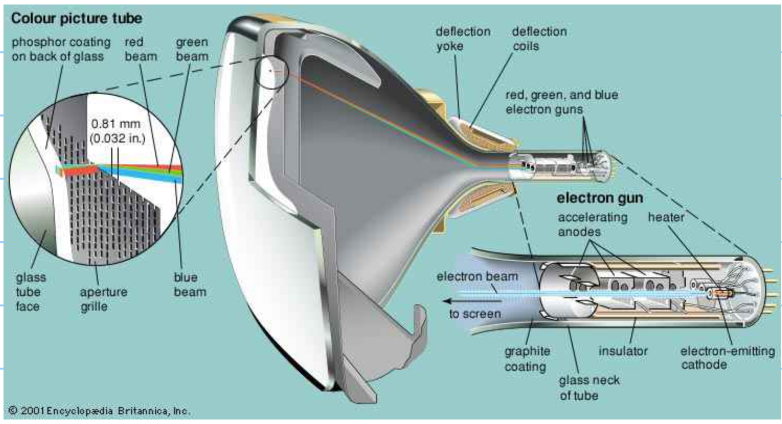
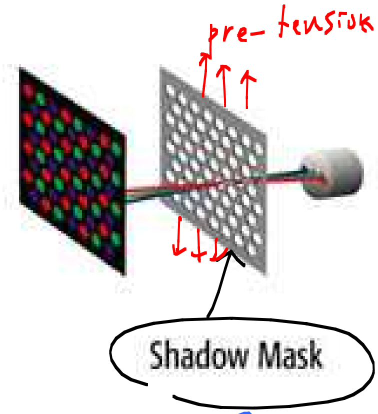
Lect 1-6

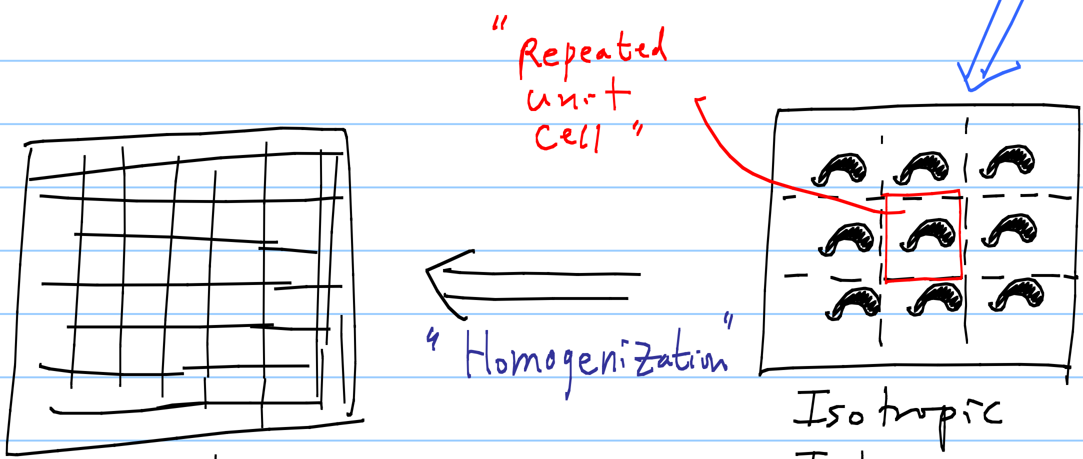
노트 제목

Homogenization Method

<Motivation 1 : For Easier Analysis>







Anisotropic Homogenized

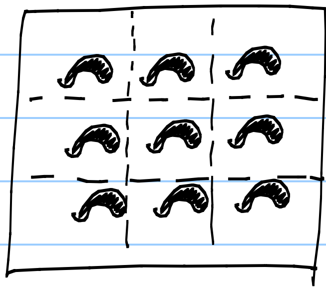
↳ Easier to work with

Isotropic Inhomogeneous

<Motivation 2: Material Microstructure Design>

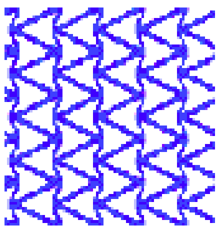
We want a certain material property!

