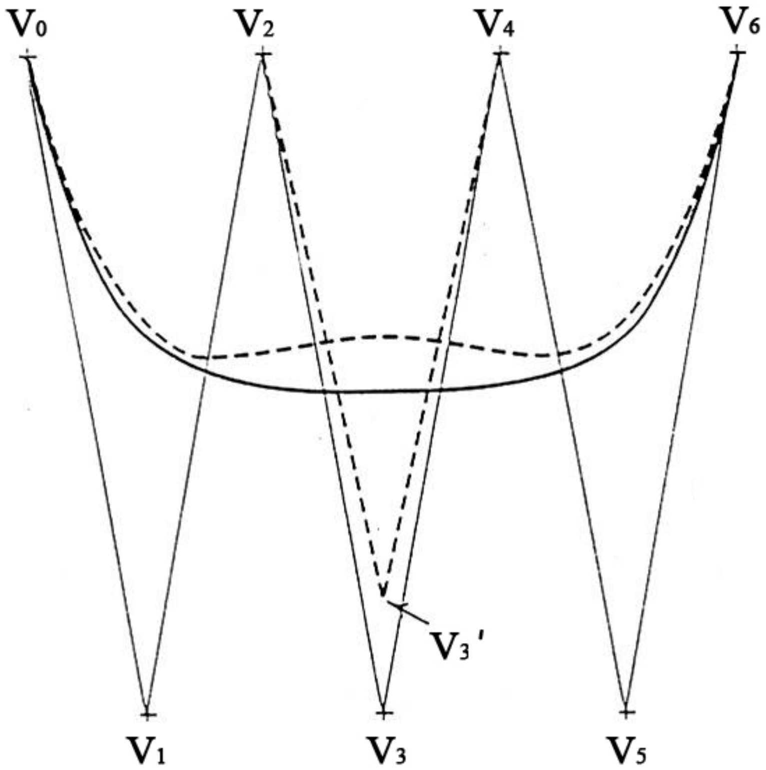
# Theory of Curve II
## (B-spline curve)

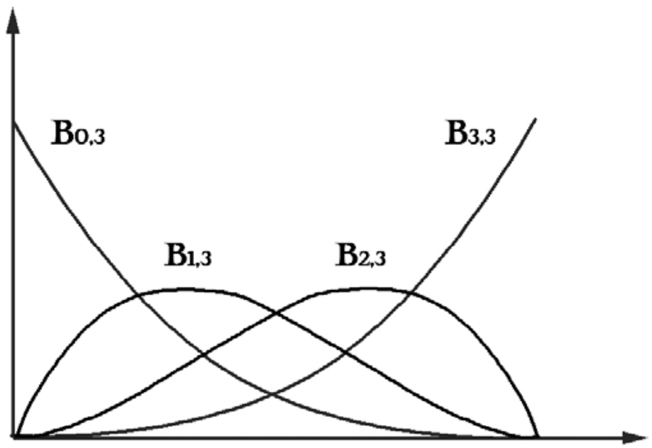
CAD Lab.

# Properties of B-spline curves

- B-spline curve:
  - Degree of curve is independent of number of control points
- Bezier curve: global modification
  - Modification of any one control point changes the curve shape everywhere
  - All the blending functions have non-zero value in the whole interval $0 \le u \le 1$

# Bezier curve: global modification

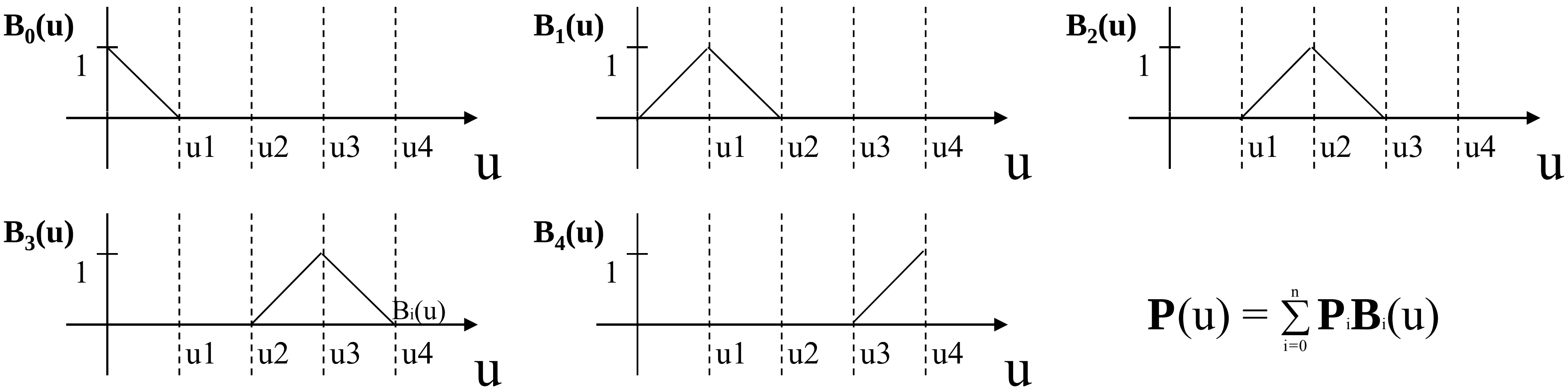



Bezier curve of degree 3

# Desired Blending Function

Consider degree 1 blending functions, and n=4




$$\mathbf{P}(u) = \sum_{i=0}^{n} \mathbf{P}_i \mathbf{B}_i(u)$$

| | | |
|---|---|---|
| $P_0$ | has an effect only for | $0 \le u \le u_1$ |
| $P_1$ | has an effect only for | $0 \le u \le u_2$ |
| $P_2$ | has an effect only for | $u_1 \le u \le u_3$ |
| $P_3$ | has an effect only for | $u_2 \le u \le u_4$ |
| $P_4$ | has an effect only for | $u_3 \le u \le u_4$ |

# Resulting Curve

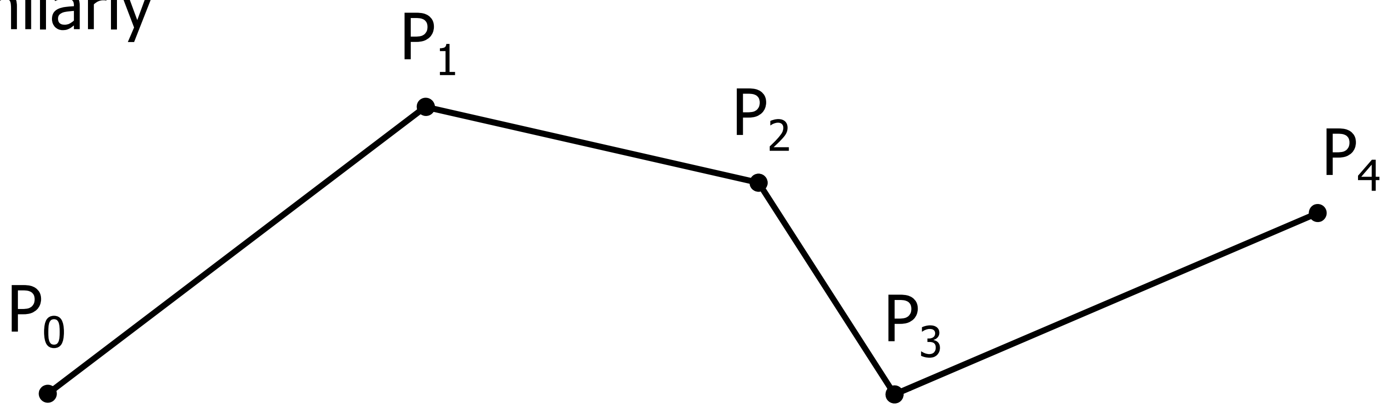
For $0 \le u \le u_1$ $\quad \mathbf{P}(u) = \mathbf{P}_0 B_0(u) + \mathbf{P}_1 \mathbf{B}_1(u) = \mathbf{P}_0(1-u) + \mathbf{P}_1 u$

