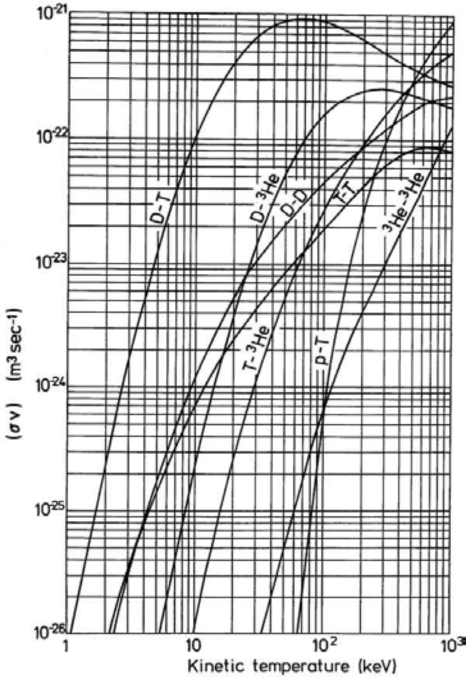
Introduction to Nuclear Fusion

Prof. Dr. Yong-Su Na

How to describe a plasma?

Description of a Plasma

- **Thermonuclear fusion with high energy particles of 10-20 keV energy → plasma**



- **Three approaches to describe a plasma**

- Single particle approach
- Kinetic theory
- Fluid theory

http://http://dz-dev.net/blog/tag/eruption-solaire

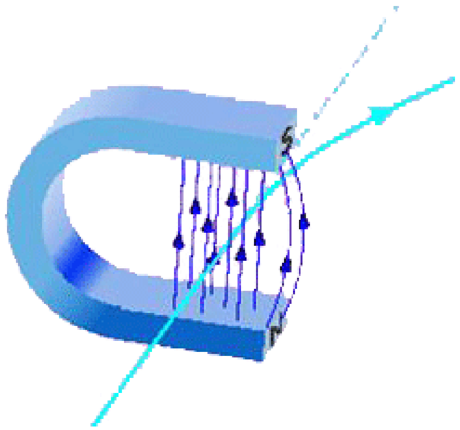
Individual Charge Trajectories

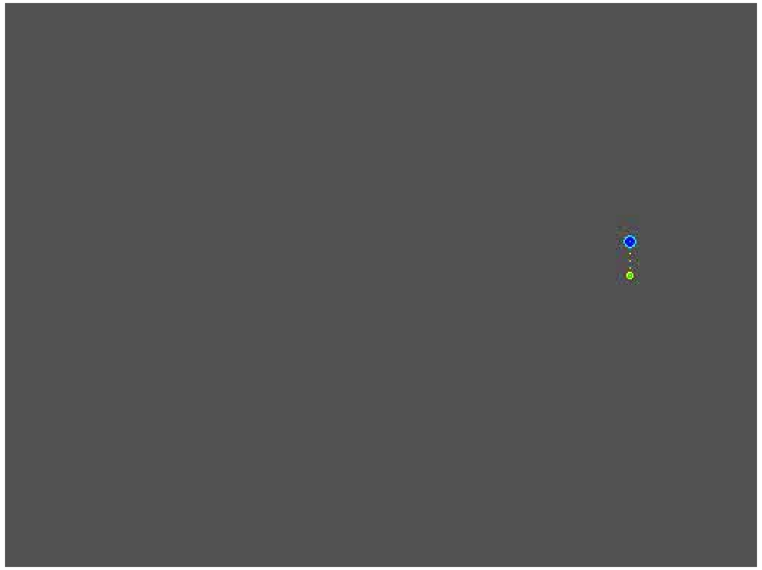
- **Equation of motion**

- Basic relation determining the motion of an individual charged particle of mass *m* and charge *q* in a combined electric (**E**) and magnetic (**B**) field

- Neglecting electromagnetic fields generated by movement of the charge itself

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$





Individual Charge Trajectories

- **Homogeneous electric field**

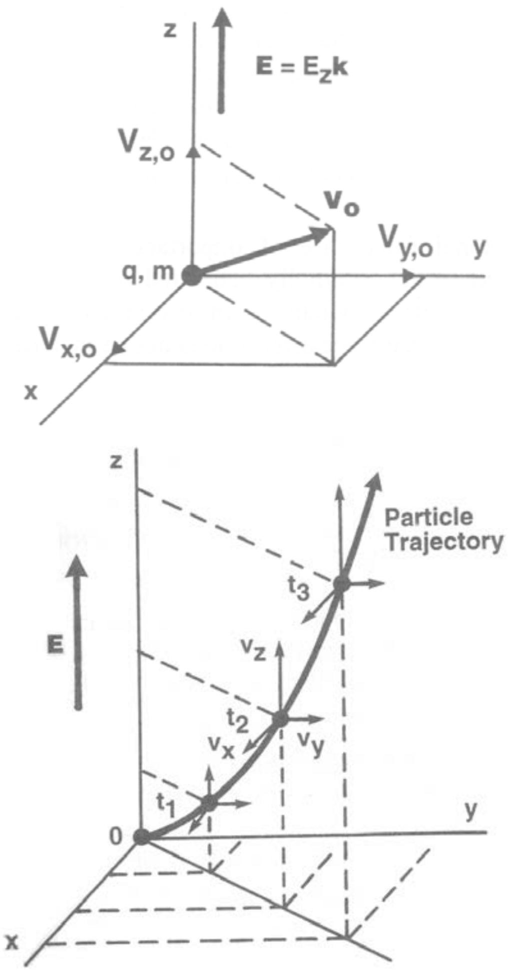
$$m\frac{d\mathbf{v}}{dt} = q\mathbf{E} = qE_z\mathbf{k}$$

$$v_x(0) = v_{x,0}, \quad v_y(0) = v_{y,0}, \quad v_z(0) = v_{z,0}$$

$$v_x(t) = v_{x,0}, \quad v_y(t) = v_{y,0}, \quad v_z(t) = v_{z,0} + \frac{q}{m}E_z t$$

$$x(t) = v_{x,0}t, \quad y(t) = v_{y,0}t,$$

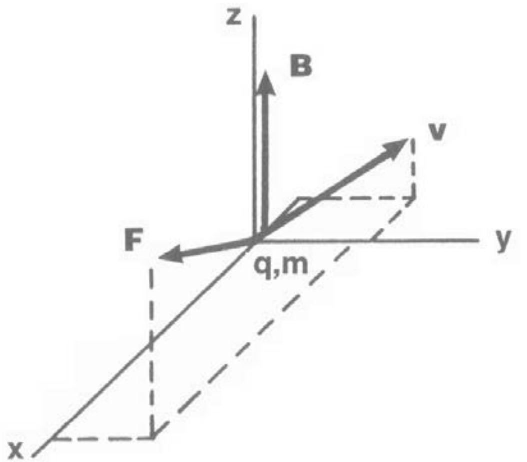
$$z(t) = v_{z,0}t + \frac{1}{2}\left(\frac{q}{m}E_z\right)t^2$$



A. A. Harms et al, "Principles of Fusion Energy", World Scientific (2000)

Individual Charge Trajectories

- **Homogeneous magnetic field**



$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{v}\times\mathbf{B}) = q(\mathbf{v}\times B_z\mathbf{k})$$

$$v_x(0) = v_{x,0},\quad v_y(0) = v_{y,0},\quad v_z(0) = v_{z,0} = v_\parallel$$

$$v_x(t) = v_\perp\cos(\omega_c t+\phi),\quad v_y(t) = \mp v_\perp\sin(\omega_c t+\phi),\quad v_z(t) = v_{z,0} = v_\parallel$$

$$v_\perp = \sqrt{v_x^2+v_y^2},\quad \tan(\phi) = \mp\frac{v_{y,0}}{v_{x,0}},\quad \omega_c = \frac{|q|B_z}{m}$$

Cyclotron frequency

$$x(t) = x_0 + \frac{v_\perp}{\omega_c}\sin(\omega_c t+\phi),\quad y(t) = y_0 \pm \frac{v_\perp}{\omega_c}\cos(\omega_c t+\phi),\quad z(t) = z_0 + v_\parallel t$$

$$(x-x_0)^2+(y-y_0)^2 = \left(\frac{v_\perp}{\omega_c}\right)^2 = \left(\frac{v_\perp m}{|q|B_z}\right)^2 = r_L^2$$

