Engineering Mathematics II

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Text book: Erwin Kreyszig, Advanced Engineering Mathematics, 9th Edition, Wiley (2006)

Ch. 11 Fourier Series, Integrals, and Transforms

Ch. 11 Fourier Series, Integrals, and Transforms

- 주기현상의 예: 모터, 회전 기계, 음파, 지구의 운동, 정상 조건하의 심장
- 푸리에 급수는 상미분 방정식과 편미분 방정식을 수반하는 문제를 해결하는 데 매우 중요한 도구
- 푸리에 급수의 실용적 관심대상: Discontinuous(불연속적)인 주기함수
- 내용: 푸리에 급수의 개념과 기법, 푸리에 적분(Fourier Integrals),

  푸리에 변환(Fourier Transform)

- 우리 주변에서 푸리에 급수는 매우 다양하게 이용됨:

파동의 간섭현상을 이용하여 소음도 없앨 수 있음.
예를 들어 여객기 밖은 엔진소음으로 매우 시끄럽지만 안은 조용할 수 있는 것은 엔진의 소음과 같은 주파수를 가지고 위상만 반대인 소음을 발생시켜 소음을 상쇄시키는 푸리에 급수 원리 덕분에 가능함.

11.1 Fourier Series (푸리에 급수)

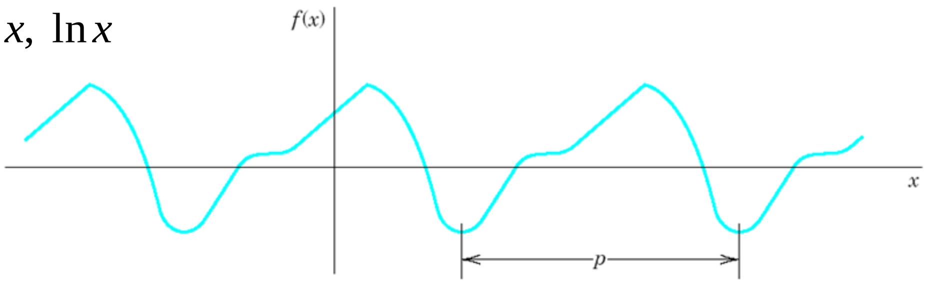
Periodic Function (주기함수)

* 모든 실수 x에 대하여 정의

* 어떤 양수 p가 존재해서, 모든 x에 대하여 $f(x+p)=f(x)$

$\Leftrightarrow$ $f(x)$를 주기함수(Periodic Function)라 하고, p를 $f(x)$의 주기(Period)라 한다.

- 주기함수인 예 : $\sin x,\ \cos x$
- 주기함수가 아닌 예 : $x,\ x^2,\ x^3,\ e^x,\ \cosh x,\ \ln x$



Properties

- 함수 $f(x)$의 주기가 p이면, 모든 x에 대하여 $f(x+np)=f(x)\quad (n=1,\ 2,\ 3,\ \cdots)$
- $f(x)$와 $g(x)$의 주기가 p이면, $af(x)+bg(x)$($a,\ b$는 임의 상수)의 주기도 p이다.

11.1 Fourier Series (푸리에 급수)

- **Trigonometric Series (삼각급수)**

- **Trigonometric System (삼각함수계)**

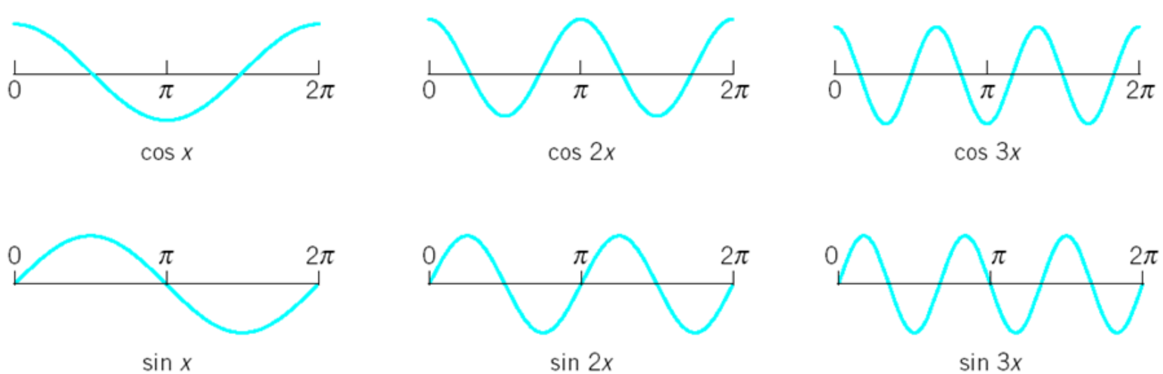
$$1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \cdots,\ \cos nx,\ \sin nx,\ \cdots$$

- **Trigonometric Series (삼각급수)**

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

상수 $a_0,\ a_1,\ b_1,\ a_2,\ b_2,\ \cdots$을 계수(Coefficients)라 한다.

- 삼각급수가 수렴한다면 그 합은 주기가 2π인 주기함수이다.



11.1 Fourier Series (푸리에 급수)

- **Fourier Series (푸리에 급수)** $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$
- **Euler Formulas (오일러 공식)**

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 1, 2, \cdots$$

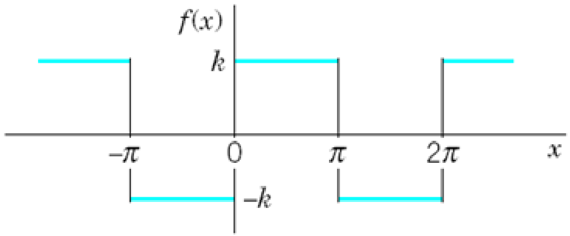
- **Fourier Coefficients (푸리에 계수):** 오일러 공식에 의해 주어진 값
- **Fourier Series:** 푸리에 계수를 갖는 삼각급수

11.1 Fourier Series (푸리에 급수)

■ **Ex. 1 Rectangular Wave (주기적인 직사각형파)**

Find the Fourier coefficients of the periodic function,

$$f(x)=\begin{cases}-k & (-\pi<x<0)\\ k & (0<x<\pi)\end{cases} \quad \text{and} \quad f(x+2\pi)=f(x)$$



Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc.

11.1 Fourier Series (푸리에 급수)

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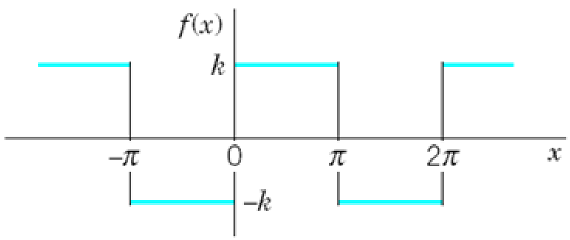


Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc.

$$*\ \int_0^{\pi} f(x)dx = k\cdot\pi \text{이고} \ \int_{-\pi}^{0} f(x)dx=-k\cdot\pi\text{이므로} \ \int_{-\pi}^{\pi} f(x)dx=\int_0^{\pi} f(x)dx+\int_{-\pi}^{0} f(x)dx=0 \quad \therefore\ a_0=0$$