… straight line from $P_0$ to $P_1$

For $u_1 \le u \le u_2$ $\quad \mathbf{P}(u) = \mathbf{P}_1 \mathbf{B}_1(u) + \mathbf{P}_2 \mathbf{B}_2(u) = \mathbf{P}_1(2-u) + \mathbf{P}_2(u-1)$

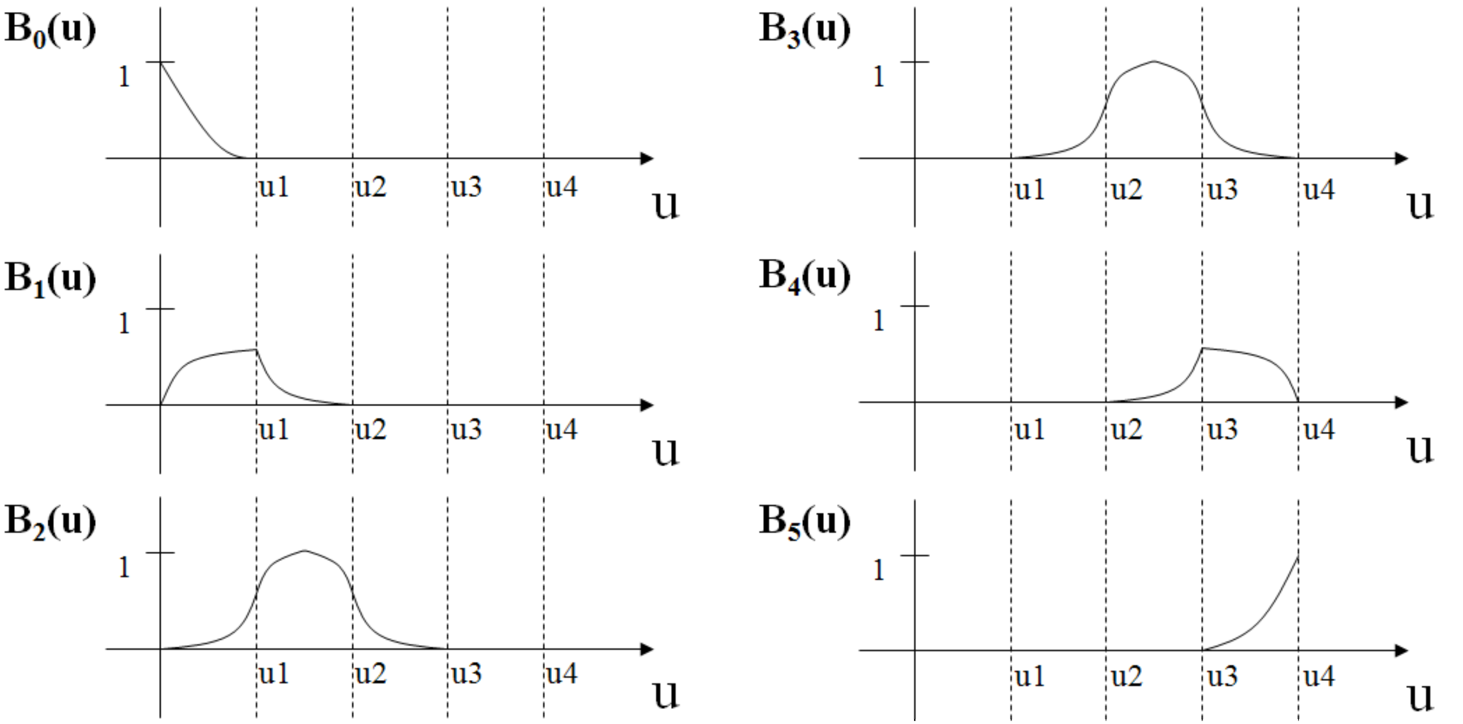
… straight line from $P_1$ to $P_2$

Similarly



# Blending Function

Consider degree 2 blending functions, and n=5




$P_0$ has an effect only for $0 \le u \le u_1$

$P_1$ has an effect only for $0 \le u \le u_2$

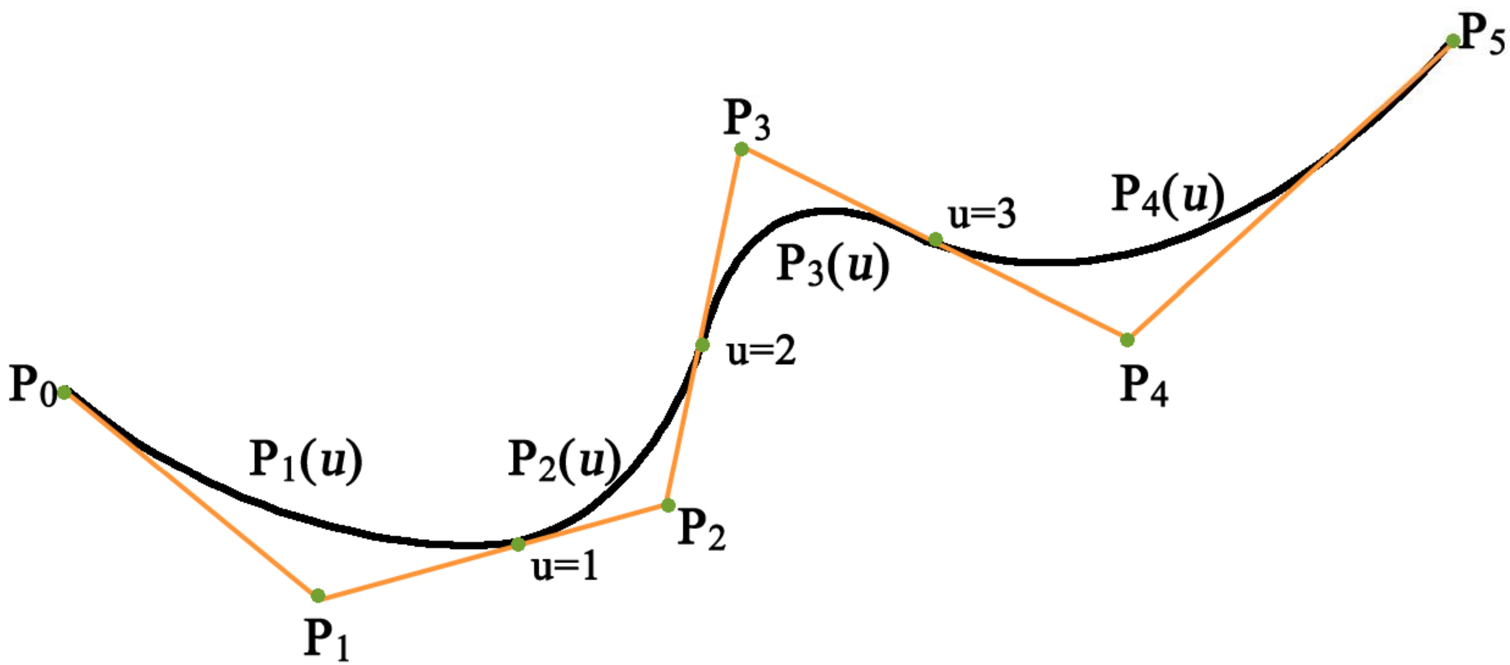
$P_2$ has an effect only for $0 \le u \le u_3$

