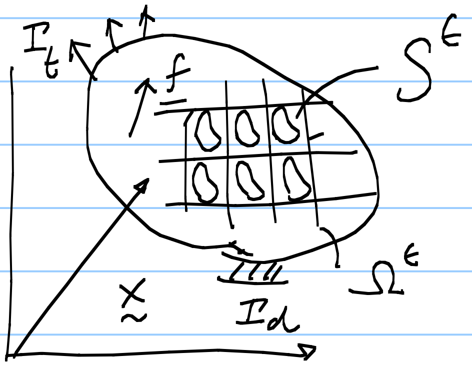
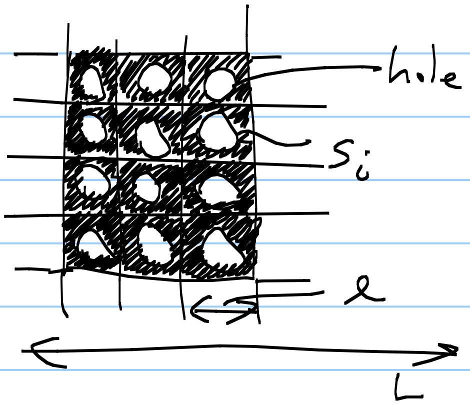
Lecture 1-8 : Homogenization for 2-3 D Elasticity problems





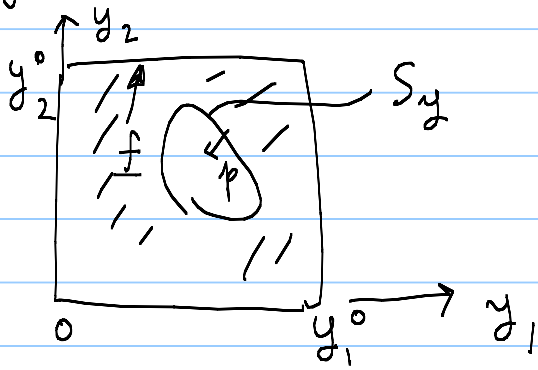
$$\Gamma = \Gamma_d \cup \Gamma_t$$

$$\Gamma \cap S^\epsilon = \phi \ , \ \underline{S^\epsilon = \cup S_i}$$

$$\underset{\sim}{X} \begin{cases} \underset{\sim}{x} = \underset{\sim}{X} \\ \underset{\sim}{y} = \underset{\sim}{X}/\epsilon \end{cases} \qquad (\epsilon = l/L)$$

- volume element : $d\Omega = dV_X = dV_x = \epsilon^3 dV_y$
- Surface element : $dS_X = dS_x = \epsilon^2 dS_y$
- volume of unit cell : $V_X^c = V_x^c = \epsilon^3 V_y^c$

※ Every unit cell looks like



$$\boxed{Y = \bar{Y} + \text{Hole}}$$

Elastic Region without hole

$$Y = [y_1^L, y_1^u = y_1^L + Y_1] \times [y_2^L, y_2^u = y_2^L + Y_2] \times [y_3^L, y_3^u = y_3^L + Y_3]$$

Given Original Problem:

find $\underset{\sim}{u}^{\epsilon} \in \{ u_i \in H^1(\Omega^\epsilon) \mid u_i|_{\Gamma_d} = \bar{u}_i \}$

such that

$$(W)\quad \boxed{\begin{aligned}\int_{\Omega^\epsilon} C_{ijkl}\frac{\partial u_k^\epsilon}{\partial x_l}\frac{\partial v_i}{\partial x_j}\,d\Omega &= \int_{\Omega^\epsilon} f_i v_i \, d\Omega + \int_{\Gamma_t} t_i v_i \, d\Gamma \\ &\quad + \int_{S^\epsilon} p_i v_i \, dS\end{aligned}} \quad (*)$$

term not appearing in 1-D

for any $\underline{v} \in \{ v_i \in H^1(\Omega^\epsilon) \mid v_i|_{\Gamma_d} = 0 \}$.

Recall the usual relation:

$$\sigma_{ij}^\epsilon = C_{ijkl}\,\varepsilon_{kl}^\epsilon$$

$$\varepsilon_{kl}^\epsilon = \frac{1}{2}\left(\frac{\partial u_k^\epsilon}{\partial x_l} + \frac{\partial u_l^\epsilon}{\partial x_k} \right)$$

- **Perturbation expansion**

$$\begin{aligned}\underset{\sim}{u}^\epsilon(\underset{\sim}{x}) &= \underset{\sim}{u}^\epsilon(\underline{x}, \underline{y}) \\ &= \underline{u}^0(\underline{x}, \underline{y}) + \epsilon\,\underline{u}^1(x,y) + \epsilon^2\,\underline{u}^2(x,y) \\ &\quad + \cdots\end{aligned} \qquad (a)$$

$$\underline{v}(\underset{\sim}{X}) = \underset{\sim}{v}(\underline{x}, \underline{y}) \qquad (b)$$

↑ no expansion on ϵ is needed because $\underline{v}$ can be arbitrary as long as $v_i|_{P_d} = 0$.

As in 1-D case, we use;

① $$\frac{\partial \phi(\underline{X})}{\partial X_i} = \frac{\partial}{\partial X_i}\phi(\underline{x}, \underline{y})$$

$$= \frac{\partial \phi(\underline{x}, \underline{y})}{\partial x_i} + \frac{1}{\epsilon}\frac{\partial \phi(\underline{x}, \underline{y})}{\partial y_i} \qquad (A)$$

② $$\lim_{\epsilon \to 0}\int_{\Omega^\epsilon} \phi(\underline{X})\, \underset{\overset{\parallel}{dV_X}}{d\Omega} = \lim_{\epsilon \to 0}\int_{\Omega^\epsilon} \phi(\underline{x}, \underline{y})\, d\underline{V}_X$$

$$\left(= \int_{\Omega} \underbrace{\left[\int_{\boxed{\underline{Y}}} \phi(\underline{x}, \underline{y})\, \underbrace{\epsilon^3 dV_y}_{dV_x} \Big/ \underbrace{\epsilon^3 V_y^c}_{V_x^c} \right]}_{\text{Cell volume average of } \phi(\underline{x}, \underline{y}) \text{ over } V_x^c \to \text{ ftn of } \underline{x} \text{ only}} dV_X \right)$$

$$= \lim_{\epsilon \to 0}\int_{\Omega} \left[\frac{1}{V_y^c} \int_{\boxed{\underline{Y}}} \phi(\underline{x}, \underline{y})\, dV_y \right] dV_x \qquad (B)$$

③ $$\lim_{\epsilon\to0}\int_{S^\epsilon}\phi(\underline{x})\,dS_x = \lim_{\epsilon\to0}\int_{S^\epsilon}\phi(\underline{x},\underline{y})\,dS_x$$

$$= \lim_{\epsilon\to0}\int_\Omega \boxed{\frac{1}{V_x^c}\left[\int_{S^\epsilon}\phi(x,y)\,dS_x\right]}\,dV_x$$

