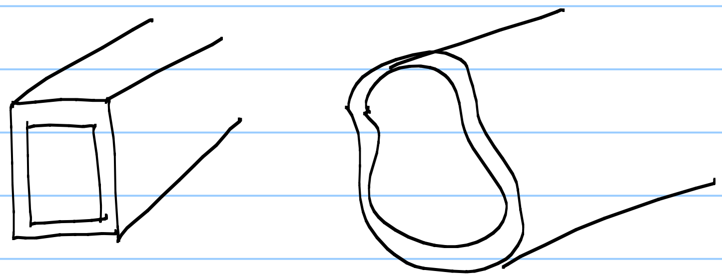
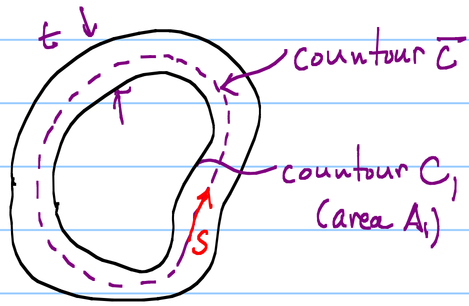
①

Lecture 3-5. Torsion of Thin-Walled Closed-Sectioned Bars

[Case of Uniform Torsion]

노트 제목





$\bar{A}$: area enclosed by $\bar{C}$

$(t^2 \ll \bar{A}$, but t may vary$)$

Will show

① $q \triangleq \tau t$ (shear flow) = CONSTANT for all s

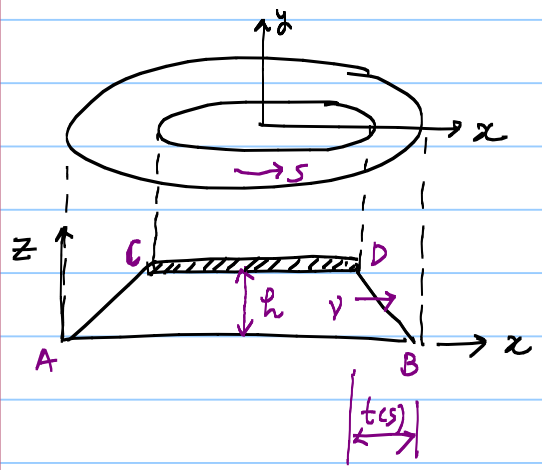
$= M_z / 2\bar{A}$

② D (torsional rigidity)

$$= \frac{M_t}{\alpha} = \frac{4\mu \bar{A}^2}{\oint_{\bar{C}} ds/t}$$

Analysis ⇐ Use the Membrane Analogy

$(\tau = -\mu\alpha \, d\psi/d\nu)$



$$\Rightarrow \begin{cases} \psi_{AB} = 0 \\ \psi_{CD} \triangleq h \end{cases}$$

$$\tau(s) = -\mu\alpha \frac{d\psi}{d\nu}$$

$$\approx -\mu\alpha \frac{\psi_{AB} - \psi_{CD}}{t(s)}$$

$$= \mu\alpha \frac{h}{t(s)} \qquad (*)$$

h

From (*)

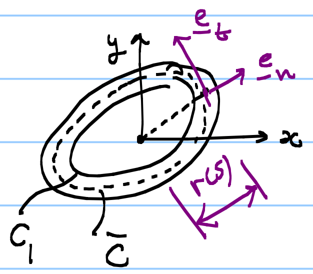
$$q \triangleq \tau(s)\,t(s) \qquad (1)$$

$$= \mu \alpha h \quad \Leftarrow \text{CONSTANT for all } s\text{'s}$$

(However, $\tau(s) \neq$ CONST if $t(s) \neq$ CONST)

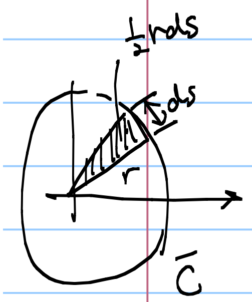
※ Must compute h to calculate $\tau(s)$ or q.