$\Longrightarrow$ Inverse Homogenization

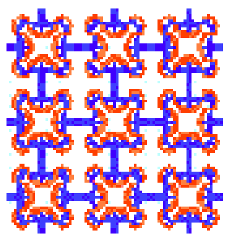


Example: Microstructure Design for

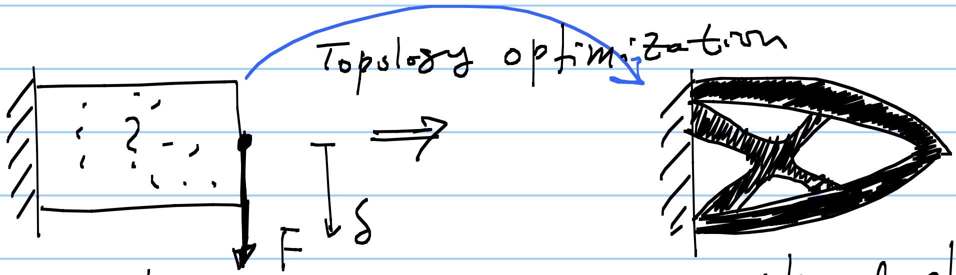
① Negative Poisson's ratio



② zero thermal expansion



< Motivation 3 : Topology optimization >



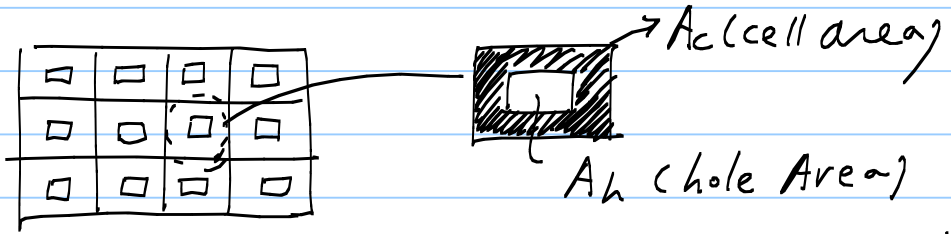
optimal structure

30% mass

find the stiffest structure (i.e., minimizing δ)

Theoretical Background

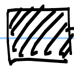
1. Discretization into small cells (or elements) and consider a hole of variable size in each cell

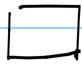


$$\rho \triangleq (A_c - A_h)/A_c$$

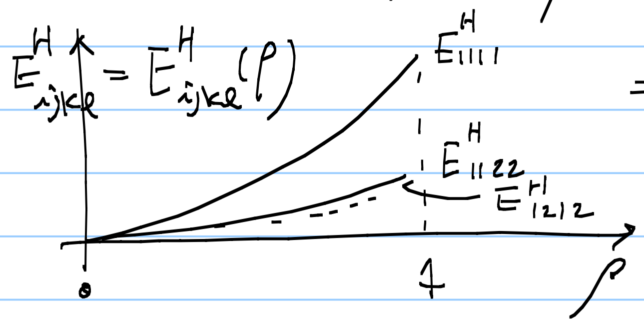
$$0 < \rho \leq 1$$

※ limiting case

as $\rho \to 1$  $\to$ solid (existence of material)

as $\rho \to 0$  $\to$ void (no material in the cell)

② Homogenization of holey cell

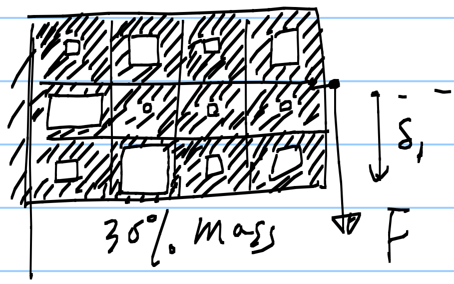


$\Rightarrow$ Anisotropic material, in general

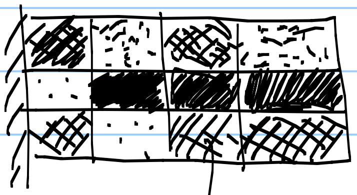
may approximate E^H_{ijkl} as some polynomial function of ρ.

