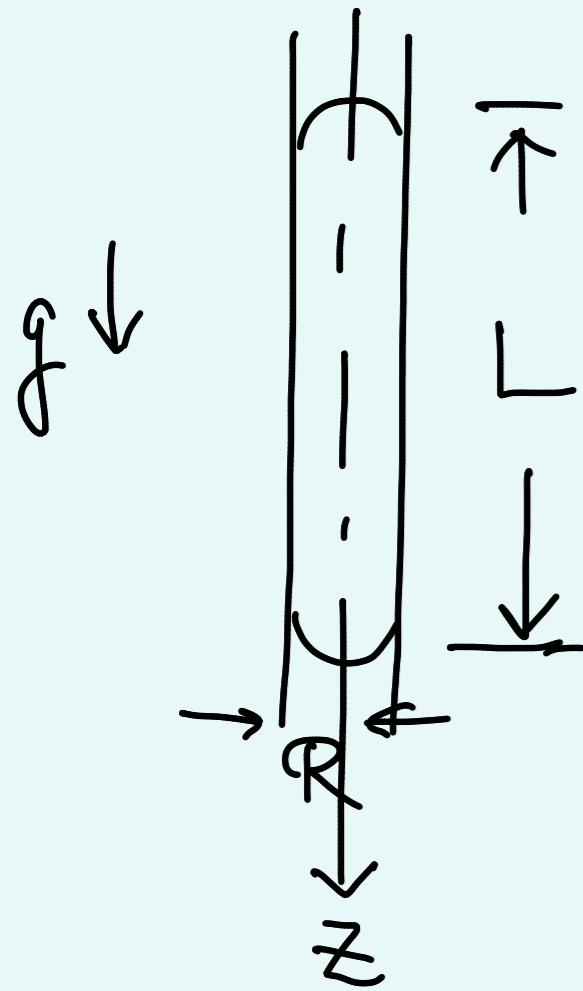


\* Slug motion in capillary



(1) Simplest case ' Poiseuille flow'

N. S. eq.

$$0 = - \frac{\partial P}{\partial z} + \mu \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) + \rho g$$

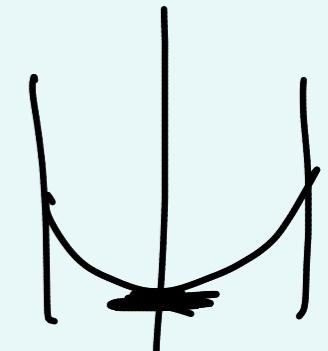
$$\mu \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) = \frac{\partial}{\partial z} \underbrace{(P - \rho g z)}_{= P}$$

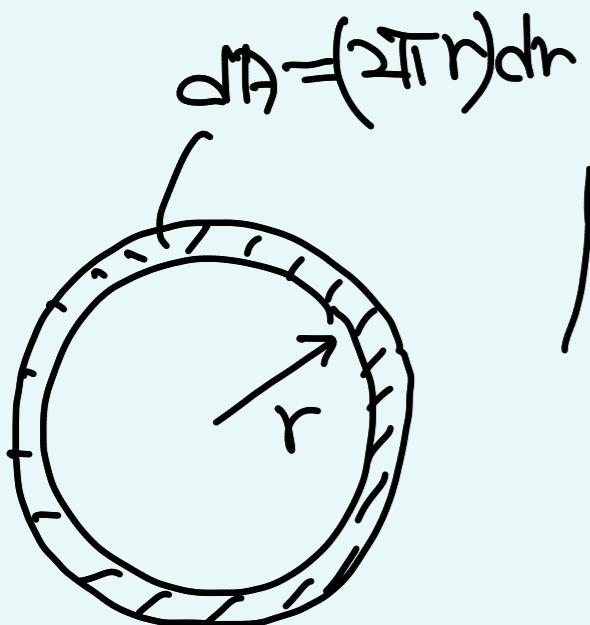
$$\frac{\mu}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = \frac{dP}{dz}$$