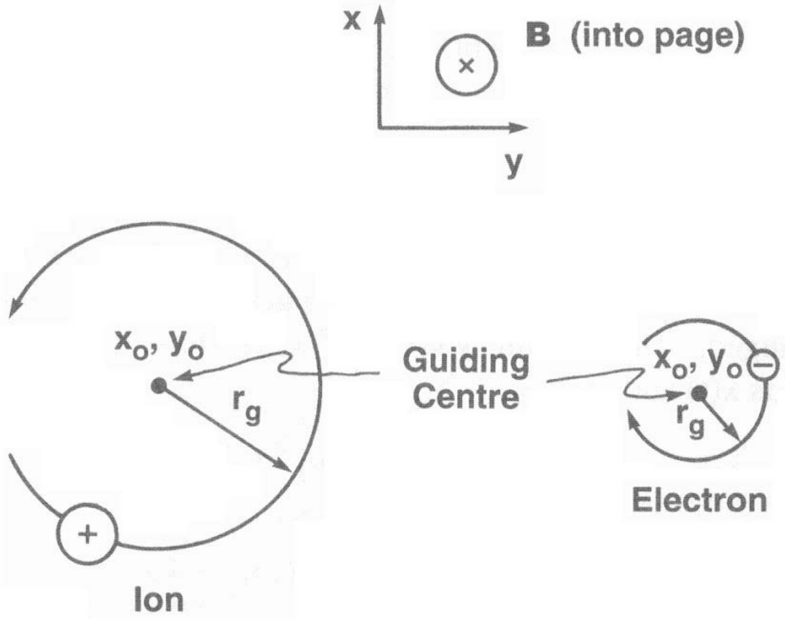
Larmor radius

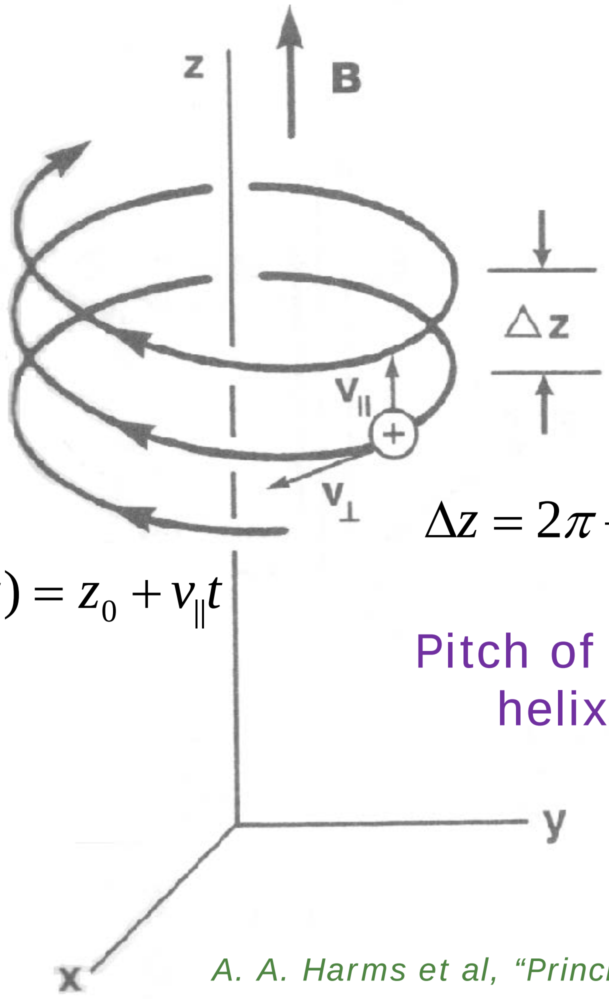
5 T, 10 keV
r_L = 0.05 mm for e
r_L = 2.9 mm for d
r_L = 3.5 mm for t

A. A. Harms et al, “Principles of Fusion Energy”, World Scientific (2000)

Individual Charge Trajectories

- **Homogeneous magnetic field**



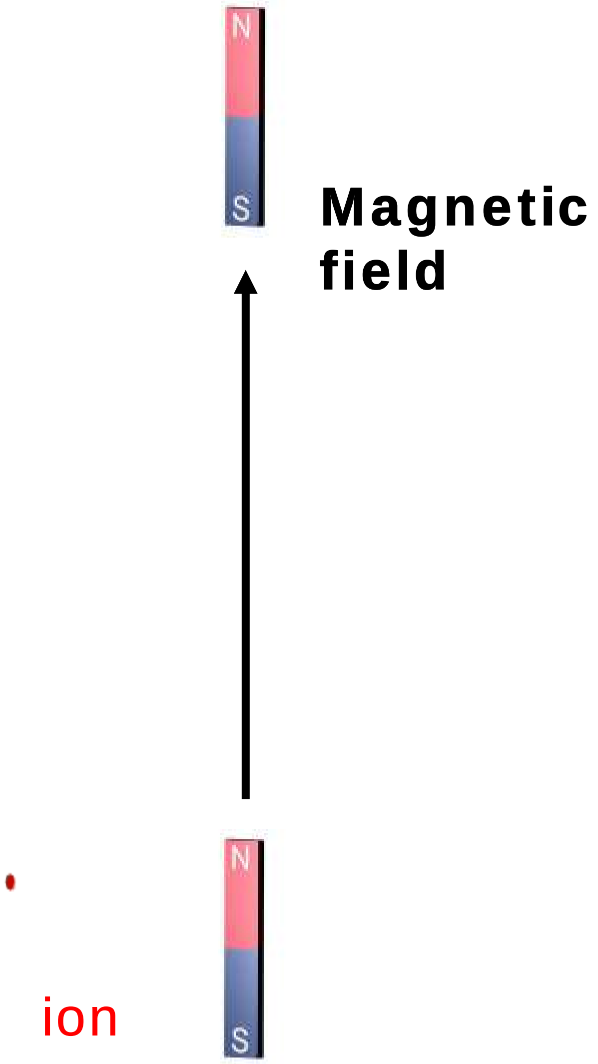


$$z(t) = z_0 + v_\parallel t$$

$$\Delta z = 2\pi \frac{v_\parallel}{\omega_c}$$

Pitch of the helix

$$(x - x_0)^2 + (y - y_0)^2 = \left(\frac{v_\perp m}{|q| B_z}\right)^2 = r_L^2$$



A. A. Harms et al, "Principles of Fusion Energy", World Scientific (2000)

Individual Charge Trajectories

- **Combined homogeneous electric and magnetic field**

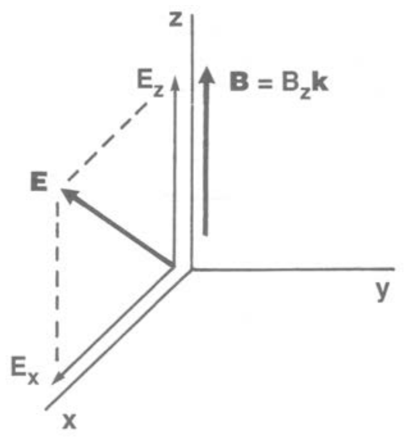
$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = q\left[(E_x\mathbf{i} + E_z\mathbf{k}) + \mathbf{v}\times B_z\mathbf{k}\right]$$

$$v_x(0) = v_{x,0}, \quad v_y(0) = v_{y,0}, \quad v_z(0) = v_{z,0} = v_{\parallel}$$

$$v_x(t) = v_\perp^* \cos(\omega_c t + \phi^*), \quad v_y(t) = \mp v_\perp^* \sin(\omega_c t + \phi^*) - \frac{E_x}{B_z},$$

$$v_z(t) = v_\parallel + \left(\frac{qE_z}{m}\right)t$$

$$v_\perp^* = \sqrt{v_x^2 + \left(v_y + \frac{E_x}{B_z}\right)^2} = \frac{v_\perp \cos\phi}{\cos\phi^*}, \quad \tan(\phi^*) = \mp\frac{v_{y,0} + \dfrac{E_x}{B_z}}{v_{x.0}}$$