$P_3$ has an effect only for $u_1 \le u \le u_4$

$P_4$ has an effect only for $u_2 \le u \le u_4$

$P_5$ has an effect only for $u_3 \le u \le u_4$

# Resulting Curve



# B-spline curve equation – cont'

$$\mathbf{P}(u) = \sum_{i=0}^{n} \mathbf{P}_i N_{i,k}(u) \qquad t_{k-1} \le u \le t_{n+1} \qquad \text{(a)}$$

$$N_{i,k}(u) = \frac{(u - t_i) N_{i,k-1}(u)}{t_{i+k-1} - t_i} + \frac{(t_{i+k} - u) N_{i+1,k-1}(u)}{t_{i+k} - t_{i+1}} \qquad \left(\frac{0}{0} = 0\right) \qquad \text{(b)}$$

$$N_{i,1}(u) = \begin{cases} 1 & t_i \le u < t_{i+1} \\ 0 & \text{otherwise} \end{cases} \qquad \text{(c)}$$

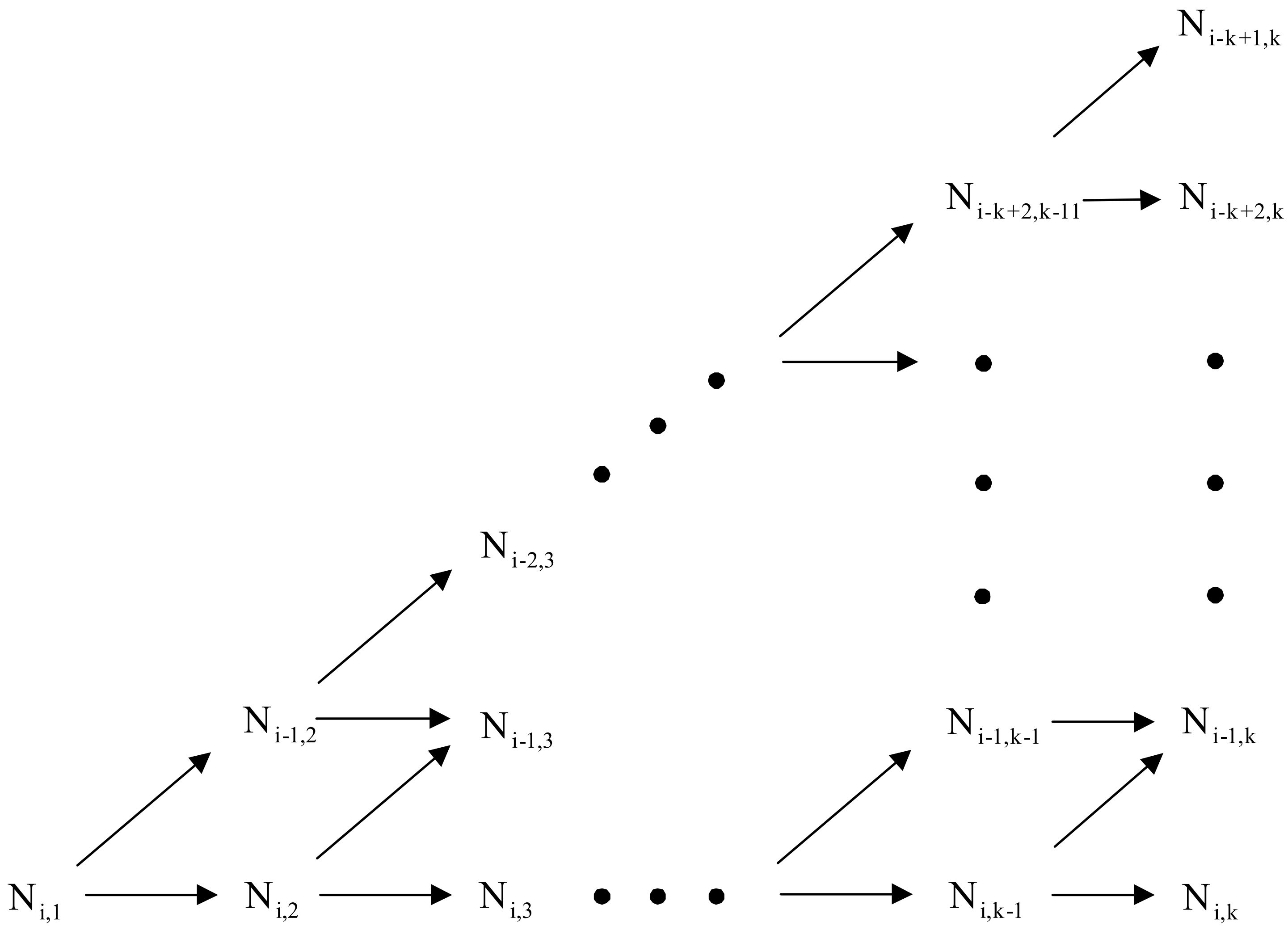
→ At any value of u, there should be only one non-zero $N_{i,1}(u)$

- $N_{i,k}$ : degree (k-1) of u, k : order (independent of number of control points n)
- $t_i$: knot values, boundary of non-zero range of each blending function
- $t_0$ (for i=0)to $t_{n+k}$ (for i=n) are needed (n+k+1 values)

# B-spline curve equation – cont'

- Only the differences in $t_i$ (i=0, ⋯, n+k) is important in (b)
- Can be shifted as a whole, parameter range should be shifted together
- A portion of B-spline curve is affected by a limited number of control points
- For u in $[t_l, t_{l+1}]$
  - Control points associated with blending functions that are non-zero in $[t_l, t_{l+1}]$ have effect
  - $N_{l,1}(u)$ is nonzero in $[t_l, t_{l+1}]$ among $N_{i,1}(u)$
  - Substitute $N_{l,1}(u)$ into the right-hand side of (b)
  - $N_{l,2}(u)$, $N_{l-1,2}(u)$ can be non-zero
  - Apply recursively
  - From $N_{l,2}(u)$, $N_{l,3}(u)$ and $N_{l-1,3}(u)$ can be non-zero
  - From $N_{l-1,2}(u)$, $N_{l-1,3}(u)$ and $N_{l-2,3}(u)$can be non-zero