$$\mu r \frac{\partial u}{\partial r} = \frac{r^2}{2} \frac{dP}{dz} + C_1$$

$$\text{at } r=0 : \frac{\partial u}{\partial r} = 0 , \quad C_1 = 0$$

$$\mu \frac{\partial u}{\partial r} = \frac{r}{2} \frac{dP}{dz}$$

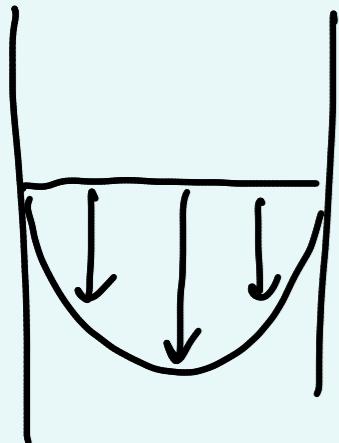




$$\mu u = \frac{r^2}{4} \frac{dP}{dz} + C_2$$

$$\text{at } r=R, u=0$$

$$C_2 = -\frac{R^2}{4} \frac{dP}{dz}$$

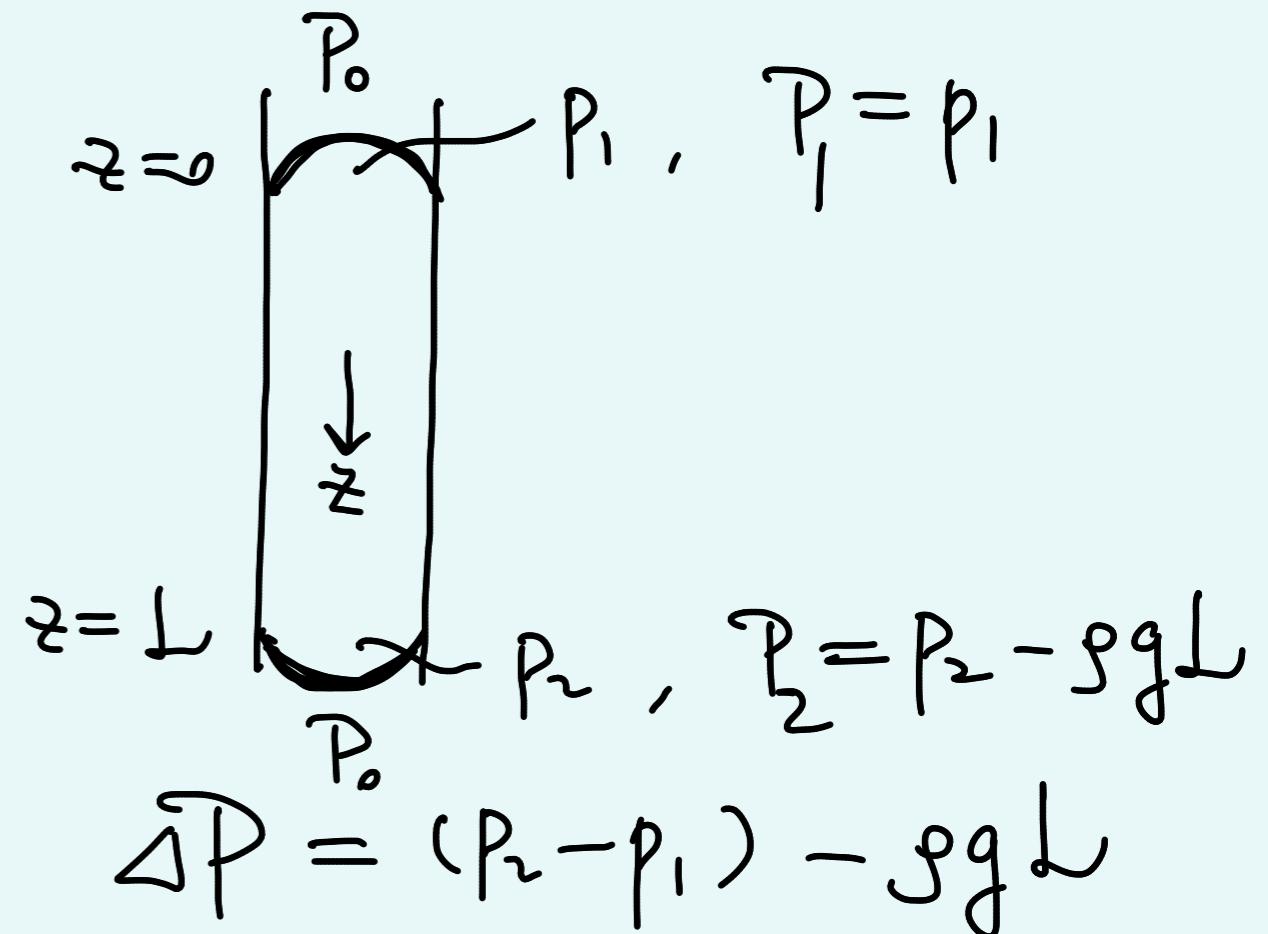


$$u = \frac{1}{\mu} \left( -\frac{dp}{dz} \right) \frac{1}{4} (R^2 - r^2)$$

$\parallel$   
 $gq$

$$u(r) = \frac{gq}{4\mu} (R^2 - r^2)$$

$$U = \frac{1}{\pi R^2} \int_0^R u(2\pi r) dr = \frac{gq R^2}{8\mu}$$



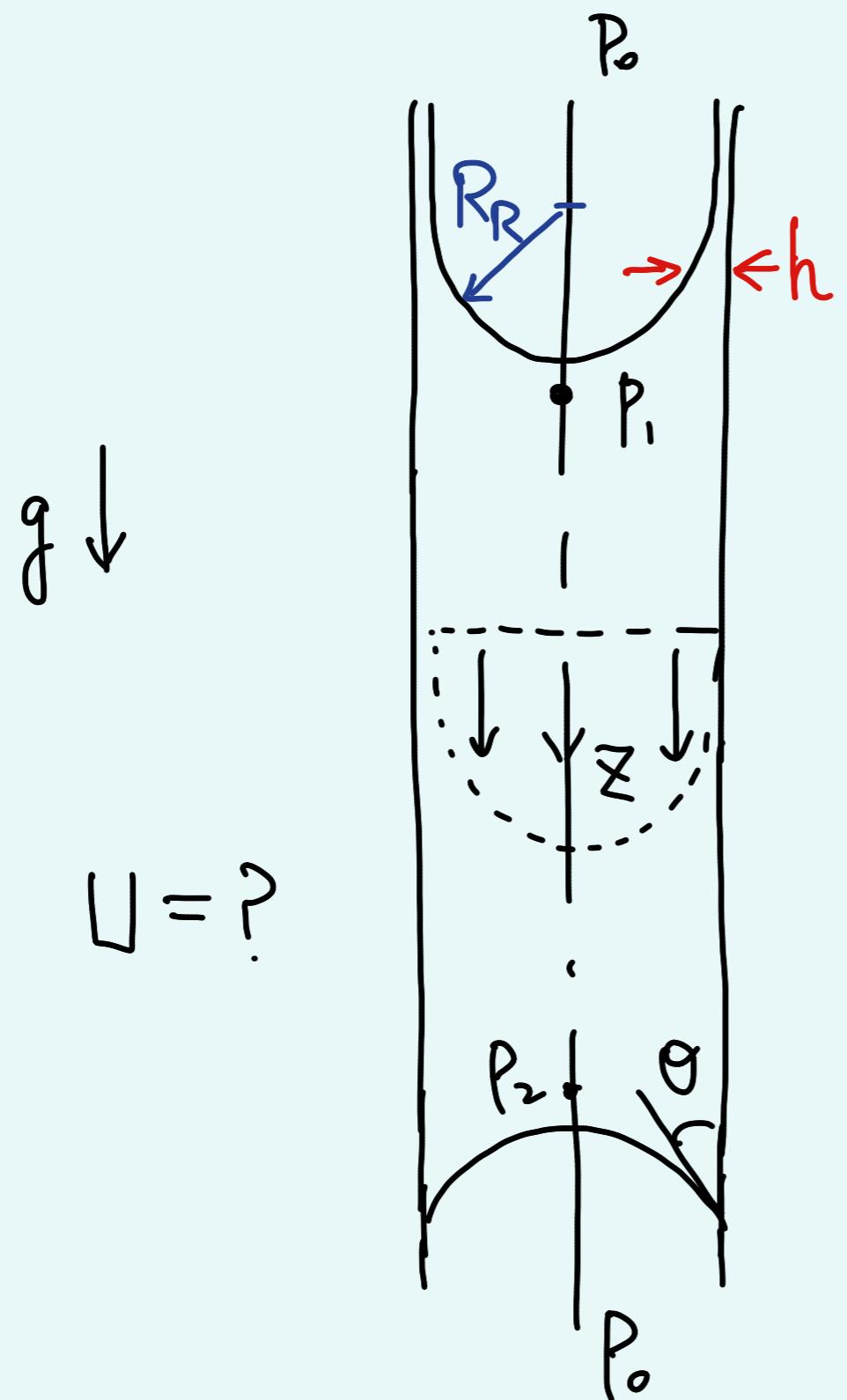
$$\frac{\Delta P}{L} = \frac{P_2 - P_1}{L} - gq$$