A. A. Harms et al, "Principles of Fusion Energy", World Scientific (2000)

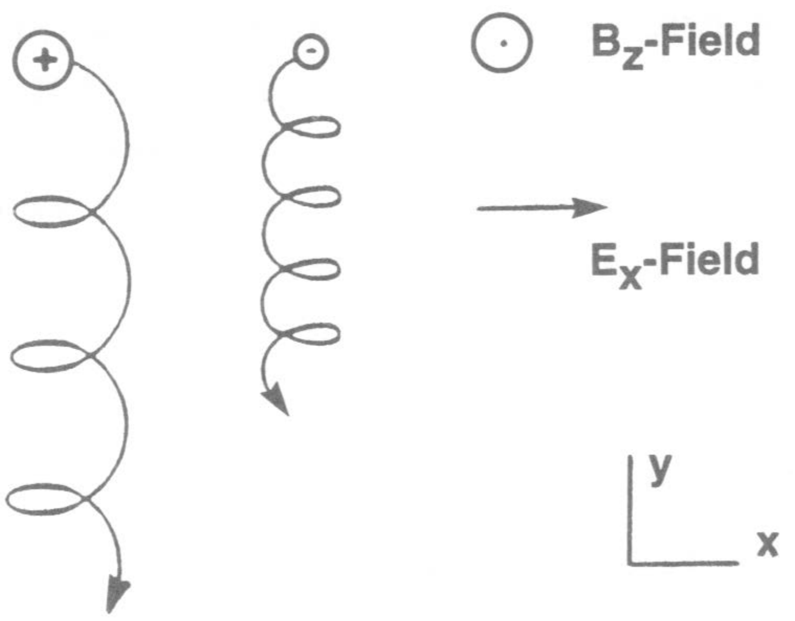
Individual Charge Trajectories

- **Combined homogeneous electric and magnetic field**

$$x(t) = x_0 + \frac{v_\perp^*}{\omega_c}\sin(\omega_c t + \phi^*), \quad y(t) = y_0 \pm \frac{v_\perp^*}{\omega_c}\cos(\omega_c t + \phi^*) - \frac{E_x}{B_z}t,$$

$$z(t) = z_0 + v_\parallel t + \frac{1}{2}\left(\frac{qE_z}{m}\right)t^2$$

$$(x - x_0)^2 + \left(y - y_0 + \frac{E_x}{B_z}t\right)^2 = \left(\frac{v_0^*}{\omega_c}\right)^2$$

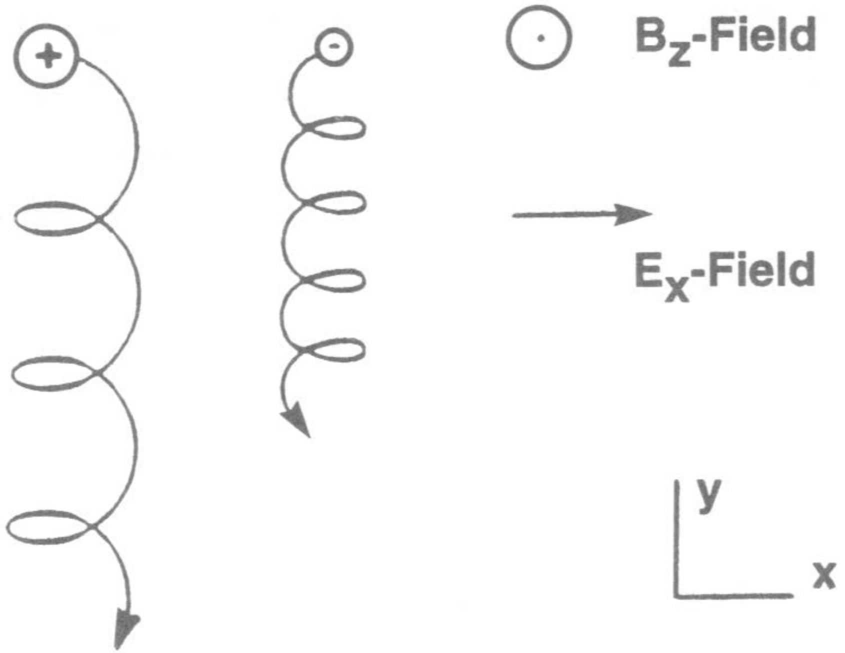


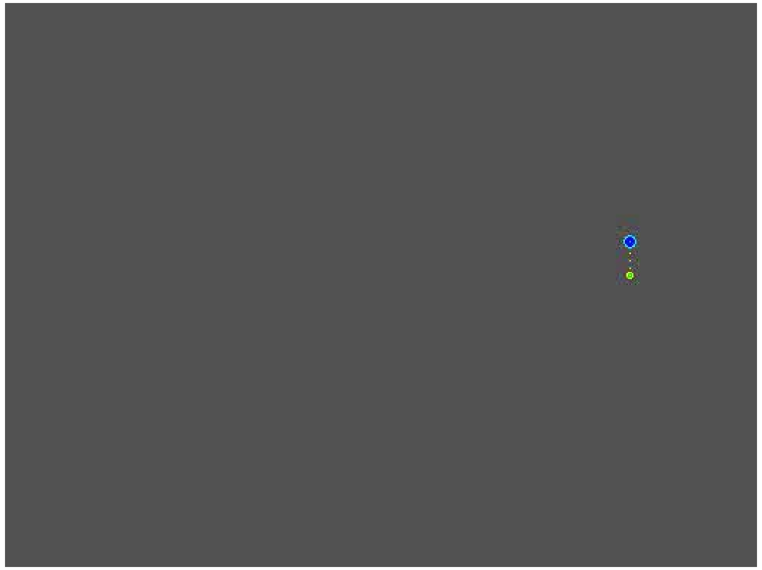
A. A. Harms et al, "Principles of Fusion Energy", World Scientific (2000)

Individual Charge Trajectories

- **Combined homogeneous electric and magnetic field**

$\mathbf{v} = \mathbf{v}_{gc} + \mathbf{v}_g$: Guiding centre + Gyro motion





Individual Charge Trajectories

- **Combined homogeneous electric and magnetic field**

$$\mathbf{v} = \mathbf{v}_{gc} + \mathbf{v}_g$$ **: Guiding centre + Gyro motion**

2.2 IN MAGNETIC FIELDS 17

Because of this change of momentum, the *centre of curvature* of the projection on the *xy*-plane of the path is displaced a distance

$$\mathbf{D} = \boldsymbol{\rho}' - \boldsymbol{\rho} \quad (9)$$

or, because of (6)–(8),

$$\mathbf{D} = -\frac{c}{eH^2}[\mathbf{H}\Delta\mathbf{p}_\perp], \quad (10)$$

$$\mathbf{D} = -\frac{c}{eH^2}\left[\mathbf{H}\int \mathbf{f}_\perp\, dt\right]. \quad (11)$$

This formula holds for a single collision, but of course also for a series of collisions. If **f** is a continuous force, **D** gives the displacement of that point where the centre of curvature would be if **f** vanished for a moment. This point shall be called the *guiding centre*. If **f** is continuous, *the guiding centre drifts with the velocity*

$$\mathbf{U}_\perp = \frac{d\mathbf{D}}{dt} = -\frac{c}{eH^2}[\mathbf{H}\mathbf{f}]. \quad (12)$$

H. Alfvén, Cosmical Electrodynamics, Oxford (1950)

Hannes Alfvén
(1908-1995)

"Nobel prize in Physics (1970)"

Wikipedia

Individual Charge Trajectories

- **Combined homogeneous electric and magnetic field**

$$\mathbf{v} = \mathbf{v}_{gc} + \mathbf{v}_g$$: Guiding centre + Gyro motion

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} + q(\mathbf{v} \times \mathbf{B})$$

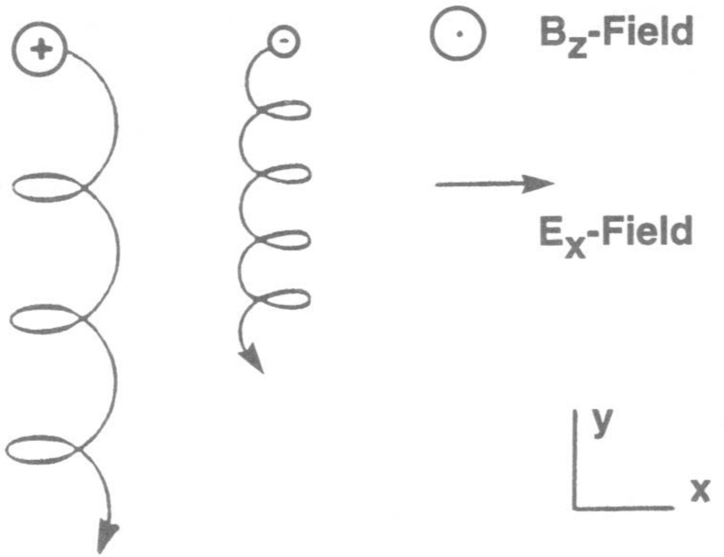
$$m\frac{d\mathbf{v}_{gc}}{dt} + m\frac{d\mathbf{v}_g}{dt} = \mathbf{F} + q(\mathbf{v}_{gc} \times \mathbf{B}) + q(\mathbf{v}_g \times \mathbf{B})$$

$$v_{gc,\parallel}(t) = v_{gc,\parallel}(0) + \frac{1}{m}\int F_\parallel dt$$

$$\bar{\mathbf{v}}_{gc,\perp} = \frac{\bar{\mathbf{F}} \times \mathbf{B}}{qB^2} = \mathbf{v}_{DF}$$