Cell volume density of $\int_{S^\epsilon}(\ )\,dS_x$

(☒ Use ① $V_x^c = \epsilon^3 V_y^c$

② $dS_x = \epsilon^2 dS_y$)

$$= \lim_{\epsilon\to0}\int_\Omega \boxed{\left[\frac{1}{\epsilon^3 V_y^c}\int_{S^\epsilon}\phi(x,y)\,\epsilon^2 dS_y\right]}\,dV_x$$

$$= \lim_{\epsilon\to0}\frac{1}{\epsilon}\int_\Omega\left[\frac{1}{V_y^c}\int_{S^\epsilon}\phi(\underline{x},\underline{y})\,dS_y\right]dV_x \quad (C)$$

<Analysis>

- (a) (b) → (*) with (A) :

$$\int_{\Omega^\epsilon} C_{ijkl} \Big\{ \frac{1}{\epsilon^2} \frac{\partial u_k^0}{\partial y_l}\frac{\partial v_i}{\partial y_j} + \frac{1}{\epsilon}\Big[\Big(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l}\Big)\frac{\partial v_i}{\partial y_j} + \frac{\partial u_k^0}{\partial y_l}\frac{\partial v_i}{\partial x_j}\Big] + \Big[\Big(\frac{\partial u_k^0}{\partial x_l}+\frac{\partial u_k^1}{\partial y_l}\Big)\frac{\partial v_i}{\partial x_j} + \Big(\frac{\partial u_k^1}{\partial x_l} + \frac{\partial u_k^2}{\partial y_l}\Big)\frac{\partial v_i}{\partial y_j}\Big] + \epsilon(\cdots) + \cdots \Big\} d\Omega$$

$$= \int_{\Omega^\epsilon} f_i v_i \, d\Omega + \int_{\Gamma_t} t_i v_i \, d\Gamma + \int_{S^\epsilon} p_i v_i \, dS$$

- Apply (B) and (C) to the above equation:

(i.e, Apply the cell-volume averaging technique)

$$\frac{1}{\epsilon^2}\int_{\Omega} \frac{1}{V_y^c}\Big[\int_{Y} C_{ijkl}\frac{\partial u_k^0}{\partial y_l}\frac{\partial v_i}{\partial y_i}\, dV_y\Big]\, dV_x$$

$$+ \frac{1}{\epsilon} \int_{\Omega} \frac{1}{V_y^c} \left[\int_{Y} C_{ijkl} \left\{ \left(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right) \frac{\partial v_i}{\partial y_j} + \frac{\partial u_k^0}{\partial y_l} \frac{\partial v_i}{\partial x_j} \right\} \right] dV_y \, dV_x$$

$$+ \int_{\Omega} \frac{1}{V_y^c} \left[\int_{Y} C_{ijkl} \left\{ \left(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right) \frac{\partial v_i}{\partial x_j} + \left(\frac{\partial u_k^1}{\partial x_l} + \frac{\partial u_k^2}{\partial y_l} \right) \frac{\partial v_i}{\partial y_j} \right] dV_y \, dV_x\right.$$

$$+ O(\epsilon)$$

$$= \int_{\Omega} \frac{1}{V_y^c} \int_{Y} f_i \, dV_y \, dV_x + \int_{\Gamma_t} t_i v_i \, d\Gamma$$

$$+ \frac{1}{\epsilon} \int_{\Omega} \frac{1}{V_y^c} \left[\int_{S} p_i \, v_i \, dS_y \right] dV_x$$

- Collect terms of the same order:

[1] $O(\frac{1}{\epsilon^2})$ term:

$$\int_{\Omega} \frac{1}{V_y^c} \left[\int_{Y} \underbrace{C_{ijkl} \frac{\partial u_k^0}{\partial y_l} \frac{\partial v_i}{\partial y_j}}_{\frac{\partial}{\partial y_j}\left[C_{ijkl} \frac{\partial u_k^0}{\partial y_l} v_i \right] - \frac{\partial}{\partial y_i}\left(C_{ijkl} \frac{\partial u_k^0}{\partial y_l} \right) v_i} dV_y \right] dV_x = 0 \qquad (\alpha)$$

Using the divergence Theorem,

$$\left(\int_{\bar{Y}} \frac{\partial}{\partial y_j} h_j \, dV_y = \int_S n_j f_j \, dS_y \right)$$

(α) becomes

$$\int_{\Omega} \frac{1}{V_y c} \left[\int_S n_j C_{ijkl} \frac{\partial u_k^0}{\partial y_l} v_i \, dS_y - \int_{\bar{Y}} \frac{\partial}{\partial y_j} \left(C_{ijkl} \frac{\partial u_k^0}{\partial y_l} \right) v_i \, dV_y \right] dV_x = 0 \qquad (\alpha)'$$

Since (α)′ must hold for any admissible v's, we have

$$\begin{bmatrix} -\frac{\partial}{\partial y} \left(C_{ijkl} \frac{\partial u_k^0}{\partial y_l} \right) = 0 \quad \text{in } \bar{Y} \\ n_j C_{ijkl} \frac{\partial u_k^0}{\partial y_l} = 0 \quad \text{on } S \end{bmatrix} \qquad (**)$$

If we define

$$\tilde{\sigma}_{ij} = C_{ijkl} \varepsilon_{kl}^0$$

$$\tilde{\varepsilon}_{kl} = \frac{1}{2} \left(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_l^0}{\partial x_k} \right)$$

$$(\ast\ast) \iff \begin{cases} \dfrac{\partial}{\partial y_j}\tilde{\sigma}_{\underset{\sim}{i}j} = 0 \text{ in } \bar{Y} \\ n_j \tilde{\sigma}_{\underset{\sim}{i}j} = 0 \text{ on } S \end{cases}$$

$\therefore$ Solution $\tilde{\sigma}_{\underset{\sim}{i}j}$, $\tilde{\varepsilon}_{\underset{\sim}{i}j}$, u°_k is not dependent on $\underset{-}{y}$.

$$\Longrightarrow \quad \boxed{u^\circ_k(\underset{-}{x}, \underset{-}{y}) = u^\circ_k(\underset{-}{x})} \qquad (1)$$

$\uparrow$ function of $\underset{-}{x}$ only

[2] $O\left(\frac{1}{\epsilon}\right)$ term

$$\int_\Omega \frac{1}{V_y^c}\left[\int_{\bar{Y}} C_{ijkl}\left(\frac{\partial u^\circ_k(x)}{\partial x_l} + \frac{\partial u^1_k}{\partial y_l}\right)\frac{\partial v_i}{\partial y_j}\, dV_y - \int_S p_i v_i \, dS_y\right] dV_x = 0 \qquad (\ast\ast\ast)$$

i) If we choose $\underline{v}(x,y) = \underline{v}(\underline{y})$;