← capillary effect

$$\left\{ \begin{array}{l} \text{if } \frac{P_2 - P_1}{L} \ll gq \\ \frac{dP}{dz} = -gq \end{array} \right.$$

: average vel.

(2) Fully wetting liquid slug in dry tube



$$P_1 = P_0 - \frac{2\sigma}{R_R}$$

$$= P_0 - \frac{2\sigma}{R-h}$$

$$R_R = R-h$$

$$P_2 = P_0 - \frac{2\sigma}{R_A}$$

$$= P_0 - \frac{2\sigma \cos \theta}{R}$$

$$R_A \cos \theta = R$$

$$u = \frac{1}{4\mu} \left( -\frac{dP}{dz} \right) (R^2 - r^2). \quad P = p - \rho g z$$

$$\frac{dP}{dz} = \frac{1}{L} (\tilde{P}_2 - \rho g L - \tilde{P}_1)$$

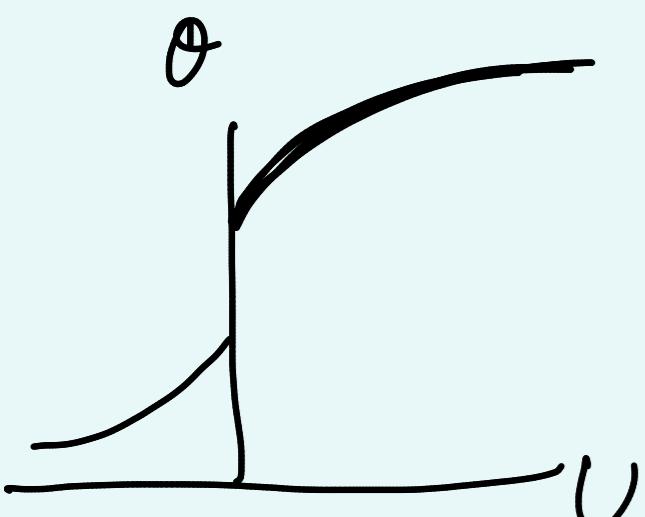
$$\frac{dP}{dz} = \frac{1}{L} \left( -\frac{\omega_0 \cos \theta}{R} - \rho g L + \frac{\omega_0}{R-h} \right)$$

$$U = \frac{R^2}{8\mu L} \left( \frac{\omega_0 \cos \theta}{R} + \rho g L - \frac{\omega_0}{R-h} \right)$$

$$\begin{aligned} & (1+\epsilon)^n \\ & \approx 1+n\epsilon \\ & (\epsilon \ll 1) \end{aligned}$$

(2-1) Limiting case for (2) :  $\theta \ll 1$

$$\cos \theta \approx 1 - \frac{\theta^2}{2}$$



$$\frac{1}{R-h} = \frac{1}{R} \frac{1}{1-h/R} = \frac{1}{R} \left(1 - \frac{h}{R}\right)^{-1} \approx \frac{1}{R} \left(1 + \frac{h}{R}\right)$$

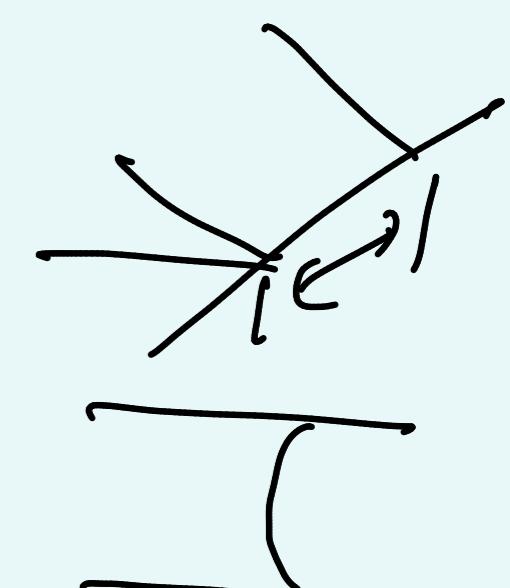
Then

$$U = \frac{R^2}{8\mu L} \left[ \rho g L - \frac{\omega_0}{R} \left( \frac{\theta^2}{2} + \frac{h}{R} \right) \right].$$

