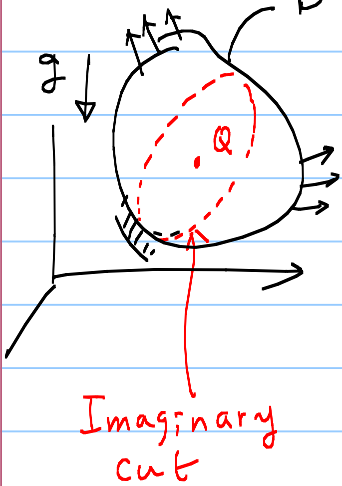
Lecture 1-4 Stress and Equilibrium Equations

노트 제목



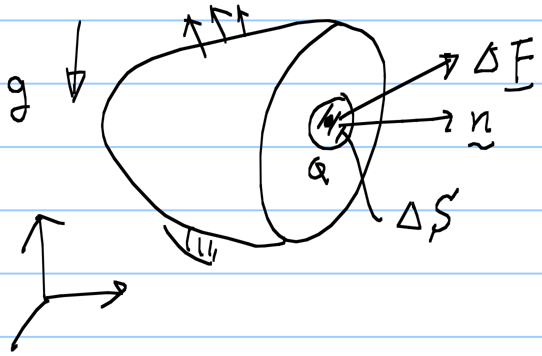
B (elastic Body)

under force

Body force : gravity
magnetic force
(no physical contact)

Surface or Contact force
(physical contact)

Now consider a imaginary cut around Q



$$\boxed{\lim_{\Delta S \to 0} \frac{\Delta \underset{\sim}{F}}{\Delta S} \triangleq \underset{\sim}{t}^{(n)}}$$

"traction vector"

※ At same Q, depending on $\underset{\sim}{n}$
$\underline{t}^{(n)}$ will differ!

Cauchy says:

$$\boxed{\begin{array}{l} \underline{t}^{(n)} \text{ at } Q \text{ for any } \underline{n} \text{ can be determined if } \underline{t}^{(1)}, \underline{t}^{(2)}, \underline{t}^{(3)} \\ \text{are Known.} \\ \left(\underset{\sim}{t}^{(i)} = \text{traction for } \underline{n} = \underline{e}_i \right) \end{array}}$$

* $\underline{t}^{(i)}$ is a vector → Let's write $\underline{t}^{(i)}$ as

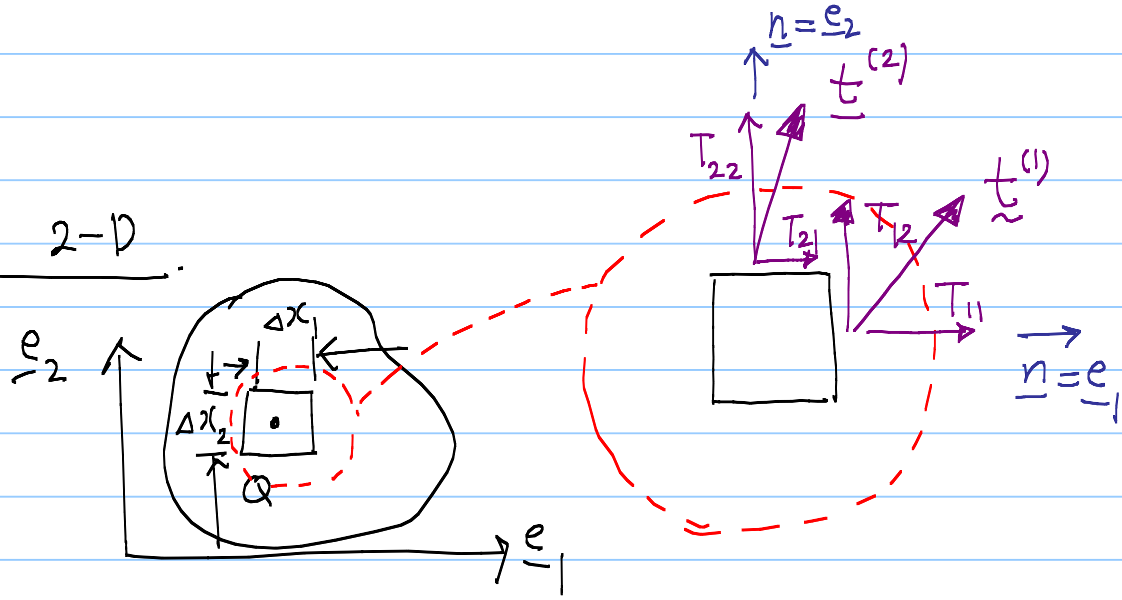
$$\begin{cases} \underline{t}^{(1)} \triangleq T_{11}\,\underline{e}_1 + T_{12}\,\underline{e}_2 + T_{13}\,\underline{e}_3 \\ \underline{t}^{(2)} \triangleq T_{21}\,\underline{e}_1 + T_{22}\,\underline{e}_2 + T_{23}\,\underline{e}_3 \\ \underline{t}^{(3)} \triangleq T_{31}\,\underline{e}_1 + T_{32}\,\underline{e}_2 + T_{33}\,\underline{e}_3 \end{cases}$$

or

$$\underline{t}^{(i)} \triangleq T_{ik}\,\underline{e}_k$$

Meaning of T_{ik} (and $\underset{\sim}{t}^{(i)}$)