Eqm Condition / Single-valuedness



a) Eqm Condition

$$M_z = \oint_{\bar{C}} r(s) \times (\tau t)\, ds$$

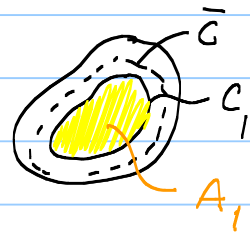


$$\underset{(1)}{=} \mu \alpha h \oint_{\bar{C}} r\, ds$$

$$= 2\mu\alpha h \bar{A} \qquad \left(\because \frac{1}{2}\oint_{\bar{C}} r\,ds = \bar{A}\right)$$

still unknown $\cdots (2)$

b) Single-valuedness



$$\oint_{C_1} \frac{d\psi}{d\nu}\, ds = -2A_1$$

i.e $$\oint_{C_1} \left(\frac{-h}{t}\right) ds = -2A_1$$

Since $h =$ CONST

$$h \oint_{C_1} \frac{ds}{t(s)} = 2A_1$$

Approximation: $C_1 \to \bar{C}$, $A_1 \to \bar{A}$

$$\boxed{h = \frac{2\bar{A}}{\oint_{\bar{C}} \frac{ds}{t}}} \qquad (3)$$

Eq. (3) $\to$ Eq. (2)

$$M_z = 2\mu \, 2\bar{A} \Big/ \left(\oint_{\bar{C}} ds/t\right) \cdot \alpha \bar{A}$$

$$= \frac{4\mu \bar{A}^2}{\oint \frac{ds}{t(s)}} \alpha$$

$\therefore$

$$\boxed{\begin{aligned} M_z &= C_t \, \alpha \qquad (4) \\ C_t &= \frac{4\mu \bar{A}^2}{\oint_{\bar{C}} \frac{ds}{t}} \qquad (5) \end{aligned}}$$

(cf $\sigma_{\theta z} = \mu \alpha r$ in circular section)

Finally (3) $\to$ (1)

$$\tau(s) = \mu\alpha \frac{h}{t(s)} = \mu\alpha \frac{2\bar{A}}{t(s) \oint_{\bar{C}} \frac{ds}{t(s)}}$$

$$\triangleq \mu \alpha r_n \qquad (6)$$

where

$$r_n = \frac{2\bar{A}}{t(s) \oint_{\bar{C}} \frac{ds}{t(s)}} \underset{t=\text{CONST}}{=} \frac{2\bar{A}}{\oint_{\bar{C}} ds} \qquad (7)$$

(Note $r_m \Rightarrow$ mean radius, because $\bar{A} = (\oint_{\bar{c}} ds) \times \frac{r_m}{2}$)

($2\pi r_m$ for circular $\bar{c}$)

Summary:

- $q = \tau(s)\, t(s)$

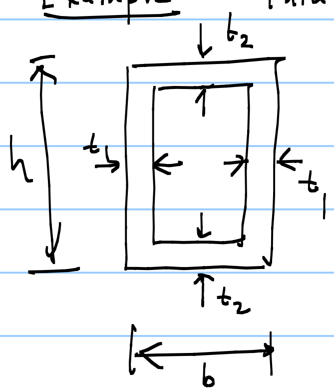
  $= \mu \alpha h = \frac{M_z}{2\bar{A}}$ ← by Eq. (2) (8)

- $\tau(s) = \frac{q}{t(s)} = \frac{M_z}{2\bar{A}\, t(s)}$

- $C_t = 4\mu \bar{A}^2 / \oint_{\bar{c}} ds/t(s)$

for thin-walled closed sections under twisting

Example Thin-walled rectangular tube



$t_1, t_2 \ll b, h$

$$C_t = \frac{4\mu \bar{A}^2}{\oint ds/t(s)} = ?$$

$$\bar{A} = b h$$

$$\oint ds/t(s) = 2\left[\int_0^h \frac{ds}{t_1} + \int_0^b \frac{ds}{t_2}\right]$$