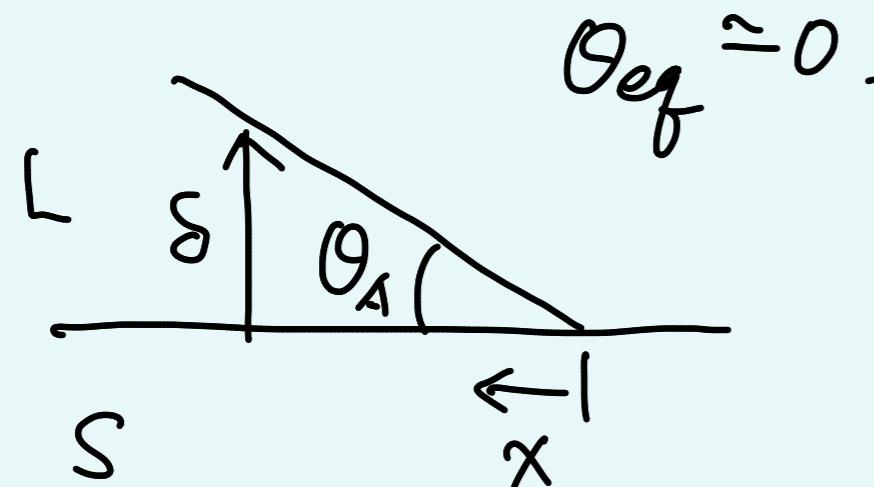
$$\left[ \begin{array}{l} \frac{h}{R} = f_n(\zeta_a) \leftarrow \text{Bretherton's law} \\ \theta = f_n(\zeta_a) \leftarrow \text{Hoffman's law} \end{array} \right.$$

. Bretherton's law :  $\frac{h}{R} = 2.9 \frac{h_\infty}{R} = 3.88 C_a^{2/3}$   $\rightarrow h < h_\infty$   
 . Hoffman's law  $\theta = (6\Gamma C_a)^{1/3}$

$$U = \frac{R^2}{8\mu L} \left[ \rho g L - \frac{25}{R} \beta C_a^{2/3} \right]$$

$$\beta = \frac{1}{2} (6\Gamma)^{2/3} + 3.88$$


Hoffman-Tanner's law



Driving force  $\sim \gamma \underbrace{(\cos \theta_{eq} - \cos \theta_A)}_{length}$

$$\sim \gamma (1 - \cos \theta_A)$$

$$\sim \gamma \theta_A^2 \quad (\theta_A \ll 1)$$

Resisting force  $\frac{\text{force}}{\text{length}} \sim \int_0^L \tau(x) dx$

$$\sim \int_0^L \mu \frac{U}{\delta(x)} dx$$

$$\delta(x) \approx x \tan \theta_A$$

$$\approx x \theta_A$$

$$\sim \int_0^\lambda \mu \frac{U}{x \theta_A} dx$$

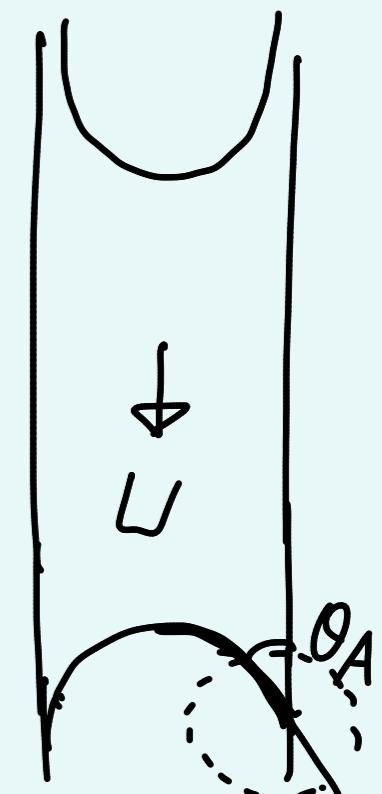
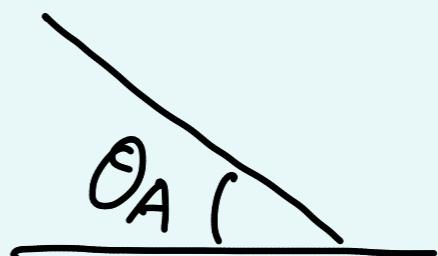
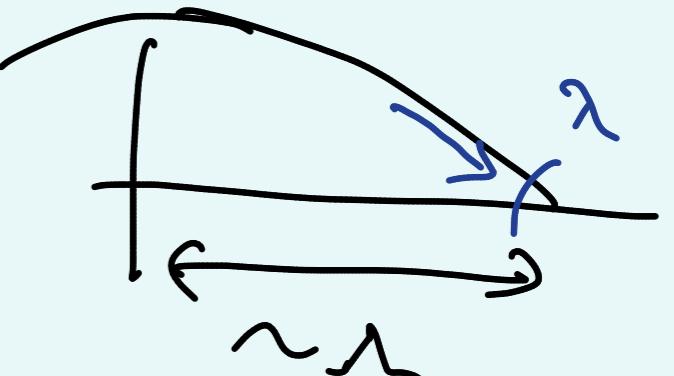
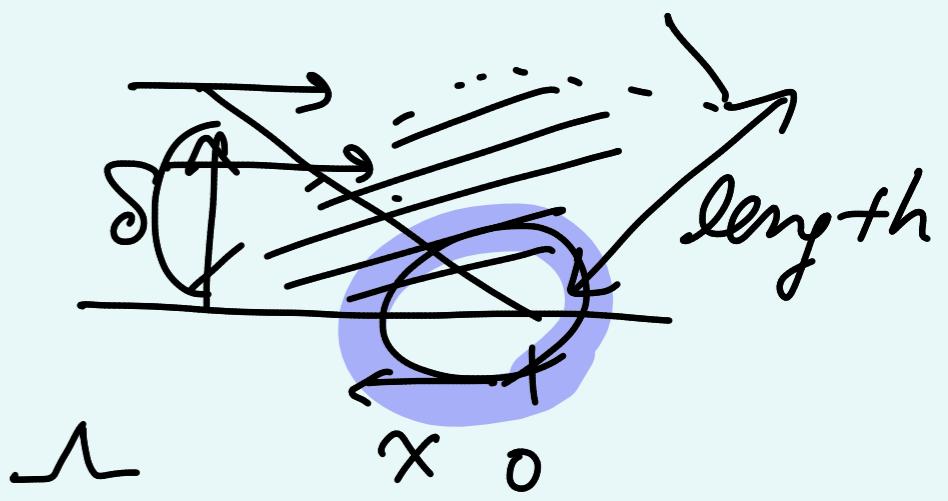
cutoff distance