in 2-D

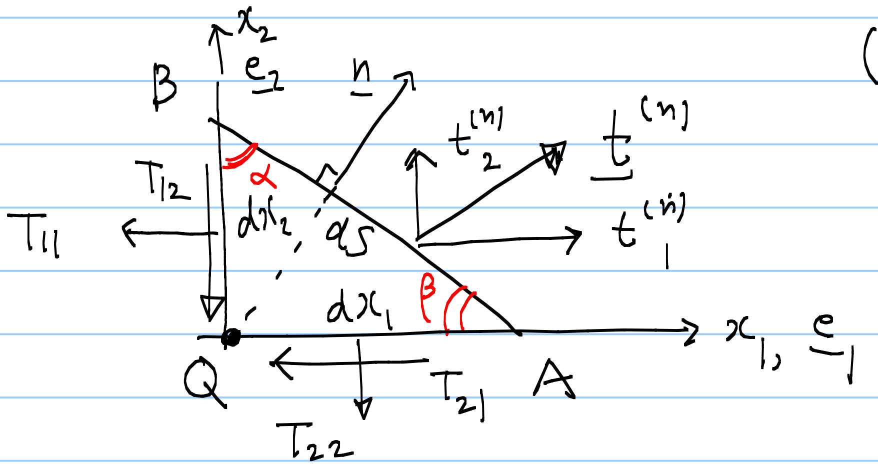


☆☆ Cauchy shows

$$\boxed{\begin{aligned} \underset{\sim}{t}^{(n)} &= \underset{\sim}{n} \cdot \underset{\sim}{T} \quad (\text{for any } \underset{\sim}{n}) \\ \underset{\sim}{T} &= \underbrace{T_{ij}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j}_{\text{"Cauchy Stress Tensor"}} \end{aligned}}$$

$\Rightarrow$ It says, "if you know 9 T_{ij}'s (actually 6 because $T_{ij} = T_{ji}$ as shall be shown), you know traction $\underline{t}^{(n)}$ for any $\underline{n}$"

2-D version of Cauchy's proof



(In the limit of Δx_i)

(1) Geometric Relation

$$\underset{\sim}{n} = (n_1, n_2)^T, \quad n_1 = \cos\alpha, \quad n_2 = \cos\beta$$

$$\begin{cases} dx_1 = ds\cos\beta = n_2\,ds \\ dx_2 = ds\cos\alpha = n_1\,ds \end{cases}$$

(2) Equilibrium (as $\delta x_i \to 0$)

$\sum F_1 = 0$:

$$t_1^{(n)} ds = T_{11}\, dx_2 + T_{21}\, dx_1$$
$$= T_{11}\, n_1\, ds + T_{21}\, n_2\, ds$$
$$= n_1 T_{11}\, ds + n_2 T_{21}\, ds$$

$$\hookrightarrow t_1^{(n)} = n_1 T_{11} + n_2 T_{21}$$

$\sum F_2 = 0$;

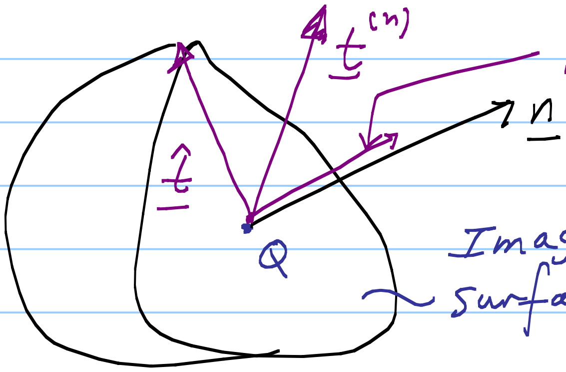
$$\hookrightarrow t_2^{(n)} = n_1 T_{12} + n_2 T_{22}$$

$$\Rightarrow \quad t_i^{(n)} = n_j T_{ji}$$

$$t_i^{(n)} \underset{\sim}{e}_i = n_j T_{ji} \underline{e}_i$$
$$= n_k \underbrace{\delta_{kj}}_{\underline{e}_k \cdot \underline{e}_j} T_{ji} \underline{e}_i$$
$$= (n_k \underline{e}_k) \cdot (T_{ji} \underline{e}_j \otimes \underline{e}_i)$$

i.e. $\underline{t}^{(n)} = \underline{n} \cdot \underset{\sim}{T}$ //

Normal and Tangential Components



Normal component:

$\underline{t}^{(n)} \cdot \underline{n} = \underline{n} \cdot \underset{\sim}{T} \cdot \underset{\sim}{n}$

Imaginary surface S

(In 3-D; $\hat{\underset{\sim}{t}}$ be any unit vector lying on S)

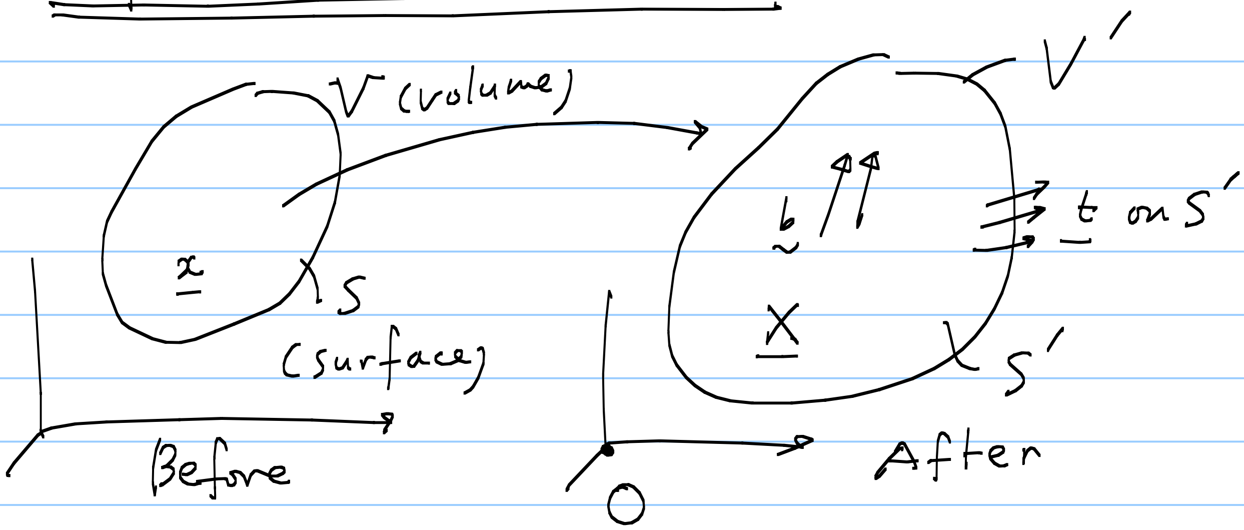
Remark: $\underline{t}^{(n)} \cdot \underset{\sim}{m}$ = Component of traction $\underline{t}^{(n)}$ in the direction of $\underset{\sim}{m}$