③ Consider a candidate layout

<Layout 1>



④ Compute δ_1 using $E^H_{ijkl}(\rho)$



$\xrightarrow[\text{FEM}]{\text{may use}}$ $\delta_1 = 3\times10^{-3}\,\text{cm}$

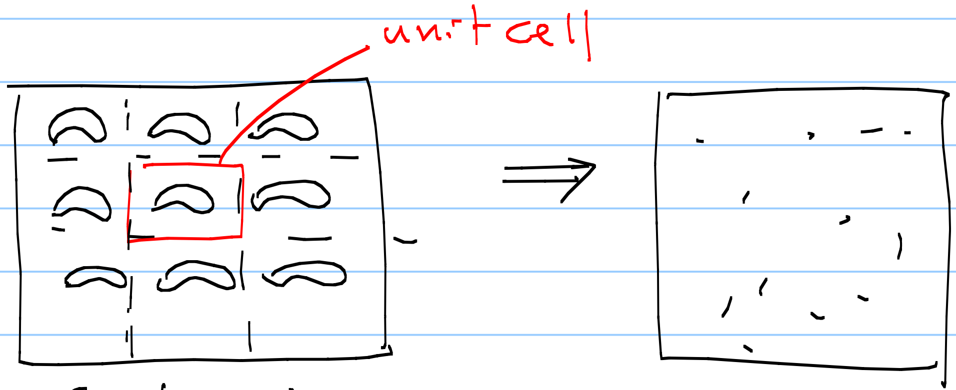
$E^H_{ijkl} \leftarrow$ ftn of ρ
(Easily computed)

⑤ Consider another candidate layout
utilizing Layout 1

$\Rightarrow$ Use Optimization Alogorithm
(e.g. Genetic Alogorithm, Steepest
descent method, etc)

6 Repeat 3, 4, 5
until the optimal solution is
found.

Back to Homogenization



(Isotropic)
Inhomogeneous

(Anisotropic)
Homogenized

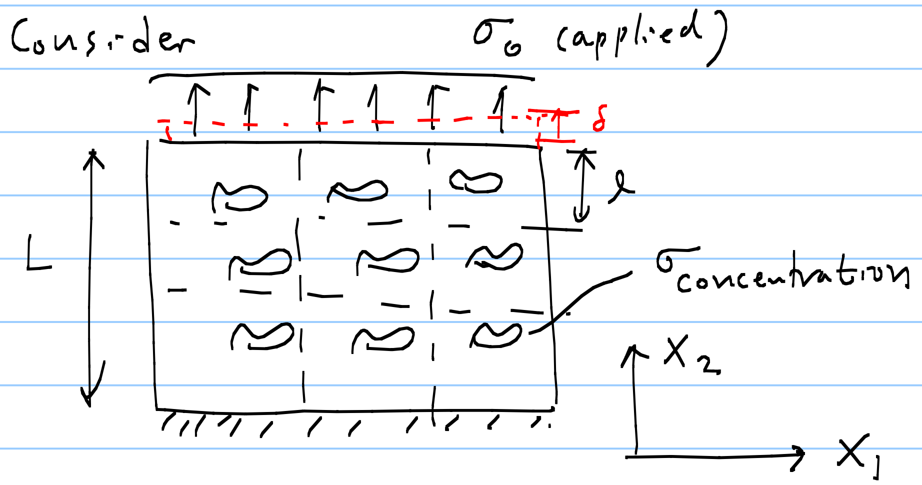
Condition for Homogenization

① same unit cell pattern must repeat
or the variation of pattern must
be small

② the size of unit cell must be
much smaller than the global
structure size

⇒ We need two scales to distinguish
between macroscopic (global) and
microscopic (local) mechanical behavior.