# B-spline curve equation – cont'



# B-spline curve equation – cont'

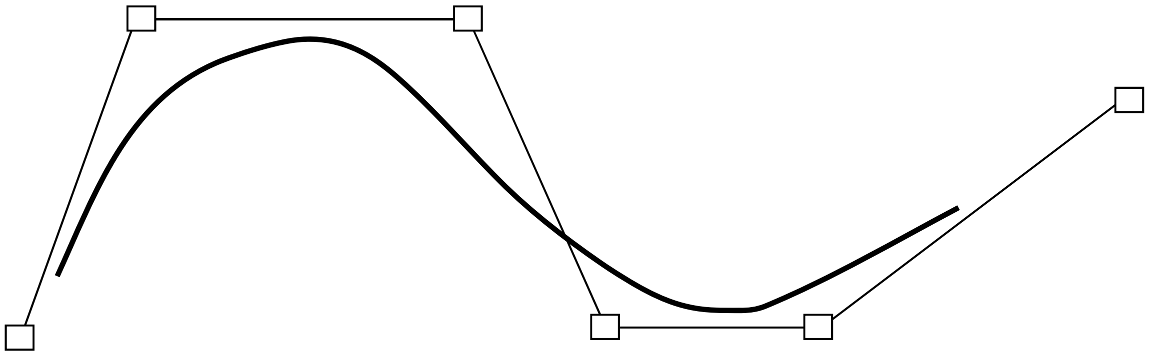
- Control points that have influence in the region $[t_l, t_{l+1}]$ are
  - $P_{l-k+1}, P_{l-k+2}, \cdots, P_l$ k control points
  - Control points modify: example

# B-spline curve - Knot

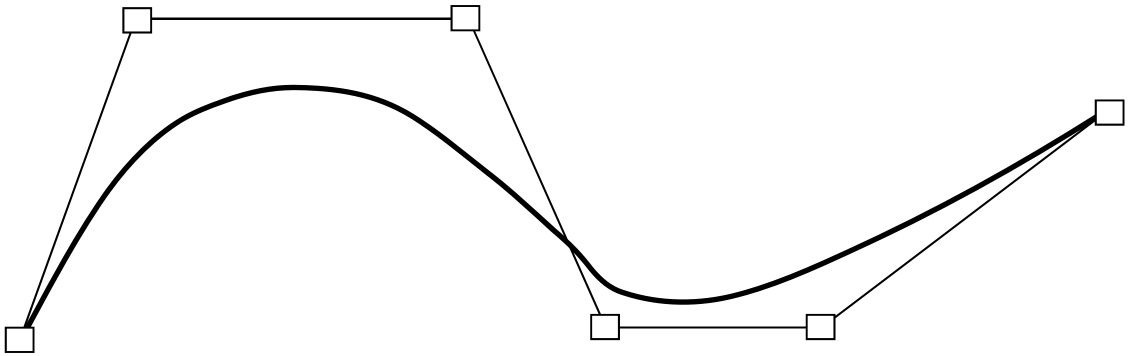
- Knot: $t_0, t_1, \cdots, t_{n+k}$
  - parameter range is determined by knots
  - Periodic knots
    - $t_i = i-k \qquad 0 \le i \le n+k$
  - Non-periodic knots

$$t_i = \begin{cases} 0 & 0 \le i < k & \text{duplicates k times} \\ i-k+1 & k \le i \le n & \\ n-k+2 & n < i \le n+k & \text{duplicates k times} \end{cases} \qquad \text{(d)}$$

# Periodic vs. Non-periodic



Periodic knot



Non-periodic knot

# B-spline curve - Knot

- By duplicating knot  k times at the ends
  - Curve passes through first control point and last control point
- Periodic knot
  - First control point and last control point do not pass through curve as other control points
  - non-periodic knots are used in most CAD systems
- knot interval is uniform in (d)
  - uniform  B-spline (vs. non-uniform B-spline)
- During manipulation of curve shape, knots are added or removed
  - non-uniform knot → non-uniform B-spline curve

# Example program

- Knot Insertion
  - Example

# Expansion of curve equation

- Ex)
  - $K=3$, $P_0$, $P_1$, $P_2$ non-periodic uniform B-spline

    $t_0=0,\ t_1=0,\ t_2=0,\ t_3=1,\ t_4=1,\ t_5=1$

$$\underset{(=t_2)}{\underset{\uparrow}{0}} \le u \le \underset{(=t_3)}{\underset{\uparrow}{1}}$$

# Expansion of curve equation – cont'

$$N_{0,1}(u) = \begin{cases} 1 & t_0 \le u \le t_1 \\ 0 & \text{otherwise} \end{cases} \qquad (u = 0)$$

$$N_{1,1}(u) = \begin{cases} 1 & t_1 \le u \le t_2 \\ 0 & \text{otherwise} \end{cases} \qquad (u = 0)$$

$$N_{2,1}(u) = \begin{cases} 1 & t_2 \le u \le t_3 \\ 0 & \text{otherwise} \end{cases} \qquad (0 \le u \le 1)$$

$$N_{3,1}(u) = \begin{cases} 1 & t_3 \le u \le t_4 \\ 0 & \text{otherwise} \end{cases} \qquad (u = 1)$$

$$N_{4,1}(u) = \begin{cases} 1 & t_4 \le u \le t_5 \\ 0 & \text{otherwise} \end{cases} \qquad (u = 1)$$

# Expansion of curve equation – cont'

- At u=0, select $N_{2,1}(u)$ to be non-zero among $N_{0,1}(0)$, $N_{1,1}(0)$, $N_{2,1}(0)$
- Selection of any one is O.K.
- At u=1, select $N_{2,1}(u)$ similarly
- Only $N_{2,1}(u)$ needs to be considered among blending functions of order 1

# Expansion of curve equation – cont'

$$N_{1,2}(u) = \frac{(u - t_1)N_{1,1}}{t_2 - t_1} + \frac{(t_3 - u)N_{2,1}}{t_3 - t_2} = \frac{(1-u)N_{2,1}}{1} = (1-u)$$

$$N_{2,2}(u) = \frac{(u - t_2)N_{2,1}}{t_3 - t_2} + \frac{(t_4 - u)N_{3,1}}{t_4 - t_3} = \frac{uN_{2,1}}{1} = u$$

$$N_{0,3}(u) = \frac{(u - t_0)N_{0,2}}{t_2 - t_0} + \frac{(t_3 - u)N_{1,2}}{t_3 - t_1} = \frac{(1-u)N_{1,2}}{1} = (1-u)^2$$

$$N_{1,3}(u) = \frac{(u - t_1)N_{1,2}}{t_3 - t_1} + \frac{(t_4 - u)N_{2,2}}{t_4 - t_2} = u(1-u) + (1-u)u = 2u(1-u)$$

$$N_{2,3}(u) = \frac{(u - t_2)N_{2,2}}{t_4 - t_2} + \frac{(t_5 - u)N_{3,2}}{t_5 - t_3} = u^2$$

$$\therefore \mathbf{P}(u) = (1-u)^2\mathbf{P}_0 + 2u(1-u)\mathbf{P}_1 + u^2\mathbf{P}_2$$