$$\mathbf{F} = \mathbf{E}q$$

$$\mathbf{v}_{D,E} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

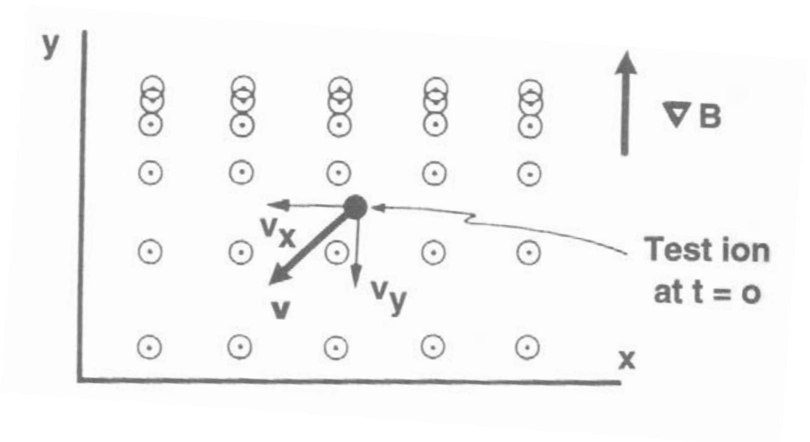


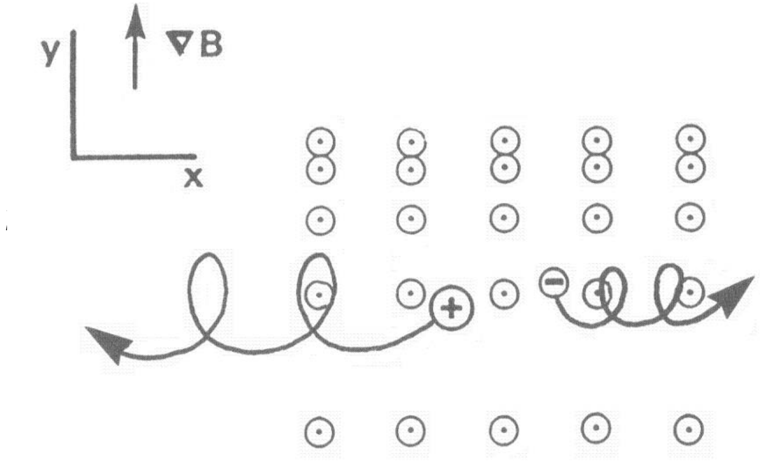
Individual Charge Trajectories

- **Spatially varying magnetic field**

$$m\frac{d\mathbf{v}}{dt} = q\left[\mathbf{v}\times\mathbf{B}(r)\right]$$

$$\nabla B = \frac{\partial B_z}{\partial y}\mathbf{j}, \quad v_{\parallel} = 0$$





$$\overline{\mathbf{F}} \approx -|q|\left(\frac{dB}{dy}\right)\left(\frac{v_\perp^2}{2\omega_c}\right)\mathbf{j} = -|q|\frac{v_\perp^2}{2\omega_c}\nabla B$$

$$= -\frac{|q|}{2}v_\perp r_L \nabla B = -\frac{mv_\perp^2/2}{B}\nabla B$$

$$\mathbf{v}_{D,\nabla B} = \frac{\overline{\mathbf{F}}\times\mathbf{B}}{qB^2}$$

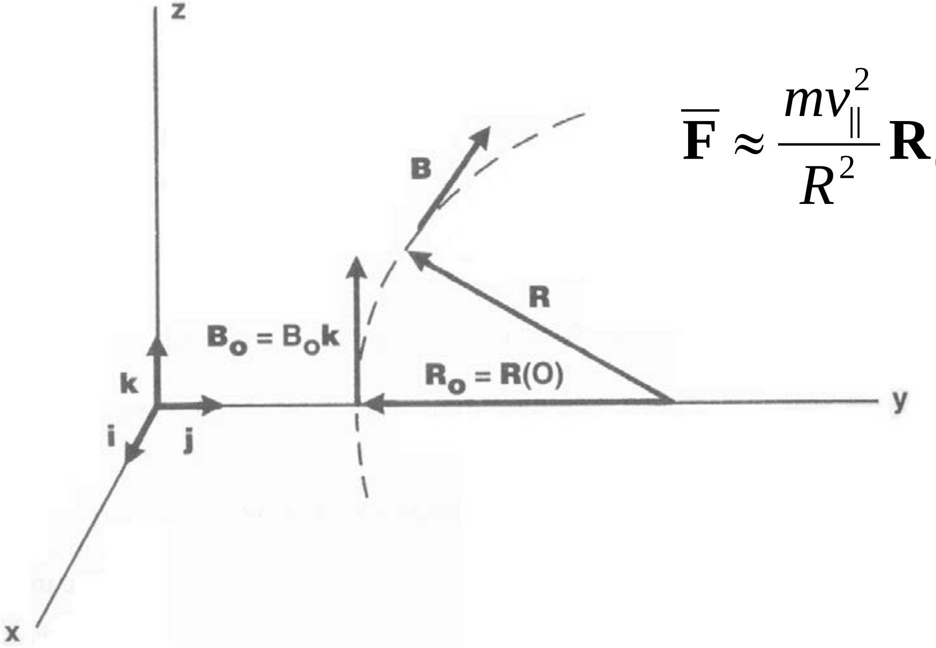
$$= \pm\frac{v_\perp^2}{2\omega_c}\frac{\mathbf{B}\times\nabla B}{B^2} = \pm\frac{1}{2}v_\perp r_L\frac{\mathbf{B}\times\nabla B}{B^2}$$

A. A. Harms et al, "Principles of Fusion Energy", World Scientific (2000)

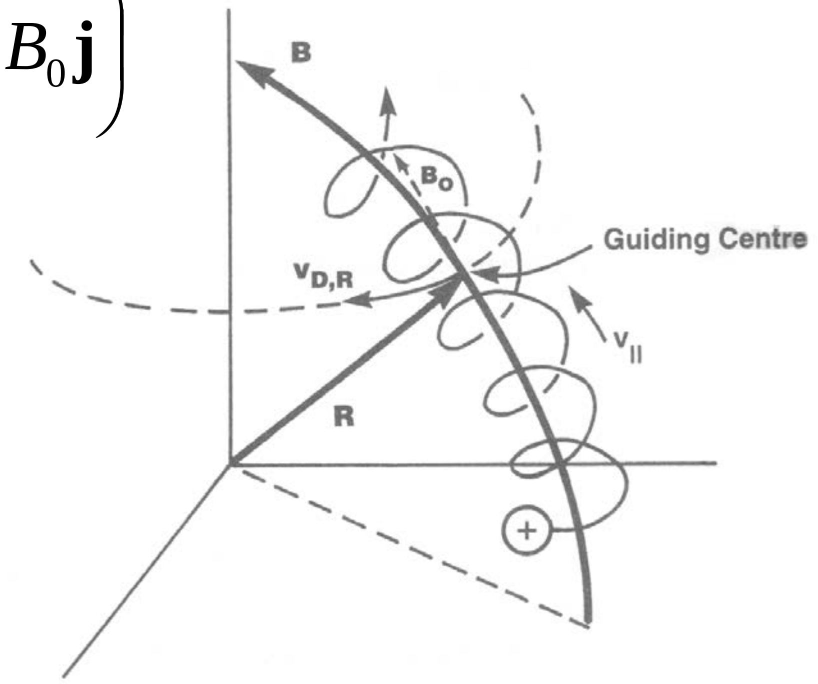
Individual Charge Trajectories

- **Curvature drift**

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{v}\times\mathbf{B}) = q(\mathbf{v}\times B_0\mathbf{k}) + q\left(\mathbf{v}\times\frac{z}{R}B_0\mathbf{j}\right)$$



$$\overline{\mathbf{F}} \approx \frac{mv_{\parallel}^2}{R^2}\mathbf{R}_0$$



$$\mathbf{v}_{D,R} = \overline{\mathbf{v}}_{gc,x} = \frac{mv_{\parallel}^2}{qB_0^2}\frac{\mathbf{R}_0\times\mathbf{B}_0}{R^2}$$