Consider



δ : governed by macroscopic behavior

$\sigma_{concentration}$: governed by microscopic behavior

$\Rightarrow$ Structural behavior :

observed in $\begin{cases} \text{macroscopic scale } (L) \\ \text{microscopic scale } (\ell) \end{cases}$

Thus, two scales or two coordinate systems may be introduced for easier analysis in the limit of $\varepsilon = \ell / L \to 0$

$\underset{\sim}{X}$ behavior $\rightarrow$ $\underset{\sim}{x} = \underset{\sim}{X}$

$\rightarrow$ $\underset{\sim}{y} = \dfrac{\underset{\sim}{X}}{\varepsilon}$ (zoomed coord.)

<Two systems>

$\varepsilon = 0$	$\varepsilon \ll 1$, but $\varepsilon \neq 0$
original system	perturbed system by ε

$\Rightarrow$ perturbed system analyzed by "Perturbation Analysis"

① Regular perturbation ?

② Singular perturbation ?

perturbation analysis:
approximate analysis technique
built from original system

Basic Concept on Perturbation

A. Regular Perturbation

ex) unperturbed system:

$$x^3 - 4x = 0 \quad \text{------} \quad (1)$$

Sol: $x_u = 0$, $x_u = 2$, $x_u = -2$

perturbed system:

$$x^3 - 4.001x + 0.002 = 0 \quad \text{---} \quad (2)$$

(expectation $x_p \simeq x_u$)

$\uparrow$ Regular Behavior

★ Introduce a small ε

$$x^3 - (4+\epsilon)x + 2\epsilon = 0$$

(In this case: $\epsilon = 0.001$)

Analysis:

expecting $x_p \simeq x_u$

Let

$$\boxed{x_p = a_0 + a_1 \epsilon + a_2 \epsilon^2 + \cdots} \quad (3)$$

(power series in ϵ)

$$(a_0 + a_1 \epsilon + a_2 \epsilon^2 + \cdots)^3$$
$$- (4 + \epsilon)(a_0 + a_1 \epsilon + a_2 \epsilon^2 + \cdots) + 2\epsilon = 0$$

Expand:

$$(a_0^3 + 3a_0^2 \cdot a_1 \epsilon + 3a_0^2 a_2 \epsilon^2$$
$$+ 3a_0 a_1^2 \epsilon^2 + \cdots)$$

up to ϵ^2-term

$$- (4a_0 + 4a_1 \epsilon + 4a_2 \epsilon^2$$
$$+ a_0 \epsilon + a_1 \epsilon^2 + \cdots) + 2\epsilon = 0$$

up to ϵ^2-term

$$O(\epsilon^0 = 1): \quad a_0^3 - 4a_0 = 0 \qquad (a)$$

$O(\epsilon^1)$; $3a_0^2 a_1 - 4a_1 - a_0 + 2 = 0$ (b)

$O(\epsilon^2)$; $3a_0^2 a_2 + 3a_0 a_1^2 - 4a_2 - a_1 = 0$ (c)

$O(\epsilon^3)$; $\cdots = 0$

$\Rightarrow$

Eq (a) = Eq. (1)

i.e; $a_0 = -2, 0, 2$

If $a_0 = -2$ is chosen,

(b) $\rightarrow$ $12a_1 - 4a_1 - (-2) + 2 = 0$

$a_1 = -\frac{1}{2}$

$(c) \rightarrow \quad 12a_2 + 3(-2)\left(-\frac{1}{2}\right)^2 - 4a_2 - \left(-\frac{1}{2}\right) = 0$

$$8a_2 - 1 = 0$$

$$\therefore a_2 = \frac{1}{8}$$

Thus

$$\boxed{\begin{aligned} x_p &= a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots \\ &= -2 - \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + \cdots \\ &\qquad (\text{near } x = -2) \end{aligned}} \qquad (4)$$

