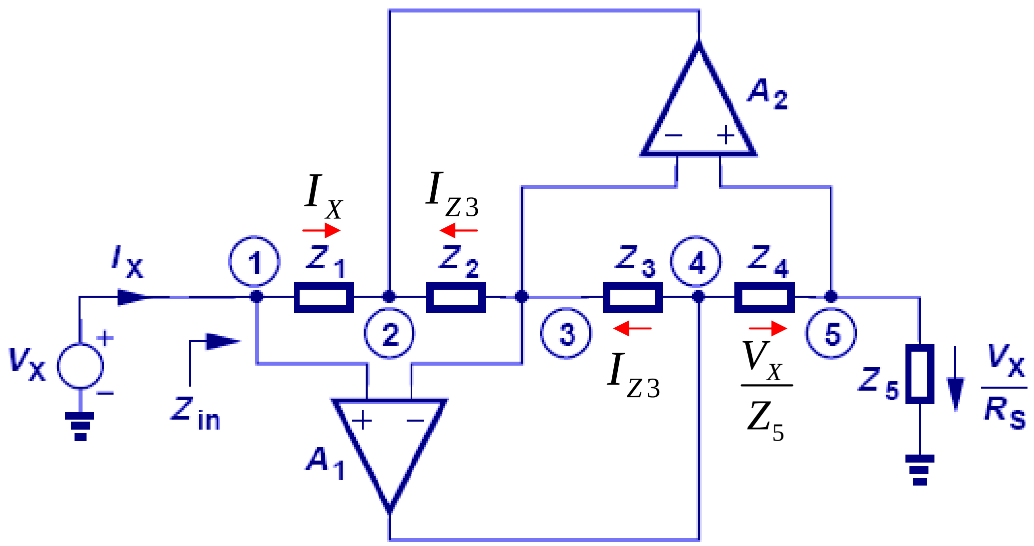
$$\omega_n = \frac{1}{\sqrt{R_2 R_4 C_1 C_2}}$$

Adjusted by R_2 or R_4

$$Q^{-1} = \frac{1}{R_3}\sqrt{\frac{R_2 R_4 C_2}{C_1}}$$

Adjusted by R_3

Antoniou General Impedance Converter



$$Z_{in} = \frac{Z_1 Z_3}{Z_2 Z_4} Z_5$$

$$V_1 = V_3 = V_5 = V_X$$

$$V_4 = \frac{V_X}{Z_5} Z_4 + V_X$$

$$I_{Z3} = \frac{V_4 - V_3}{Z_3} = \frac{V_X}{Z_5} \cdot \frac{Z_4}{Z_3}$$

$$V_2 = V_3 - Z_2 I_{Z3}$$

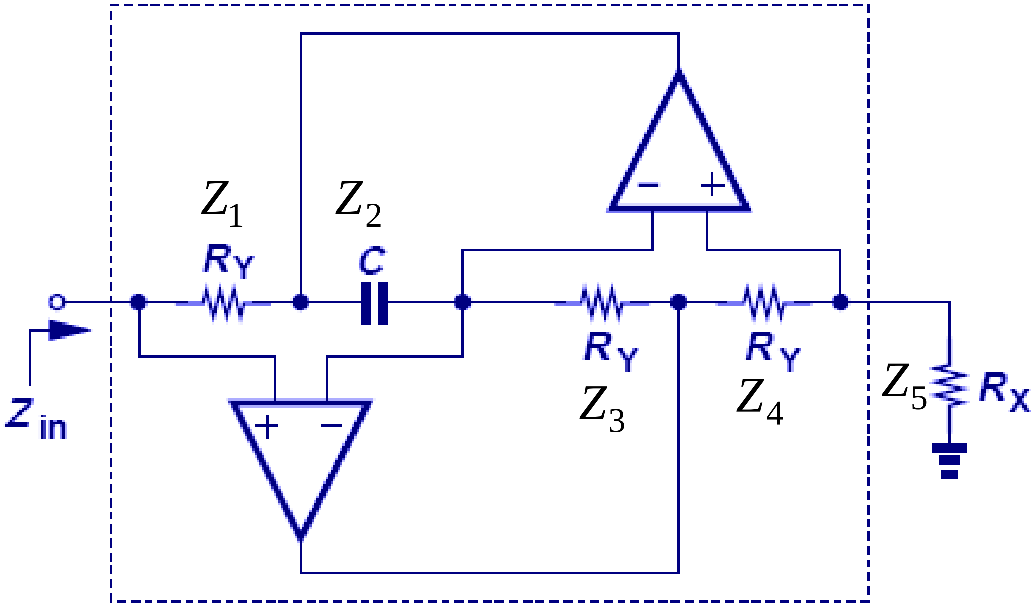
$$= V_X - Z_2 \cdot \frac{V_X}{Z_5} \cdot \frac{Z_4}{Z_3}$$

$$I_X = \frac{V_X - V_2}{Z_1}$$

$$= V_X \frac{Z_2 Z_4}{Z_1 Z_3 Z_5}$$

- **It is possible to simulate the behavior of an inductor by using active circuits in feedback with properly chosen passive elements.**

Simulated Inductor



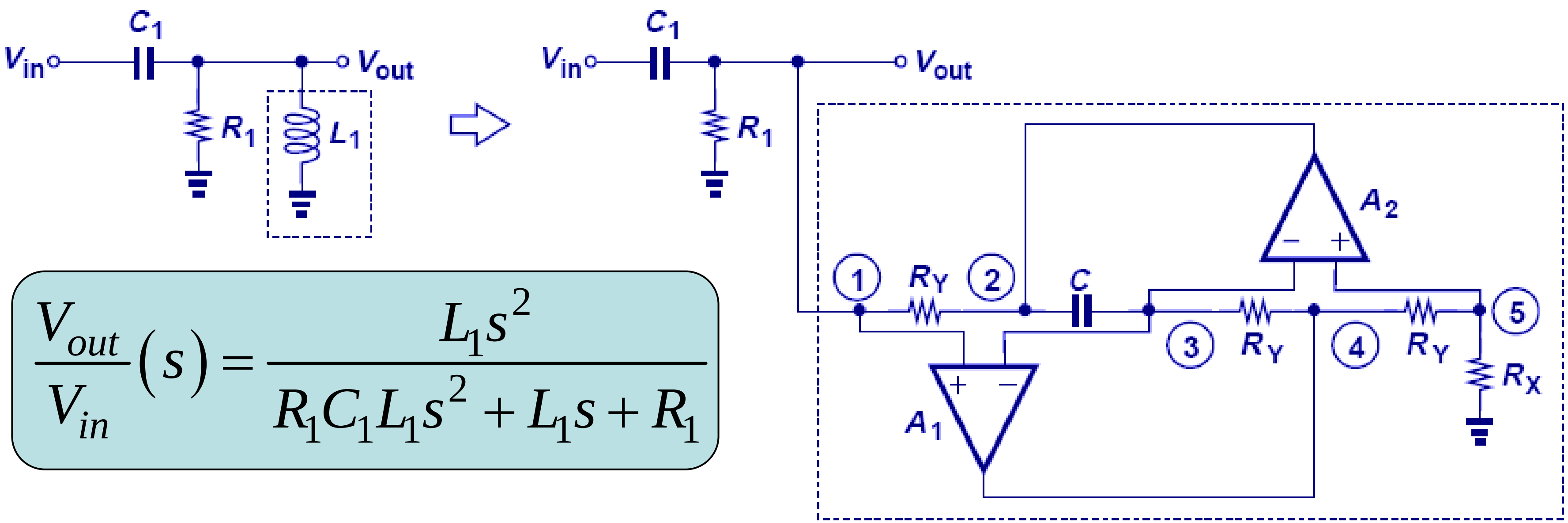
$$\because Z_{in} = \frac{Z_1 Z_3}{Z_2 Z_4} Z_5$$