$$\boxed{\underline{t}^{(n)} \cdot \underline{m} = \underline{n} \cdot \underset{\sim}{T} \cdot \underset{\sim}{m} = T_{nm}}$$

for any $\underline{n}, \underset{\sim}{m}$ $\|\underline{n}\| = \|\underline{m}\| = 1$.

(Simplest Case: $T_{12} = \underline{t}^{(1)} \cdot \underline{e}_2$)

Equilibrium Equation



① Force equilibrium

$$\int_{S'} \underline{t}\, dS + \int_{V'} \underline{b}\, dV = 0$$

$$\int_{S'} \underline{n} \cdot \underset{\sim}{T}\, dS = \int_{V'} \underline{\nabla}_X \cdot \underset{\sim}{T}\, dV$$

Div. Th (1.1.49b)

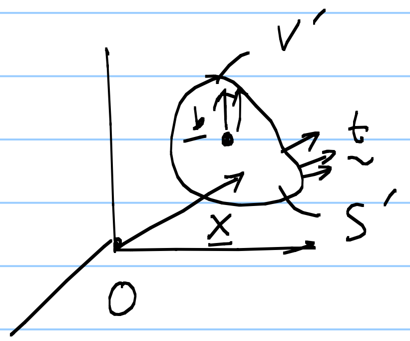
Based on Deformed State

$$\therefore \int_{V'} (\underline{\nabla}_X \cdot \underset{\sim}{T} + \underset{\sim}{b})\, dV = 0$$

$$\Rightarrow \quad \underline{\nabla}_X \cdot \underset{\sim}{T} + \underset{\sim}{b} = \underline{0}$$

$$\left(T_{ij,i} + b_j = 0 \quad \text{wrt } \underline{X} \right)$$

② Moment Equilibrium



$$\int_{S'} \underset{\sim}{X} \times \underset{\sim}{t}\, dS + \int_{V'} \underline{X} \times \underline{b}\, dV = 0 \qquad (a)$$

Note

$$\int_{S'} \underline{X} \times \underline{t}\, dS = \int_{S'} \underline{X} \times \underbrace{(\underline{n} \cdot \underset{\sim}{T})}_{\text{vector}} dS$$

$$= -\int_{V'} \underbrace{(n \cdot \underset{\sim}{T}) \times \underset{\sim}{X}}_{\underline{n} \cdot (\underset{\sim}{T} \times \underset{\sim}{X})} dS$$

$$\underset{\text{Div}}{=} -\int_{V'} \underline{\nabla}_X \cdot (\underset{\sim}{T} \times \underline{X})\, dV$$

$$= \int_{V'} \left[\underset{\sim}{X} \times (\nabla_X \cdot \underset{\sim}{T}) + \underline{e} : \underset{\sim}{T}\right] dV \quad (6)$$

(To show the last equation, let's use the Cartesian tensor;

$$\nabla_X \cdot (\underline{T} \times \underset{\sim}{X})$$

$$= \underline{e}_i \cdot \frac{\partial}{\partial X_i} (\underline{T} \times \underline{X})$$

$$= \underbrace{e_i \cdot \frac{\partial T}{\partial X_i}}_{\underline{\nabla}_X \cdot \underset{\sim}{T}} \times \underset{\sim}{X} + \underbrace{\underline{e}_i \cdot \underset{\sim}{T}}_{\underline{e}_i \cdot T_{jk} e_j \otimes \underline{e}_k \,=\, T_{ik} \underline{e}_k} \times \frac{\partial X}{\partial X_i} \rightarrow \underline{e}_i$$

using $\underline{e}_k \times \underline{e}_i = e_{kim} \underline{e}_m$

$$= (\underline{\nabla}_x \cdot \underset{\sim}{T}) \times \underset{\sim}{X} + T_{ik} \underbrace{e_{kim}}_{} \underline{e}_m$$

$$- T_{ik}\, e_{ikm}\, \underline{e}_m$$

$$\underbrace{\overset{\shortparallel}{e_{mik}}}_{}$$

$$- \underline{e}_m \underbrace{e_{mik} T_{ik}}_{}$$

$$- \underline{e} : \underset{\sim}{T}$$

$$= - \underset{\sim}{X} \times (\underline{\nabla}_x \cdot \underset{\sim}{T}) - \underline{e} : \underset{\sim}{T}$$