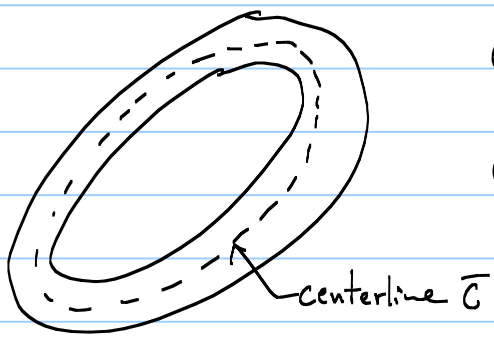
$$= 2\left(\frac{h}{t_1} + \frac{b}{t_2}\right)$$

$$C_t = \frac{4\mu (bh)^2}{2\left(\frac{h}{t_1} + \frac{b}{t_2}\right)} = \frac{2\mu b^2 h^2 t_1 t_2}{b t_1 + h t_2}$$

(Note: Given M_z, stress $\propto \frac{1}{\bar{A}}$ and $\frac{1}{t(s)}$)

Warping Displacement

completely different from open section!



① shear stress along the centerline $\bar{C} \neq 0$

② $\tau(s)$ is given by (or determined as)

$$\tau(s) = \alpha r_n$$

Approach: Use the same approach used for open-sections except $\epsilon^{\circ}_{sz} \neq 0$

① Use shear strain along the centerline

$$2\epsilon^{\circ}_{sz} = \tau(s)/\mu = \frac{\partial w}{\partial s} + \frac{\partial v}{\partial z} \quad (*)$$

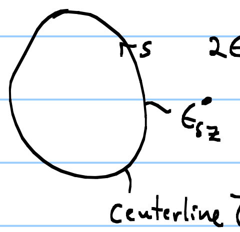
calculated already

② Use in-plane tangential displacement field

$$v = r(s)\,\Phi(z)$$

③ Integrate (*) for w with $\oint_{\bar{C}} w(s)\,ds = 0$

i)



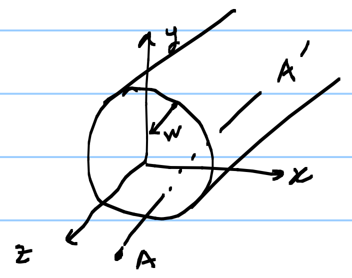
$$2\epsilon^{\circ}_{sz}(s,z) = \frac{\partial w}{\partial s} + \frac{\partial v}{\partial z}$$

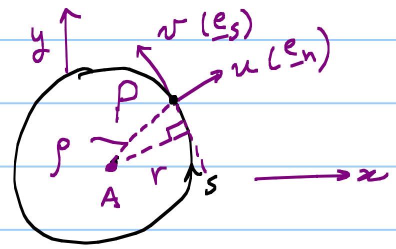
tangential disp

$$\equiv \frac{\tau(s)}{\mu} = \frac{1}{\mu}(\mu \alpha r_n)$$

$$= \alpha r_n \qquad (9)$$

ii)





$$v(s,z) = \left(\Theta(z)\,\underline{e}_z \times \underline{r}_{AP}\right) \cdot \underline{e}_s$$

$$= r(s)\;\Theta(z) \qquad (10)$$

iii) Eq (10) $\rightarrow$ Eq. (9)

$$\frac{\partial w}{\partial s} = -\frac{\partial v}{\partial z} + \alpha r_n$$

extra term compared with open-section

$$= -\alpha\, r(s) + \alpha\, r_n$$

$$= -\alpha\left(r(s) - r_n\right)$$

$$\therefore \quad w = -\alpha \int_0^s [r(s) - r_n]\, ds + w^0$$

$$= \underbrace{\alpha\left[-\int_0^s [r(s) - r_n]\, ds + \frac{w^0}{\alpha}\right]}_{\triangleq\ \omega_s(s)}, \qquad \frac{w^0}{\alpha} \triangleq \omega_s^0$$