$$Z_{in} = R_X R_Y C s$$

$$\text{Thus, } L_{eq} = R_X R_Y C$$

- By proper choices of Z_1-Z_4, Z_{in} has become an impedance that increases with frequency, simulating inductive effect.

High-Pass Filter with SI

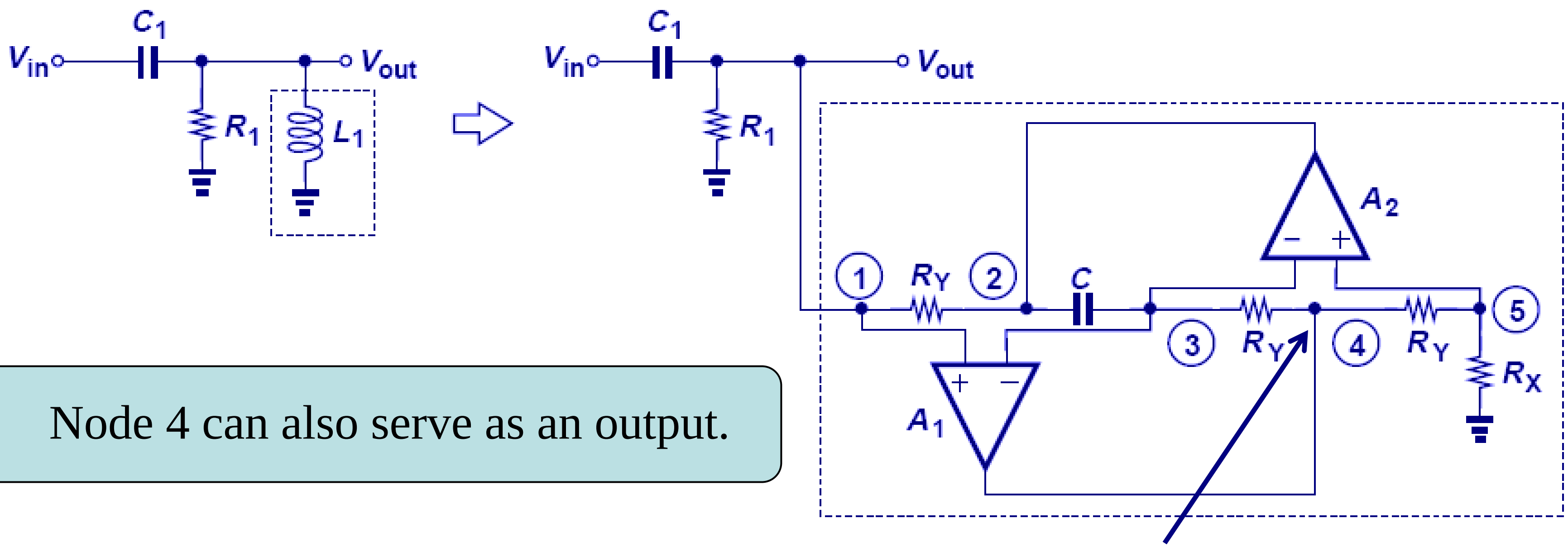


$$\frac{V_{out}}{V_{in}}(s) = \frac{L_1 s^2}{R_1 C_1 L_1 s^2 + L_1 s + R_1}$$

- With the inductor simulated at the output, the transfer function resembles a second-order high-pass filter.

Example 14.22: High-Pass Filter with SI

$$V_{out} = V_1 = V_3 = V_5$$



Node 4 can also serve as an output.

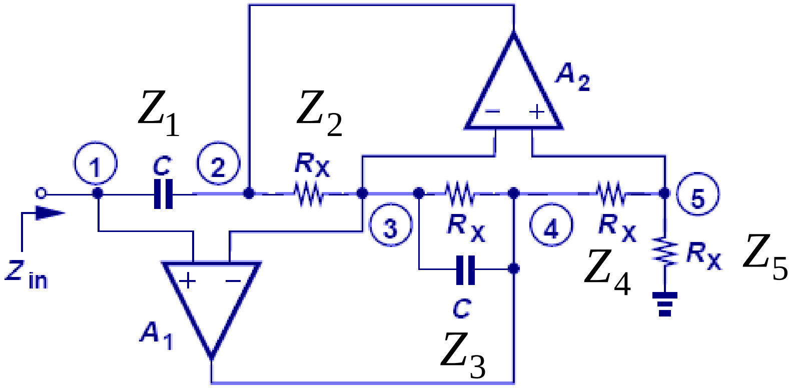
$$V_5 = \frac{R_X}{R_Y + R_X} V_4 \Rightarrow V_4 = \left(1 + \frac{R_Y}{R_X}\right) V_5 \Rightarrow$$

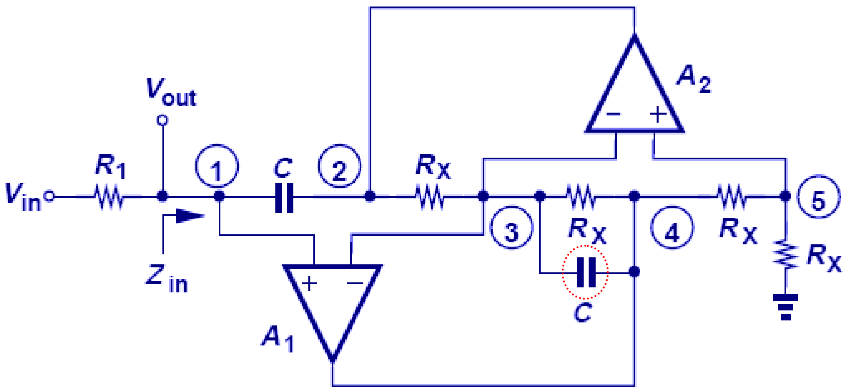
$$V_4 = V_{out}\left(1 + \frac{R_Y}{R_X}\right)$$

- V_4 is better than V_{out} since the output impedance is lower.