$$*\ \int_{-\pi}^{\pi} f(x)\cos x dx=0 \qquad \therefore\ a_n=0$$

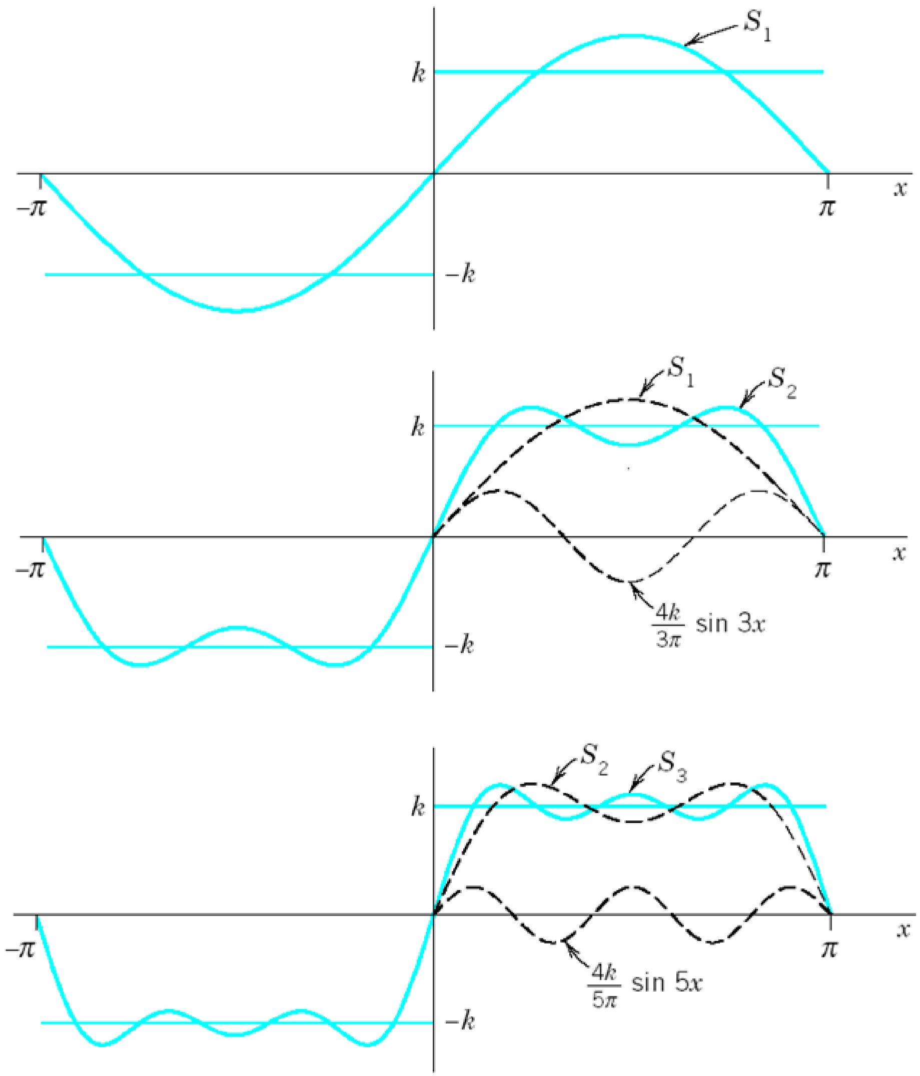
$$*\,b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx=\frac{2}{\pi}\int_0^{\pi} k\sin nx dx=\frac{2}{\pi}k\left.\frac{-\cos nx}{n}\right|_0^{\pi}=\frac{2k}{n\pi}(1-\cos n\pi)=\begin{cases}\dfrac{4k}{n\pi} & n\text{이 홀수}\\ 0 & n\text{이 짝수}\end{cases}$$

$$\therefore\ \text{푸리에 급수} : \frac{4k}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\cdots\right)$$

$$f\left(\frac{\pi}{2}\right)=k=\frac{4k}{\pi}\left(1-\frac{1}{3}+\frac{1}{5}-+\cdots\right) \quad\Rightarrow\quad 1-\frac{1}{3}+\frac{1}{5}-+\cdots=\frac{\pi}{4}$$

Leibniz (1673)

11.1 Fourier Series (푸리에 급수)



- 함수 $f(x)$가 x=0과 x=π에서 불연속이므로, 모든 부분합은 이 점에서 값이 0이 되며, 이 값은 극한값인 $-k$와 k의 산술평균임.

11.1 Fourier Series (푸리에 급수)

- **Orthogonality of the Trigonometric Systems:**

  삼각함수 계는 구간 -π ≤ x ≤ π에서 직교한다.

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = 0 \qquad (n \neq m)$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = 0 \qquad (n \neq m)$$

Prove!

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0 \qquad (n \neq m \text{ 또는 } n = m)$$

- **Representation by a Fourier Series (푸리에 급수에 의한 표현)**

* 주기가 2π 인 주기함수

* 구간 $-\pi \le x \le \pi$ 에서 Piecewise Continuous (구분연속)

* 각 점에서 Left-hand Derivative(좌도함수)와 Right-hand Derivative(우도함수)를 갖는다.

$\Rightarrow$ $f(x)$의 푸리에 급수는 수렴한다.

급수의 합

* 불연속인 점을 제외한 모든 점에서 급수의 합 = $f(x)$

* 불연속인 점에서 급수의 합 = $f(x)$의 좌극한과 우극한의 평균

11.1 Fourier Series (푸리에 급수)

PROBLEM SET 11.1

HW: 16, 28, 29

11.2 Functions of Any Period p=2L (임의의 주기 p=2L을 가지는 함수)

- **Fourier Series of the Function *f* (*x*) of period 2*L***

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

- **Fourier Coefficients of *f* (*x*)**

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 1, 2, \cdots$$

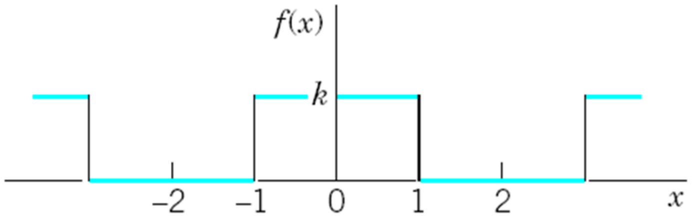
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, \cdots$$

11.2 Functions of Any Period p=2L (임의의 주기 p=2L을 가지는 함수)

■ **Ex. 1 Periodic Rectangular Wave (주기적인 직사각형파)**

Find the Fourier series of the function,

$$f(x) = \begin{cases} 0 & (-2 < x < -1) \\ k & (-1 < x < 1) \\ 0 & (1 < x < 2) \end{cases} \qquad p = 2L = 4, \quad L = 2$$

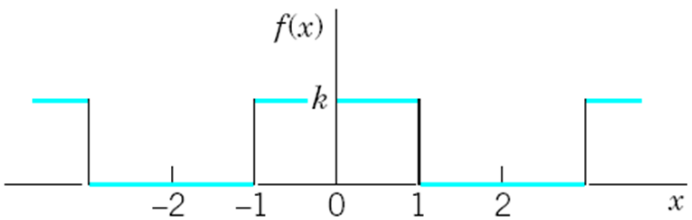


11.2 Functions of Any Period p=2L (임의의 주기 p=2L을 가지는 함수)

■ Ex. 1 Periodic Rectangular Wave (주기적인 직사각형파)

Find the Fourier series of the function,

$$f(x)=\begin{cases} 0 & (-2<x<-1) \\ k & (-1<x<1) \\ 0 & (1<x<2) \end{cases} \qquad p=2L=4, \quad L=2$$



$$* \quad a_0=\frac{1}{4}\int_{-2}^{2} f(x)dx=\frac{1}{4}\int_{-1}^{1} kdx=\frac{1}{4}\cdot 2k=\frac{k}{2}$$

$$* \quad a_n=\frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx=\frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}dx=\frac{2k}{n\pi}\sin\frac{n\pi}{2}=\begin{cases} 0 & n\text{이 짝수} \\ \dfrac{2k}{n\pi} & n=1,\ 5,\ 9,\ \cdots \\ -\dfrac{2k}{n\pi} & n=3,\ 7,\ 11,\ \cdots \end{cases}$$

$$* \quad \int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx=0 \quad \therefore\ b_n=0$$

∴ 푸리에 급수 : $f(x)=\dfrac{k}{2}+\dfrac{2k}{\pi}\left(\cos\dfrac{\pi}{2}x-\dfrac{1}{3}\cos\dfrac{3\pi}{2}x+\dfrac{1}{5}\cos\dfrac{5\pi}{2}x-+\cdots\right)$

11.2 Functions of Any Period p=2L
(임의의 주기 p=2L을 가지는 함수)

Example 2

Example 3

11.2 Functions of Any Period p=2L
(임의의 주기 p=2L을 가지는 함수)

PROBLEM SET 11.2

HW: 13, 16

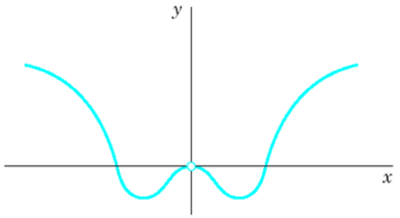
11.3 Even and Odd Functions. Half-Range Expansions (우함수와 기함수. 반구간 전개)

Fourier Cosine Series, Fourier Sine Series (푸리에 코사인, 사인 급수)

- **Fourier Cosine Series:** 주기가 $2L$인 우함수의 푸리에 급수

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

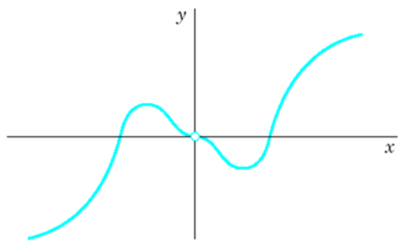
11.2 Example 1



$$\text{Fourier Coefficients } a_0 = \frac{1}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}xdx, \quad n = 1, 2, \cdots$$

- **Fourier Sine Series:** 주기가 $2L$인 기함수의 푸리에 급수

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$$



$$\text{Fourier Coefficients } b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}xdx, \quad n = 1, 2, \cdots$$

Sum and Scalar Multiple (합과 스칼라곱)

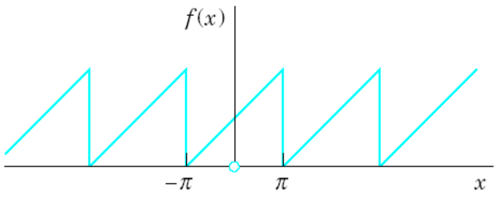
- 함수의 합의 푸리에 계수는 각각에 해당하는 푸리에 계수의 합과 같다.
- 함수 cf 의 푸리에 계수는 f 에 해당하는 푸리에 계수에 c 를 곱한 것과 같다.