B. Singular Perturbation

Consider Duffing equation

$$m \frac{d^2 y}{dt^2} + k(y - \alpha y^3) = 0 \qquad (*)$$

if α small,
weakly nonlinear

Assuming
$\omega_0 = \sqrt{k/m} = 1$, rewrite $(*)$ as

$$\frac{d^2 y}{dt^2} + (y - \epsilon y^3) = 0 \qquad (5)$$

with $\quad y(0) = 1, \quad \frac{dy}{dt}(0) = 0 \qquad$ (6a, b)

physics:
if $\epsilon = 0$, $\omega_0 = 1$
if $1 \gg \epsilon > 0$, $\omega^* < \omega_0$

As in the earlier example,
Let us attempt a regular expansion:

$$y(t) = y_1(t) + \epsilon y_2(t) + \epsilon^2 y_3(t) + \cdots \quad (7)$$

Eq. (7) $\rightarrow$ Eq. (5), Eq (6)

- $$\frac{d^2}{dt}\left(y_1 + \epsilon y_2 + \epsilon^2 y_3 + \cdots\right) + \left(y_1 + \epsilon y_2 + \cdots\right) - \epsilon\left(y_1 + \epsilon y_2 + \cdots\right)^3 = 0$$

- $$y_1(0) + \epsilon y_2(0) + \cdots = 1$$

- $$\frac{dy_1(0)}{dt} + \epsilon \frac{dy_2(0)}{dt} + \cdots = 0$$

① For first Approx., $y_1(t)$

$$\frac{d^2y_1}{dt^2} + y_1 = 0 \; ; \quad y_1(0) = 1, \; \frac{dy_1}{dt} = 0$$

$$y_1 = A\cos t + B\sin t$$
$$= \cos t$$

② For 2nd Approximation

WATCH OUT!

$$\frac{d^2y_2}{dt^2} + y_2 = y_1^3$$
$$= \cos^3 t$$
$$= \frac{1}{4}(\cos 3t + 3\cos t) \; ;$$

$$y_2(0) = 0, \quad \frac{dy_2(0)}{dt} = 0 \qquad (*)$$

Sol: $y_2(t) = y_2^c(t) + y_2^p(t)$

$$= (a\cos t + b\sin t)$$
$$+ a_p \cos 3t + b_p \text{[redacted]} \sin t + c_p \text{[redacted]} \cos t) \qquad (**)$$

(**) → (*) yields

$$y_2(t) = \frac{1}{32}\left\{(\cos t - \cos 3t) + 12t\sin t\right\}$$

Thus

$$y(t) = y_1(t) + \epsilon y_2(t) + \cdots$$

$$= \cos t + \frac{\epsilon}{32}\left\{(\cos t - \cos 3t) + \underline{12t\sin t}\right\} + \cdots$$

"secular term"

→ It blows up as $t \to \infty$
especially when $t > O\left(\frac{1}{\epsilon}\right)$

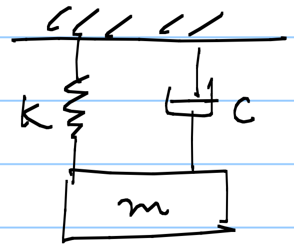
* If we ignore ϵ-term,

$y(t) = \cos t$ → "trivial" (no correction)

$$\boxed{\begin{array}{l}\Rightarrow \text{Regular Perturbation method does not work} \\ \quad \Leftrightarrow \text{Try Singular Perturbation}\end{array}}$$

What's going on?