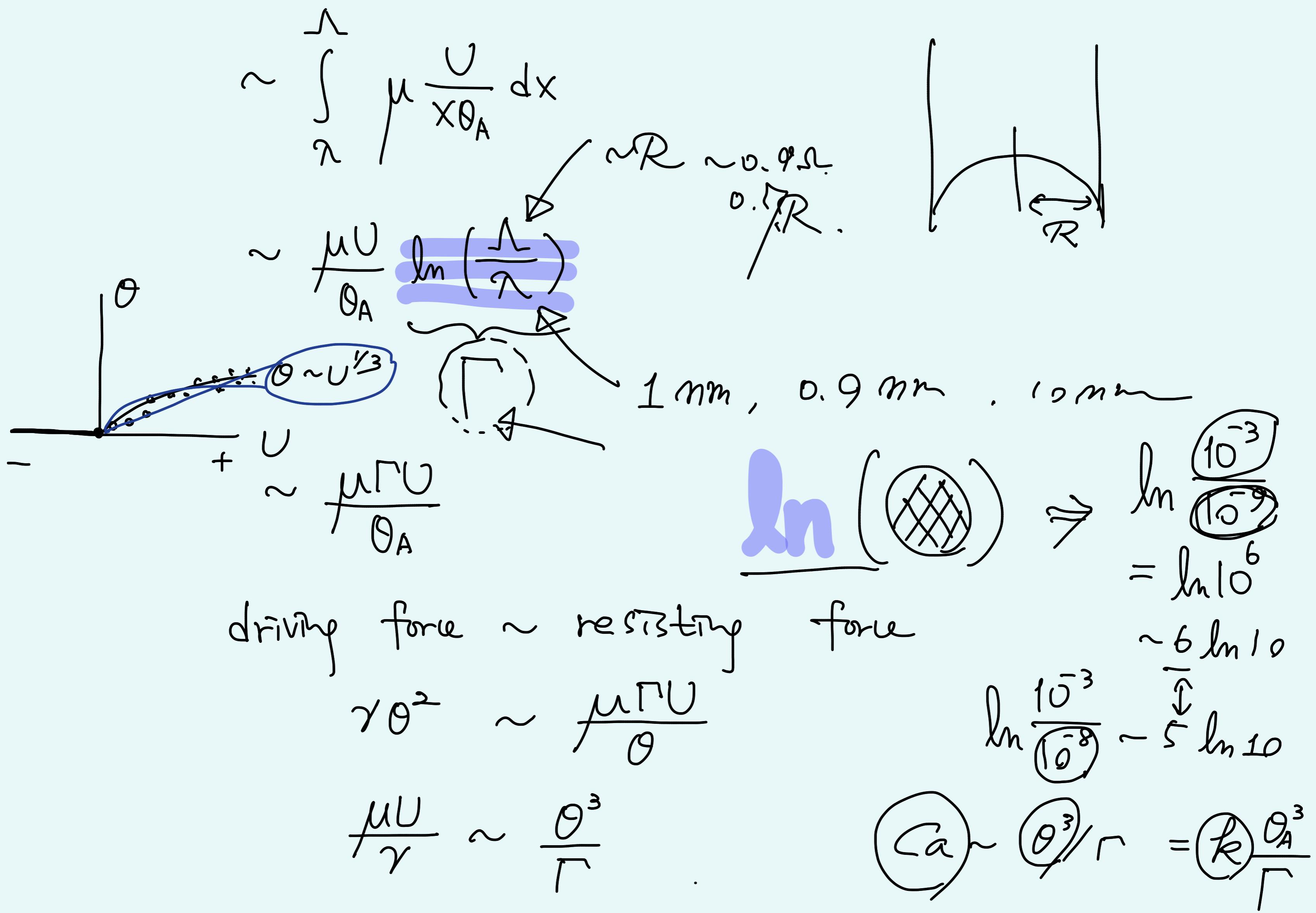
$$\lim_{x \rightarrow 0} \ln x = -\infty.$$