# Expansion of curve equation – cont'

- Consider Bezier curve defined by $P_0$, $P_1$, $P_2$

$$\mathbf{P}(u) = \binom{2}{0} u^0 (1-u)^2 \mathbf{P}_0 + \binom{2}{1} u^1 (1-u)^1 \mathbf{P}_1 + \binom{2}{2} u^2 (1-u)^0 \mathbf{P}_2$$

- Non-periodic B-spline curve having k (order) control points ends in Bezier curve
- Bezier curve is a special case of B-spline curve

# Example

- Ex)

$K=3$, $P_0, P_1, P_2, P_3, P_4, P_5$

$t_0=0,\ t_1=0,\ t_2=0,\ t_3=1,\ t_4=2,\ t_5=3,\ t_6=4,\ t_7=4,\ t_8=4$

$0 \le u \le 4$

$$N_{2,1}(u) = \begin{cases} 1 & 0 \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{3,1}(u) = \begin{cases} 1 & 1 \le u \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{4,1}(u) = \begin{cases} 1 & 2 \le u \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{5,1}(u) = \begin{cases} 1 & 3 \le u \le 4 \\ 0 & \text{otherwise} \end{cases}$$

# Example – cont'

$$N_{1,2}(u) = \frac{(u - t_1)N_{1,1}}{t_2 - t_1} + \frac{(t_3 - u)N_{2,1}}{t_3 - t_2} = (1 - u)N_{2,1}$$

$$N_{2,2}(u) = \frac{(u - t_2)N_{2,1}}{t_3 - t_2} + \frac{(t_4 - u)N_{3,1}}{t_4 - t_3} = u \quad N_{2,1} + (2 - u)N_{3,1}$$

$$N_{3,2}(u) = \frac{(u - t_3)N_{3,1}}{t_4 - t_3} + \frac{(t_5 - u)N_{4,1}}{t_5 - t_4} = (u - 1)N_{3,1} + (3 - u)N_{4.1}$$

$$N_{4,2}(u) = \frac{(u - t_4)N_{4,1}}{t_5 - t_4} + \frac{(t_6 - u)N_{5,1}}{t_6 - t_5} = (u - 2)N_{4,1} + (4 - u)N_{5.1}$$

$$N_{5,2}(u) = \frac{(u - t_5)N_{5,1}}{t_6 - t_5} + \frac{(t_7 - u)N_{6,1}}{t_7 - t_6} = (u - 3)N_{5,1}$$

# Example – cont'

$$N_{0,3}(u) = \frac{(u-t_0)N_{0,2}}{t_2-t_0} + \frac{(t_3-u)N_{1,2}}{t_3-t_1} = (1-u)N_{1,2} = (1-u)^2 N_{2,1}$$

$$N_{1,3}(u) = \frac{(u-t_1)N_{1,2}}{t_3-t_1} + \frac{(t_4-u)N_{2,2}}{t_4-t_2} = u \quad N_{1,2} + \frac{2-u}{2} N_{2,2}$$

$$= \left[ u(1\text{-}u) + \frac{(2\text{-}u)u}{2} \right] N_{2,1} + \frac{(2\text{-}u)^2}{2} N_{3.1}$$

$$N_{2,3}(u) = \frac{(u-t_2)N_{2,2}}{t_4-t_2} + \frac{(t_5-u)N_{3,2}}{t_5-t_3} = \frac{u}{2} N_{2,2} + \frac{3-u}{2} N_{3,2}$$

$$= \frac{u^2}{2} N_{2,1} + \left[ \frac{u(2-u)}{2} + \frac{(3-u)(u-1)}{2} \right] N_{3,1} + \frac{(3-u)^2}{2} N_{4,1}$$

$$N_{3,3}(u) = \frac{(u-t_3)N_{3,2}}{t_5-t_3} + \frac{(t_6-u)N_{4,2}}{t_6-t_4} = \frac{u-1}{2} N_{3,2} + \frac{4-u}{2} N_{4,2}$$

$$= \frac{(u\text{-}1)^2}{2} N_{3,1} + \left[ \frac{(u\text{-}1)(3\text{-}u)}{2} + \frac{(4\text{-}u)(u\text{-}2)}{2} \right] N_{4,1} + \frac{(4\text{-}u)^2}{2} N_{5,1}$$

# Example – cont'

$$N_{4,3}(u) = \frac{(u - t_4)N_{4,2}}{t_6 - t_4} + \frac{(t_7 - u)N_{5,2}}{t_7 - t_5} = \frac{u-2}{2}N_{4,2} + (4-u)N_{5,2}$$

$$= \frac{(u-2)^2}{2}N_{4,1} + \left[\frac{(u-2)(4-u)}{2} + (4-u)(u-3)\right]N_{5,1}$$

$$N_{5,3}(u) = \frac{(u - t_5)N_{5,2}}{t_7 - t_5} + \frac{(t_8 - u)N_{6,2}}{t_8 - t_6} = (u-3)N_{5,2} = (u-3)^2 N_{5,1}$$

# Example – cont'

$$\therefore \mathbf{P}(\mathrm{u}) = (1-\mathrm{u})^2 N_{2,1}\mathbf{P}_0 + \left\{\left[\mathrm{u}(1-\mathrm{u}) + \frac{(2-\mathrm{u})\mathrm{u}}{2}\right]N_{2,1} + \frac{(2-\mathrm{u})^2}{2}N_{3,1}\right\}\mathbf{P}_1$$