(***) becomes

$$\boxed{\begin{aligned}&\int_{\bar{V}} \underbrace{C_{ijkl}\left(\frac{\partial u_k^0(\underline{x})}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l}\right)\frac{\partial v_i(\underline{y})}{\partial y_j}}_{\frac{\partial}{\partial y_j}\left[C_{ijkl}\left(\frac{\partial u_k^0}{\partial x_l}+\frac{\partial u_k^1}{\partial y_l}\right)v_i\right] - \frac{\partial}{\partial y_j}\left[C_{ijkl}\left(\frac{\partial u_k^0}{\partial x_l}+\frac{\partial u_k^1}{\partial y_l}\right)\right]v_i} dV_y \\ &\quad = \int_S \underset{\sim}{p}\, v_i(\underline{y})\, dS_y\end{aligned}}$$

(2)

- Let us consider the strong form equivalent to (2):

$$\int_S n_j C_{ijkl}\left(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l}\right) v_i(y)\, dS_y$$

$$-\int_{\bar{Y}} \frac{\partial}{\partial y_j}\left[C_{ijkl}\left(\frac{\partial u_k^0}{\partial x_l}+\frac{\partial u_k^1}{\partial y_l}\right)\right] v_i = \int_S p_i\, v_i(\underline{y})\, dS_y$$

Thus

(5)

- $$\frac{\partial}{\partial y_j}\left(C_{ijkl}\frac{\partial u_k^1(x,y)}{\partial y_l}\right) = -\frac{\partial}{\partial y_j}\left(C_{ijkl}\frac{\partial u_k^0(\underline{x})}{\partial x_l}\right)$$

for later convenience
$j \to p,\ k \to q,\ l \to r$

$$= -\boxed{\frac{\partial C_{ipqr}}{\partial y_p}\ \frac{\partial u_q^0(\underline{x})}{\partial x_r}} \quad \text{in } \bar{Y}$$

- $$C_{ijkl}\frac{\partial u_k^1}{\partial y_l} n_j = -C_{ipqr}\frac{\partial u_q^0(\underline{x})}{\partial x_r} n_p + p_i \quad \text{on } S$$

(⊕ periodic condition such as
$u_k^1(y_i^L) = u_k^1(y_i^u = y_i^L + Y_i)$)

(4)

(2)'

ii) if $\underset{\sim}{v} = \underset{\sim}{v}(\underset{\sim}{x})$ is chosen in (***),

(***) becomes

$$0 = \int_{\Omega} \frac{1}{V_y^c} \left[\int_S p_i \, v_i(\underset{\sim}{x}) \, dS_y \right] dV_x$$

thus

$$\boxed{0 = \int_S p_i(x,y) \, dS_y} \qquad (3)$$

** ⇒ applied traction on S must be self-equilibrated. (Example: just holes without any traction)

< Back to (2)′ >

It says: u_k' is solution (in $\bar{Y}$)

$$\left\{ \begin{array}{l} \text{for body force} = \dfrac{\partial}{\partial y_p} C_{ipqr} \underbrace{\dfrac{\partial u_q^0(x)}{\partial x_r}}_{\text{ftn of } \underset{\sim}{x}} \\[2em] \text{and traction} = -C_{ipqr} \dfrac{\partial u_q^0(\underset{\sim}{x})}{\partial x_r} n_p + p_i \end{array} \right.$$

Let us write u_k^1 as

$$u_k^1(\underline{x},\underline{y}) = \underset{\substack{\uparrow \\ \text{particular} \\ \text{solution}}}{u_k^{1p}(\underline{x},\underline{y})}$$

$$+\ \underset{\substack{\uparrow \\ \text{complementary} \\ \text{solution}}}{u_k^{1c}(\underline{x},\underline{y})} + \underset{\substack{\uparrow \\ \text{Integration} \\ \text{constant}}}{\tilde{u}_k^1(\underline{x})}$$

i) Choose $u_k^{1p}(x,y)$ to satisfy

$$\boxed{\left\{\begin{array}{ll} \dfrac{\partial}{\partial y_j}\left[C_{ijkl}\dfrac{\partial u_k^{1p}}{\partial y_l}\right] = \dfrac{\partial C_{ipqr}}{\partial y_p}\left(-\dfrac{\partial u_q^0}{\partial x_r}\right) & \text{in } \bar{Y} \\[2ex] C_{ijkl}\dfrac{\partial u_k^{1p}}{\partial y_l}n_j = -C_{ipqr}\dfrac{\partial u_q^0}{\partial x_r}n_p & \text{on } S \end{array}\right.} \quad (\beta)$$

If we select

$$\boxed{u^{1p}_{k}(\underset{\sim}{x},\underset{\sim}{y}) = -\chi^{qr}_{k}(\underset{\sim}{x},\underset{\sim}{y})\,\frac{\partial u^{0}_{q}(\underset{\sim}{x})}{\partial x_r}} \quad (\gamma)$$

Then governing equatn for χ^{qr}_{k}:
(using β, γ)

$$\boxed{\begin{cases} \dfrac{\partial}{\partial y_i}\left[C_{ijkl}\dfrac{\partial \chi^{qr}_{k}}{\partial y_l}\right] = \dfrac{\partial C_{ipqr}}{\partial y_p} \text{ in } \bar{Y} & ① \\ C_{ijkl}\dfrac{\partial \chi^{qr}_{k}}{\partial y_l} n_j = C_{ipqr} n_p \text{ on } S & ② \end{cases} \\ \oplus \text{ periodicity on } \chi^{qr}_{k}\,)} \quad (4)$$

(physical meaning of (4) will be given later)

Let's consider weak form of χ_k^{qr}

$$-\int ① v_i(y)\,dV_y + \int ② v_i(y)\,dS_y$$

$\Rightarrow$

$$\boxed{\int_{\bar{Y}} C_{ijkl} \frac{\partial \chi_k^{qr}}{\partial y_l} \frac{\partial v_i(y)}{\partial y_j} dV_y = \int_{\bar{Y}} C_{ipqr} \frac{\partial v_i(y)}{\partial y_p} dV_y} \quad (4)'$$

⊕ periodicity on χ_k^{qr}

ii) Choose $u_k^{ic}(x,y) \hat{=} -\psi_k(x,y)$ to satisfy

$$\boxed{\begin{cases} \dfrac{\partial}{\partial y_j}\left[C_{ijkl} \dfrac{\partial \psi_k}{\partial y_l}\right] = 0 \quad \text{in } \bar{Y} \quad ③ \\ C_{ijkl} \dfrac{\partial u_k^{ic}(x,y)}{\partial y_l} n_j = -p_i \quad \text{on } S \quad ④ \end{cases}} \quad (5)$$