11.3 Even and Odd Functions. Half-Range Expansions (우함수와 기함수. 반구간 전개)

- **Ex. 3 Sawtooth Wave (톱니파)**

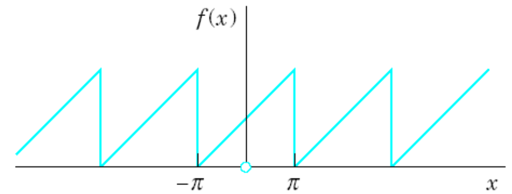
  Find the Fourier series of the function

  $f(x) = x + \pi \quad (-\pi < x < \pi)$ and $f(x + 2\pi) = f(x)$



11.3 Even and Odd Functions. Half-Range Expansions (우함수와 기함수. 반구간 전개)

Ex. 3 Sawtooth Wave (톱니파)



Find the Fourier series of the function

$$f(x)=x+\pi \quad (-\pi < x < \pi) \quad \text{and} \quad f(x+2\pi)=f(x)$$

$f_1 = x$와 $f_2 = \pi$라하면 $f = f_1 + f_2$이다.

* f_1의 푸리에 급수

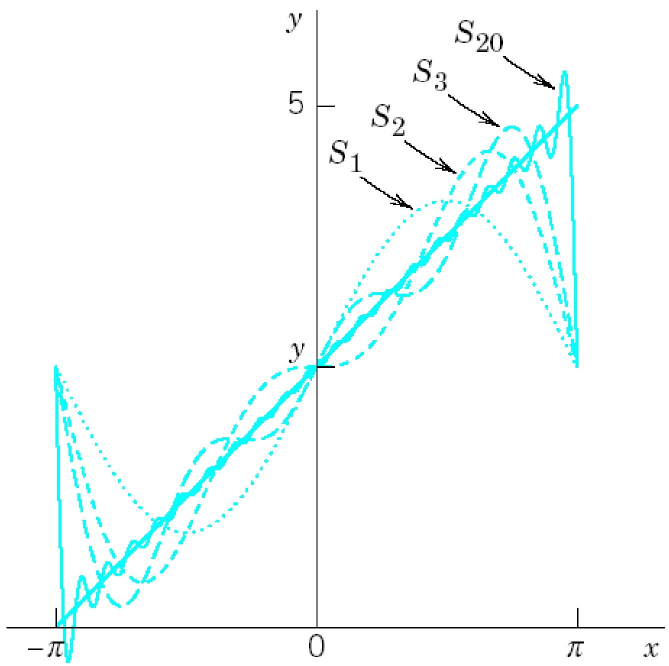
f_1은기함수이므로 $a_n = 0 \, (n = 0, 1, 2, \cdots)$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\left.\frac{-x\cos nx}{n}\right|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right] = -\frac{2}{n}\cos n\pi$$

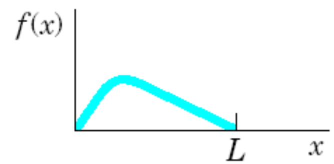
* $f_2 = \pi$

$$\therefore \; f = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots\right)$$

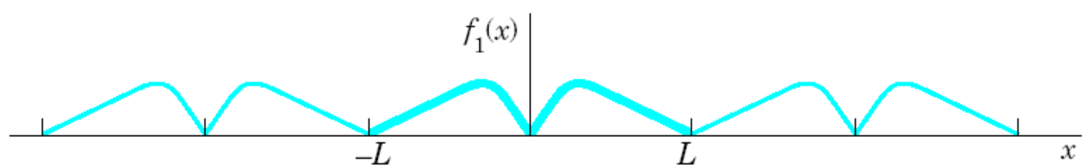


11.3 Even and Odd Functions. Half-Range Expansions (우함수와 기함수. 반구간 전개)

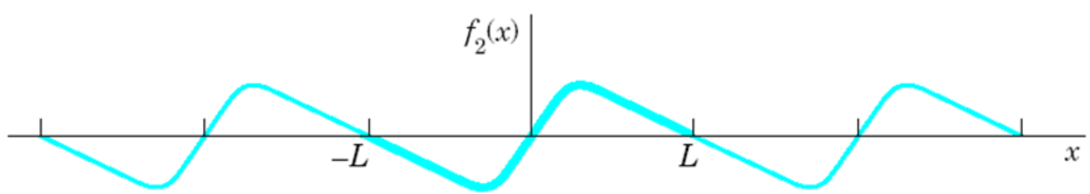
- **Half Range Expansions (반구간 전개)**
  - Even Periodic Extension (주기적인 우함수로 확장)
  - Odd Periodic Extension (주기적인 기함수로 확장)



(a) The given function $f(x)$



(b) $f(x)$ extended as an ***even*** periodic function of period $2L$

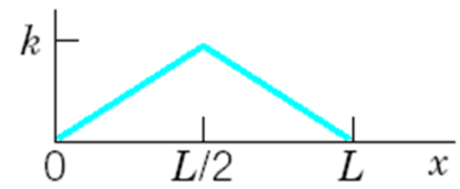


(c) $f(x)$ extended as an ***odd*** periodic function of period $2L$

11.3 Even and Odd Functions. Half-Range Expansions (우함수와 기함수. 반구간 전개)

■ **Ex. 4 "Triangle" and Its Half-Range Expansions**

Find the two half-range expansions of the function

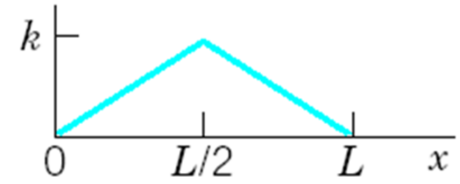


$$f(x) = \begin{cases} \dfrac{2k}{L}x & \left(0 < x < \dfrac{L}{2}\right) \\ \dfrac{2k}{L}(L - x) & \left(\dfrac{L}{2} < x < L\right) \end{cases}$$

11.3 Even and Odd Functions. Half-Range Expansions (우함수와 기함수. 반구간 전개)

■ **Ex. 4 "Triangle" and Its Half-Range Expansions**

Find the two half-range expansions of the function



$$f(x) = \begin{cases} \frac{2k}{L}x & \left(0 < x < \frac{L}{2}\right) \\ \frac{2k}{L}(L - x) & \left(\frac{L}{2} < x < L\right) \end{cases}$$

1. 주기적 우함수로 확장.

$$a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2} x\,dx + \frac{2k}{L}\int_{L/2}^{L}(L - x)dx\right] = \frac{k}{2}$$

$$a_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} x\cos\frac{n\pi}{L}x\,dx + \frac{2k}{L}\int_{L/2}^{L}(L - x)\cos\frac{n\pi}{L}x\,dx\right] = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{L} - \cos n\pi - 1\right)$$

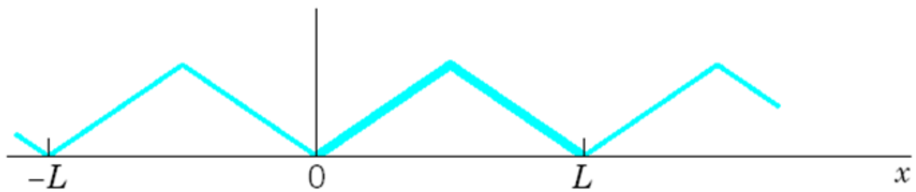
$$\therefore\ f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \cdots\right)$$

11.3 Even and Odd Functions. Half-Range Expansions (우함수와 기함수. 반구간 전개)

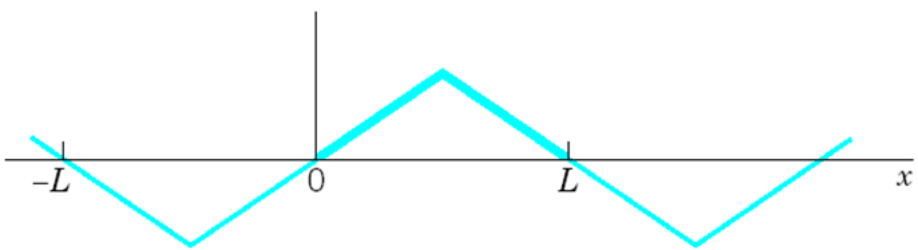
2. 주기적 기함수로 확장.