(b) $\rightarrow$ (a) :

$$\int_{V'} \underset{\sim}{X} \times \big(\underbrace{\underline{\nabla}_x \cdot \underset{\sim}{T} + \underline{b}}_{\overset{\shortparallel}{0}\ \text{by Force eqm}} \big)\, dV + \int_{V'} \underline{e} : \underset{\sim}{T}\, dV = 0$$

$$\therefore\ \underline{e} : \underset{\sim}{T} = 0$$

or $\quad e_{mik} T_{ik} = 0$

$$\Rightarrow \underbrace{\tfrac{1}{2}\left(e_{mik} - e_{mki} \right)}_{e_{mik}} T_{ik} = 0$$

$$\therefore \quad e_{mik} T_{ik} - \underbrace{e_{mki} T_{ik}}_{} = 0$$

$$\underset{i \leftrightarrow k}{=} e_{mik} T_{ki}$$

$$\Rightarrow e_{mik} \left(T_{ik} - T_{ki} \right) = 0$$

Thus $\quad T_{ik} = T_{ki} \iff \underset{\sim}{T} = \underset{\sim}{T}^T$

Summary: $\quad \underset{\sim}{\nabla}_x \cdot \underset{\sim}{T} + \underset{\sim}{b} = 0$

$$\underset{\sim}{T} = \underset{\sim}{T}^T$$

* applies to nonlinear deformation
* Eqm defined in deformed state,

Remark: if Stress Tensor is defined in the undeformed coord, it is Piola-Kirchhoff Stress Tensor $\underset{\sim}{S}$ (1st & 2nd)

(Eqm Eq: more complex in PK).

	Strain		Stress
Undeformed	E (Lagrangian)	$\xleftrightarrow{\text{Const. Rel.}}$	T Cauchy
Deformed	E^* (Eulerian)	$\xleftrightarrow{\text{Const. Rel}}$	S P-K

For small deformation $\left(\frac{\partial}{\partial x_i} \approx \frac{\partial}{\partial X_i} \right)$

$$\underset{\sim}{E} \approx \underset{\sim}{E}^* \longrightarrow \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{T} \approx \underset{\sim}{S} \longrightarrow \underset{\sim}{\sigma}$$

☑ Thus for linear Elasticity, the following equilibrium equation is used

$$\boxed{\begin{aligned} &\left\{\begin{aligned} &\nabla \cdot \underset{\sim}{\sigma} + \underset{\sim}{b} = 0 \qquad (\nabla = \text{wrt } \underset{\sim}{x} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \uparrow \\ &\qquad\qquad\qquad\qquad\qquad\quad \text{more convenient}) \\ &\frac{\partial \sigma_{ji}}{\partial x_i} + b_j = 0 \\ &\qquad\qquad (j = 1, 2, 3) \end{aligned}\right. \\ &\oplus\ \underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T \end{aligned}}$$

for proof (or plasticity problems)

Sometimes, useful to use

$$\begin{cases} \text{deviatoric stress tensor } \underset{\sim}{\sigma}' & (\text{"traceless"}) \\ \quad " \quad \text{strain} \quad " \quad \underset{\sim}{\varepsilon}' & (\text{"traceless"}) \end{cases}$$

Such that

$$(*) \quad \underset{\sim}{\sigma}' = \underset{\sim}{\sigma} \underbrace{- \frac{1}{3} \operatorname{tr} \underset{\sim}{\sigma}}_{\equiv p} \underset{\sim}{\mathbb{1}}$$

$$\left(\operatorname{tr} \underset{\sim}{\sigma}' = \operatorname{tr} \underset{\sim}{\sigma} - \frac{1}{3} \operatorname{tr} \underset{\sim}{\sigma} \operatorname{tr} \underset{\sim}{\mathbb{1}} = 0 \right)$$

Then

$$\boxed{\underset{\sim}{\sigma}' = \underset{\sim}{\sigma} + p \underset{\sim}{\mathbb{1}} \quad \text{or} \quad \underset{\sim}{\sigma} = \underset{\sim}{\sigma}' - p \underset{\sim}{\mathbb{1}}} \quad (\alpha)$$

no net mean pressure

$$\Updownarrow$$

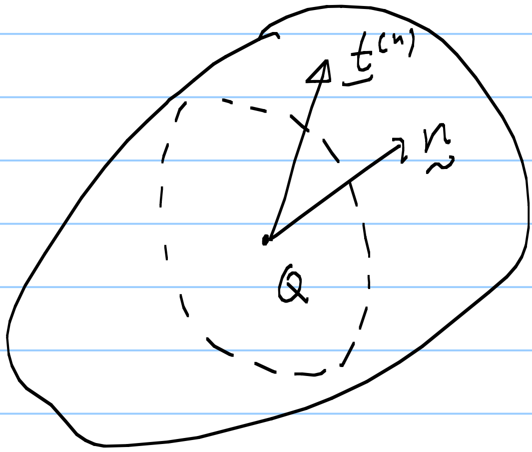
$$\boxed{\underset{\sim}{\varepsilon}' = \underset{\sim}{\varepsilon} - \frac{1}{3} \operatorname{tr} \underset{\sim}{\varepsilon} \, \underset{\sim}{\mathbb{1}} \quad \text{or} \quad \underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}' + \frac{1}{3} e \, \underset{\sim}{\mathbb{1}}} \quad (\beta)$$

no net volume change

$e \Rightarrow$ volume change rate

(see problem 1.2-4)

Principal Stress and Direction



for some $\underset{\sim}{n}$ at Q

$\underset{\sim}{t}^{(n)} \parallel \underset{\sim}{n}$

$\Rightarrow$ "$\underset{\sim}{n}$" : principal Direction

Thus, we have

$$\begin{cases} \underset{\sim}{t}^{(n)} = \lambda \underset{\sim}{n} \rightarrow \underset{\sim}{n} \cdot \underset{\sim}{\sigma} = \lambda \underset{\sim}{n} \\ \underset{\sim}{\sigma}^T \cdot \underset{\sim}{n} = \lambda \underset{\sim}{n} \rightarrow \underbrace{\underset{\sim}{\sigma} \cdot \underset{\sim}{n} = \lambda \underset{\sim}{n}}_{\text{"eigenvalue problem"}} \end{cases}$$