Weak form of (ε)

$$-\int ③ \, v_i(y)\,dV_y + \int ④ \, v_i(y)\,dS_y$$

$$\Rightarrow \boxed{\begin{gathered}\int_{\bar{Y}} C_{ijkl} \frac{\partial \psi_k}{\partial y_l} \frac{\partial v_i(\underline{y})}{\partial y_j}\, dV_y = \int_S p_i v_i(y)\, dS_y \quad (5)' \\ \oplus \text{ periodic condition}\end{gathered}}$$

Summarizing:

"still undetermined"

$$\boxed{\begin{aligned} u_k^1(\underline{x},\underline{y}) &= u_k^{1p}(x,y) + u_k^{1c}(x,y) + \tilde{u}_k^1(\underline{x}) \\ &= -\chi_k^{qr}(\underline{x},\underline{y}) \frac{\partial u_q^0(\underline{x})}{\partial x_r} - \psi_k(\underline{x},\underline{y}) + \tilde{u}_k^1(\underline{x}) \end{aligned}}$$

(6)

where $\begin{cases} \chi_k^{qr}(\underline{x},\underline{y}) \text{ satisfies ④ (or ④')} \\ \psi_k(\underline{x},\underline{y}) \text{ satisfies ⑤ (or ⑤')} \end{cases}$

Remark: $O(\frac{1}{\varepsilon^2})$, $O(\frac{1}{\varepsilon})$ terms alone

→ $u_k^0(\underset{\sim}{x})$ is not yet determined.

(others: χ_k^{qr}, ψ_k = determined)

[3] $O(1)$ term:

$$\int_{\Omega} \frac{1}{V_y^c} \int_{\bar{Y}} C_{ijkl} \left\{ \left(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right) \frac{\partial v_i}{\partial x_j} + \left(\frac{\partial u_k^1}{\partial x_l} + \frac{\partial u_k^2}{\partial y_l} \right) \frac{\partial v_i}{\partial y_j} \right\} dV_y \Big] dV_x$$

$$= \int_{\Omega} \frac{1}{V_y^c} \int_{\bar{Y}} f_i v_i \, dV_y \, dV_x + \int_{\Gamma_t} t_i v_i \, d\Gamma \qquad (****)$$

If we take $\underset{\sim}{v} = \underset{\sim}{v}(\underset{\sim}{x})$ in $(****)$,
we obtain

$$\boxed{\int_{\Omega} \frac{1}{V_y^c} \left[\int_{\bar{Y}} C_{ijkl} \left(\frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right) dV_y \right] \frac{\partial v_i(\underset{\sim}{x})}{\partial x_j} \, dV_x = \int_{\Omega} \left(\frac{1}{V_y^c} \int f_i \, dV_y \right) v_i(\underset{\sim}{x}) \, dV_x + \int_{\Gamma_t} t_i v_i(\underset{\sim}{x}) \, d\Gamma} \qquad (7)$$

"Macroscopic eqm"

If $v = \underset{\sim}{v}(y)$ is used in (****),

$$\int_{\Omega} \frac{1}{V_{yc}} \left[\int_{\bar{Y}} C_{ijkl} \left(\frac{\partial u_k^1}{\partial x_l} + \frac{\partial u_k^2}{\partial y_l} \right) \frac{\partial v_i(y)}{\partial y_j} \, dV_y \right] dV_x$$

$$= \int_{\Omega} \frac{1}{V_{yc}} \int_{\bar{Y}} f_i \, v_i(\underset{\sim}{y}) \, dV_y \, dV_x$$

Thus

$$\boxed{\begin{aligned} &\int_{\bar{Y}} C_{ijkl} \left(\frac{\partial u_k^1}{\partial x_l} + \frac{\partial u_k^2}{\partial y_l} \right) \frac{\partial v_i(\underset{\sim}{y})}{\partial y_j} \, dV_y \\ &\qquad = \int_{\bar{Y}} f_i \, v_i(\underset{\sim}{y}) \, dV_y \end{aligned}} \quad (8)$$

- Eqm in microscopic sense
- used to determine u_k^2 from u_k^1

Now Eq (6) → Eq. (7):

$$\int_{\Omega}\left[\frac{1}{V_{yc}}\int_{\tilde{Y}}\left(C_{ijkl}-C_{ijpm}\frac{\partial \chi_p^{kl}}{\partial y_m}\right)dV_y\right]\frac{\partial u_k^0(x)}{\partial x_l}\frac{\partial v_i(\underline{x})}{\partial x_j}\,dV_x$$

$$=\int_{\Omega}\frac{1}{V_{yc}}\int_{\tilde{Y}}C_{ijkl}\frac{\partial \psi_k(\underline{x},\underline{y})}{\partial y_l}dV_y\,\frac{\partial v_i(\underline{x})}{\partial x_j}\,dV_x$$

$$+\int_{\Omega}\left(\frac{1}{V_{yc}}\int_{\tilde{Y}}f_i\,dV_y\right)v_i(\underline{x})\,dV_x+\int_{\Gamma_t}t_i\,v_i(\underline{x})\,d\Gamma \qquad (9)$$

Let use define

- $$D_{ijkl}(\underline{x})\triangleq\frac{1}{V_{yc}}\int_{\tilde{Y}}\left(C_{ijkl}-C_{ijpm}\frac{\partial \chi_p^{kl}}{\partial y_m}\right)dV_y \qquad (10)$$

→ Homogenized elasticity tensor

- $$\tau_{ij}(\underset{\sim}{x}) \triangleq \frac{1}{V_y^c} \int_{Y} C_{ijkl} \frac{\partial \psi_k}{\partial y_l} \, dV_y \qquad (11)$$

→ average residual stress within a cell due to self-equilibrated traction p applied to the inside boundary of a cell

- $$b_i(\underset{\sim}{x}) \triangleq \frac{1}{V_y^c} \int f_i \, dV_y \qquad (12)$$

→ average body force

Then (10) becomes

$$\boxed{\begin{aligned} &\int_{\Omega} D_{ijkl}(\underset{\sim}{x}) \frac{\partial u_k^{0}(\underset{\sim}{x})}{\partial x_l} \frac{\partial v_i(\underset{\sim}{x})}{\partial x_j} \, dV_x \\ &\quad = \int_{\Omega} \tau_{ij}(\underset{\sim}{x}) \frac{\partial v_i(\underset{\sim}{x})}{\partial x_j} \, dV_x \qquad (12) \\ &\quad + \int_{\Omega} b_i(\underset{\sim}{x}) \, v_i(\underset{\sim}{x}) \, dV_x + \int_{\Gamma_t} t_i(\underset{\sim}{x}) \, v_i(x) \, d\Gamma \end{aligned}}$$

⇒ "Homogenized, macroscopic equilibrium equation"