$$b_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} x\sin\frac{n\pi}{L}x\,dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\sin\frac{n\pi}{L}x\,dx\right] = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore\ f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - + \cdots\right)$$



(a) Even extension



(b) Odd extension

11.3 Even and Odd Functions. Half-Range Expansions (우함수와 기함수. 반구간 전개)

PROBLEM SET 11.3

HW: 10, 11, 19

11.4 Complex Fourier Series (복소 푸리에 급수)

- **Complex Fourier Series (복소 푸리에 급수):** $f(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx}$

- **Complex Fourier Coefficient (복소 푸리에 계수):**

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \qquad n=0,\ \pm 1,\ \pm 2,\ \cdots$$

주기가 2L인 복소 푸리에 급수:

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

$$c_n=\frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx, \qquad n=0,\ \pm 1,\ \pm 2,\ \cdots$$

11.4 Complex Fourier Series (복소 푸리에 급수)

■ **Ex. 1 Complex Fourier Series**

Find the complex Fourier series of $f(x) = e^x$ if $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$ and obtain from it the usual Fourier series

11.4 Complex Fourier Series (복소 푸리에 급수)

■ **Ex. 1 Complex Fourier Series**

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$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}dx = \frac{1}{2\pi}\frac{1}{1-in}e^{x-inx}\Big|_{x=-\pi}^{\pi} = \frac{1}{2\pi}\frac{1}{1-in}\left(e^{\pi}-e^{-\pi}\right)(-1)^n = \frac{\sinh\pi}{\pi}\frac{1+in}{1+n^2}(-1)^n$$

$$\Rightarrow \quad \text{Fourier Series: } e^x = \frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{1+in}{1+n^2}e^{inx} \quad (-\pi < x < \pi)$$

$$(1+in)e^{inx} = (1+in)(\cos nx + i\sin nx) = (\cos nx - n\sin nx) + i(n\cos nx + \sin nx)$$

$$(1-in)e^{-inx} = (1-in)(\cos nx - i\sin nx) = (\cos nx - n\sin nx) - i(n\cos nx + \sin nx)$$

$$\Rightarrow \quad (1+in)e^{inx} + (1-in)e^{-inx} = 2(\cos nx - n\sin nx)$$

$$\Rightarrow \quad e^x = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \frac{1}{1+1^2}(\cos x - \sin x) + \frac{1}{1+2^2}(\cos 2x - 2\sin 2x) - + \cdots\right]$$

11.5 Forced Oscillations (강제진동)

Section 2.8 Revisited

- **Free Motion (자유운동):** 외력이 없는 경우의 운동지배방정식

$$my''+cy'+ky=0$$

- **Forced Motion (강제운동):** 외부로부터의 힘이 물체에 작용하는 경우의 운동지배방정식

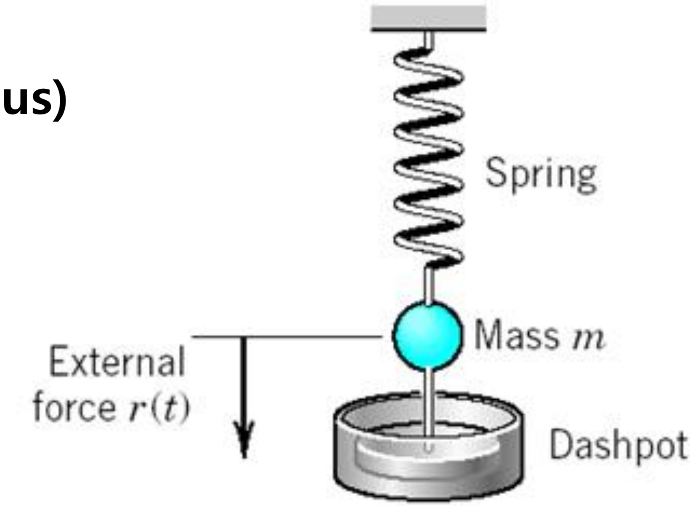
$$my''+cy'+ky=r(t)$$

- **Driving Force (입력이나 구동력):** $r(t)$
- **Response (출력 또는 구동력에 대한 시스템의 응답):** $y(t)$

11.5 Forced Oscillations (강제진동)

$$my''+cy'+ky=r(t)$$

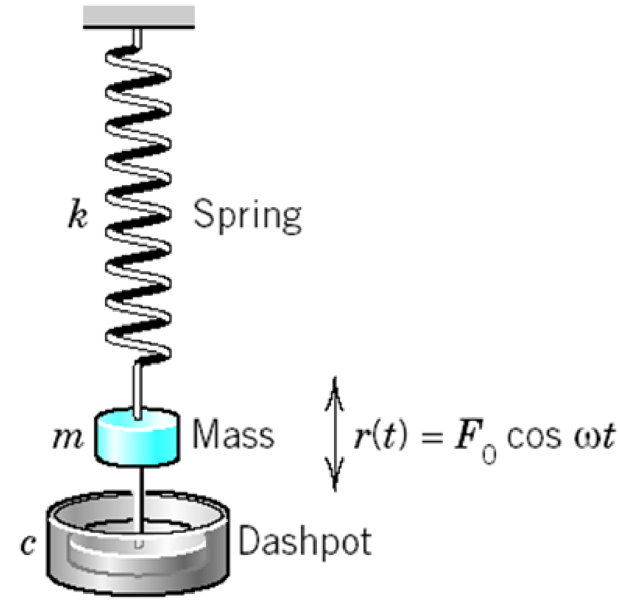
- $y = y(t)$: displacement from rest
- c : damping constant
- k : spring constant (spring modulus)
- $r(t)$: external force



2.8 Modeling: Forced Oscillations. Resonance (모델화: 강제진동. 공진)

- With Periodic external forces:

$$my''+cy'+ky=F_0\cos\omega t$$



- 미정계수법에 의한 y_p 결정

$$y_p = a\cos\omega t + b\sin\omega t$$

$$a = F_0 \frac{m\left(\omega_0^{\ 2} - \omega^2\right)}{m^2\left(\omega_0^{\ 2} - \omega^2\right)^2 + \omega^2 c^2}, \quad b = F_0 \frac{\omega c}{m^2\left(\omega_0^{\ 2} - \omega^2\right)^2 + \omega^2 c^2}$$

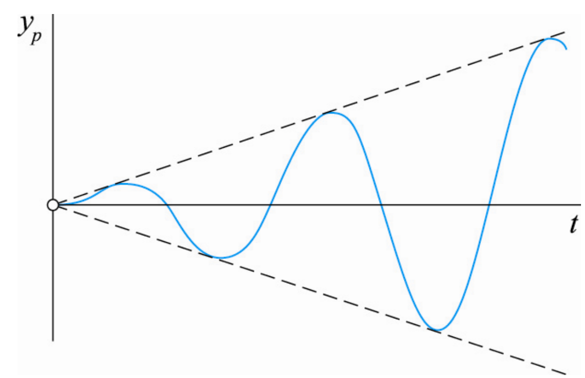
$$\omega_0 = \sqrt{k/m}$$

2.8 Modeling: Forced Oscillations. Resonance (모델화: 강제진동. 공진)

- **Case 1. Undamped Forced Oscillations (비감쇠 강제진동)**

$$c = 0 \quad \Rightarrow \quad y_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \quad \Rightarrow \quad y = C \cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

- 이 출력은 두 개의 조화진동의 중첩을 나타냄.
  - 고유주파수 : $\frac{\omega_0}{2\pi} \left[\text{cycles}/\text{sec} \right]$
  - 구동력의 주파수 : $\frac{\omega}{2\pi} \left[\text{cycles}/\text{sec} \right]$



- **Resonance (공진) :** 입력주파수와 고유주파수가 정합됨으로써

$(\omega = \omega_0)$ 발생하는 큰 진동의 여기현상

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega_0 t \longrightarrow y_p = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