Examine a simpler problem



$$m\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky=0$$

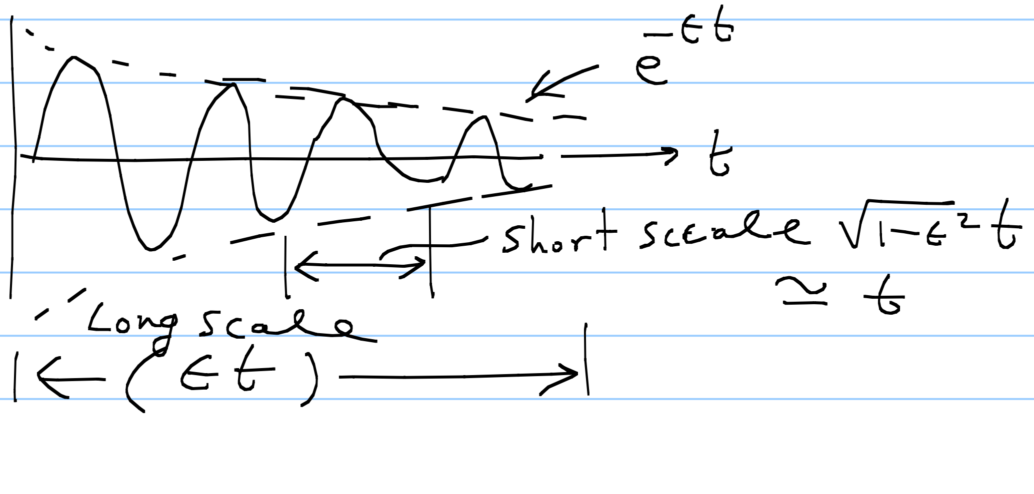
or
$$\frac{d^2y}{dt^2}+2\epsilon\frac{dy}{dt}+y=0$$

$$\left(\text{with } y(0)=0,\ \frac{dy}{dt}(0)\neq 0\right)$$

Solution:

$$y=e^{-\epsilon t}\left\{\cancel{A\cos\sqrt{1-\epsilon^2}t}+B\sin\sqrt{1-\epsilon^2}t\right\}$$

$$=Be^{-\epsilon t}\sin\sqrt{1-\epsilon^2}t$$



※ There are two time scales:

t and $\underbrace{\epsilon t}_{\tau}$ slowly varying

Thus, for perturbation analysis of this kind, we need to make a correct approximation

$$\boxed{t = t(\tau, T)} \tag{8}$$

where

$$\begin{cases} \tau = t \\ T = \varepsilon t \quad \text{"contracted" time scale} \end{cases} \tag{9a, b}$$

· Let

$$y(t) = y_1(\tau, T) + \epsilon y_2(\tau, T) + \cdots \tag{10}$$

☆ Differentiation

$$\frac{d}{dt} = \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} + \frac{\partial T}{\partial t}\frac{\partial}{\partial T} = \frac{\partial}{\partial \tau} + \varepsilon\frac{\partial}{\partial T} \tag{11}$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial \tau^2} + 2\varepsilon \frac{\partial^2}{\partial \tau \partial T} + \epsilon^2 \frac{\partial^2}{\partial T^2} \qquad (12)$$

(*) Eqs (10, 11, 12) $\rightarrow$ Eq (5) (Duffing Eq.)

$$\left(\frac{\partial^2}{\partial \tau^2} + 2\epsilon \frac{\partial^2}{\partial \tau \partial T} + \epsilon^2 \frac{\partial^2}{\partial T^2}\right)(y_1 + \epsilon y_2 + \epsilon^2 y_3 + \cdots)$$

$$+ (y_1 + \epsilon y_2 + \epsilon^2 y_3 + \cdots) - \epsilon (y_1 + \epsilon y_2 + \epsilon^2 y_3 + \cdots)^3 = 0$$

$$\boxed{\begin{aligned} &O(\epsilon^0 = 1): \quad \frac{\partial^2 y_1}{\partial \tau^2} + y_1 = 0 && (13) \\ &O(\epsilon^1): \quad \frac{\partial^2 y_2}{\partial \tau^2} + y_2 = y_1^3 - 2\frac{\partial^2 y_1}{\partial \tau \partial T} && (14) \end{aligned}}$$