Finally,

$$u^{\epsilon}(x,y) = u^{0}(\underline{x},\underline{y}) + \epsilon\, u^{1}(x,y) + \cdots$$

$$= u^{0}(\underline{x}) - \epsilon\Big[\underline{\chi}^{kl}(\underline{x},\underline{y})\,\frac{\partial u_k^{0}(\underline{x})}{\partial x_l}$$

$$+ \underline{\psi}(\underline{x},\underline{y}) + \tilde{\underset{\sim}{u}}^{1}(\underline{x})\Big] + \cdots \qquad (13)$$

if we write

$$\underline{\sigma}^{\varepsilon}(\underset{\sim}{X}) = \underline{\sigma}^{\varepsilon}(\underline{x},\underline{y})$$

$$= \underline{\sigma}^{0}(\underline{x},\underline{y}) + \varepsilon\,\underline{\sigma}^{1}(\underline{x},\underline{y}) + \cdots$$

$$= \underline{C} : \underline{\varepsilon}^{0}(x,y) + \varepsilon\,\underline{C} : \underline{\varepsilon}^{1}(\underline{x},\underline{y}) \cdots$$

where $\qquad (14)$

$$\underline{\varepsilon}^{\epsilon}(\underset{\sim}{X}) = \underline{\varepsilon}^{\epsilon}(\underline{x},\underline{y})$$

$$\varepsilon_{ij}^{\varepsilon}(\underline{x},\underline{y}) = \frac{1}{2}\left(\frac{\partial u_j^{\varepsilon}}{\partial X_i} + \frac{\partial u_i^{\varepsilon}}{\partial X_j}\right)$$

$$= \frac{1}{2}\left\{\left(\frac{\partial u_j^{\varepsilon}}{\partial x_i} + \frac{\partial u_i^{\epsilon}}{\partial x_j}\right) + \frac{1}{\epsilon}\left(\frac{\partial u_j^{\epsilon}}{\partial y_i} + \frac{\partial u_i^{\epsilon}}{\partial y_j}\right)\right\} \qquad (15)$$

Substituting $u_i^{\epsilon} = u_i^{0} + \epsilon\, u_i^{1} + \cdots$

$$\varepsilon_{ij}^{\epsilon}(x,y) = \varepsilon_{ij}^{0} + \epsilon\,\varepsilon_{ij}^{1} + \cdots \qquad (16)$$

we find

$$\varepsilon^{0}_{ij} = \frac{1}{2}\left(\frac{\partial u^{0}_{i}}{\partial x_{j}} + \frac{\partial u^{0}_{j}}{\partial x_{i}}\right) + \frac{1}{2}\left(\frac{\partial u'_{i}}{\partial y_{j}} + \frac{\partial u'_{j}}{\partial y_{i}}\right) \qquad (17)$$

Thus

$$\sigma^{0}_{ij}(x,y) = C_{ijkl}\,\varepsilon^{0}_{kl}(x,y)$$

$$= C_{ijkl}\left(\frac{\partial u^{0}_{i}}{\partial x_{j}} + \frac{\partial u^{1}_{i}}{\partial y_{i}}\right)$$

$$\left(\because \quad C_{ijkl}\underset{\uparrow}{S_{ij}} = C_{ijkl}\,S^{sym}_{ij}\right)$$

sym tensor

∴

$$\boxed{\begin{aligned}\sigma^{0}_{ij}(\underline{x},\underline{y}) &\underset{(13)}{=} \left(C_{ijkl} - C_{ijpm}\frac{\partial \chi^{kl}_{p}}{\partial y_{m}}\right)\frac{\partial u^{0}_{k}(\underline{x})}{\partial x_{l}} \\ &\quad - C_{ijkl}(\underline{x},\underline{y})\,\frac{\partial \psi_{k}(x,\underline{y})}{\partial y_{l}} \\ &\rightarrow \text{Can compute local stress}\end{aligned}} \qquad (18)$$

Now consider the average of $\sigma^0(\underline{x},\underline{y})$ over a unit cell:

$$\frac{1}{V_y^c}\int_{\bar{Y}} \sigma^0_{ij}(x,y)\,dV_y$$

$$= \frac{1}{V_y^c}\int_{\bar{Y}} \left[C_{ijkl} - C_{ijpm}\frac{\partial \chi_p^{kl}}{\partial y_m}\right] dV_y \,\frac{\partial u_k^0}{\partial x_l}$$

$$- \frac{1}{V_y^c}\int C_{ijkl}(\underline{x},\underline{y})\,\frac{\partial \psi_k}{\partial y_l}(\underline{x},\underline{y})\,dV_y$$

$$\triangleq \sigma_{ij}^h(\underline{x}) - \sigma_{ij}^r(\underline{r})$$

$\Uparrow$ Homogenized stress

$\Uparrow$ residual stress ($\equiv 0$ if $\psi = 0$)

Then

$$\boxed{\sigma_{ij}^h(\underline{x}) = D_{ijkl}\,\varepsilon_{ij}^h(x)} \qquad (19)$$

where

$$\varepsilon_{ij}^h \triangleq \frac{1}{2}\left(\frac{\partial u_i^0}{\partial x_j} + \frac{\partial u_j^0}{\partial x_i}\right) \qquad (20)$$

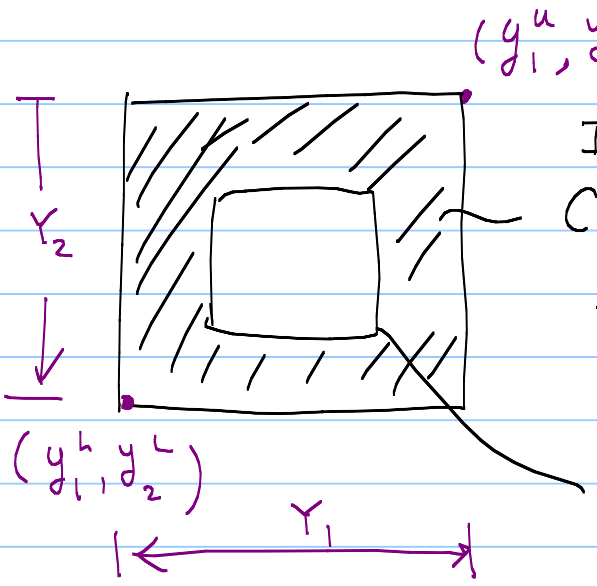
$$= \tilde{\varepsilon}_{ij}(\underline{x})$$

▨ Some words on $\chi_k^{qr}(\underline{x},\underline{y})$:

$$\begin{cases} \dfrac{\partial}{\partial y_j}\left[C_{ijkl} \dfrac{\partial \chi_k^{qr}}{\partial y_l} \right] = \dfrac{\partial C_{ipqr}}{\partial y_p} \quad \text{in } \overline{Y} \\ \underbrace{C_{ijkl} \dfrac{\partial \chi_k^{qr}}{\partial y_l} n_j}_{\underset{\sim}{t}^{qr}} = C_{ipqr} n_p \quad \text{on } S \end{cases}$$