A. A. Harms et al, "Principles of Fusion Energy", World Scientific (2000)

Individual Charge Trajectories

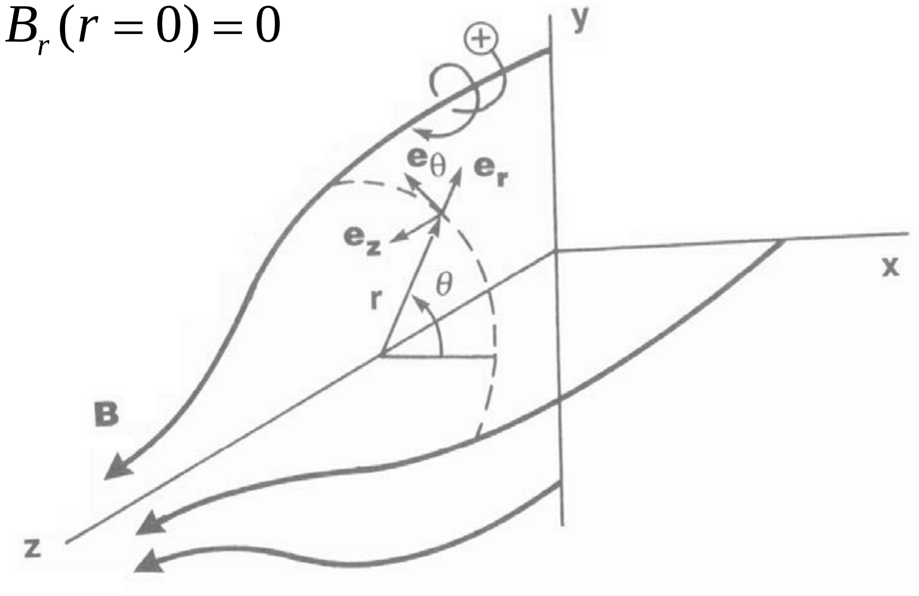
- **Axial field variation**

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{v}\times B_z\mathbf{e}_z\right) + q\left(\mathbf{v}\times B_r\mathbf{e}_r\right)$$

$$\nabla\cdot\mathbf{B} = 0 \rightarrow B_r \approx -\frac{r}{2}\frac{\partial B_z}{\partial z} \qquad B_\theta = 0,\quad B_r(r=0) = 0$$

$$\mathbf{F}_\parallel = -\frac{1}{2}\frac{mv_\perp^2}{B}\nabla_\parallel B = -\mu\nabla_\parallel B$$

μ : magnetic moment of the gyrating particle



A. A. Harms et al, “Principles of Fusion Energy”, World Scientific (2000)

Individual Charge Trajectories

- **Invariant of motion**

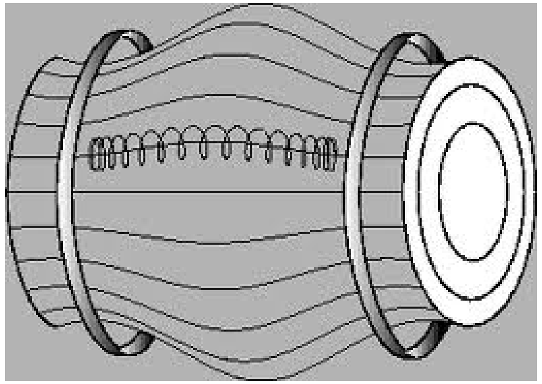
$$\frac{d}{dt}E_0 = \frac{d}{dt}\left(\frac{1}{2}mv_\perp^2 + \frac{1}{2}mv_\parallel^2\right) = 0 \qquad \mu = \frac{mv_\perp^2/2}{B} \qquad m\frac{dv_\parallel}{dt} = -\frac{\mu}{v_\parallel}\frac{dB}{dt} \qquad mv_\parallel\frac{dv_\parallel}{dt} = -\mu\frac{dB}{dt}$$

$$\mathbf{F}_\parallel = m\frac{d\mathbf{v}_\parallel}{dt} = -\mu\nabla_\parallel\mathbf{B} = -\mu\frac{\partial\mathbf{B}}{\partial s} = -\mu\frac{\partial\mathbf{B}}{\partial s}\cdot\frac{ds}{dt}\cdot\frac{dt}{ds} = -\mu\frac{\partial\mathbf{B}}{\partial s}\cdot\frac{ds}{dt}\cdot\frac{1}{v_\parallel} = -\frac{\mu}{v_\parallel}\frac{d\mathbf{B}}{dt} \longrightarrow \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu\frac{dB}{dt}$$

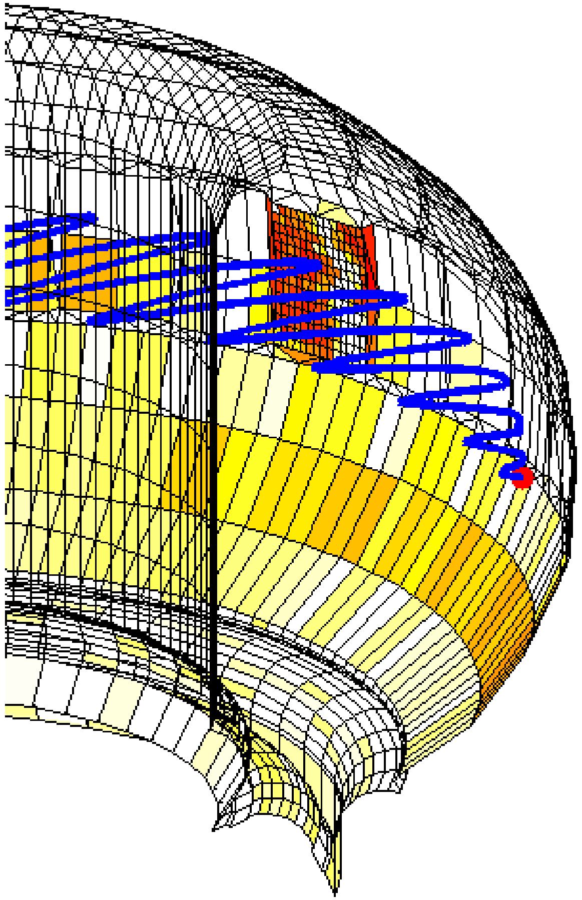
$$\longrightarrow \frac{d}{dt}\left(\frac{1}{2}mv_\perp^2 + \frac{1}{2}mv_\parallel^2\right) = \frac{d}{dt}(\mu B) + \left(-\mu\frac{dB}{dt}\right) = 0$$

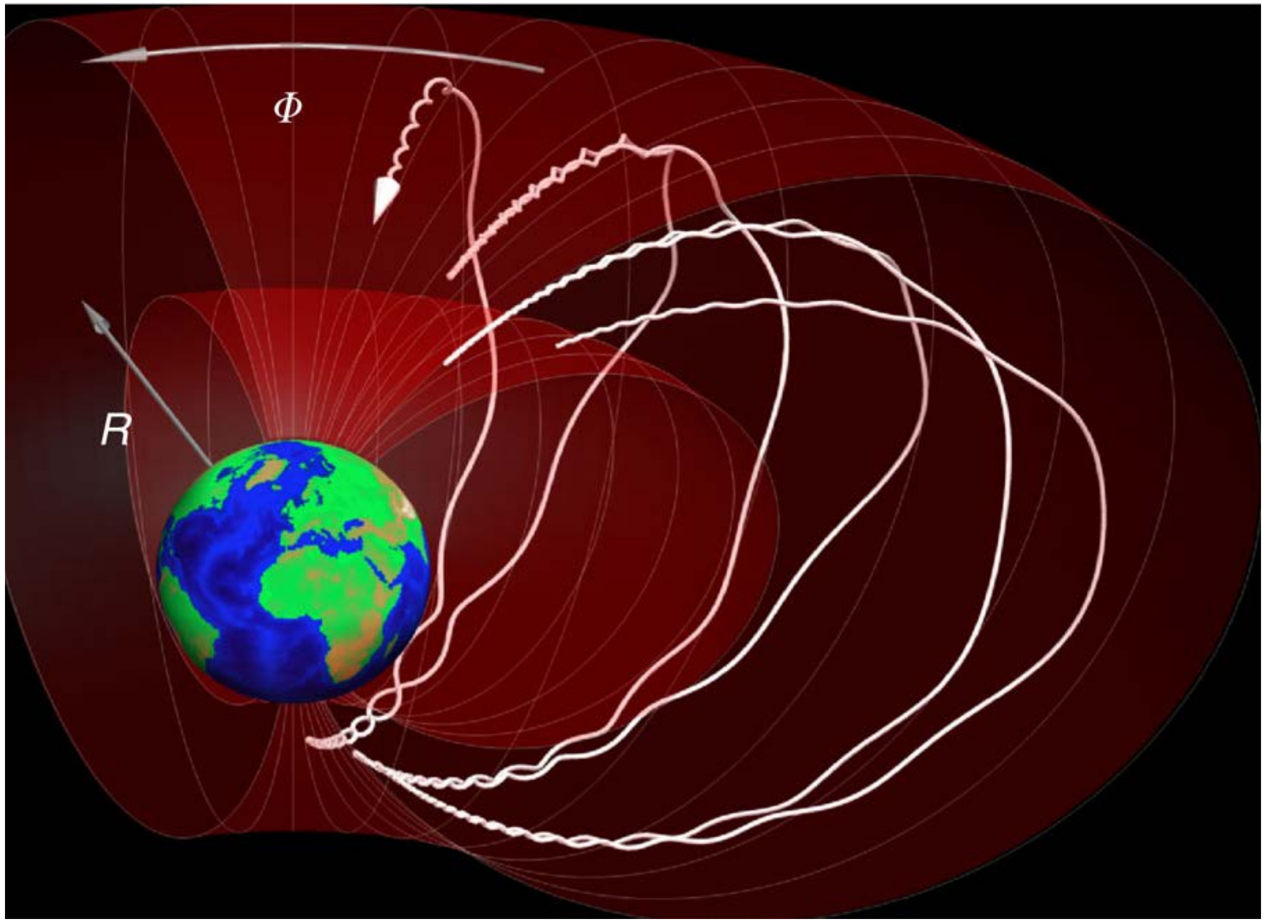
$$\frac{d}{dt}(\mu B) = \mu\frac{dB}{dt} + B\frac{d\mu}{dt}$$