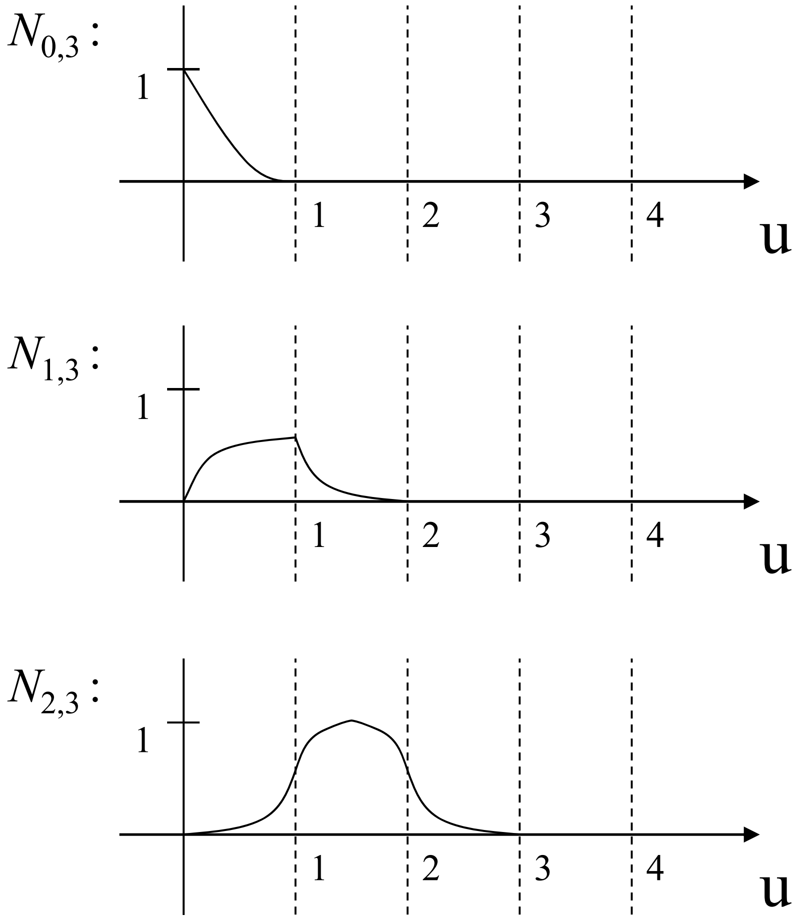
$$+\left\{\frac{\mathrm{u}^2}{2}N_{2,1} + \left[\frac{\mathrm{u}(2-\mathrm{u})}{2} + \frac{(3-\mathrm{u})(\mathrm{u}-1)}{2}\right]N_{3,1} + \frac{(3-\mathrm{u})^2}{2}N_{4,1}\right\}\mathbf{P}_2$$

$$+\left\{\frac{(\mathrm{u}-1)^2}{2}N_{3,1} + \left[\frac{(\mathrm{u}-1)(3-\mathrm{u})}{2} + \frac{(4-\mathrm{u})(\mathrm{u}-2)}{2}\right]N_{4,1} + \frac{(4-\mathrm{u})^2}{2}N_{5,1}\right\}\mathbf{P}_3$$

$$+\left\{\frac{(\mathrm{u}-2)^2}{2}N_{4,1} + \left[\frac{(\mathrm{u}-2)(4-\mathrm{u})}{2} + (4-\mathrm{u})(\mathrm{u}-3)\right]N_{5,1}\right\}\mathbf{P}_4$$

$$+(\mathrm{u}-3)^2 N_{5,1} \quad \mathbf{P}_5$$

# Example – cont'



# Example – cont'

- For each knot interval, coefficients of certain control points = 0
  - only subset of control points has influence

- For 0≤u≤1, all $N_{i,1}$ except $N_{2,1}$ are 0

$$\therefore \mathbf{P}_1(u) = (1-u)^2\mathbf{P}_0 + \left[u(1-u) + \frac{(2-u)u}{2}\right]\mathbf{P}_1 + \frac{u^2}{2}\mathbf{P}_2$$

# Example – cont'

- Similarly

$$1 \le u \le 2$$

$$\mathbf{P}_2(u) = \frac{(2-u)^2}{2}\mathbf{P}_1 + \left[\frac{u(2-u)}{2} + \frac{(3-u)(u-1)}{2}\right]\mathbf{P}_2 + \frac{(u-1)^2}{2}\mathbf{P}_3$$

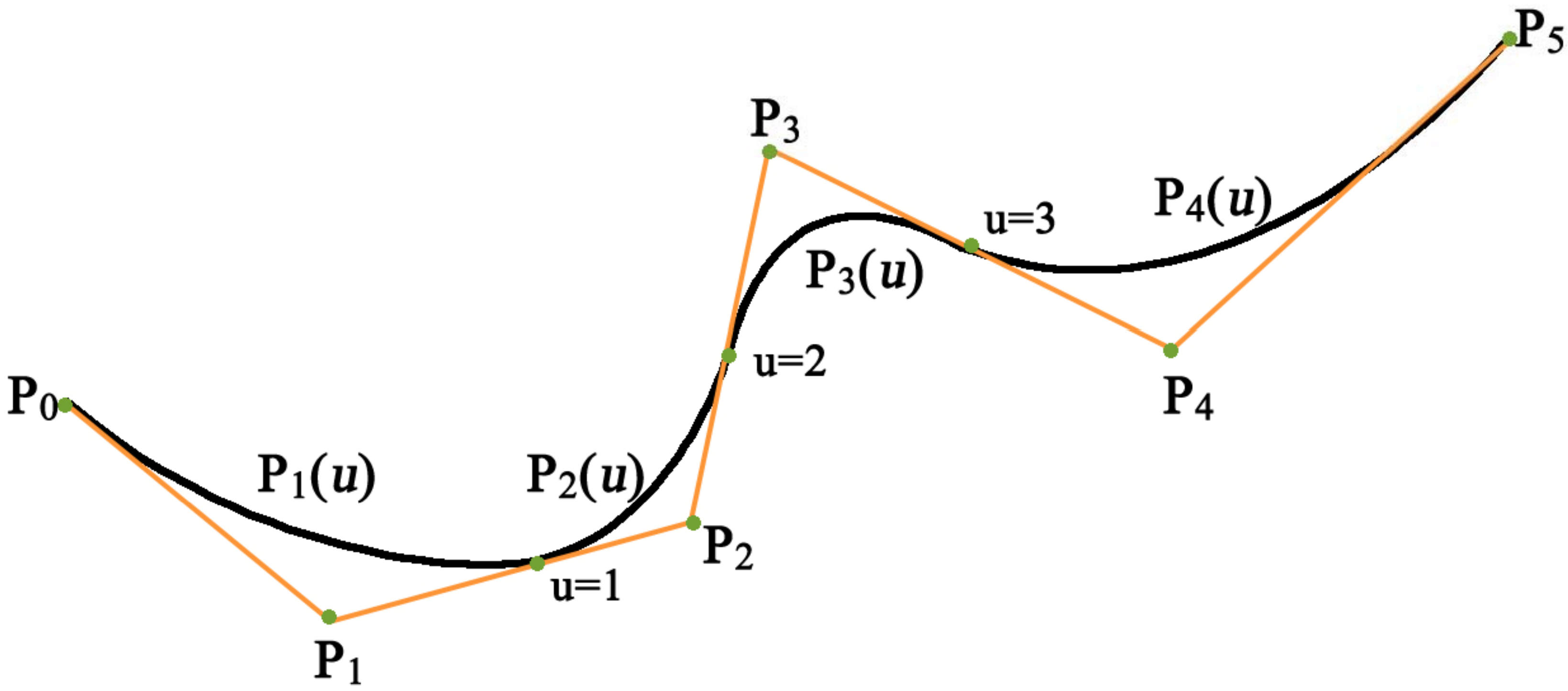
$$2 \le u \le 3$$

$$\mathbf{P}_3(u) = \frac{(3-u)^2}{2}\mathbf{P}_2 + \frac{1}{2}(-2u^2 + 10u - 11)\mathbf{P}_3 + \frac{(u-2)^2}{2}\mathbf{P}_4$$

$$3 \le u \le 4$$

$$\mathbf{P}_4(u) = \frac{(4-u)^2}{2}\mathbf{P}_3 + \frac{1}{2}(-3u^2 + 20u - 32)\mathbf{P}_4 + (u-3)^2\mathbf{P}_5$$