⊕ periodic condition

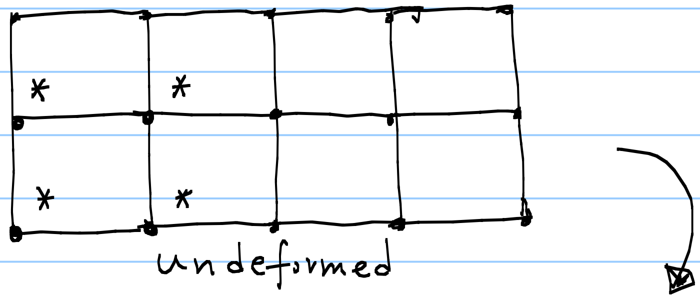
< 2-D Case with a hole >



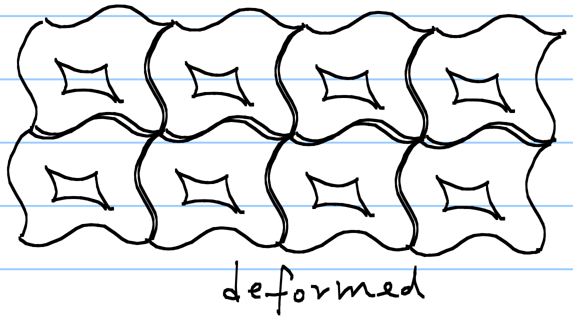
Isotropic case
$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

$p_i = 0$
(traction-free)

⊙ periodicity



*: same displacement



$$\chi^{qr}_k(y_1, y_2) = \chi^{qr}_k(y_1+Y_1, y_2)$$

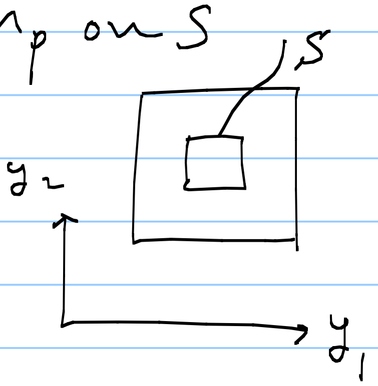
$$= \chi^{qr}_k(y_1+Y_1, y_2+Y_2)$$

$$= \chi^{qr}_k(y_1, y_2+Y_2)$$

② PDE + Internal Boundary Condition

$$\left\{ \begin{array}{l} \frac{\partial}{\partial y_j}\left[C_{ijkl}\frac{\partial \chi_k^{qr}}{\partial y_l}\right] = \cancelto{0}{\frac{\partial E_{ipqr}}{\partial y_p}} = 0 \quad \text{in } \bar{Y} \\ \underbrace{C_{ijkl}\frac{\partial \chi_k^{qr}}{\partial y_l}}_{t_i^{qr}} n_j = C_{ipqr} n_p \quad \text{on } S \end{array}\right.$$

(traction)

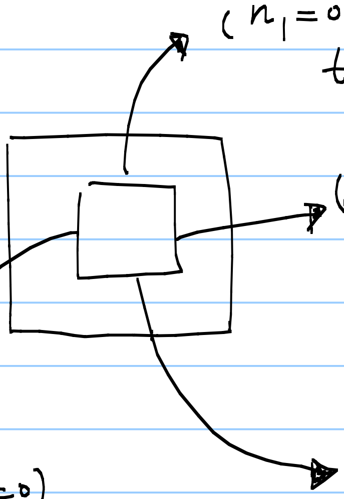


(1) $(qr) = (11)$

* BC on S: $t_i^{(11)} = C_{ip11} n_p \quad (i = 1, 2)$

$$t_1^{(11)} = C_{1p11} n_p = C_{1111} n_1 = (\lambda + 2\mu) n_1$$

$$t_2^{(11)} = C_{2p11} n_p = C_{2211} n_2 = \lambda n_2$$

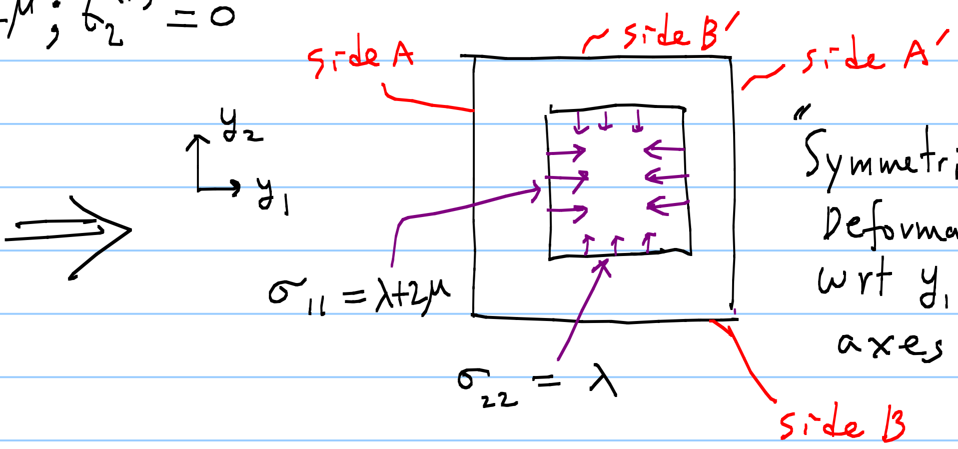


$(n_1=0,\ n_2=-1)$
$t_1^{(11)}=0;\ t_2^{(11)}=-\lambda$

$(n_1=-1,\ n_2=0)$
$t_1^{(11)}=-(\lambda+2\mu);\ t_2^{(11)}=0$

$(n_1=0,\ n_2=1)$
$t_1^{(11)}=0;\ t_2^{(11)}=\lambda$

$(n_1=1,\ n_2=0)$
$t_1^{(11)}=\lambda+2\mu;\ t_2^{(11)}=0$



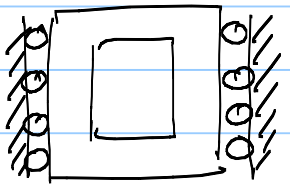
"Symmetric" Deformation wrt y_1, y_2 axes

<• Boundary condition for side A & A'>

- periodicity: $u_1(y_1^L, y_2) = u_1(y_1^u, y_2)$
  $u_2(y_1^L, y_2) = u_2(y_2^L, y_2)$

- symmetry: $u_1(y_1^L, y_2) = -u_1(y_1^u, y_2)$
  $u_2(y_1^L, y_2) = u_2(y_2^L, y_2)$

Thus, $\begin{cases} u_1(y_1^L, y_2) = 0 \\ u_1(y_1^u, y_2) = 0 \end{cases}$

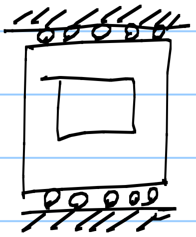


(a)