For nontrivial sol;

$$|\sigma_{ij} - \lambda \delta_{ij}| = 0$$

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0$$

$$\begin{cases} I_1 = \operatorname{tr} \underset{\sim}{\sigma} \\ I_2 = \frac{1}{2}\left[I_1^{\,2} - \underbrace{\underset{\sim}{\sigma} : \underset{\sim}{\sigma}}_{\operatorname{tr} \underline{\sigma}^T \underline{\sigma}} \right] \\ I_3 = \det \underline{\sigma} \end{cases}$$

if $\begin{cases} \text{principal stress; } \sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)} \\ \quad \text{〃} \quad \text{direction; } \underline{n}^{(1)}, \underline{n}^{(2)}, \underline{n}^{(3)} \end{cases}$

then

$$\underline{\sigma} = \sum_{i=1}^{3} \sigma^{(i)} \underline{n}^{(i)} \otimes \underline{n}^{(i)}$$

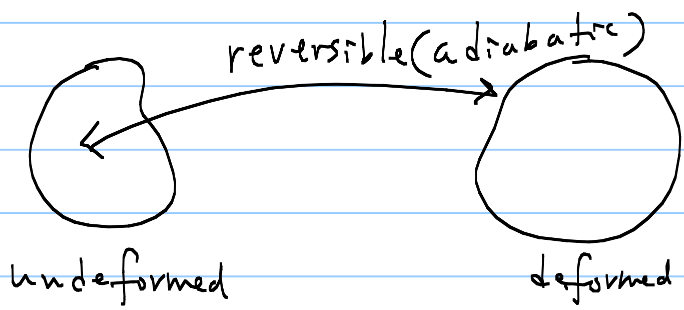
$$I_1 = \sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)}$$

$$I_2 = \sigma^{(1)}\sigma^{(2)} + \sigma^{(2)}\sigma^{(3)} + \sigma^{(3)}\sigma^{(1)}$$

$$I_3 = \sigma^{(1)}\sigma^{(2)}\sigma^{(3)}$$

Constitutive Relation

(Relation between $\underline{\varepsilon}$ and $\underline{\sigma}$
(This is <u>material</u> property.



{ External Work Input ($W_{external}$)
{ Internal Energy Stored (U_{system})

Thus we will show

$$dW_{ext} = dU_{sys}$$ ← "Internal strain energy"

Where

$$dW_{ext} = \int_V (\underline{b}\, dV) \cdot d\underline{u} + \int_S (\underline{t}\, ds) \cdot d\underline{u}$$

(a)

Note: $\int_S \underline{t} \cdot d\underline{u}\, dS$

$$= \int_S \underline{n} \cdot \left(\underline{\sigma} \cdot d\underline{u}\right) dS$$

$$\underset{\text{Div.}}{=} \int_V \underline{\nabla} \cdot (\underline{\sigma} \cdot d\underline{u})\, dV$$

$$= \int_V \left(\underline{\nabla} \cdot \underline{\sigma} \cdot d\underline{u} + \underline{\sigma} : \underline{\nabla} d\underline{u} \right) dV \qquad (6)$$

(Let's check the last result:

$$\underline{e}_i \frac{\partial}{\partial x_i} \cdot \Big(\underbrace{\sigma_{jk}\, \underline{e}_j \otimes \underline{e}_k \cdot du_m\, \underline{e}_m}_{\sigma_{jk}\, du_k\, \underline{e}_j} \Big) \qquad \delta_{mk}$$

$$= \left(\sigma_{jk,i}\, du_k + \sigma_{jk}\, du_{k,i} \right) \delta_{ij}$$

$$= \sigma_{jk,j}\, du_k + \sigma_{jk}\, du_{k,j}$$

Sym Tensor

$$= (\underline{\nabla}\cdot\underline{\sigma})\cdot d\underline{u} + \underline{\sigma} : \underline{\nabla}\, d\underline{u} \quad)$$

sym

Using $\underline{\sigma} : \nabla d\underset{\sim}{u} = \underline{\sigma} : d\,\underline{\nabla}\underline{u}$

$$= \underset{\sim}{\sigma} : d\,\underset{\sim}{\varepsilon} \qquad (c)$$

↑ only sym part

[Using $\underset{\sim}{T}^{sym} : \underset{\sim}{S}$

$$= \underset{\sim}{T}^{sym} : (\underset{\sim}{S}^{sym} + \underset{\sim}{S}^{anti})$$

$$= \underset{\sim}{T}^{sym} : \underset{\sim}{S}^{sym}$$

$$\Big(\because \quad \underline{T}^{sym} : \underline{S}^{antisym} = T^{sym}_{ij} S^{anti}_{ij}$$

$$\underset{(i\leftrightarrow j)}{=} T^{sym}_{ji} S^{anti}_{ji} = - T^{sym}_{ji} S^{anti}_{ij}$$

$$= - T^{sym}_{ij} S^{anti}_{ij}$$

$$2\, T^{sym}_{ij} S^{anti}_{ij} = 0 \quad \Big) \quad]$$

Using (a, b, c)

$$dW_{ext} = \int_V \underline{b} \cdot d\underline{u}\, dV + \int_S \underline{t} \cdot d\underline{u}\, dS$$

$$= \int_V (\underbrace{\cancel{\underline{b} + \nabla \cdot \underline{\sigma}}}_{\text{by eqm}}) \cdot d\underline{u}\, dV + \int_V \underline{\sigma} : d\underline{\varepsilon}\, dV$$

Let $dU_{sym} = \int_V \underline{\sigma} : d\underline{\varepsilon}\, dV$

Then $\underline{dW_{ext} = dU_{sym}}$

<u>For reversible deforming process;</u>

U_{sym} = path Independent
(loading-process Independent)
= only depends on finial deformed state!