Low-Pass Filter with Super Capacitor

- How to build a floating inductor to derive a low-pass filter? Not possible. So use a super capacitor.





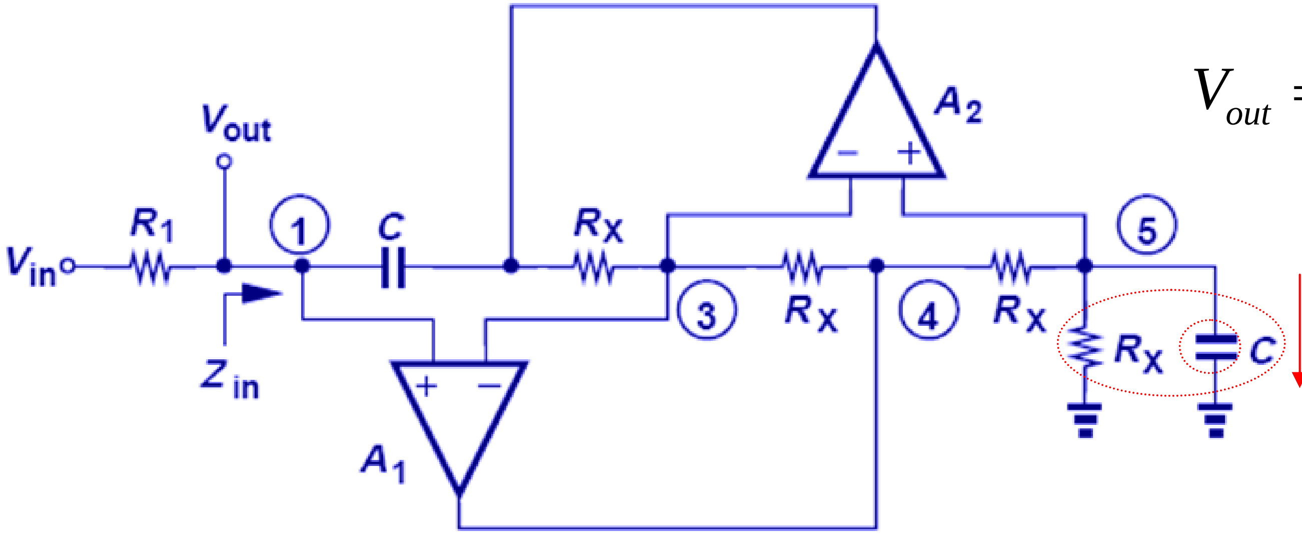
$$Z_{in} = \frac{1}{Cs\left(R_X Cs + 1\right)}$$

$$H(s) = \frac{\gamma}{s^2 + \frac{\omega_n}{Q}s + \omega_n^2}$$

$$\frac{V_{out}}{V_{in}} = \frac{Z_{in}}{Z_{in} + R_1}$$

$$= \frac{1}{R_1 R_X C^2 s^2 + R_1 Cs + 1}$$

Example 14.23: Poor Low-Pass Filter



$$V_{out} = V_1 = V_3 = V_5$$

$$V_{out}\left(\frac{1}{R_X} + Cs\right)$$

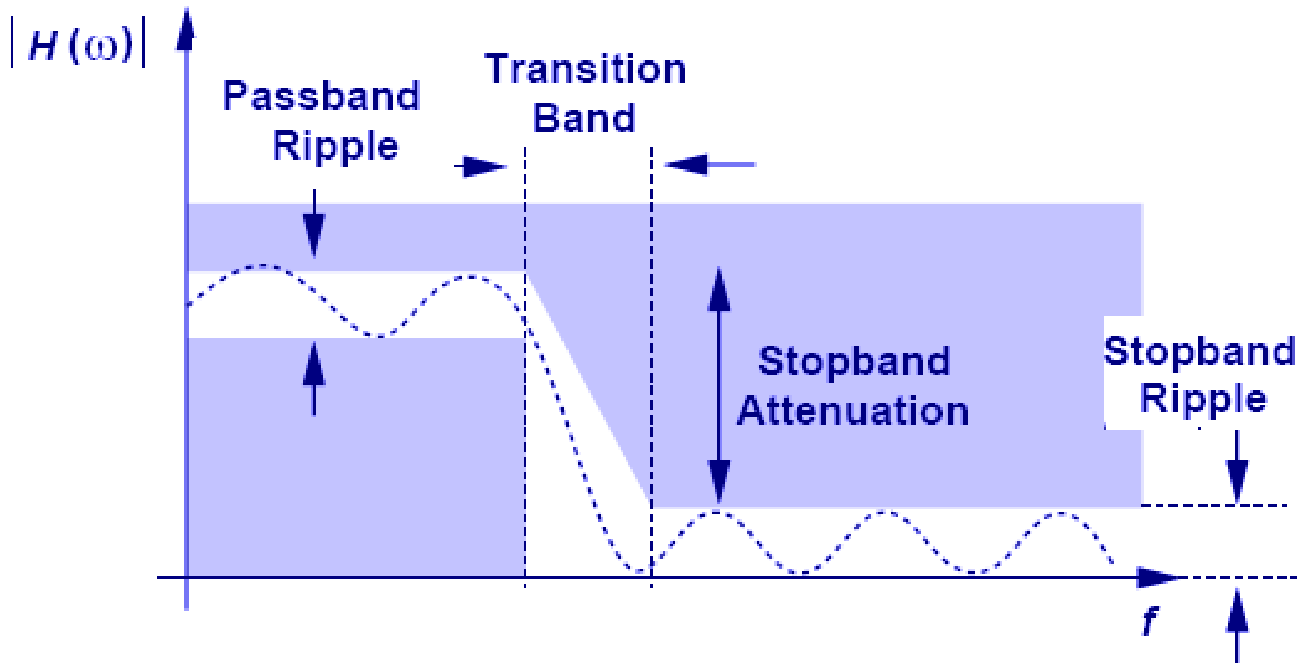
$$V_4 = \left[V_{out}\left(\frac{1}{R_X} + Cs\right)\right]R_X + V_{out} = V_{out}\left(2 + R_X Cs\right)$$

- Node 4 is no longer a scaled version of the V_{out}. Therefore the output can only be sensed at node 1, suffering from a high impedance.