< Boundary Condition for side B & B' >

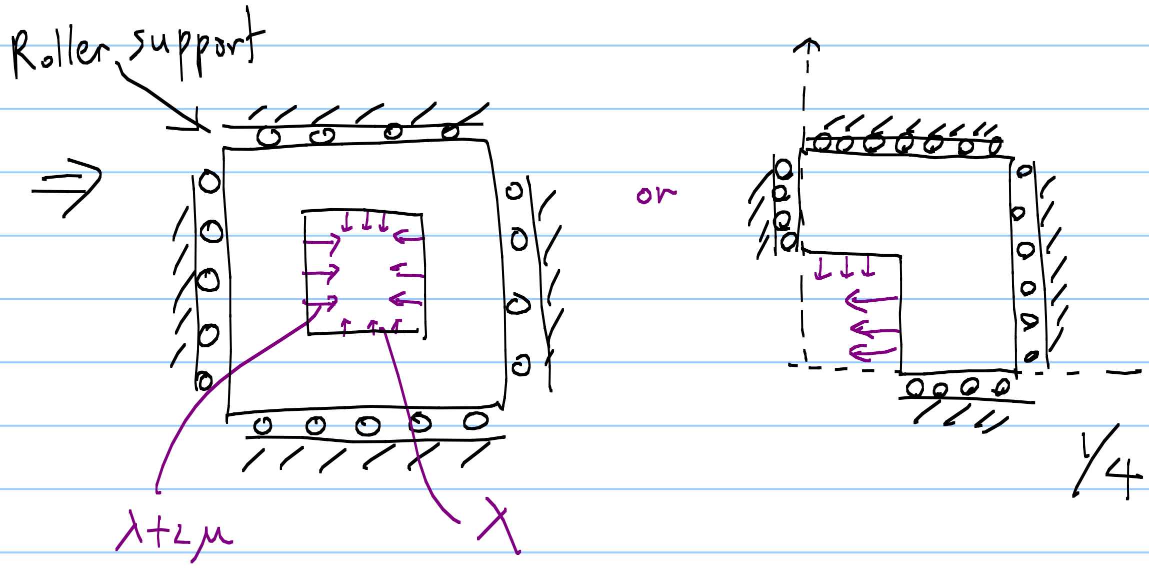
- periodicity : $u_1(y_1, y_2^L) = u_1(y_1, y_2^u)$
  $u_2(y_1, y_2^L) = u_2(y_1, y_2^u)$
- Symmetry : $u_1(y_1, y_2^L) = u_1(y_1, y_2^u)$
  $u_2(y_1, y_2^L) = - u_2(y_1, y_2^u)$

Thus $\begin{cases} u_2(y_1, y_2^L) = 0 \\ u_2(y_1, y_2^u) = 0 \end{cases}$

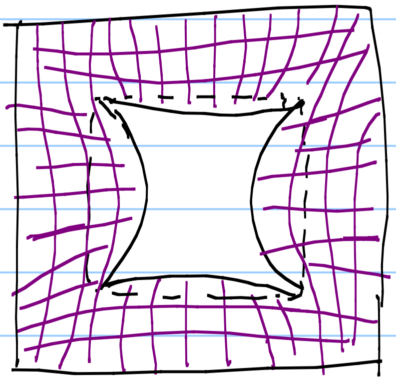


(b)