$$dU_{sym} = \int \underline{\sigma} : d\underline{\varepsilon}\, dV$$

$$= \int_V dU(\underline{\varepsilon})\, dV$$

only depends on $\underline{\varepsilon}$

U : Strain energy density
(defined on unit volume)

In this case

$$dU(\varepsilon) = \underline{\sigma} : d\underline{\varepsilon}$$

$$\Longleftrightarrow \quad \underset{\sim}{\sigma} = \frac{\partial U}{\partial \underset{\sim}{\varepsilon}} \quad \left(\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} \right)$$

Because U is ftn of ε_{ij}, use the Taylor expansion for small ε_{ij}

$$U(\varepsilon_{ij}) = U_0 + B_{ij}\varepsilon_{ij} + \frac{1}{2} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$$

choose $=0$

$$+ O(\varepsilon_{ij}^3)$$

$$U \approx \underset{\sim}{B} : \underset{\sim}{\varepsilon} + \frac{1}{2} \underline{\varepsilon} : \underset{\sim}{C} : \underline{\varepsilon} \qquad (A)$$

$$\sigma_{ij} = B_{ij} + C_{ijkl}\,\varepsilon_{kl} \qquad (B)$$

pre-stress
(Let $B_{ij} = 0$ for simplicity)

Elasticity Tensor denoting elastic coefficient (4th-order Tensor)

$$\underset{\sim}{C} = C_{ijkl}\, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l$$

$$\Rightarrow \begin{cases} \underset{\sim}{\sigma} = \underline{C} : \underset{\sim}{\varepsilon} \\ \sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \end{cases}$$

$3 \times 3 \times 3 \times 3 = 81$ coefficients
but only 21 independent.

$\boxed{\text{Constitutive Relation}}$

Basic Symmetry in C_{ijkl}

① $C_{ijkl} = C_{klij}$

$$C_{ijkl} \underset{(A)}{=} \frac{\partial^2 U}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = \frac{\partial^2 U}{\partial \epsilon_{kl} \partial \varepsilon_{ij}} = C_{klij}$$

② $C_{ijkl} = C_{jikl}$

$$\sigma_{ij} = C_{ijkl} \, \varepsilon_{kl}$$

$$\sigma_{ji} = C_{jikl} \, \varepsilon_{kl}$$

Since $\sigma_{ij} = \sigma_{ji} \rightarrow C_{ijkl} = C_{jikl}$

③ $C_{ijkl} = C_{ijlk}$

$$\sigma_{ij} = C_{ijkl} \, \varepsilon_{kl} \quad \text{— } (\alpha)$$

$$= C_{ijkl} \, \varepsilon_{lk}$$

$$\underset{k \leftrightarrow j}{=} C_{ijlk} \, \varepsilon_{kl} \quad \text{— } (\beta)$$

$\therefore C_{ijkl} = C_{ijlk}$

Thus, it is convenient to express the constitutive relation ($\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$) as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1113} & C_{1123} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2213} & C_{2223} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3313} & C_{3323} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1213} & C_{1323} \\ C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1313} & C_{1323} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2313} & C_{2323} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{Bmatrix}$$

$\Uparrow$

stiffness matrix.

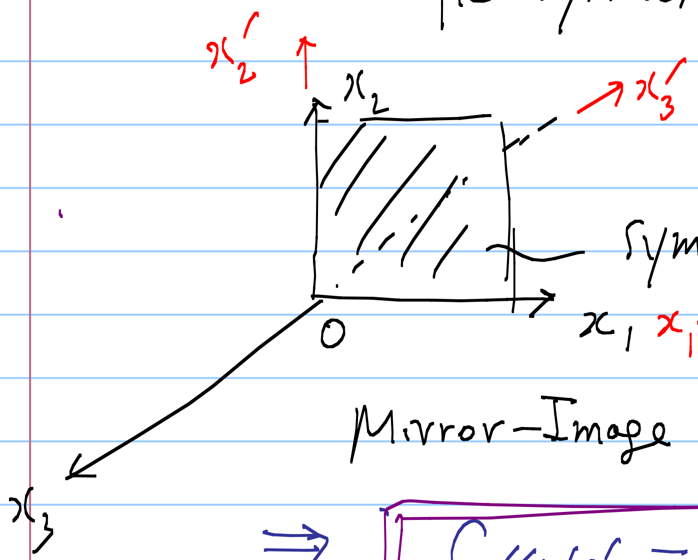
symmetric matrix due to 1

Indep coeff: $1+2+\cdots+6 = \frac{6\times 7}{2} = 21$

Most Engineering Materials have more Symmetries in material responses

- i) Monoclinic (single Symmetric plane)
- ii) orthotropic (Two " ") $\Leftrightarrow$ same as Three " ")
- iii) Isotropic (2 indep.: E, ν)

i) Monoclinic Material
(Single Symmetric plane)



Mirror-Image Material response

$$\Rightarrow \boxed{\begin{array}{l} C_{i'j'k'l'} = C_{ijkl} \\ \text{for the Coord. Sys shown in fig.} \end{array}} \quad (*)$$

OLD $\qquad$ New

$(\underline{e}_1, \underline{e}_2, \underline{e}_3) \Rightarrow (\underline{e}_1' = \underline{e}_1, \ \underline{e}_2' = \underline{e}_2, \ \underline{e}_3' = -\underline{e}_3)$

Thus $\quad \beta_{11'} = \beta_{1'1} = 1, \quad \beta_{22'} = \beta_{2'2} = 1$

$\beta_{33'} = \beta_{3'3} = -1, \quad$ other $\beta_{ij} = 0$

(Recall $\beta_{ij'} = \beta_{j'i} = \underline{e}_{i'} \cdot \underline{e}_j$)