Summary

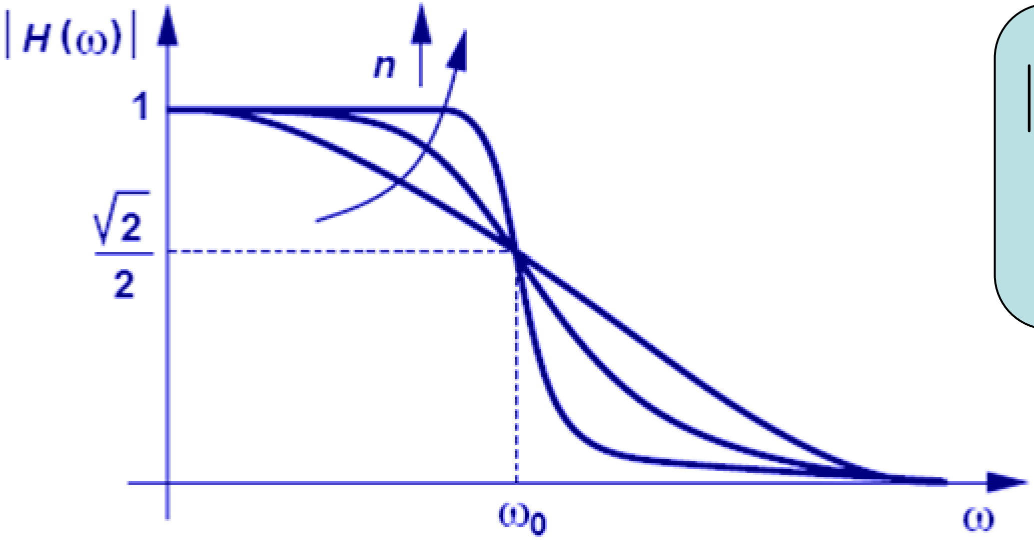
- Continuous-time passive second-order filters employ RSC sections, but they become impractical at very low frequencies (because of large physical size of inductors and capacitors).
- Active filters employ op amps, resistors, and capacitors to create the desired frequency response. The Sallen and Key topology is an example.
- Second-order active (biquad) sections can be based on integrators. Examples include the KHN biquad and the Tow-Thomas biquad.
- Biquads can also incorporate simulated inductors, which are derived from the “general impedance converter” (GIC). The GIC can yield large inductor or capacitor values through the use of two op amps.

Frequency Response Template



- **With all the specifications on pass/stop band ripples and transition band slope, one can create a filter template that will lend itself to transfer function approximation.**

Butterworth Response



$$|H(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_0}\right)^{2n}}}$$

$\omega_{-3dB}=\omega_{0,}$ **for all n.**

- **The Butterworth response completely avoids ripples in the pass/stop bands at the expense of the transition band slope.**

To obtain the poles, we make a reverse substitution, $\omega = s/j$, and set the denominarot to zero:

Butterworth Response

$$1+\left(\frac{s}{j\omega_0}\right)^{2n}=0$$

$$s^{2n}+(j\omega_0)^{2n}=0 \Rightarrow s^{2n}+(j)^{2n}(\omega_0)^{2n}=0$$

$$s^{2n}+(-j)^{n}(\omega_0)^{2n}=0$$

This polynomial has 2n roots given by

$$p_k=\omega_0\exp\frac{j\pi}{2}\exp\left(j\frac{2k-1}{2n}\right),\ k=1,2,\ldots,2n$$

For 2nd order filter,

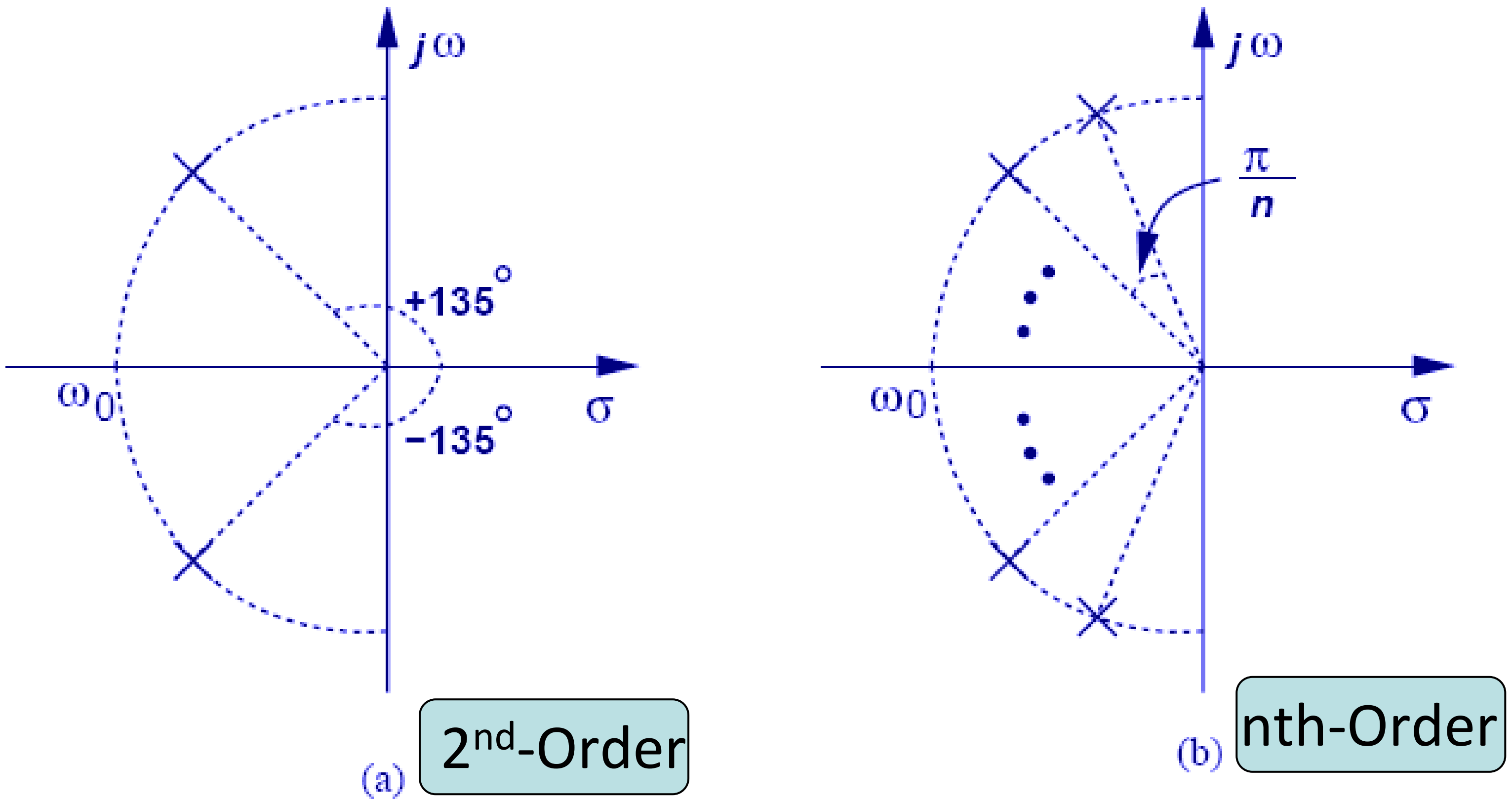
$$\exp\frac{-j\pi}{4} \qquad \exp\frac{j\pi}{4}$$