Combining all Boundary Condition

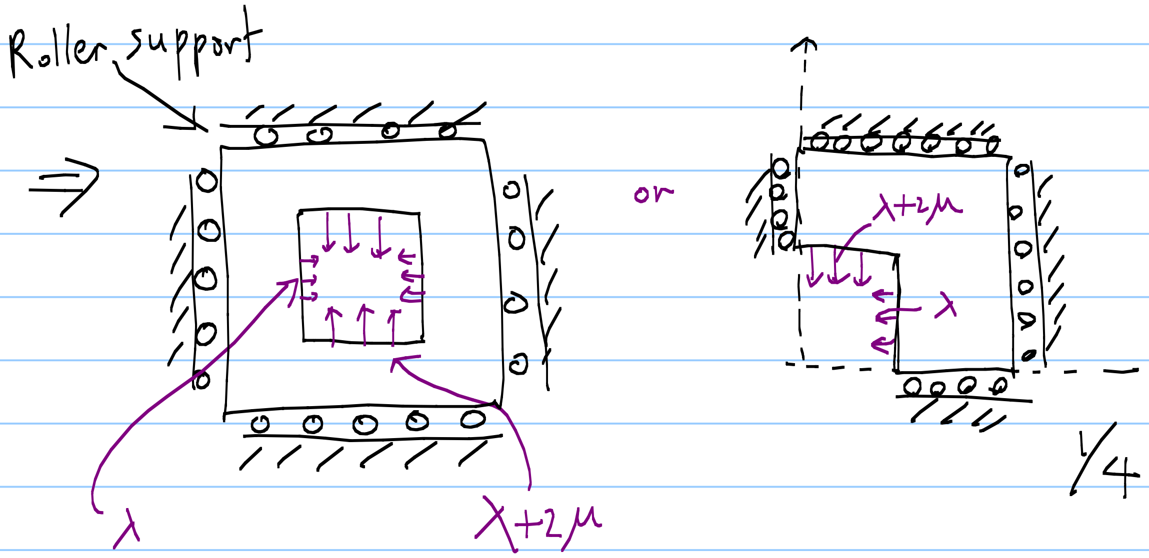


Deformed Shape for $\chi_k^{(11)}$

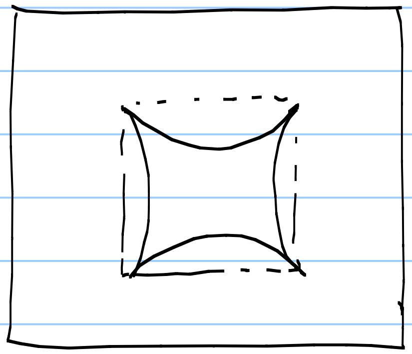


(2) $(q r) = (22)$

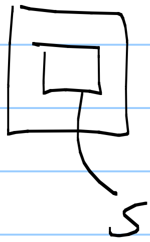
same procedure as for $(qr) = (11)$



Deformed shape for $\chi_k^{(22)}$



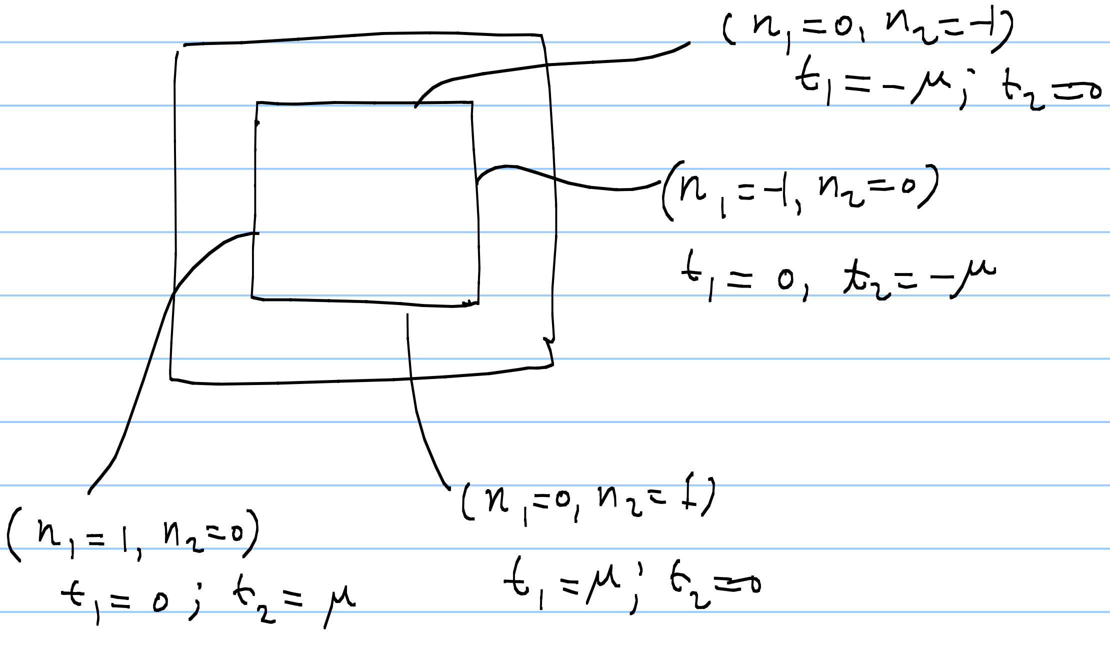
(3) $(qr) = (12) = (21)$

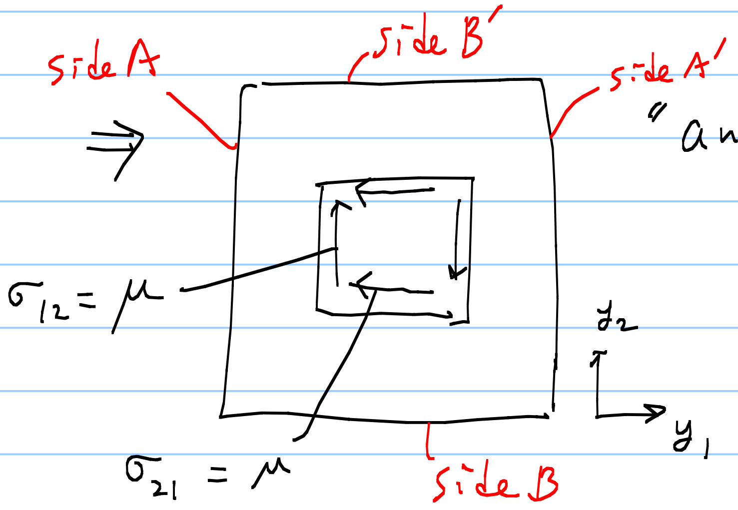


* BC on S: $\underline{t_i^{(12)} = C_{ip12} n_p}$ $(i=1,2)$

$$t_1^{(11)} = C_{1p12} n_p = C_{1212} n_2$$
$$= \mu n_2$$

$$t_2^{(11)} = C_{2p12} n_p = C_{2112} n_1$$
$$= \mu n_1$$





"antisymmetric" deformation wrt y_1 and y_2 axes

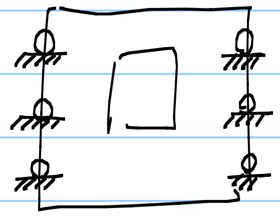
< Boundary Condition for side A and A'>

- periodicity: $u_1(y_1^L, y_2) = u_1(y_1^u, y_2)$
  $u_2(y_1^L, y_2) = u_2(y_2^L, y_2)$

- antisymmetry: $u_1(y_1^L, y_2) = u_1(y_1^u, y_2)$
  $u_2(y_1^L, y_2) = -u_2(y_2^L, y_2)$

Thus

$$\begin{cases} u_2(y_1^L, y_2) = 0 \\ u_2(y_1^u, y_2) = 0 \end{cases}$$



< Boundary Condition for side B & B' >

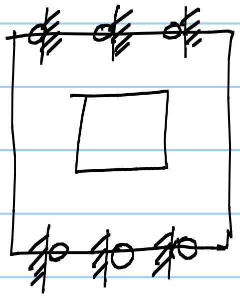
- periodicity : $u_1(y_1, y_2^L) = u_1(y_1, y_2^u)$

  $u_2(y_1, y_2^L) = u_2(y_1, y_2^u)$

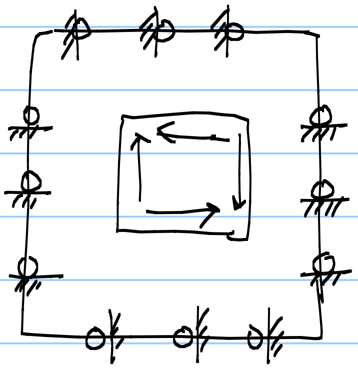
- Symmetry : $u_1(y_1, y_2^L) = -u_1(y_1, y_2^u)$

  $u_2(y_1, y_2^L) = u_2(y_1, y_2^u)$

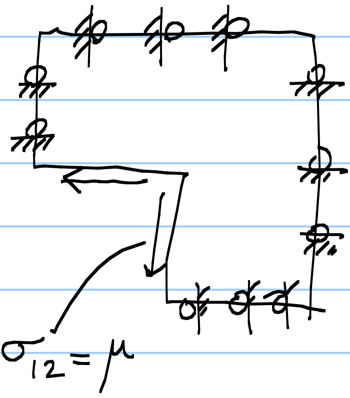
Thus $\begin{cases} u_1(y_1, y_2^L) = 0 \\ u_2(y_1, y_2^L) = 0 \end{cases}$



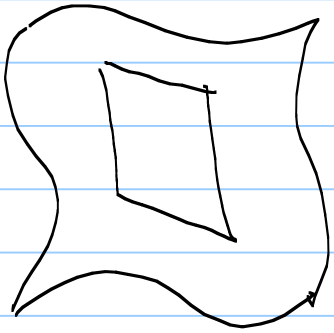
Combining all Boundary Conditions



or



Deformed shape of $\chi_k^{(12)} = \chi_k^{(21)}$



< Remark >

- Revist the weak form for χ_k^{qr}

$$\int_{\bar{Y}} C_{ijkl} \frac{\partial \chi_k^{qr}}{\partial y_l} \frac{\partial v_i(\underline{y})}{\partial y_j} dV_y = \underbrace{\int_{\bar{Y}} C_{ipqr} \frac{\partial v_i(\underline{y})}{\partial y_p} dV_y}_{}$$

boundary traction term! for the present case.