(*) Eqs (10, 11, 12) $\rightarrow$ Eq (6) (BC's)

$$y_1(0,0) + \epsilon y_2(0,0) + \cdots = 1$$

$$\left[\left(\frac{\partial}{\partial \tau} + \epsilon \frac{\partial}{\partial T}\right)(y_1 + \epsilon y_2 + \epsilon^2 y_3 + \cdots)\right]_{(\tau = T = 0)} = 0$$

$O(\epsilon^0 = 1)$: $\boxed{y_1 = 1 \; ; \; \frac{\partial y_1}{\partial \tau} = 0 \text{ at } \tau = T = 0}$ (15)

$O(\epsilon^1)$: $\boxed{y_2 = 0 \; ; \; \frac{\partial y_2}{\partial \tau} = -\frac{\partial y_1}{\partial T} \text{ at } \tau = T = 0}$ (16)

i) To find $y_1(\tau, T)$,

solve

$$\frac{\partial^2 y_1}{\partial \tau^2} + y_1 = 0$$

$$y_1 = 1, \quad \frac{\partial y_1}{\partial \tau} = 0 \quad \text{at} \quad \tau = 0, \ T = 0$$

$$\rightarrow \quad y_1(\tau, T) = A(T)\cos\tau + B(T)\sin\tau \tag{17}$$

using BC's

$$y_1(0,0) = A(0) = 1$$

$$\frac{\partial y_1}{\partial \tau}(0,0) = B(0) = 0 \tag{18a,b}$$

☆☆ → However, the function forms of $A(T)$ and $B(T)$ are not known.

(→ characteristics of singular perturbation)

ii) Let's consider for y_2

$$\frac{\partial^2 y_2}{\partial \tau^2} + y_2 = y_1^3 - 2\frac{\partial^2 y_1}{\partial \tau \partial T}$$

$$= (A\cos\tau + B\sin\tau)^3$$
$$-2(-A'\sin\tau + B'\cos\tau)$$

$$= A^3\cos^3\tau + 3A^2B\cos^2\tau\sin\tau$$
$$3AB^2\cos\tau\sin^2\tau + B^3\sin^3\tau$$
$$+ 2A'\sin\tau - 2B'\cos\tau$$

$$= A^3\cos^3\tau + 3A^2B(1-\sin^2\tau)\sin\tau$$
$$+3AB^2\cos\tau(1-\cos^2\tau) + B^3\sin^3\tau$$
$$+2A'\sin\tau - 2B'\cos\tau$$

$$= (A^3 - 3AB^2)\cos^3\tau + (B^3 - 3A^2B)\sin^3\tau$$
$$+3A^2B\sin\tau + 3AB^2\cos\tau$$
$$+2A'\sin\tau - 2B'\cos\tau$$

$$\left(\text{Now use: } \begin{aligned} \cos^3\alpha &= \frac{1}{4}(3\cos\alpha + \cos 3\alpha) \\ \sin^3\alpha &= \frac{1}{4}(3\sin\alpha - \sin 3\alpha) \end{aligned} \right)$$

$$= \left\{ \frac{3}{4}(A^3 - 3AB^2) + 3AB^2 - 2B' \right\} \cos\tau$$

$$+ \frac{3}{4}(B^3 - 3A^2B) + 3A^2B + 2A' \Big\} \sin\tau$$

$$+ (- -) \cos 3\tau + (- - -) \sin 3\tau$$

$\therefore$

$$\frac{\partial^2 y_2}{\partial \tau^2} + y_2 = \left(\frac{3}{4}A^3 + \frac{3}{4}AB^2 - 2B' \right) \cos\tau$$