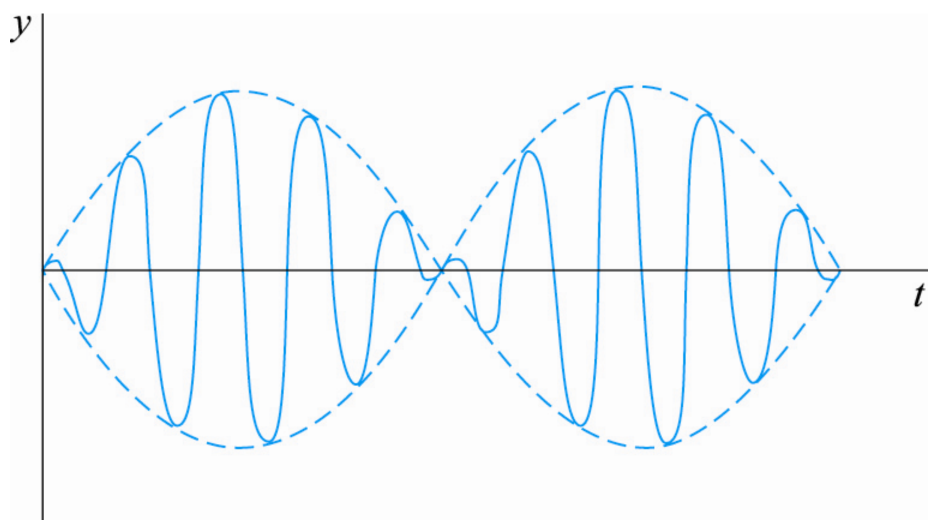
2.8 Modeling: Forced Oscillations. Resonance (모델화: 강제진동. 공진)

Tacoma Narrows Bridge
(1940. 11. 4)

2.8 Modeling: Forced Oscillations. Resonance (모델화: 강제진동. 공진)

- **Beats (맥놀이)**: 입력주파수와 고유주파수의 차가 적을 때의 강제 비감쇠진동 ($w \rightarrow w_0$)

$$y = \frac{F_0}{m(\omega_0{}^2 - \omega^2)}(\cos \omega t - \cos \omega_0 t) = \frac{2F_0}{m(\omega_0{}^2 - \omega^2)} \sin\left(\frac{\omega_0 + \omega}{2}t\right) \sin\left(\frac{\omega_0 - \omega}{2}t\right)$$



2.8 Modeling: Forced Oscillations. Resonance (모델화: 강제진동. 공진)

- **Case 2. Damped Forced Oscillations (감쇠 강제진동)**

- **Transient Solution (과도해)**: 비제차 방정식의 일반해 (y)
- **Steady-State Solution (정상상태해)**: 비제차 방정식의 특수해 (y_p)

  순수 사인파 형태의 구동력이 주어지는 감쇠진동 시스템의 출력은 충분하게 긴 시간 후에 실제적으로 주파수가 입력주파수인 조화진동이 됨.

❖ 과도해는 정상상태해로 접근한다.

- **Practical Resonance**: 비감쇠의 경우 ω가 ω_0 에 접근할 때 y_p 의 진폭이 무한대로 접근하는 반면에, 감쇠의 경우에는 이와 같은 현상은 발생하지 않는다.
이 경우에는 진폭은 항상 유한하나, c에 의존하는 어떤 ω 에 대해 최대값을 가질 수 있다.

$$y_p = a\cos\omega t + b\sin\omega t \qquad y_p(t) = C^*\cos(\omega t - \eta)$$

$$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}, \quad b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}$$

2.8 Modeling: Forced Oscillations. Resonance (모델화: 강제진동. 공진)

- y_p 의 진폭 (ω의 함수로 표현): $C^*(\omega) = \sqrt{a^2 + b^2} = \dfrac{F_0}{\sqrt{m^2({\omega_0}^2 - \omega^2)^2 + \omega^2 c^2}} = \dfrac{F_0}{R}$

$$\tan \eta(\omega) = \frac{b}{a} = \frac{\omega c}{m({\omega_0}^2 - \omega^2)}$$

$$\frac{dC^*}{d\omega} = F_0\left(-\frac{1}{2}R^{-3/2}\right)[2m^2(\omega_0^2 - \omega^2)(-2\omega) + 2\omega c^2]$$

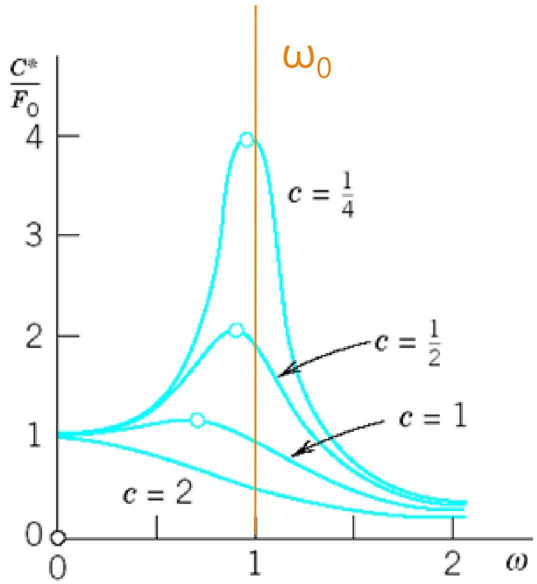
$$= -F_0\left(\frac{1}{2}R^{-3/2}\right)[2\omega(c^2 - 2mk + 2m^2\omega^2)]$$

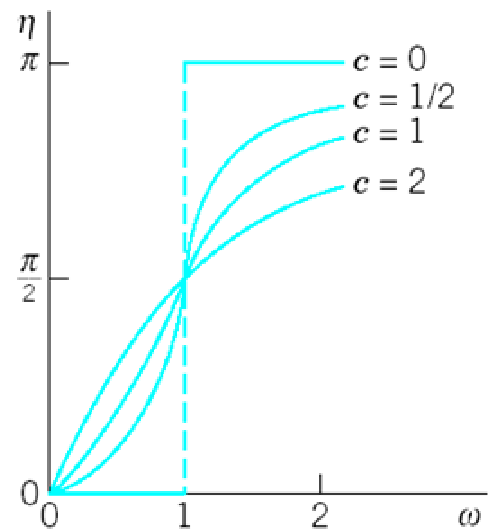
← $c^2 > 2mk$ 일 때 $C^*(\omega)$는 w가 증가함에 따라 단조 감소함.

$$\omega_{\max}^2 = \frac{k}{m} - \frac{c^2}{2m^2} = \omega_0^2 - \frac{c^2}{2m^2}$$

$$C^*(\omega_{\max}) = \sqrt{a^2 + b^2} = \frac{2mF_0}{c\sqrt{4m^2{\omega_0}^2 - c^2}}$$

2.8 Modeling: Forced Oscillations. Resonance (모델화: 강제진동. 공진)





- 의미
- $c > 0$ 일 때 $C^*(\omega_{\max})$는 유한하다는 것을 알 수 있다.
- $c^2 < 2mk$ 일 때 $C^*(\omega_{\max})$의 값은

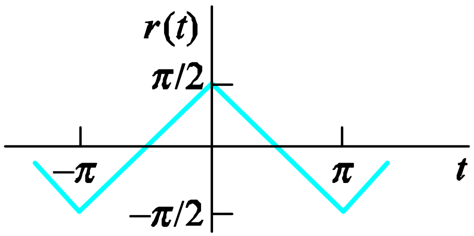
  c가 감소함에 따라 증가하고, c가 0에 접근함에 따라 무한대로 접근한다.

11.5 Forced Oscillations (강제진동)

■ Ex. 1 **Forced Oscillations under a Nonsinusoidal Periodic Driving Force**

Find the steady-state solution $y(t)$.

$$y''+0.05y'+25y=r(t), \quad r(t)=\begin{cases} t+\dfrac{\pi}{2} & (-\pi<t<0) \\ -t+\dfrac{\pi}{2} & (0<t<\pi) \end{cases} \quad r(t+2\pi)=r(t)$$



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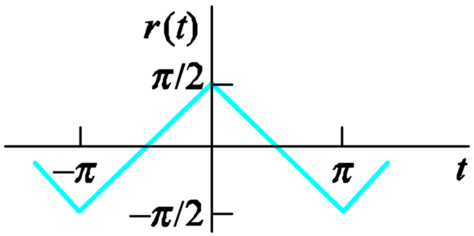


$r(t)$의 푸리에 급수 : $r(t)=\dfrac{4}{\pi}\left(\cos t+\dfrac{1}{3^2}\cos 3t+\dfrac{1}{5^2}\cos 5t+\cdots\right)$

FourierCosineSeries: 주기가 $2L$인 우함수의 푸리에 급수

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{L}x$$

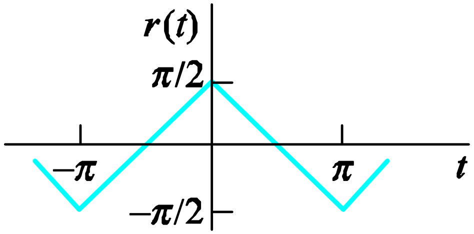
FourierCoefficients $a_0=\dfrac{1}{L}\displaystyle\int_0^L f(x)dx,\quad a_n=\dfrac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}xdx,\quad n=1,\ 2,\ \cdots$