$$p_{k=1,2,3,4}=\underbrace{\omega_0\exp\frac{j3\pi}{4},\omega_0\exp\frac{j5\pi}{4}}_{\text{negative real}},\underbrace{\omega_0\exp\frac{j7\pi}{4},\omega_0\exp\frac{j9\pi}{4}}_{\text{positive real}}$$

But only the roots having a negative real part are acceptable

$$p_k=\omega_0\exp\frac{j\pi}{2}\exp\left(j\frac{2k-1}{2n}\pi\right),k=1,2,\cdots,n$$

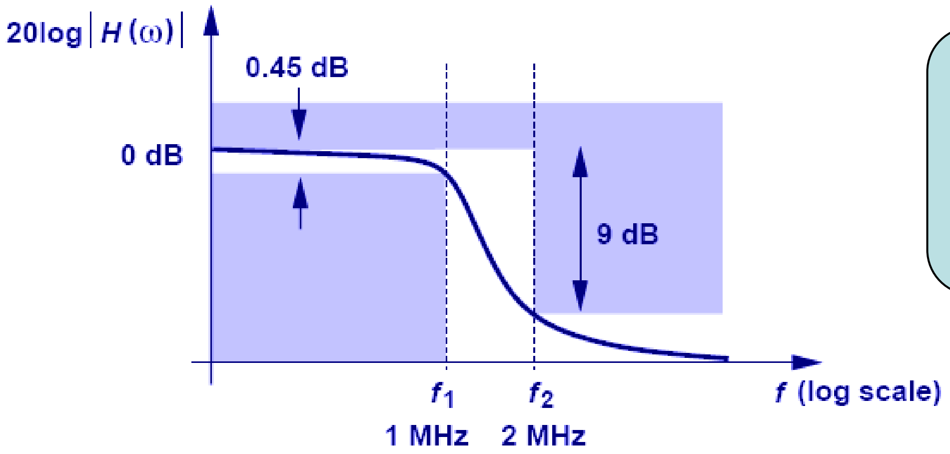
Poles of the Butterworth Response



$$H(s) = \frac{(-p_1)(-p_2)\cdots(-p_n)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

where the factor in the numerator is included to yield *H*(*s*=0)=1

Example 14.24: Order of Butterworth Filter



Specification: passband flatness of 0.45 dB for f < f_1=1 MHz, stopband attenuation of 9 dB at f_2=2 MHz.

$$|H(f_1 = 1\text{MHz})| = 0.95 \approx -0.45\text{ dB}$$

$$\frac{1}{1+\left(\frac{2\pi f_1}{\omega_0}\right)^{2n}} = 0.95^2$$

$$|H(f_2 = 2\text{MHz})| = 0.355 \quad \approx -9\text{ dB}$$

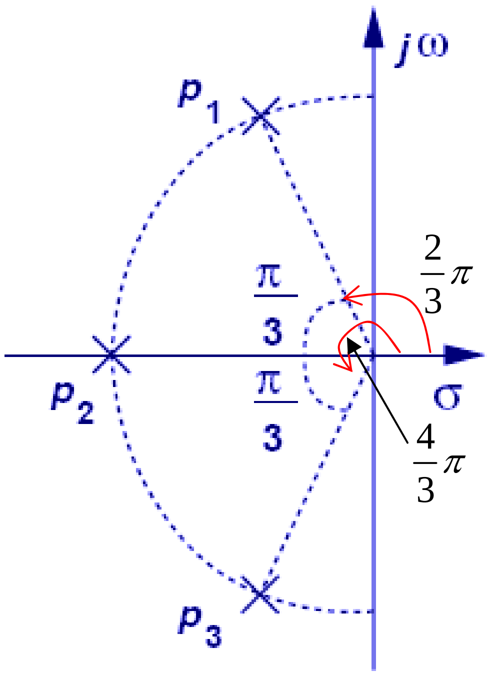
$$\frac{1}{1+\left(\frac{2\pi f_2}{\omega_0}\right)^{2n}} = 0.355^2$$

$$\left(\frac{f_2}{f_1}\right)^{2n} = 64.2 \Rightarrow f_2 = 2f_1 \Rightarrow n = 3, \quad \omega_0 = 2\pi \times (1.45\text{MHz})$$

- **The minimum order of the Butterworth filter is three.**

Example 14.25: Butterworth Response

Using a Sallen and Key topology, design a Butterworth filter for the response derived in Example 14.24.



$$p_{k=1,2,3} = \omega_0 \exp\frac{j2\pi}{3}, \omega_0 \exp\frac{j3\pi}{3}, \omega_0 \exp\frac{j4\pi}{3}$$

$$= -\omega_0$$

$$p_1 = 2\pi * (1.45MHz) * \left(\cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3}\right)$$