# Example – cont'

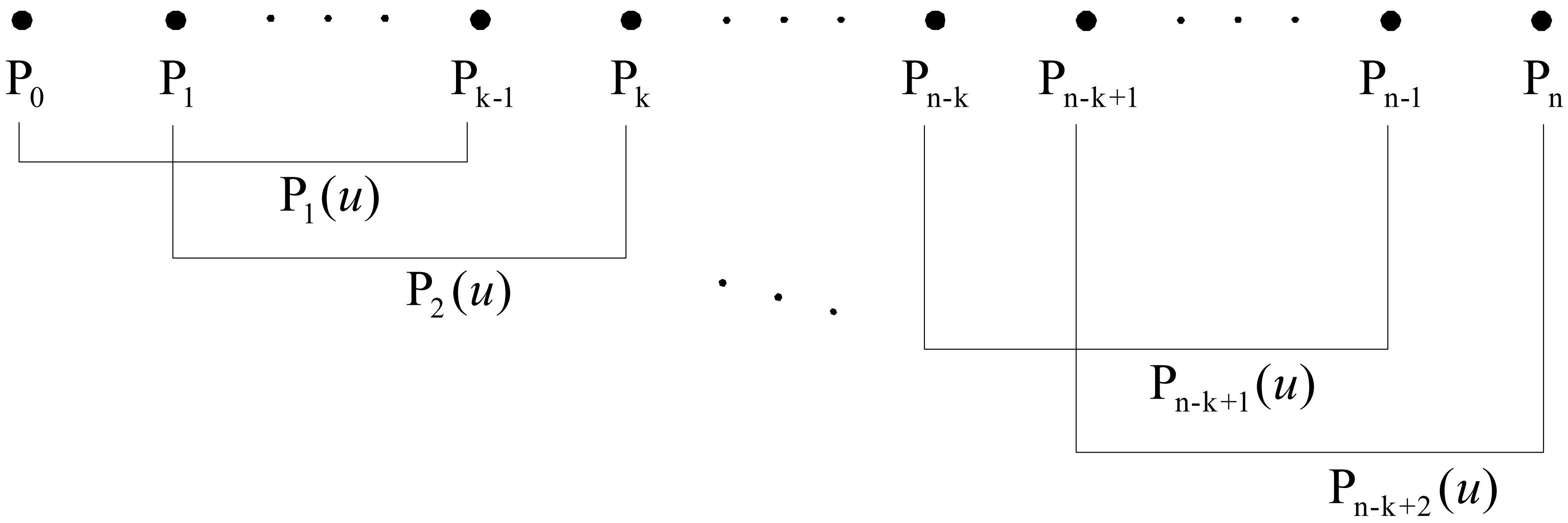


# Example – cont'

- $\mathbf{P}'_1(1) = \mathbf{P}'_2(1), \quad \mathbf{P}'_2(2) = \mathbf{P}'_3(2), \quad \mathbf{P}'_3(3) = \mathbf{P}'_4(3) \quad \mathrm{C}^1\text{continuiy}$
- C2 continuity is not satisfied. ( ∵ k=3, degree 2)
  - For curve of order k, neighboring curves have same derivatives up to (k-2)-th derivative at the common knot
- Each curve segment is defined by k control points.
- Any one control point can influence up to maximum k  curve segments.

# Example – cont'

count curve segment including $\mathbf{P}_{k-1}$



# Intersection between curves

- $\mathbf{P}(u) - \mathbf{Q}(v) = 0$
- 3 scalar equations, two unknowns
  - $P_x(u) - Q_x(v) = 0$
  - $P_y(u) - Q_y(v) = 0$
- Use Newton Raphson method
  - Derivative of $P_x$, $Q_x$, $P_y$, $Q_y$ need to be calculated
  - $$\begin{aligned} f_1(x_1,\ldots,x_n) &= 0 \\ f_2(x_1,\ldots,x_n) &= 0 \\ &\vdots \\ f_n(x_1,\ldots,x_n) &= 0 \end{aligned}$$

# Intersection between curves – cont'

$$f_1(x_1+\Delta x_1, x_2+\Delta x_2, \cdots, x_n+\Delta x_n) = f_1(x_1,\cdots,x_n) + \frac{\partial f_1}{\partial x_1}\Delta x_1 + \cdots + \frac{\partial f_1}{\partial x_n}\Delta x_n$$

$$\vdots$$

$$f_n(x_1+\Delta x_1, x_2+\Delta x_2, \cdots, x_n+\Delta x_n) = f_n(x_1,\cdots,x_n) + \frac{\partial f_n}{\partial x_1}\Delta x_1 + \cdots + \frac{\partial f_n}{\partial x_n}\Delta x_n$$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} \cdots \frac{\partial f_2}{\partial x_n} \\ \vdots \\ \frac{\partial f_n}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_n \end{bmatrix} = \begin{bmatrix} -f_1 \\ -f_2 \\ \vdots \\ -f_n \end{bmatrix}$$

# Intersection between curves – cont'

- If initial values of u, v are too far from real solution, the iteration diverges.
- Hard to find all the intersection points.
- Cannot handle the case of overlapping curves.
- Two curves are regarded to intersect each other if they lie within numerical tolerance.
- Control polygons are approximated to the curve by subdivision and initial values of u,v can be provided closely by intersecting control polygons
- Better to detect special situation in advance before resorting to numerical solution.
- Tuning tolerance values is necessary

# Straight line vs. curve

- $P(u) = P_0 + u(P_1 - P_0)$
- $Q(v) = P_0 + u(P_1 - P_0)$ (a)
- Apply dot product $(P_0 \times P_1)$ to both sides of eq(a) gives

$$(P_0 \times P_1) \cdot Q(v) = 0$$

non-linear equation of v

# Non-uniform Rational B-spline (NURBS) curve

- Use same Blending functions as B-spline
- Control points are given in homogeneous coordinates

$$(x_i, \quad y_i, \quad z_i) \Rightarrow (x_i \bullet h_i, \quad y_i \bullet h_i, \quad z_i \bullet h_i, \quad h_i)$$

$$x \cdot h = \sum_{i=0}^{n} (h_i \cdot x_i) \quad N_{i,k}(u)$$

$$y \cdot h = \sum_{i=0}^{n} (h_i \cdot y_i) \quad N_{i,k}(u)$$

$$z \cdot h = \sum_{i=0}^{n} (h_i \cdot z_i) \quad N_{i,k}(u)$$

$$h = \sum_{i=0}^{n} h_i \quad N_{i,k}(u)$$

# Non-uniform Rational B-spline (NURBS) curve – cont'

$$P(u) = \frac{\sum_{i=0}^{n} h_i P_i N_{i,k}(u)}{\sum_{i=0}^{n} h_i \quad N_{i,k}(u)}$$

Passes through the 1st and the last control points

( When non-periodic knots are used )