force  $\rightarrow \infty$

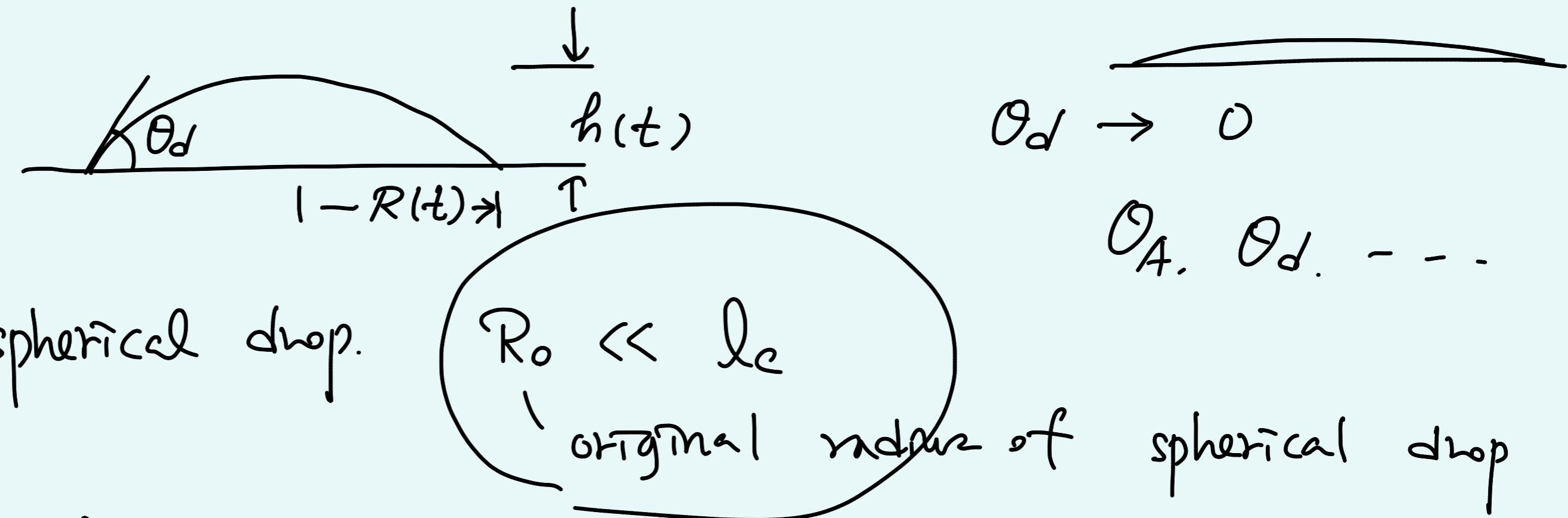
: contact line singularity .

$$\sim \mu \frac{2U}{\delta}$$





- Tanner's law  $R \sim f_m(t)$



$$\left. \begin{aligned} h &= \frac{1}{2} R \theta_d \\ \Omega &= \frac{\pi}{2} h R^2 \quad (\text{drop volume}) \end{aligned} \right\}$$

Using Hoffman - Tanner's law

$$\frac{\mu U}{\sigma} = C \theta_d^3.$$

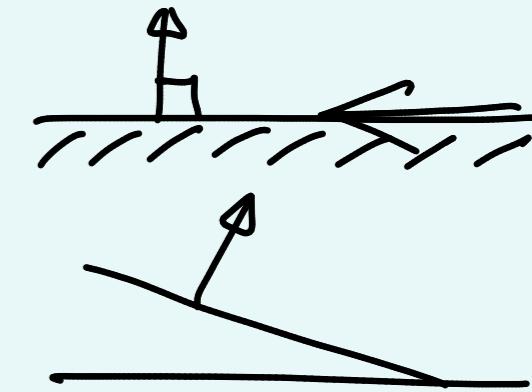
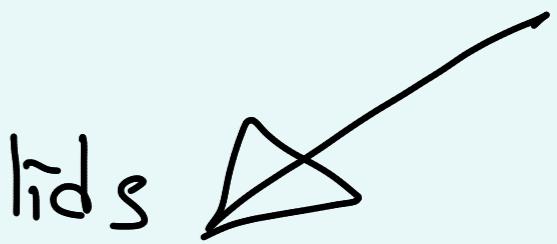
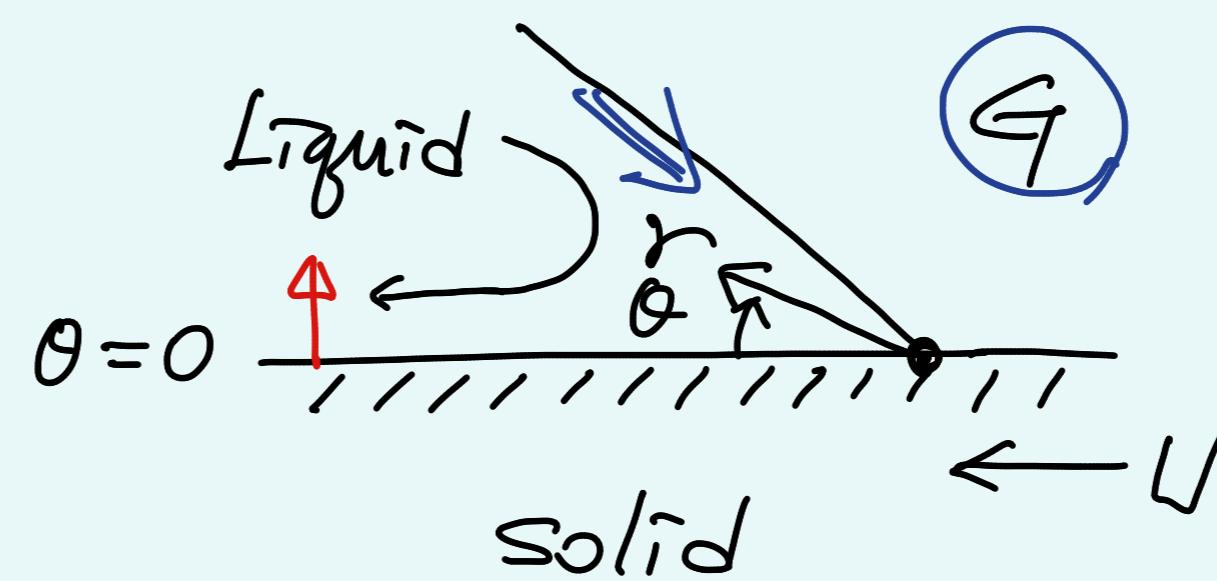
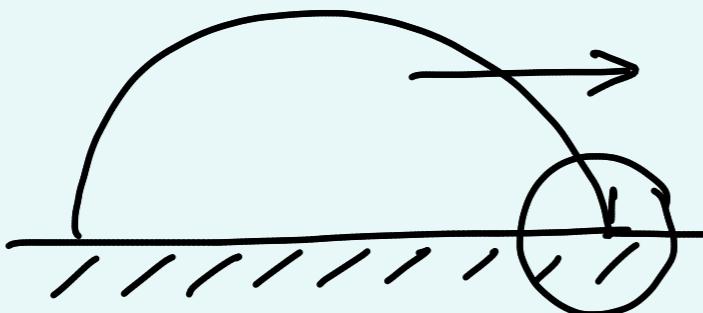
$$\frac{\mu}{\sigma} \frac{dR}{dt} = C \left( \frac{4}{\pi} \right)^3 \frac{\Omega^3}{R^9},$$