$$\longrightarrow \frac{d\mu}{dt} = 0$$: adiabatic invariant



Individual Charge Trajectories

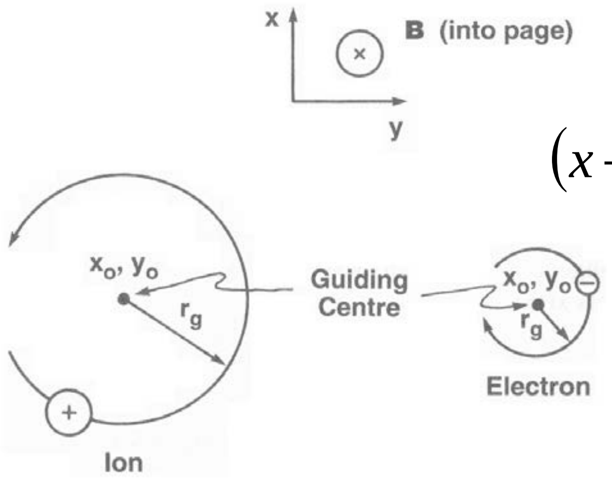




http://www.physics.ucla.edu/icnsp/Html/spong/spong.htm

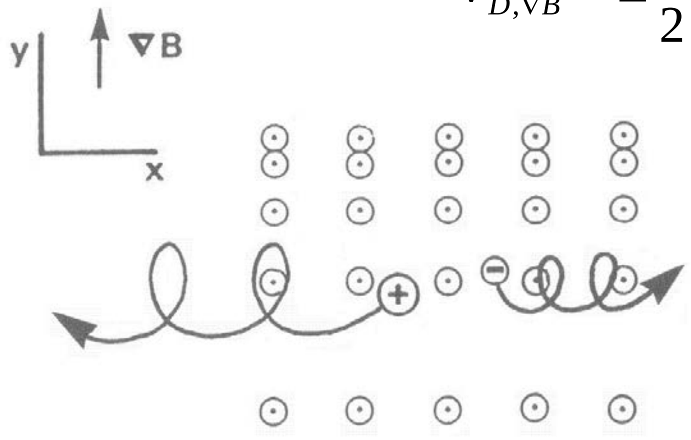
*J. P. Graves et al, Nature Communications **3** 624 (2012)*

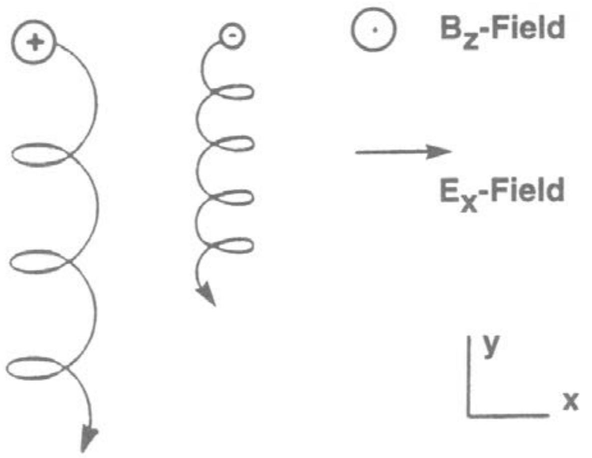
Individual Charge Trajectories



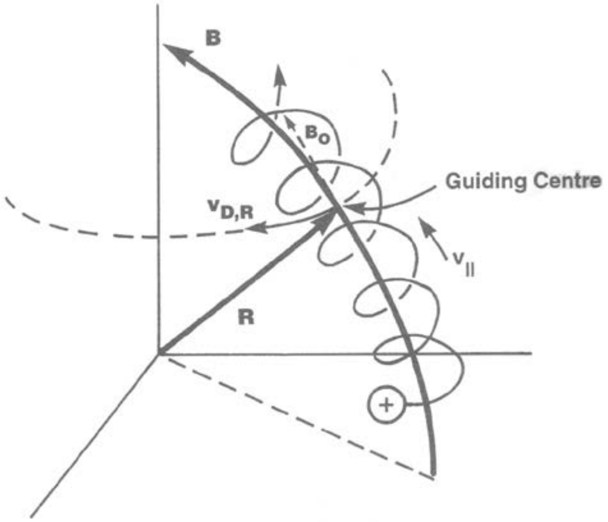
$$(x-x_0)^2+(y-y_0)^2=r_L^2$$

$$\mathbf{v}_{D,\nabla B}=\pm\frac{1}{2}v_\perp r_L\frac{\mathbf{B}\times\nabla B}{B^2}$$





$$\mathbf{v}_{D,E}=\frac{\mathbf{E}\times\mathbf{B}}{B^2}$$

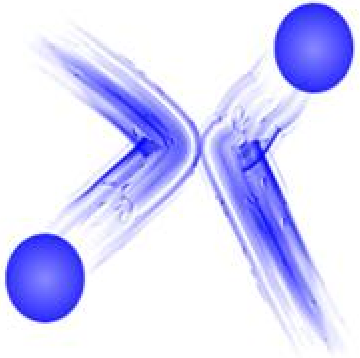


$$\mathbf{v}_{D,R}=\frac{mv_\parallel^2}{qB_0^2}\frac{\mathbf{R}_0\times\mathbf{B}_0}{R^2}$$

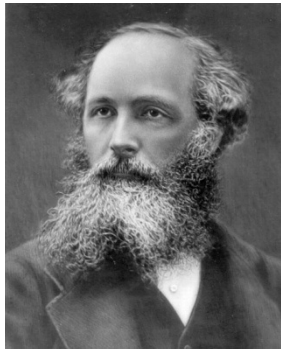
Kinetic Approach of Plasmas

- **Boltzmann equation**

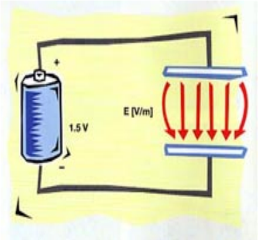
$$\frac{df_\alpha}{dt} = \frac{\partial f_\alpha}{\partial t} + \vec{v}\cdot\nabla f_\alpha + \vec{a}\cdot\nabla_{\vec{v}} f_\alpha = \frac{\partial f_\alpha}{\partial t} + \vec{v}\cdot\nabla f_\alpha + \frac{q_\alpha}{m_\alpha}(\vec{E}+\vec{v}\times\vec{B})\cdot\nabla_{\vec{v}} f_\alpha = \left(\frac{\partial f_\alpha}{\partial t}\right)_c$$

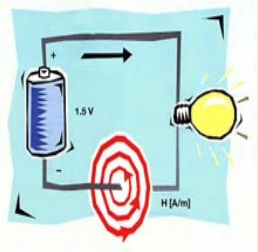


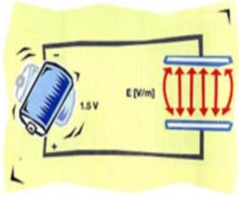
Ludwig Boltzmann
(1844-1906)



James Clerk Maxwell
(1831-1879)