$+$ [redacted]

$$+ (- -) \cos 3\tau + (- - -) \sin 3\tau \qquad (19)$$

Because $y_2^c \in \{ \cos\tau, \sin\tau \}$

$$\begin{cases} \frac{3}{4}A^3 + \frac{3}{4}AB^2 - 2B' \equiv 0 & (20) \\ \frac{3}{4}B^3 + \frac{3}{4}A^2B + 2A' \equiv 0 & (21) \end{cases}$$

To avoid the appearance of secular term ($\tau\cos\tau$, $\tau\sin\tau$) in y_2

To solve (20, 21)

$$\frac{dA}{dT} = -\frac{3}{8}B(A^2+B^2) \quad ①$$

$$\frac{dB}{dT} = \frac{3}{8}A(A^2+B^2) \quad ②$$

$$\frac{①}{②} \rightarrow \frac{dA}{dB} = -\frac{B}{A} \rightarrow AdA + BdB = 0$$

$$\therefore A^2 + B^2 = \text{CONST}$$

Using Initial condition (18)
for $A(0)$ and $B(0)$; $A(0)=1$, $B(0)=1$,

$$A^2+B^2=1. \qquad (3)$$

(3) → (1)

$$\frac{dA}{dT} = -\frac{3}{8}B \qquad (1)'$$

(3) → (2)

$$\frac{dB}{dT} = \frac{3}{8}A \qquad \text{—} (2)'$$

$\frac{d}{dT}(1)' - \left(\frac{3}{8}\right)(2)$:

$$\frac{d^2A}{dT^2} = -\left(\frac{3}{8}\right)^2 A \qquad \text{—}(1)''$$

Initial condition

$$\underline{\underset{(18a)}{A(0) = 1}, \qquad \frac{dA(0)}{dT} \overset{}{\underset{(1)'}{=}} 0}$$

Thus ;

$$\underline{A(T) = \cos\frac{3T}{8}} \qquad (22)$$

$\frac{d}{dT}$ ②$'$ + $\left(\frac{3}{8}\right)$① ;

$$\frac{d^2B}{dT^2} = -\left(\frac{3}{8}\right)^2 A \qquad ②''$$

Initial Condition

$$B(0) \underset{(18b)}{=} 0 \;, \quad \frac{dB(0)}{dT} = \frac{3}{8}$$

Thus

$$\underline{B(T) = \sin\frac{3T}{8}} \qquad (23)$$

Finally, using (17) (22) (23),

$$y(t) = y(\tau, T)$$
$$= \cos\frac{3T}{8}\cos\tau + \sin\frac{3T}{8}\sin\tau + O(\epsilon)$$

$$= \cos\left(\frac{3T}{8} - \tau\right) + O(\epsilon)$$

$$= \cos\left(1 - \frac{3\epsilon}{8}\right)t + O(\epsilon^2 t \cos t)$$

(not good if $t \geq O\left(\frac{1}{\epsilon^2}\right)$)

Observation:

i) Regular perturbation fails when $t \geq O\left(\frac{1}{\epsilon}\right)$

ii) "this" singular perturbation fail when $t \geq O\left(\frac{1}{\epsilon^2}\right)$

For more accurate solutions,

$$T_1 = t, \quad T_2 = \epsilon t, \quad T_3 = \epsilon^3 t, \cdots$$

$$y(t) = y(T_1, T_2, T_3, \cdots)$$

$$= y_1 + \epsilon y_2 + \epsilon^2 y_3 + \cdots$$

$$\Rightarrow \quad y = \cos\left(\left(1 - \frac{3\epsilon}{8} - \frac{15\epsilon^2}{256} + \cdots\right)t\right)$$

(Nayfeh suggested

$$T = \varepsilon t$$

$$\tau = (1 + \epsilon^2 + \epsilon^3 + \cdots)\, t$$
)