Before using (*),
let's transform C_{ijkl} to $C_{i'j'k'l'}$ by
coordinate transformation rule:

$$C_{i'j'k'l'} = \beta_{i'i}\,\beta_{j'j}\,\beta_{k'k}\,\beta_{l'l}\,C_{ijkl}$$

$$C_{1111} = \beta_{1'i}\,\beta_{1'j}\,\beta_{1'k}\,\beta_{1'l}\,C_{ijkl}$$

$$= C_{1111}$$

$$\vdots$$

$$C_{1'1'3'3'} = \beta_{1'i}\,\beta_{1'j}\,\beta_{3'k}\,\beta_{3'l}\,C_{ijkl}$$

$$= \underset{(1)}{\beta_{1'1}}\,\underset{(1)}{\beta_{1'1}}\,\underbrace{\beta_{3'3}\,\beta_{3'3}}_{(-1)^2}\,C_{1133}$$

$$= C_{1133}$$

Single apperance of (3')

$$C_{1'1'2'3'} = \beta_{1'i}\,\beta_{1'j}\,\beta_{2'k}\,\beta_{3'l}\,C_{ijkl}$$

$$= \beta_{1'1}\,\beta_{1'1}\,\beta_{2'2}\,\underset{-1}{\beta_{3'3}}\,C_{1123}$$

$$= -C_{1123} \qquad -(*)$$

Because

$$C_{i'j'k'l'} \equiv C_{ijkl}$$

under the given Coord. Transf.

$$C_{1'1'2'3'} \equiv C_{1123} \qquad (**)$$

We must have

$$C_{1123} = 0$$

↑ single appearance of 3

Thus we have

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & \cancel{C_{1113}} & \cancel{C_{1123}} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & \cancel{C_{2213}} & \cancel{C_{2223}} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & \cancel{C_{3313}} & \cancel{C_{3323}} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & \cancel{C_{1213}} & \cancel{C_{1223}} \\ \cancel{C_{1311}} & \cancel{C_{1322}} & \cancel{C_{1333}} & \cancel{C_{1312}} & C_{1313} & C_{1323} \\ \cancel{C_{2311}} & \cancel{C_{2322}} & \cancel{C_{2333}} & \cancel{C_{2312}} & C_{2313} & C_{2323} \end{bmatrix} \begin{Bmatrix} \epsilon_{12} \\ \epsilon_{33} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{Bmatrix}$$

$\Rightarrow$ Only 13 Indepedent Coeff.

ii) Orthotropic case

$\Rightarrow$ two planes of symmetry

Add one more symmetry plane to the Monoclinic material case,

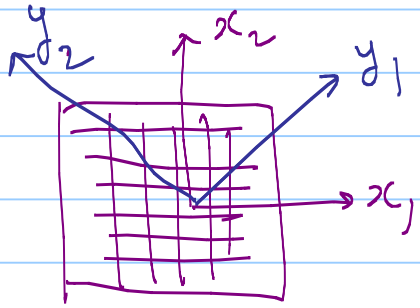
$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{1122} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{1133} & C_{2233} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{2323} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{Bmatrix}$$

* Observation:

① 9 Indep coefficient

③ has three symmetry planes

* ③ No coupling between Strech and shear (along the material axis)

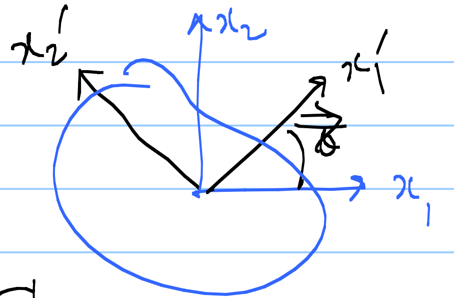


(x_1, x_2) material axis

* for (y_1, y_2),
$\Rightarrow$ ③ no longer holds.

iii) Isotropic

"Any" orthogonal axes of any orientation material axes



same material behavior for any value of θ

Show

$$
\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix}
=
\begin{bmatrix}
\lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{bmatrix}
\begin{Bmatrix} \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{Bmatrix}
$$

$\Rightarrow$ 2 Indep coeff

$$C_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu\, (\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk})$$

λ, μ : Lamé constant
($\mu = G$; sheer modulus)

Thu for Isotropic case, we have

$$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$$
$$= (\lambda\, \delta_{ij}\, \delta_{kl} + \mu\, \delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk})\, \varepsilon_{kl}$$
$$= \lambda\, \underbrace{\varepsilon_{kk}}_{\mathrm{tr}\,\underset{\sim}{\varepsilon}}\, \delta_{ij} + \mu\left(\underbrace{\delta_{ik}\, \varepsilon_{jk}}_{\varepsilon_{ji}} + \underbrace{\varepsilon_{ik}\, \delta_{jk}}_{\varepsilon_{ij}}\right)$$

$\therefore$

$$\boxed{\begin{gathered} \sigma_{ij} = \lambda\, \mathrm{tr}\,\underset{\sim}{\varepsilon}\, \delta_{ij} + 2\mu\, \varepsilon_{ij} \\ \text{or} \\ \underset{\sim}{\sigma} = \lambda\, \mathrm{tr}\,\underset{\sim}{\varepsilon}\, \underset{\sim}{\mathbb{1}} + 2\mu\, \underset{\sim}{\varepsilon} \end{gathered}}$$

or

$$\underset{\sim}{\varepsilon} = -\frac{\lambda}{2\mu(3\lambda+2\mu)} \underset{\sim}{\mathbb{1}} + \frac{1}{2\mu} \underset{\sim}{\sigma}$$