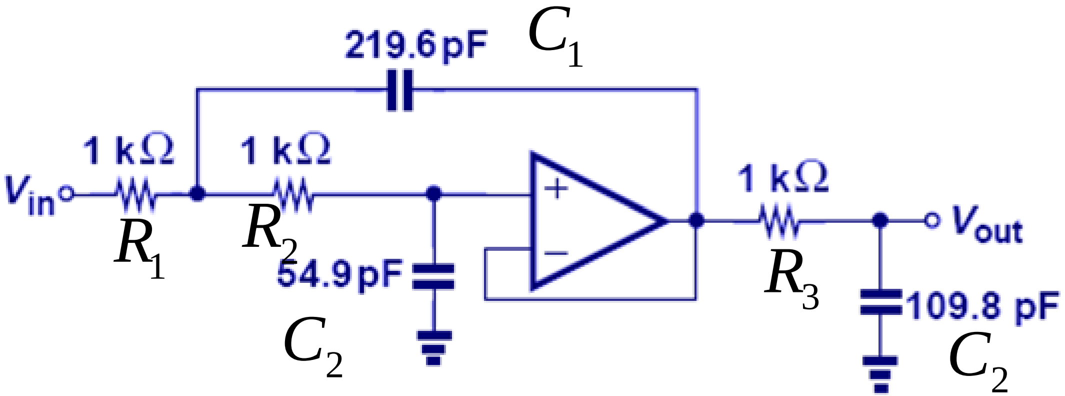
$$p_3 = 2\pi * (1.45MHz) * \left(\cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3}\right)$$

2nd-order SK

$$p_2 = 2\pi * (1.45MHz)$$

RC section

Example 14.25: Butterworth Response (cont'd)



$$H(s) = \frac{\alpha s^2 + \beta s + \gamma}{s^2 + \frac{\omega_n}{Q} s + \omega_n^2}$$

$$H_{SK}(s) = \frac{(-p_1)(-p_3)}{(s-p_1)(s-p_3)} = \frac{[2\pi \times (1.45\text{MHz})]^2}{s^2 - \left[4\pi \times (1.45\text{MHz})\cos(2\pi/3)\right]s + [2\pi \times (1.45\text{MHz})]^2}$$

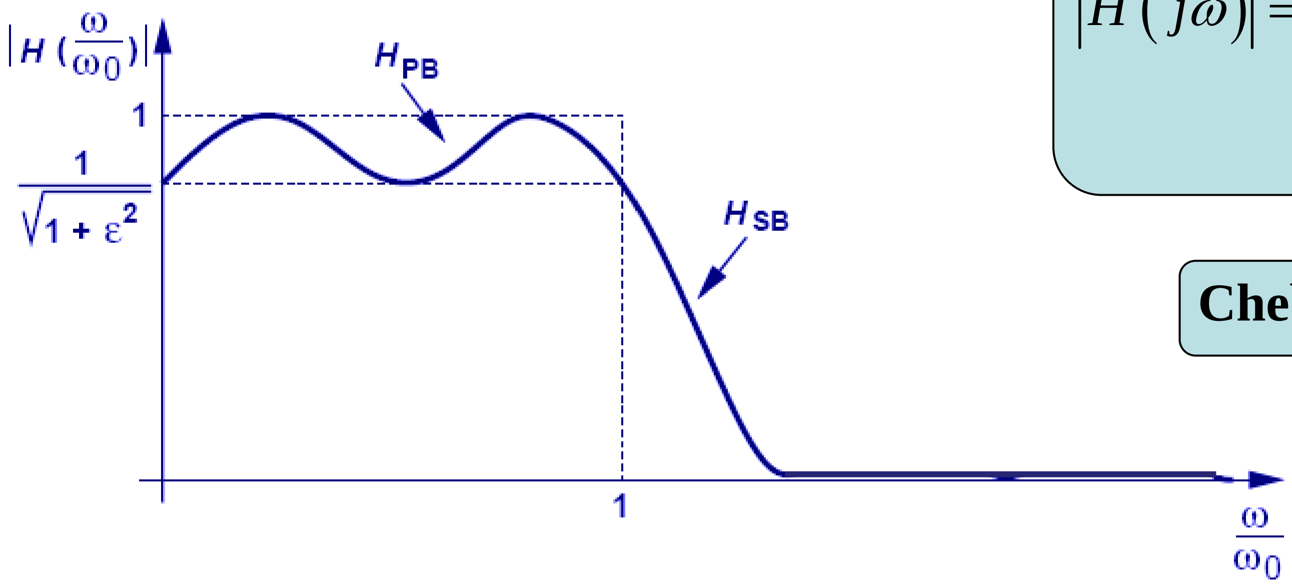
$$\omega_n = 2\pi \times (1.45\text{MHz}) \text{ and } Q = 1/\left(2\cos\frac{2\pi}{3}\right) = 1 \rightarrow \qquad \because \omega_n = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}$$

$$\because Q = \frac{1}{R_1 + R_2}\sqrt{R_1 R_2 \frac{C_1}{C_2}} \qquad R_1 = R_2 = 1\text{k}\Omega,\ C_2 = 54.9\text{pF},\ \text{and } C_1 = 4C_2$$

$$\frac{1}{R_3 C_3} = 2\pi \times (1.45\text{MHz}) \rightarrow R_3 = 1\text{k}\Omega \text{ and } C_3 = 109.8\text{pF}$$

$$\omega_n = \frac{1}{\sqrt{4R_1^2 C_1^2}}$$

Chebyshev Response



$$|H(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 C_n^2\left(\frac{\omega}{\omega_0}\right)}}$$