11.5 Forced Oscillations (강제진동)

■ Ex. 1 **Forced Oscillations under a Nonsinusoidal Periodic Driving Force**

Find the steady-state solution $y(t)$.

$$y''+0.05y'+25y = r(t), \quad r(t) = \begin{cases} t+\dfrac{\pi}{2} & (-\pi < t < 0) \\ -t+\dfrac{\pi}{2} & (0 < t < \pi) \end{cases} \quad r(t+2\pi) = r(t)$$

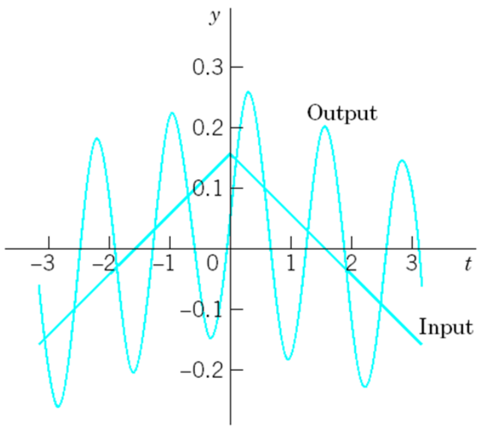


$r(t)$의 푸리에 급수 : $r(t) = \dfrac{4}{\pi}\left(\cos t + \dfrac{1}{3^2}\cos 3t + \dfrac{1}{5^2}\cos 5t + \cdots\right)$

상미분방정식 $y''+0.05y'+25y = \dfrac{4}{n^2\pi}\cos nt \quad (n = 1, 3, \cdots)$의 정상상태 해 : $y_n = A_n \cos nt + B_n \sin nt$

$A_n = \dfrac{4(25-n^2)}{n^2 \pi D_n}, \quad B_n = \dfrac{0.2}{n\pi D_n}$ 여기서 $D_n = (25-n^2)^2 + (0.05n)^2$

∴ 정상상태 해 : $y = y_1 + y_3 + y_5 + \cdots$



n=5 term is dominant.

11.5 Forced Oscillations (강제진동)

PROBLEM SET 11.5

HW: 2, 3

11.6 Approximation by Trigonometric Polynomials (삼각다항식에 의한 근사)

- **Approximation Theory (근사이론):**

푸리에 급수의 주된 응용 분야로 단순한 함수로써 어떤 함수의 근사값을 표현하는 분야

- **Idea**

$f(x)$는 주기가 2π인 푸리에 급수로 표현될 수 있는 주기함수 $(-\pi \le x \le \pi)$

$\Rightarrow$ N차 부분합은 $f(x)$에 대한 근사값

$$\therefore\ f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

- **Question**

최상의 f 의 근사인 N차 삼각다항식 $F(x) = A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx)$ (N은 고정) 구하기

11.6 Approximation by Trigonometric Polynomials (삼각다항식에 의한 근사)

- **Square Error (제곱오차)**

  $E=\int_{-\pi}^{\pi}(f-F)^2dx$: 구간 $-\pi\le x\le\pi$상에서 함수 F의 함수 f에 관한 **제곱오차(Square Error)**

- **Minimum Square Error (최소제곱오차)**

  구간 $-\pi\le x\le\pi$에서 F의 f에 관한 제곱오차는 F의 계수가 f의 푸리에 계수이면 최소가 된다. 그 최소값은 $E^*=\int_{-\pi}^{\pi}f^2dx-\pi\left[2a_0^{\ 2}+\sum_{n=1}^{N}\left(a_n^{\ 2}+b_n^{\ 2}\right)\right]$이다.

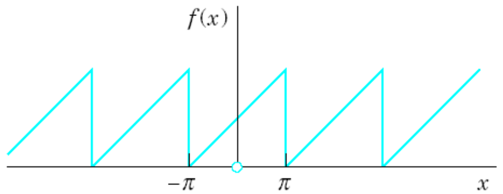
- N이 증가함에 따라 f의 푸리에 급수 부분합은 제곱오차 관점에서 점점 더 f를 잘 근사화하게 된다.

- **Bessel's Inequality:** $2a_0^{\ 2}+\sum_{n=1}^{\infty}\left(a_n^{\ 2}+b_n^{\ 2}\right)\le\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)^2dx$

- **Parseval's Identity:** $2a_0^{\ 2}+\sum_{n=1}^{\infty}\left(a_n^{\ 2}+b_n^{\ 2}\right)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)^2dx$

11.6 Approximation by Trigonometric Polynomials (삼각다항식에 의한 근사)

■ **Ex. 11.3 Sawtooth Wave (톱니파)**



Find the Fourier series of the function

$$f(x) = x + \pi \quad (-\pi < x < \pi) \quad \text{and} \quad f(x+2\pi) = f(x)$$

$f_1 = x$와 $f_2 = \pi$라 하면 $f = f_1 + f_2$이다.

* f_1의 푸리에 급수

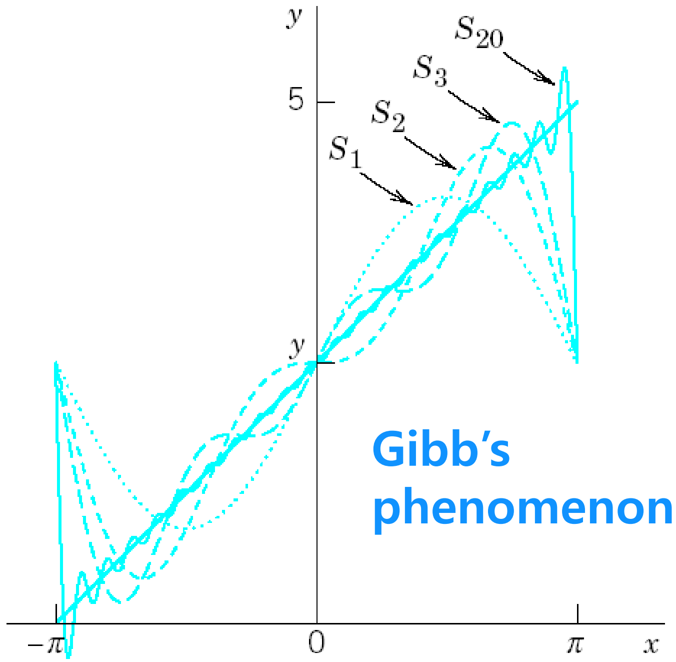
f_1은 기함수이므로 $a_n = 0 \; (n = 0, 1, 2, \cdots)$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\left.\frac{-x\cos nx}{n}\right|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right] = -\frac{2}{n}\cos n\pi$$

* $f_2 = \pi$

$$\therefore \; f = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots\right)$$



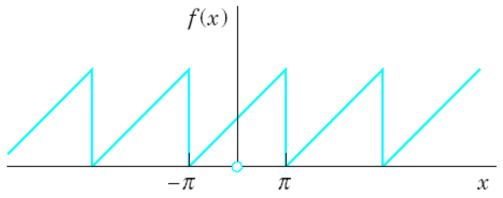
Gibb's phenomenon

11.6 Approximation by Trigonometric Polynomials (삼각다항식에 의한 근사)

■ **Ex. 11.3 Sawtooth Wave (톱니파)**

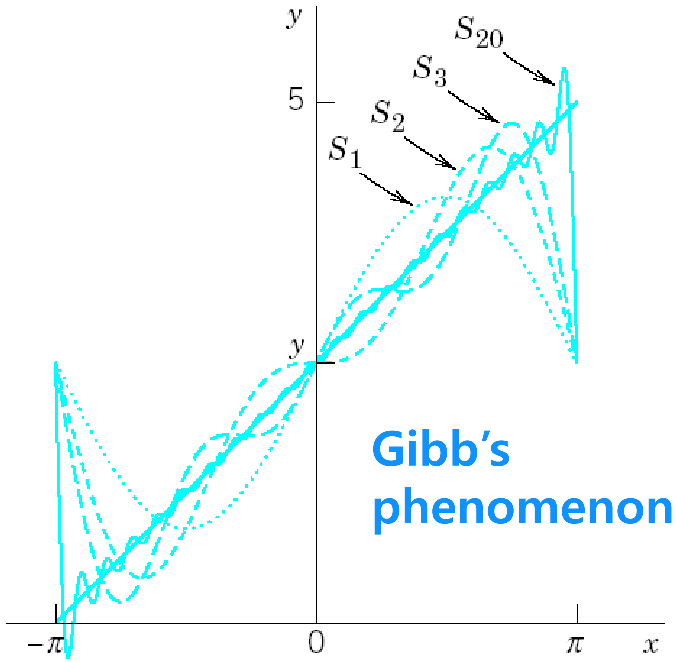
Find the Fourier series of the function

$$f(x) = x + \pi \quad (-\pi < x < \pi) \quad \text{and} \quad f(x + 2\pi) = f(x)$$



$$E^* = \int_{-\pi}^{\pi} (x+\pi)^2 dx - \pi\left[2\pi^2 + 4\sum_{n=1}^{N} \frac{1}{n^2}\right]$$

N	E*
1	8.1045
2	4.9629
10	1.1959
20	0.6129
100	0.0126



Gibb's phenomenon

11.6 Approximation by Trigonometric Polynomials (삼각다항식에 의한 근사)

PROBLEM SET 11.6

HW: 5, 14

11.7 Fourier Integral (푸리에 적분)

- Fourier series are powerful tools for problems involving functions that are periodic or are of interest on a finite interval only.
- However, many problems involve functions that are nonperiodic and are of interest on the whole x-axis.