$$R^9 \frac{dR}{dt} = C^* \frac{\sigma}{\mu} \Omega^3$$

$$R \sim t^{1/10}$$

- Wedge dissipation on partially wettable solids

$$0 < \theta < 180^\circ$$



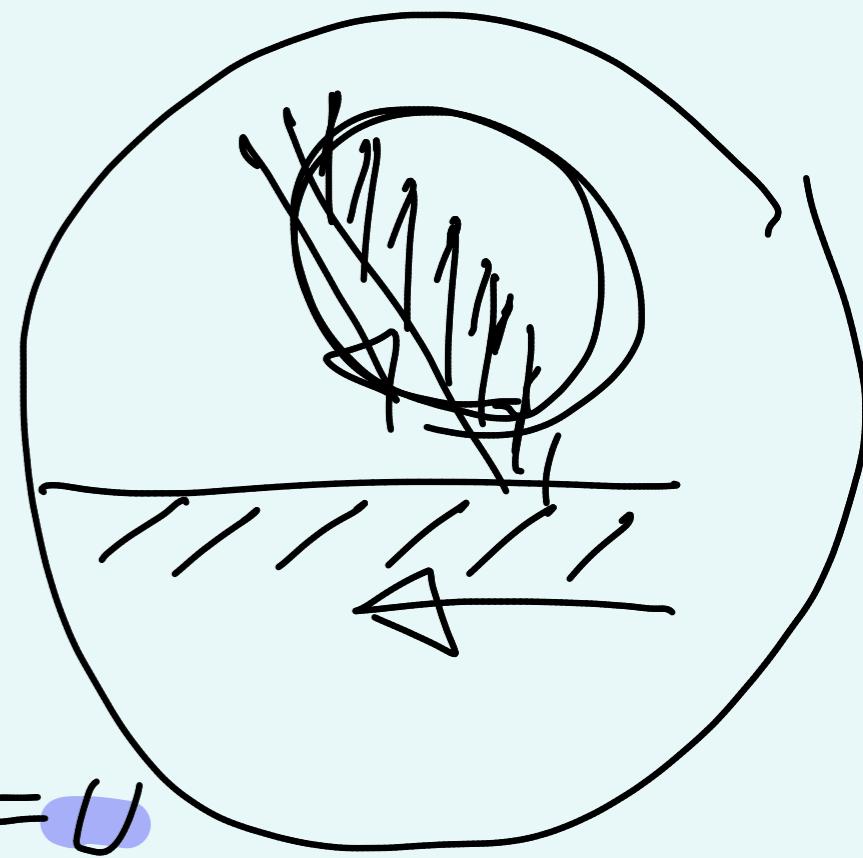
Corner flow

$$\nabla^4 \psi = 0$$

$$V_r = -\frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad V_\theta = \frac{\partial \psi}{\partial r}$$

B.C.  $\theta=0$  :  $V_\theta = 0, \quad V_r = -\frac{1}{r} \frac{\partial \psi}{\partial \theta} = U$

$\theta=\theta_0$  ( $L-G$  interface) :  $V_\theta = 0, \quad V_{r\theta} = 0.$



$$\tilde{\tau}_{r\theta} = \frac{\mu}{r} \left( \frac{\partial v_r}{\partial \theta} - v_\theta \right)$$

$$\tilde{\tau}_{r\theta} \Big|_{\theta=\theta_0} = 0 ; \quad \underbrace{\frac{\partial v_r}{\partial \theta}}_{=} = v_\theta = 0$$

$$\nabla \psi = 0 //$$

$$\frac{\partial}{\partial \theta} \left( -\frac{1}{r} \frac{\partial \psi}{\partial \theta} \right) = 0 .$$

$$\psi = \underbrace{R(r)}_{\sim r} \underbrace{\Theta(\theta)}$$

$$\psi = r \Theta(\theta) = r (a \sin \theta + b \cos \theta + c \theta \sin \theta + d \theta \cos \theta)$$

$$b = 0$$

$$\left\{ \begin{array}{l} a = \frac{-v \theta_0}{\theta_0 - \sin \theta_0 \cos \theta_0} \\ c = \frac{v \sin^2 \theta_0}{\theta_0 - \sin \theta_0 \cos \theta_0} \\ d = \frac{v \sin \theta_0 \cos \theta_0}{\theta_0 - \sin \theta_0 \cos \theta_0} \end{array} \right.$$