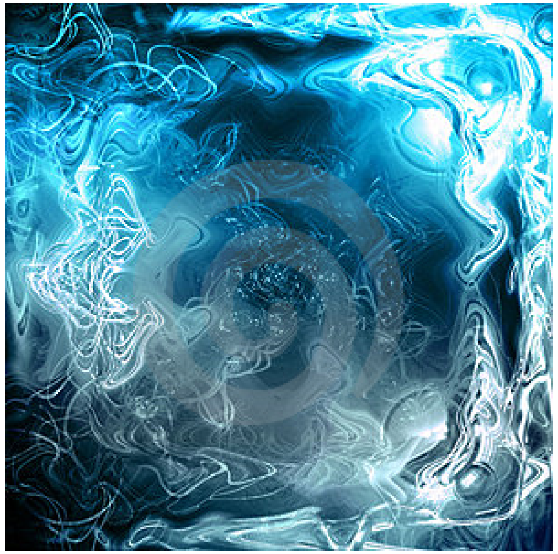


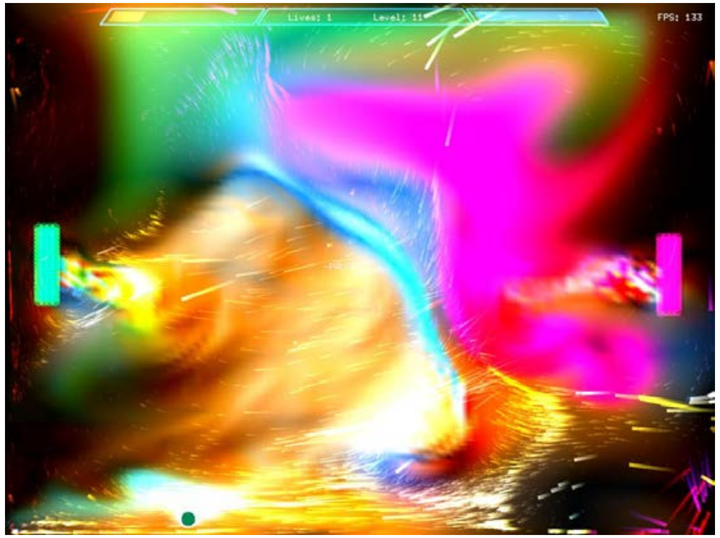


WIKIPEDIA

Fluid Approach of Plasmas

- **Plasmas as fluids**
  - The single particle approach gets to be complicated.
  - A more statistical approach can be used because we cannot follow each particle separately.
  - Introduce the concept of an **electrically charged current-carrying fluid.**





http://www.tower.com/music-label/gulan-music-studio
Album art of music OST "Plasma Pong"

Plasmas as Fluids

$$\frac{df_\alpha}{dt} = \frac{\partial f_\alpha}{\partial t} + \vec{v}\cdot\nabla f_\alpha + \frac{q_\alpha}{m_\alpha}(\vec{E}+\vec{v}\times\vec{B})\cdot\nabla_{\vec{v}} f_\alpha = \left(\frac{\partial f_\alpha}{\partial t}\right)_c$$

- **Two fluid equations**

$$\int Q_i\left[\frac{df_\alpha}{dt} - \left(\frac{\partial f_\alpha}{\partial t}\right)_c\right]d\vec{v} = 0$$

$Q_1 = 1$ mass

$Q_2 = m_\alpha \vec{v}$ momentum

$Q_3 = m_\alpha v^2/2$ energy

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_c = \sum_\beta C_{\alpha\beta}$$

$$\vec{v} = \vec{u}_\alpha(\vec{r},t) + \vec{w},\ \langle\vec{w}\rangle = 0$$

$$\left(\frac{dn_\alpha}{dt}\right)_\alpha + n_\alpha \nabla\cdot\vec{u}_\alpha = 0$$

$$n_\alpha m_\alpha\left(\frac{d\vec{u}_\alpha}{dt}\right)_\alpha = q_\alpha n_\alpha(\vec{E}+\vec{u}_\alpha\times\vec{B}) - \nabla\cdot\overleftrightarrow{P}_\alpha + \vec{R}_\alpha$$

$$\frac{3}{2}n_\alpha\left(\frac{dT_\alpha}{dt}\right)_\alpha = -\overleftrightarrow{P}_\alpha : \nabla\vec{u}_\alpha - \nabla\cdot\vec{h}_\alpha + Q_\alpha$$

$$\overleftrightarrow{P}_\alpha \equiv n_\alpha m_\alpha\langle w^2\rangle$$

$$\vec{R}_\alpha \equiv \int m_\alpha \vec{w} C_{\alpha\beta} d\vec{w}$$

$$h_\alpha \equiv \frac{1}{2}n_\alpha m_\alpha\langle w^2\vec{w}\rangle$$

$$Q_\alpha \equiv \int \frac{1}{2} m_\alpha w_\alpha^2 C_{\alpha\beta} d\vec{w}$$

Plasmas as Fluids

- **Single-fluid magnetohydrodynamics (MHDs)**

A single-fluid model of a fully ionised plasma, in which the plasma is treated as a single hydrodynamic fluid acted upon by electric and magnetic forces.

- **The magnetohydrodynamic (MHD) equation**

$$\rho = n_i M + n_e m \approx n(M+m) \approx nM$$ mass density

Hydrogen plasma, charge neutrality assumed

$$\sigma = (n_i - n_e)e$$ charge density

$$\vec{v} = (n_i M \vec{u}_i + n_e m \vec{u}_e)/\rho \approx (M\vec{u}_i + m\vec{u}_e)/(M+m) \approx \vec{u}_i + (m/M)\vec{u}_e$$ mass velocity

$$\vec{J} = e(n_i \vec{u}_i - n_e \vec{u}_e) \approx ne(\vec{u}_i - \vec{u}_e)$$

$$\vec{u}_i \approx \vec{v} + \frac{m}{M}\frac{\vec{J}}{ne}, \quad \vec{u}_e \approx \vec{v} - \frac{\vec{J}}{ne}$$

electron inertia neglected: electrons have an infinitely fast response time because of their small mass

Plasmas as Fluids

F. F. Chen, Ch. 5.7

- **Ideal MHD model**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$$

Mass continuity equation
(continuously flowing fluid)

$$\rho \frac{d\vec{v}}{dt} = \vec{J} \times \vec{B} - \nabla p$$

Single-fluid equation of motion

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

Energy equation (equation of state):
adiabatic evolution

$$\vec{E} + \vec{v} \times \vec{B} = 0$$

Ohm's law: perfect conductor → "ideal" MHD

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\nabla \cdot \vec{B} = 0$$

Maxwell equations

$\varepsilon_0 \to 0$ assumued
(Full → low-frequency Maxwell's equations)
Displacement current, net charge neglected

Plasmas as Fluids

- **Magnetohydrodynamics (MHD): 1946 by H. Alfvén**

STOCKHOLMS OBSERVATORIUMS ANNALER

(ASTRONOMISKA IAKTTAGELSER OCH UNDERSÖKNINGAR Å STOCKHOLMS OBSERVATORIUM)

BAND 14. N:o 9.

ON THE COSMOGONY OF THE SOLAR SYSTEM

III

BY

HANNES ALFVÉN

Hannes Alfvén
(1908-1995)

"Nobel prize in Physics (1970)"

WIKIPEDIA

Plasmas as Fluids

- **Magnetohydrodynamics (MHD): 1946 by H. Alfvén**

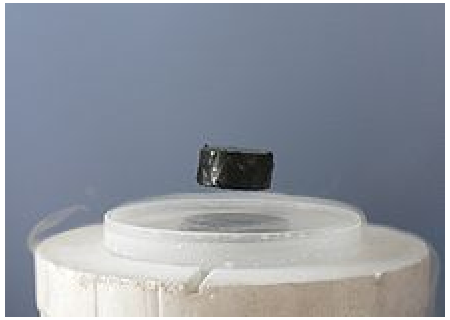
The laws found can be applied to the planetary as well as to the satellite systems (§§ 7—9) and also — as is pointed out in § 10 — to the non-solar planets newly discovered.

At last some remarks are made about the transfer of momentum from the Sun to the planets, which is fundamental to the theory (§ 11). The importance of the magneto-hydrodynamic waves in this respect is pointed out. It is possible that the Sun's general magnetic field was much stronger at the time of formation of the solar system than it is now. This must be assumed in order to explain the momentum transfer quantitatively.

Kungl. Tekniska Högskolan, Stockholm, November 1945.