* More common to use E (Young's modulus) and λ (Poisson's ratio) ;

$$\sigma_{ij} = \frac{E}{1+\nu} \varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)} \delta_{ij} \varepsilon_{kk}$$

$$\varepsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \delta_{ij} \delta_{kk}$$

Then

$$\begin{cases} \mu = \dfrac{E}{2(1+\nu)} \\ \lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)} \end{cases}$$

Restriction on values of E and ν

$$\boxed{\text{Claim:} \quad E > 0 \quad \text{and} \quad -1 < \nu < \tfrac{1}{2}}$$

Proof:

Use: "Positive work must be given to deform an elastic body"

$$\Rightarrow \quad U = \frac{1}{2} C_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl} = \frac{1}{2} \sigma_{ij}\, \varepsilon_{ij} \geq 0$$

for any nonzero $\underline{\varepsilon}$

and $U = 0$ only when $\underline{\varepsilon} = 0$

Let's write

$$U = \frac{1}{2} \sigma_{ij}\, \varepsilon_{ij}$$

$$= \frac{1}{2} (\sigma_{ij}' - p\,\delta_{ij}) (\varepsilon_{ij}' + \frac{1}{3} e\, \delta_{ij})$$

see (α, β) of page 14

$$= \frac{1}{2}\left(\sigma'_{ij}\,\varepsilon'_{ij} + \frac{1}{3}e\,\cancel{\sigma'_{ii}} - p\,\cancel{\varepsilon'_{ii}} - \frac{1}{3}p\,e\,\underbrace{\delta_{ii}}_{3}\right)$$

$$= \frac{1}{2}\left(\sigma'_{ij}\,\varepsilon'_{ij} - p\,e\right)$$

Now check the relation between

$$\begin{cases} ① & (-p) \text{ and } e \\ ② & \sigma'_{ij} \text{ and } \varepsilon'_{ij} \end{cases}$$

① $(-p)$ and e relation $\quad \left(-p = +\frac{1}{3}\operatorname{tr}\underset{\sim}{\sigma}\right)$

- $\operatorname{tr}\left\{\sigma_{ij} = \dfrac{E}{1+\nu}\varepsilon_{ij} + \dfrac{\nu E}{(1+\nu)(1-2\nu)}\delta_{ij}\,\varepsilon_{kk}\right\}$

- $$\operatorname{tr}\underset{\sim}{\sigma} = \frac{E}{1+\nu}\operatorname{tr}\underset{\sim}{\varepsilon} + \frac{3\nu E}{(1+\nu)(1-2\nu)}\operatorname{tr}\underset{\sim}{\varepsilon}$$

$$= \frac{E\{1-2\nu+3\nu\}}{(1+\nu)(1-2\nu)}\operatorname{tr}\underset{\sim}{\varepsilon}$$

$$= \frac{E}{1-2\nu}\operatorname{tr}\varepsilon$$

Thus

$$[illegible] = +\frac{1}{3}\,\mathrm{tr}\,\underset{\sim}{\sigma} = \frac{E}{3(1-2\nu)}\,\underbrace{\left(\mathrm{tr}\,\underset{\sim}{\varepsilon}\right)}_{e}$$

$$= [illegible] \qquad (*)$$

↑ Bulk modulus $= \dfrac{E}{3(1-2\nu)}$

② $\underset{\sim}{\sigma}'$ and $\underset{\sim}{\varepsilon}'$ relation

$$[illegible] = \sigma_{ij} + p\,\delta_{ij}$$

$$= \frac{E}{1+\nu}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\delta_{ij}\,e - K e\,\delta_{ij}$$

$$= \frac{E}{1+\nu}\left(\varepsilon'_{ij} + \frac{1}{3}e\,\delta_{ij}\right) + \frac{\nu E}{(1+\nu)(1-2\nu)}e\,\delta_{ij} - \frac{E}{3(1-2\nu)}e\,\delta_{ij}$$

$$= \frac{E}{1+\nu}\varepsilon'_{ij} + \frac{Ee}{3(1+\nu)(1-2\nu)}\left[(1-2\nu)+3\nu-1-\nu\right]$$

$$= \frac{E}{1+\nu}\varepsilon'_{ij} = [illegible] \qquad (*)$$

Remark: can show

$$E = \frac{9K\mu}{3K+\mu} \; ; \; \nu = \frac{3K-2\mu}{6K+2\mu} \qquad (a)$$

Thus,

$$U = \frac{1}{2}\left(\sigma'_{ij}\varepsilon'_{ij} - p e\right)$$

$$= \frac{1}{2}\left(2\mu\, \varepsilon'_{ij}\varepsilon'_{ij} + K e^2\right)$$

$$= \frac{1}{2}\Big[2\mu\big(\epsilon'^2_{11} + \epsilon'^2_{22} + \epsilon'^3_{33} + 2\epsilon'^2_{12} + 2\epsilon'^2_{23} + 2\epsilon'^2_{13}\big) + K e^2\Big] \geq 0$$

Since all $\epsilon'^2_{ij}, e^2 \geq 0$

$\mu > 0$ and $K > 0$ — ①, ②

$$\therefore \quad E \underset{(a)}{=} \frac{9K\mu}{3K+\mu} > 0$$

Then

$$\begin{cases} \mu = \dfrac{E}{2(1+\nu)} > 0 \rightarrow \nu > -1 \\ K = \dfrac{E}{3(1-2\nu)} \rightarrow \nu < +\dfrac{1}{2} \end{cases}$$

Finally,

$$\boxed{E > 0, \quad -1 < \nu < \frac{1}{2}}$$

$\nu \Rightarrow -1$

infinite shear modulus limit

$\nu \Rightarrow \frac{1}{2}$

Incompressibility limit