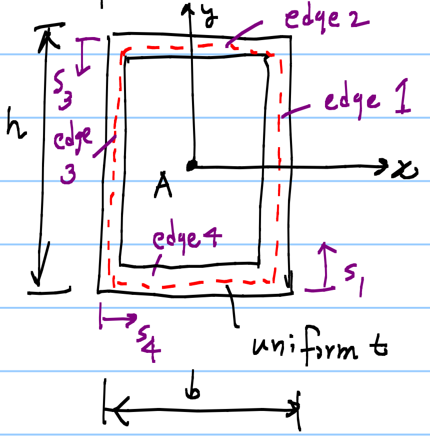
Thus

$$\boxed{\begin{aligned} w(s) &= \alpha\, \omega_s(s) && (11) \\ \omega_s(s) &= -\int_s^s (r - r_n)\, ds + \omega_s^0 && (12) \end{aligned}}$$

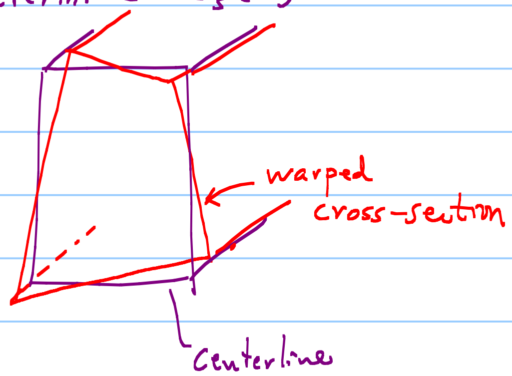
where ω_s^0 is so selected as to satisfy

$$\oint_{C} \omega_s(s)\, ds = 0 \qquad (13)$$

Example 3.8.1



Determine $\omega_s(s)$



<Analysis>

$$\left(\omega_s = -\int_0^s (r - r_n)\, ds + \omega_s^\circ\right)$$

i) Compute r_n

$$r_n = 2\bar{A} \Big/ \oint_c ds = 2bh / 2(b+h)$$

$$= bh/(b+h)$$

Thus

$$\omega_s(s) = -\int_0^s \left(r - \frac{bh}{b+h} \right) ds + \omega_s^\circ$$

ii) Convenient to introduce local edge coordinate s_i
(but need to impose continuity at all corners)

edge 1 : $r = b/2 , \quad 0 \le s_1 \le h$

$$\omega_s \triangleq \omega_{1s}(s_1) = -\frac{b(b-h)}{2(b+h)} s_1 + \omega_{1s}^\circ$$

edge 2 : $r = \frac{h}{2}, \quad 0 \le s_2 \le b$

$$\omega_s \triangleq \omega_{2s}(s_2) = - - \frac{s}{2} + \omega_{2s}^\circ$$

edge 3 : $r = b/2, \quad 0 \le s_3 \le h$

$$\omega_s(s) = \omega_{3s}(s_3) = \cdots s_3 + \omega^{\circ}_{3s}$$

edge 4 : $r = h/2, \quad 0 \le s_4 \le b$

$$\omega_s(s) = \omega_{4s}(s_4) = \cdots s_4 + \omega^{\circ}_{4s}$$

* Continuities at corners

$$\omega_{1s}(s_1 = h) = \omega_{2s}(s_2 = 0)$$
$$\omega_{2s}(s_2 = b) = \omega_{3s}(s_3 = 0)$$
$$\omega_{3s}(s_3 = h) = \omega_{4s}(s_4 = 0)$$
$$\omega_{4s}(s_4 = b) = \omega_{1s}(s_1 = 0)$$

⊕ $$\oint_C \omega_s(s)\, ds = 0$$

→ yields $\omega^{\circ}_{1s} = \omega^{\circ}_{3s} = bh(b-h)/4(b+h)$

$\omega^{\circ}_{2s} = \omega^{\circ}_{4s} = -bh(b-h)/4(b+h)$

Then one can find

$$\omega_{1s}(s) = -\frac{b}{2}\left(\frac{b-h}{b+h}\right)\left(s_1 - \frac{h}{2}\right) \quad \text{etc}$$

$$\omega_{2s}(s) = \cdots$$

simplified to as

$$\boxed{\omega(s) = C\, x(s)\, y(s)}$$