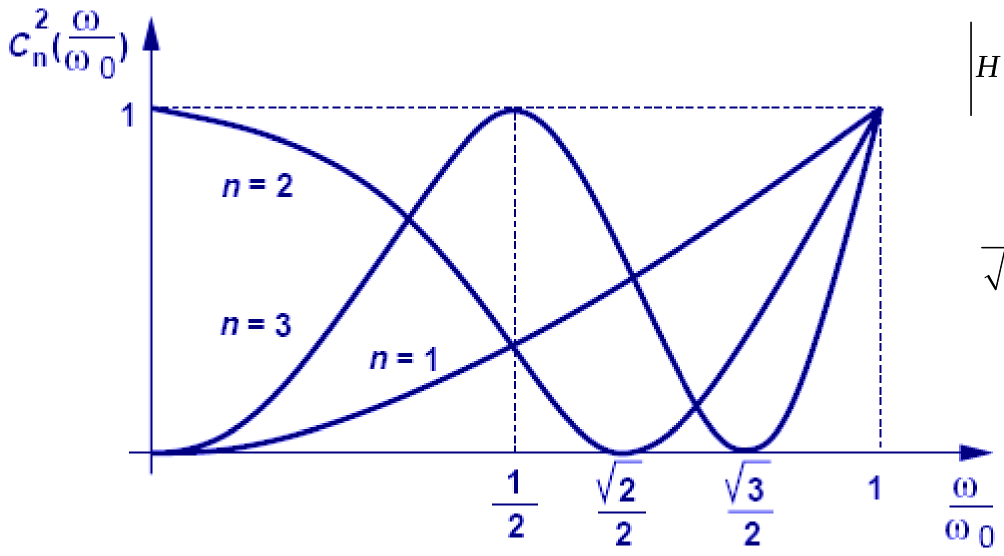
Chebyshev Polynomial

ε : the amount of ripple

$C_n^2(\omega/\omega_0)$: the "Chebyshev polynomial" of nth order

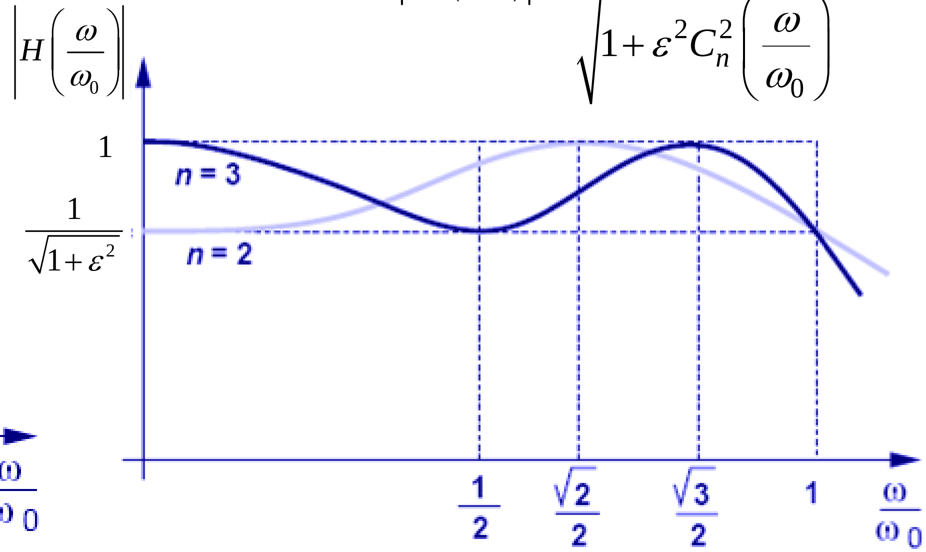
- **The Chebyshev response provides an "equiripple" pass/stop band response.**

Chebyshev Polynomial



Chebyshev polynomial for n=1,2,3

$$|H(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 C_n^2\left(\frac{\omega}{\omega_0}\right)}}$$

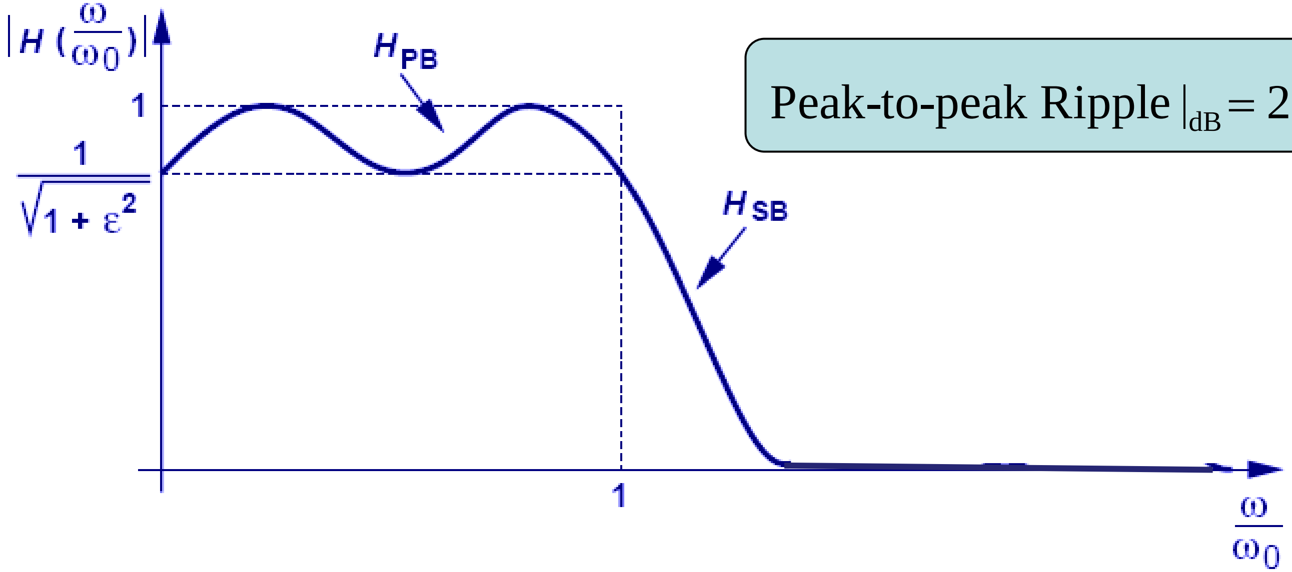


Resulting transfer function for n=2,3

$$C_n\left(\frac{\omega}{\omega_0}\right) = \cos\left(n\cos^{-1}\frac{\omega}{\omega_0}\right), \quad \omega < \omega_0$$

$$= \cosh\left(n\cosh^{-1}\frac{\omega}{\omega_0}\right), \quad \omega > \omega_0$$

Chebyshev Response

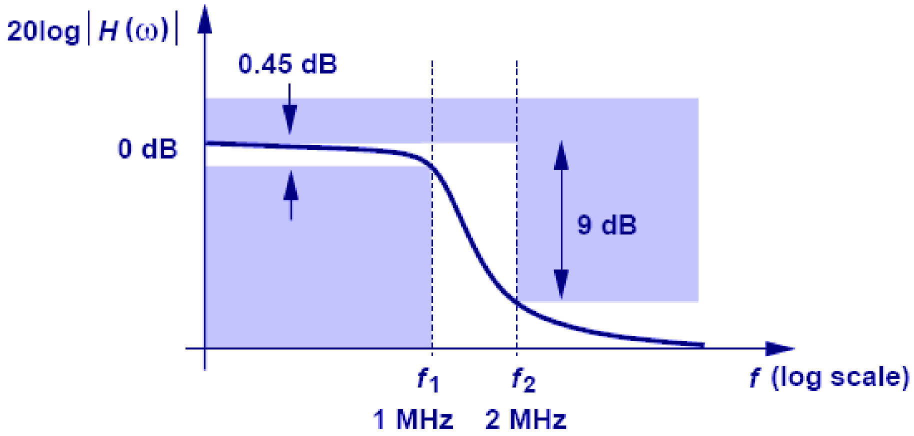


$$\text{Peak-to-peak Ripple}\,|_{\text{dB}} = 20\log\sqrt{1+\varepsilon^2}$$

$$|H_{PB}(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2\cos^2\left(n\cos^{-1}\dfrac{\omega}{\omega_0}\right)}}$$

$$|H_{SB}(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2\cosh^2\left(n\cosh^{-1}\dfrac{\omega}{\omega_0}\right)}}$$

Example 14.26: Chebyshev Response



Suppose the filter required in Example 14.24 is realized with third-order Chebyshev response. Determine the attenuation at 2MHz.