Let $L \to \infty$ (period: $2L$)

11.7 Fourier Integral (푸리에 적분)

■ Ex. 1 Rectangular Wave (직사각형파)

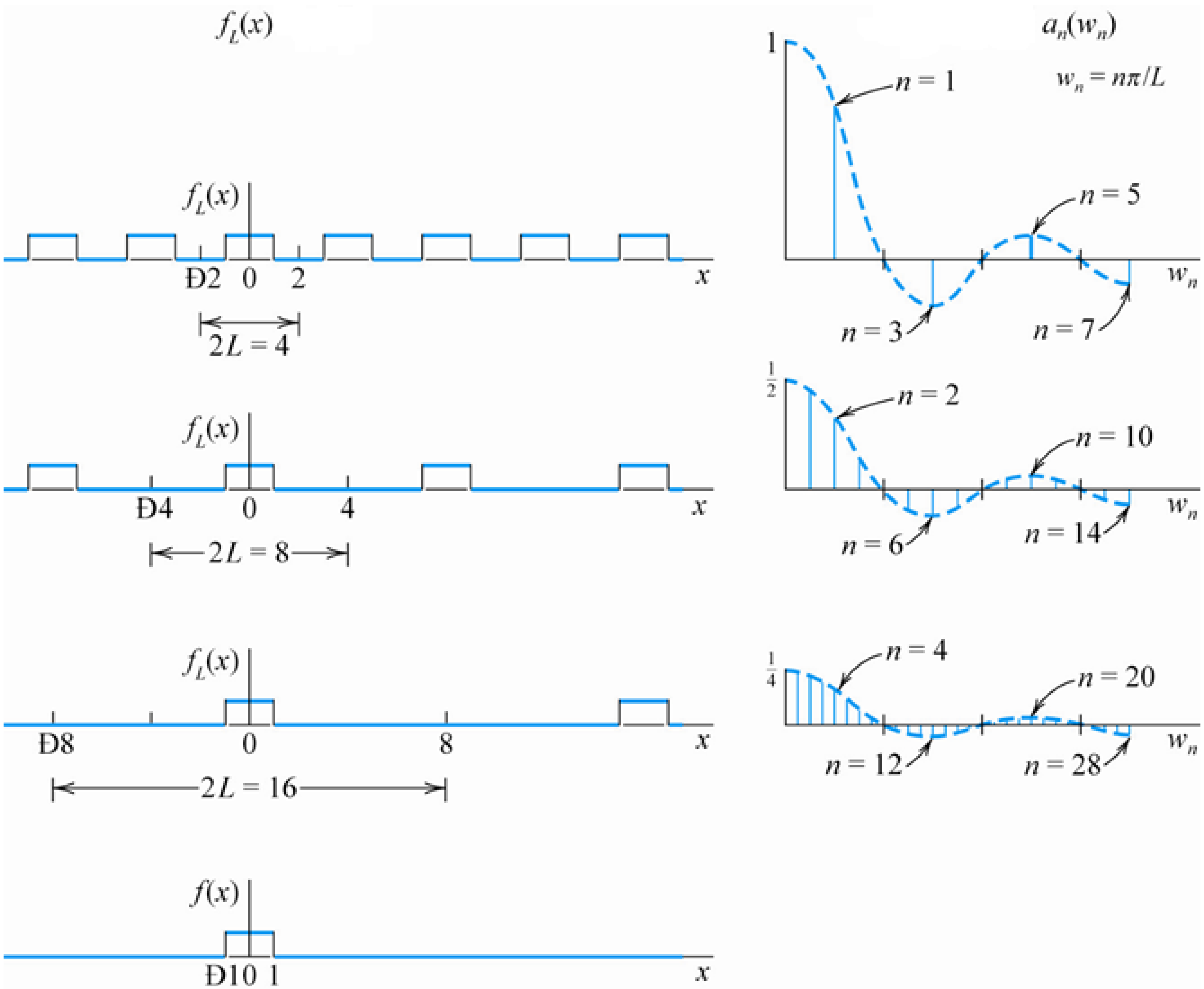
Consider the periodic rectangular wave $f_L(x)$ of period $2L>2$ given by

$$f_L(x)=\begin{cases}0 & (-L<x<-1)\\ 1 & (-1<x<1)\\ 0 & (1<x<L)\end{cases}$$

The nonperiodic function $f(x)$ obtained from f_L if we let $L \to \infty$.

$$f(x)=\lim_{L\to\infty} f_L(x)=\begin{cases}1 & (-1<x<1)\\ 0 & \text{otherwise}\end{cases}$$

11.7 Fourier Integral (푸리에 적분)



11.7 Fourier Integral (푸리에 적분)

- **From Fourier Series to Fourier Integral ($L \to \infty$)**

$$f_L(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos w_n x + b_n \sin w_n x\right), \qquad w_n = \frac{n\pi}{L}$$

$$= \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \int_{-L}^{L} f_L(v)\cos w_n v\,dv + \sin w_n x \int_{-L}^{L} f_L(v)\sin w_n v\,dv\right]$$

$$= \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos w_n x)\Delta w \int_{-L}^{L} f_L(v)\cos w_n v\,dv + (\sin w_n x)\Delta w \int_{-L}^{L} f_L(v)\sin w_n v\,dv\right]$$

$$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

11.7 Fourier Integral (푸리에 적분)

- **From Fourier Series to Fourier Integral ($L \to \infty$)**

$$f_L(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos w_n x + b_n \sin w_n x), \qquad w_n = \frac{n\pi}{L}$$

$$= \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \int_{-L}^{L} f_L(v)\cos w_n v dv + \sin w_n x \int_{-L}^{L} f_L(v)\sin w_n v dv\right]$$

$$= \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos w_n x)\Delta w \int_{-L}^{L} f_L(v)\cos w_n v dv + (\sin w_n x)\Delta w \int_{-L}^{L} f_L(v)\sin w_n v dv\right]$$

$$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$$

$$f(x) = \lim_{L\to\infty} f_L(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv dv + \sin wx \int_{-\infty}^{\infty} f(v)\sin wv dv\right]dw$$

$\Rightarrow$ 푸 리 에 적 분 : $f(x) = \int_{0}^{\infty}[A(w)\cos wx + B(w)\sin wx]dw$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv dv, \quad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv dv$$

11.7 Fourier Integral (푸리에 적분)

- **Fourier Integral**

$*$ 모든 유한구간에서 구분연속

$*$ 모든 점에서 좌도함수와 우도함수가 존재

$*$ $\lim_{a \to -\infty} \int_a^0 |f(x)| dx + \lim_{b \to \infty} \int_0^b |f(x)| dx$의 적분이 존재

$\Rightarrow$ $f(x)$는 푸리에 적분으로 표현될 수 있다.

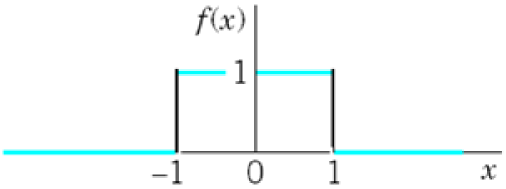
$f(x)$가 불연속인 점에서의 푸리에 적분값은 그 점에서 $f(x)$의 좌극한값과 우극한값의 평균과 같다.