Numerator is B-spline with $h_i P_i$ as contorl points

$\longrightarrow$ Z $h_0 P_0$, $h_n P_n$ at parameter boundary values

Similarly denominator has values of $h_0$, $h_n$ at parameter boundary values

# Non-uniform Rational B-spline (NURBS) curve – cont'

Directions of tangent vectors are $P_1$-$P_0$, $P_n$-$P_{n-1}$ at starting and ending points

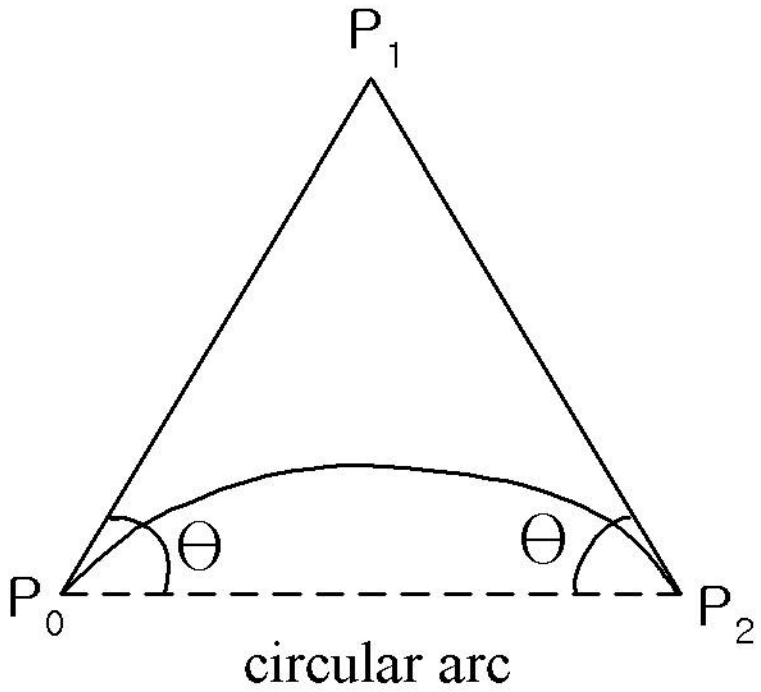
$$h_i = 1 \quad \sum_{i=0}^{n} N_{i,k}(u) = 1 \quad \Rightarrow \quad B - spline$$

B-spline curve is a special case of NURBS

# Non-uniform Rational B-spline (NURBS) curve – cont'

- Curve shape can be changed by changing weight($h_i$)
- Increasing weight has an effect of pulling curve toward associated control point
  - Example program
- Conic curve can be represented exactly
  - Reducing program coding effort

# Control points of NURBS curve equivalent to a circular arc



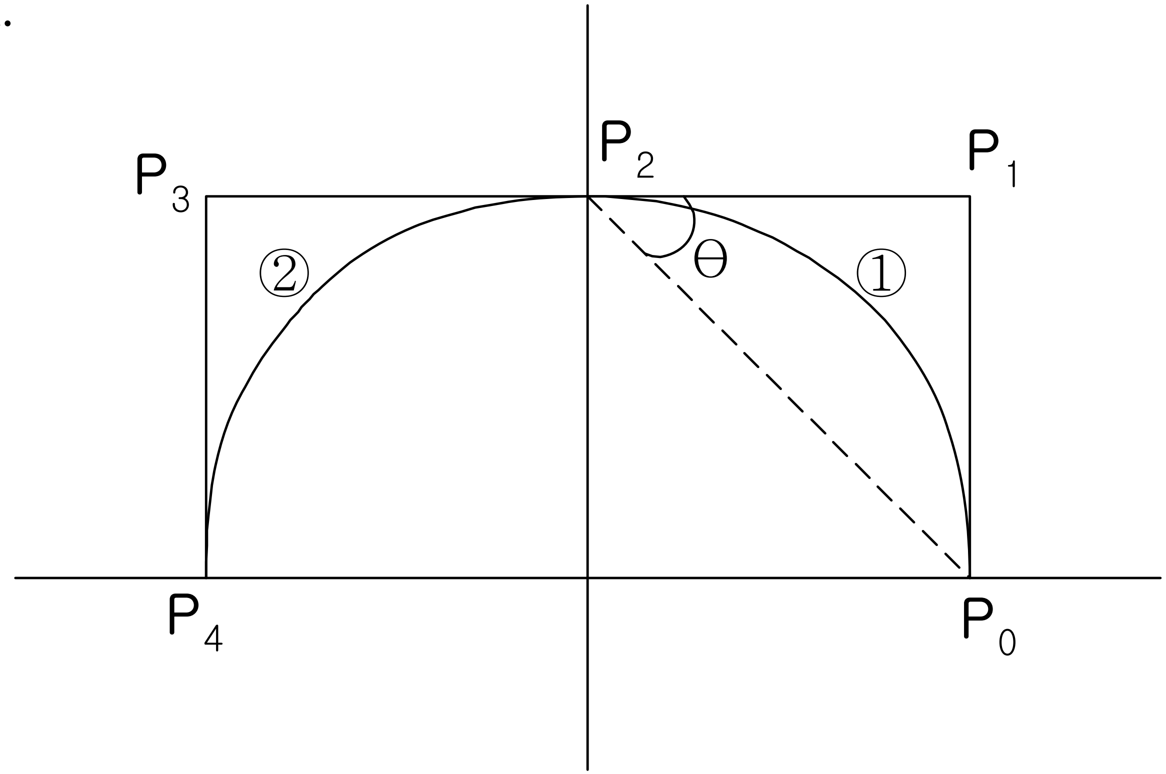

$h_0 = h_2 = 1$
$h_1 = \cos \Theta$

Can be used when center angle is less than 180°.

Arc with a center angle bigger than 180° is split into two and combined later

# Example

Ex.



# Example – cont'

$$P_0 = (1,\ \ 0), \qquad P_1 = (1,\ \ 1), \qquad P_2 = (0,\ \ 1)$$

$$h_0 = 0 \qquad h_1 = \cos 45° = \frac{1}{\sqrt{2}} \qquad h_2 = 1$$

$$knot\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ (n = 2,\ \ k = 3)$$

*Similarly* . $P_2 = (0,\ \ 1),\ \ P_3 = (-1,\ \ 1),\ \ P_4 = (-1,\ \ 0)$

$$h_2 = 1, \qquad h_3 = \frac{1}{\sqrt{2}}, \qquad h_4 = 1$$

$$knot\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ \Rightarrow\ \ 1\ \ 1\ \ 1\ \ 2\ \ 2\ \ 2$$

# Example – cont'

Composition

$$P_0 = (1,\ \ 0),\quad P_1 = (1,\ \ 1),\quad P_2 = (0,\ \ 1),$$

$$P_3 = (-1,\ \ 1),\quad P_4 = (-1,\ \ 0)$$

$$h_0 = 1 \qquad h_1 = \frac{1}{\sqrt{2}} \qquad h_2 = 1$$

$$h_3 = \frac{1}{\sqrt{2}}, \qquad h_4 = 1$$

$$knot \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 2 \quad 2 \quad 2$$