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.95 \rightarrow \varepsilon = 0.329$$

$$\omega_0 = 2\pi\ (1\text{MHz})$$

$$|H(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2\left[4\left(\frac{\omega}{\omega_0}\right)^3 - 3\frac{\omega}{\omega_0}\right]^2}}$$

$$|H(j2\pi(2\text{MHz}))| = 0.116 = -18.7\text{dB}$$

- A third-order Chebyshev response provides an attenuation of -18.7 dB a 2MHz.

Example 14.27: Order of Chebyshev Filter

Specification:

Passband ripple: 1 dB

Bandwidth: 5 MHz

Attenuation at 10 MHz: 30 dB

What's the order?

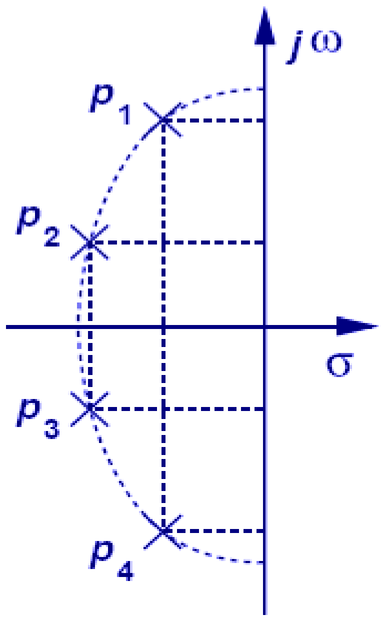
$$1\text{ dB} = 20\log\sqrt{1+\varepsilon^2} \rightarrow \varepsilon = 0.509$$

Attenuation at $\omega = 2\omega_0 = 10$ MHz: 30 dB

$$\frac{1}{\sqrt{1+0.509^2\cosh^2(n\cosh^{-1}2)}} = 0.0316$$

$$\cosh^2(1.317n) = 3862 \rightarrow n > 3.66 \rightarrow n = 4$$

Example 14.28: Chebyshev Filter Design



Using two SK stages, design a filter that satisfies the requirements in Example 14.27.

$$p_k = -\omega_0 \sin\frac{(2k-1)\pi}{2n}\sinh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right) + j\omega_0 \cos\frac{(2k-1)\pi}{2n}\cosh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right)$$

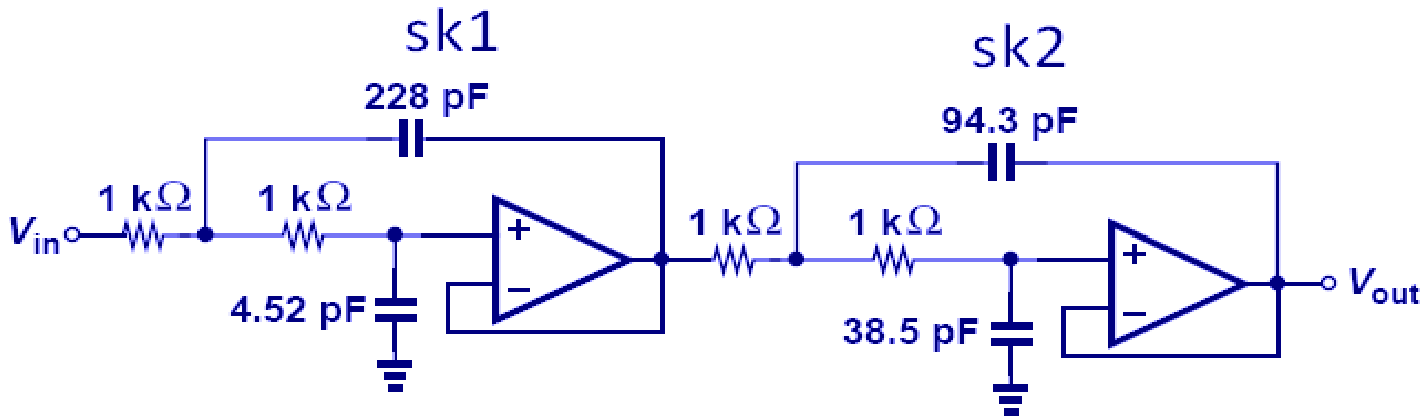
$$p_{1,4} = -0.140\omega_0 \pm 0.983\, j\omega_0$$

SK1

$$p_{2,3} = -0.337\omega_0 \pm 0.407\, j\omega_0$$

SK2

Example 14.28: Chebyshev Filter Design (cont'd)



$$H_{SK1}(s)=\frac{(-p_1)(-p_4)}{(s-p_1)(s-p_4)}$$
$$=\frac{0.986\omega_0^2}{s^2+0.28\omega_0 s+0.986\omega_0^2}$$

$$H_{SK2}(s)=\frac{(-p_2)(-p_3)}{(s-p_2)(s-p_3)}$$
$$=\frac{0.279\omega_0^2}{s^2+0.674\omega_0 s+0.279\omega_0^2}$$

$$\omega_{n1}=0.993\omega_0=2\pi\times(4.965\text{MHz})$$
$$Q_1=3.55$$

$$\omega_{n2}=0.528\omega_0=2\pi\times(2.64\text{MHz})$$
$$Q_2=0.783.$$

$$R_1=R_2=1\text{ k}\Omega,\ C_1=50.4C_2$$
$$\frac{1}{\sqrt{50.4}R_1C_2}=2\pi\times(4.965\text{MHz})$$
$$\rightarrow C_2=4.52\text{ pF},\ C_1=227.8\text{ pF}$$

$$R_1=R_2=1\text{ k}\Omega,\ C_1=2.45C_2$$
$$\frac{1}{\sqrt{2.45}R_1C_2}=2\pi\times(2.64\text{ MHz})$$
$$\rightarrow C_2=38.5\text{ pF},\ C_1=94.3\text{ pF}$$

Summary

- The desired filter response must in practice be approximated by a realizable transfer function. Possible transfer functions include Butterworth and Chebyshev responses.
- The Butterworth response contains n complex poles on a circle and exhibits a maximally-flat behavior. It is suited to applications that are intolerant of any ripple in the passband
- The Chebyshev response provides a sharper transition than Butterworth at the cost of some ripple in the passband and stopbands. It contains n complex poles on an ellipse.