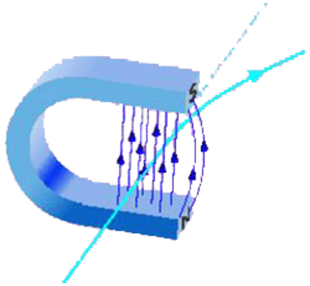
Plasmas as Fluids

- **Ideal MHD**



  - Single-fluid model
  - Ideal:

    Perfect conductor with zero resistivity



  - MHD:

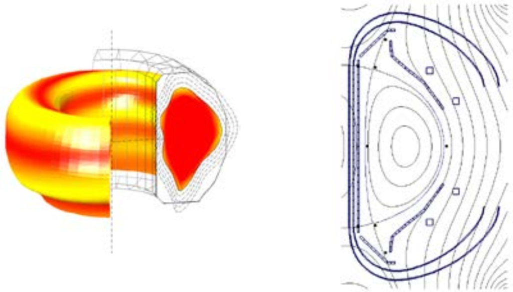
    Magnetohydrodynamic (magnetic fluid dynamic)

  - Assumptions:

    Low-frequency, long-wavelength
    collision-dominated plasma

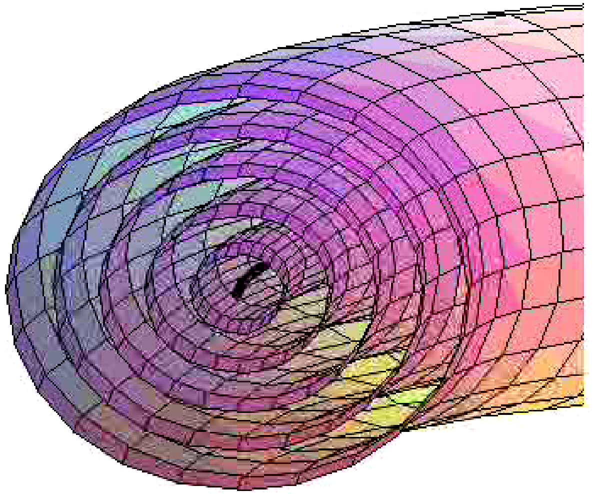
  - Applications:

    Equilibrium and stability in fusion plasmas

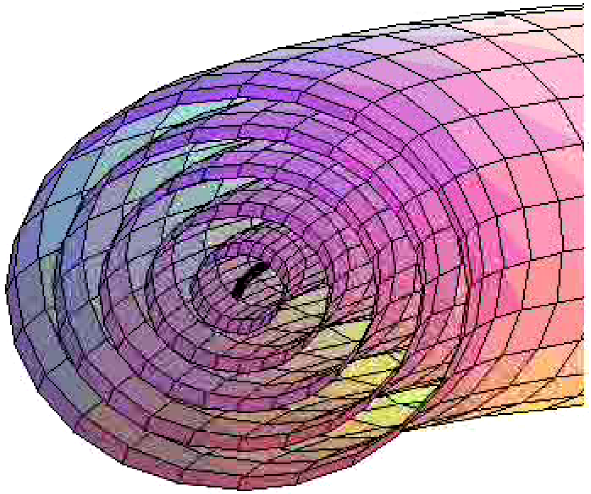


What is Ideal MHD?

- Ideal MHD: $\eta = 0$



- Resistive MHD: $\eta \neq 0$



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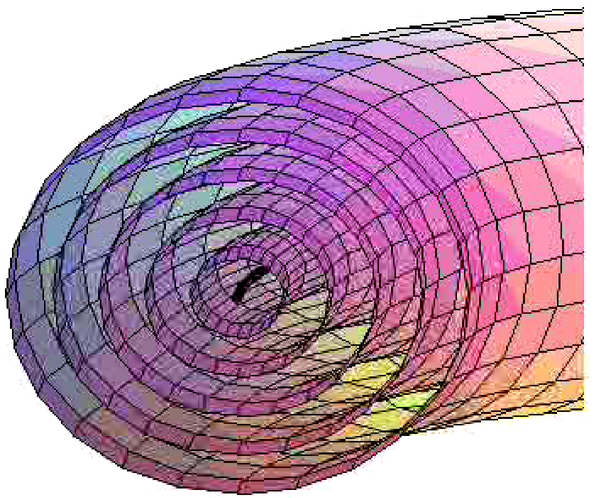

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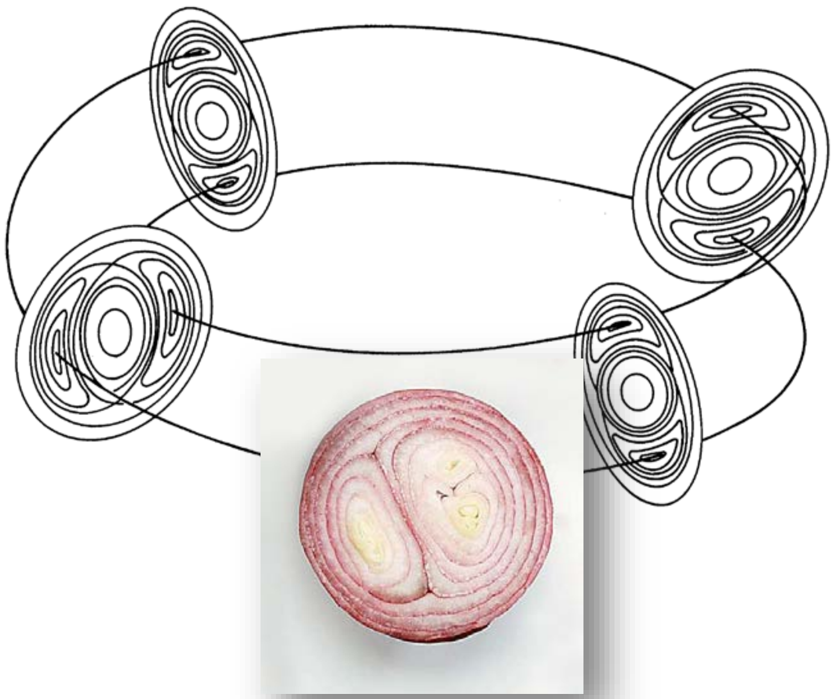
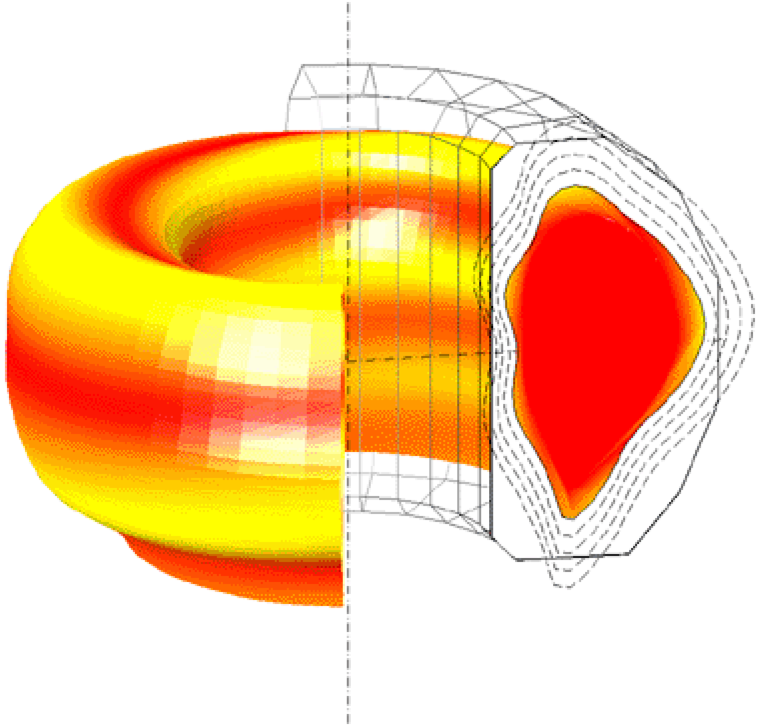
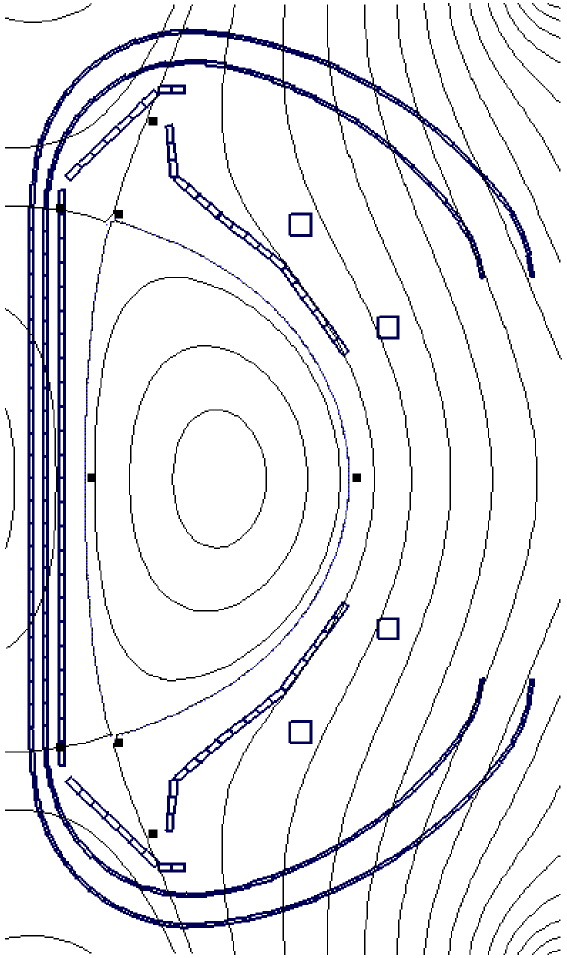


Applications





Plasma Eqauilibrium and Stability

https://fusion.gat.com/theory/File:Theory1.gif