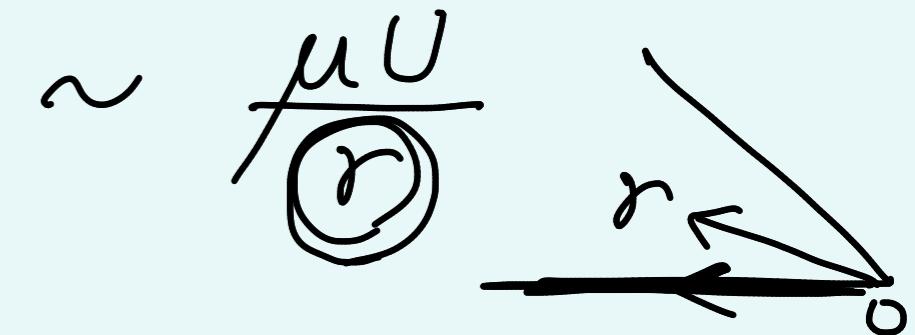
## viscous shear stress

$$\begin{aligned}\tau_{r\theta} &= \frac{\mu}{r} \left( \frac{\partial v_r}{\partial \theta} - v_\theta \right) \\ &= \frac{2\mu}{r} (-c \cos \theta + d \sin \theta)\end{aligned}$$

$$\left\{ \begin{array}{l} \tau_{rr} = 2\mu \frac{\partial v_r}{\partial r} = 0 \\ \tau_{\theta\theta} = 2\mu \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) \\ = 0 \end{array} \right.$$

at the wall

$$\underline{\tau_{r\theta} (\theta=0)} = - \frac{2\mu}{r} U \frac{\sin^2 \theta_0}{\theta_0 - \sin \theta_0 \cos \theta_0}$$



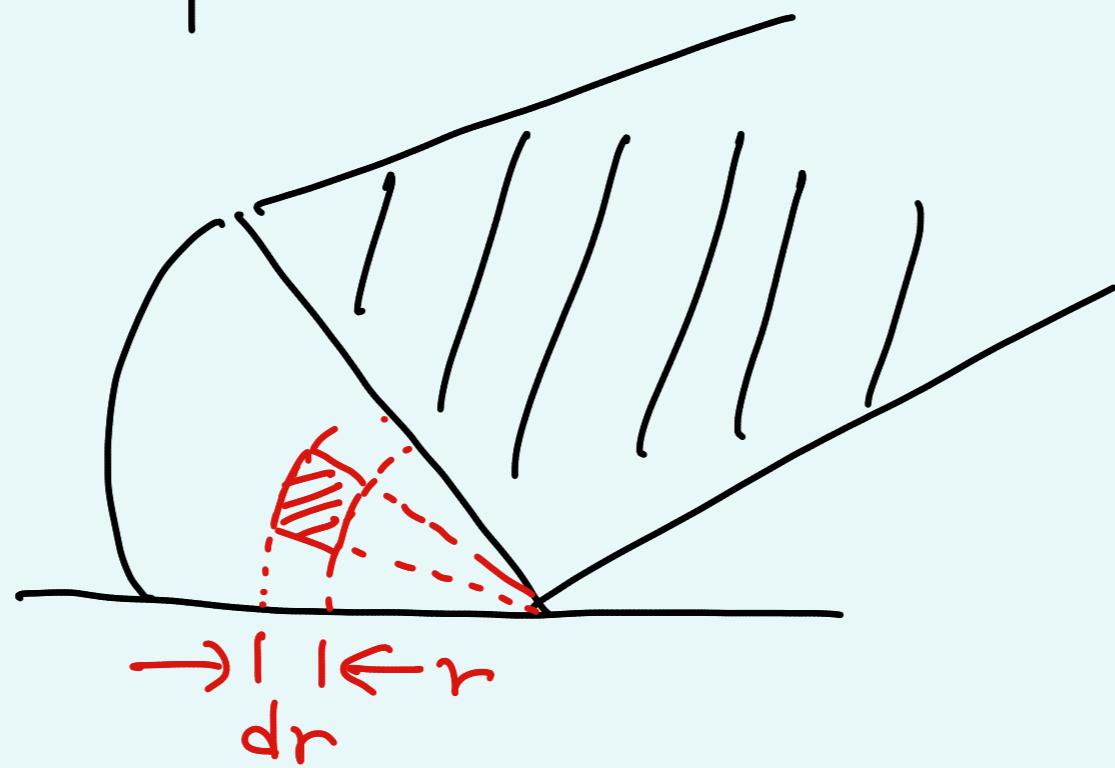
Energy dissipation  $\Phi (\text{W/m}^3)$

$$\Phi = 2\mu \left[ \left( \frac{\partial v_r}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)^2 + \frac{1}{2} \left( \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)^2 \right]$$

$$= \mu \frac{4U^2}{r^2} f(\theta_0)$$



energy dissipation over a finite volume



wedge  $\propto H^2$

$$\begin{aligned}
 \nu \int \Phi dV &= \int_0^L \int_0^{\theta_0} \mu \frac{4U^2}{r} f(\theta) d\theta dr \\
 &= 4\mu U^2 \int_0^L \frac{dr}{r} \cdot \tilde{f}(\theta_0) \\
 &= 4\mu U^2 \tilde{f}(\theta_0) \ln\left(\frac{L}{R}\right)
 \end{aligned}$$