11.7 Fourier Integral (푸리에 적분)

■ **Ex. 2 Single Pulse, Sine Integral**

Find the Fourier integral representation of the function

$$f(x)=\begin{cases}1 & (|x|<1)\\ 0 & (|x|>1)\end{cases}$$



11.7 Fourier Integral (푸리에 적분)

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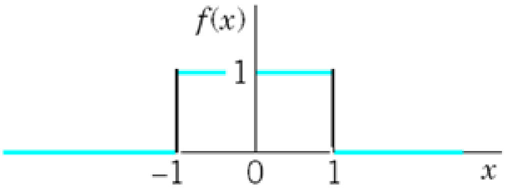
Find the Fourier integral representation of the function



$$f(x)=\begin{cases}1 & (|x|<1)\\ 0 & (|x|>1)\end{cases}$$

$$A(w)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv=\frac{1}{\pi}\int_{-1}^{1}\cos wv\,dv=\left.\frac{\sin wv}{\pi w}\right|_{-1}^{1}=\frac{2\sin w}{\pi w},$$

$$B(w)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv=\frac{1}{\pi}\int_{-1}^{1}\sin wv\,dv=0$$

$$\therefore\ f(x)=\frac{2}{\pi}\int_{0}^{\infty}\frac{\cos wx\sin w}{w}dw$$

$$\text{Dirichlet Discontinuous Factor (불연속인자)}:\ \int_{0}^{\infty}\frac{\cos wx\sin w}{w}dw=\begin{cases}\pi/2 & (0\le x<1)\\ \pi/4 & (x=1)\\ 0 & (x>1)\end{cases}$$

$$x=0\quad\Rightarrow\quad\int_{0}^{\infty}\frac{\sin w}{w}dw=\frac{\pi}{2}$$

11.7 Fourier Integral (푸리에 적분)

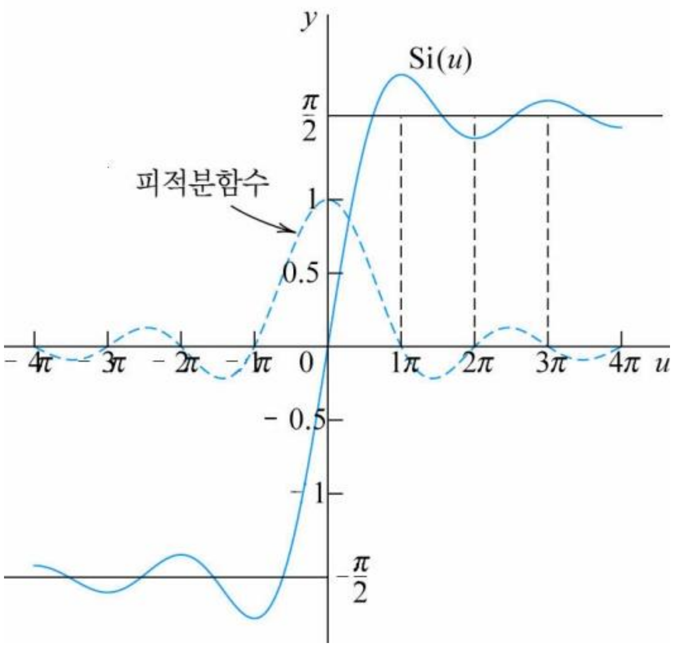
Sine Integral (사인적분): $\mathrm{Si}(u) = \int_0^u \frac{\sin w}{w} dw$

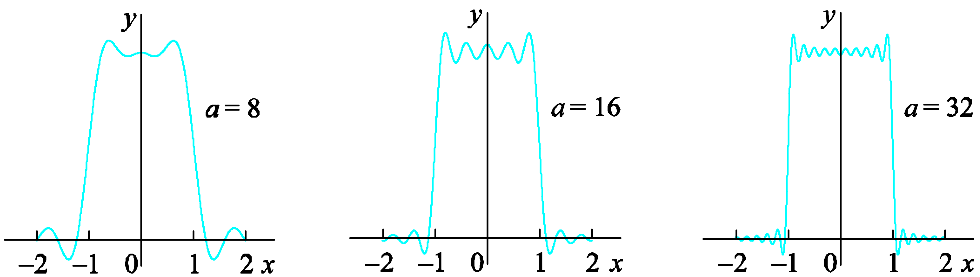
$$f(x) \approx \frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w} dw$$

$$= \frac{1}{\pi}\int_0^a \frac{\sin(w+wx)}{w} dw + \frac{1}{\pi}\int_0^a \frac{\sin(w-wx)}{w} dw$$

$$= \frac{1}{\pi}\int_0^{(x+1)a} \frac{\sin t}{t} dt - \frac{1}{\pi}\int_0^{(x-1)a} \frac{\sin t}{t} dt$$

$$= \frac{1}{\pi}\mathrm{Si}(a[x+1]) - \frac{1}{\pi}\mathrm{Si}(a[x-1])$$





11.7 Fourier Integral (푸리에 적분)

- **Fourier Cosine Integral and Fourier Sine Integral**

- **Fourier Cosine Integral**: 우함수일 때, 푸리에 적분

$$f(x)=\int_0^{\infty} A(w)\cos wx\,dw, \quad A(w)=\frac{2}{\pi}\int_0^{\infty} f(v)\cos wv\,dv$$

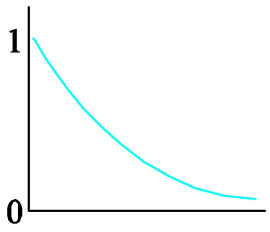
- **Fourier Sine Integral**: 기함수일 때, 푸리에 적분

$$f(x)=\int_0^{\infty} B(w)\sin wx\,dw, \quad B(w)=\frac{2}{\pi}\int_0^{\infty} f(v)\sin wv\,dv$$

11.7 Fourier Integral (푸리에 적분)

■ **Ex. 3 Laplace Integrals (라플라스 적분)**

We shall derive the Fourier cosine and Fourier sine integrals of $f(x) = e^{-kx}$, where $x > 0$ and $k > 0$. The result will be used to evaluate the so-called Laplace integrals.



11.7 Fourier Integral (푸리에 적분)

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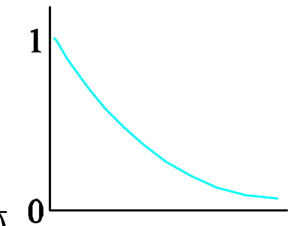


1. Fourier cosine integral

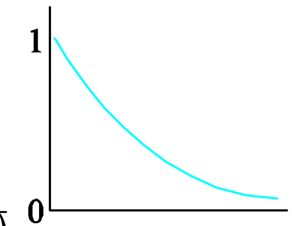
$$A(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\cos wv\,dv = \frac{2}{\pi}\left[-\frac{k}{k^2+w^2}e^{-kv}\left(-\frac{w}{k}\sin wv + \cos wv\right)\right]_0^\infty = \frac{2k}{\pi\left(k^2+w^2\right)}$$

$$\therefore\ f(x) = e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2+w^2}dw \quad \Rightarrow \quad \int_0^\infty \frac{\cos wx}{k^2+w^2}dw = \frac{\pi}{2k}e^{-kx}$$

11.7 Fourier Integral (푸리에 적분)

■ Ex. 3 Laplace Integrals (라플라스 적분)

We shall derive the Fourier cosine and Fourier sine integrals of $f(x) = e^{-kx}$, where $x > 0$ and $k > 0$. The result will be used to evaluate the so-called Laplace integrals.



1. Fourier cosine integral

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$$\therefore\ f(x) = e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2+w^2}dw \quad\Rightarrow\quad \int_0^\infty \frac{\cos wx}{k^2+w^2}dw = \frac{\pi}{2k}e^{-kx}$$

2. Fourier sine integral

$$B(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\sin wv\,dv = \frac{2}{\pi}\left[-\frac{w}{k^2+w^2}e^{-kv}\left(-\frac{k}{w}\sin wv + \cos wv\right)\right]\Bigg|_0^\infty = \frac{2w}{\pi\left(k^2+w^2\right)}$$

$$\therefore\ f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{w\sin wx}{k^2+w^2}dw \quad\Rightarrow\quad \int_0^\infty \frac{w\sin wx}{k^2+w^2}dw = \frac{\pi}{2}e^{-kx}$$

Laplace integrals

11.7 Fourier Integral (푸리에 적분)

PROBLEM SET 11.7

HW: 20

11.8 Fourier Cosine and Sine Transforms (푸리에 코사인 및 사인 변환)

- **Integral Transform (적분변환):**

  주어진 함수를 다른 변수에 종속하는 새로운 함수로 만드는 적분 형태의 변환

  예) Laplace Transform (Ch. 6)

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)dt$$

- **Fourier Cosine Transform (푸리에 코사인 변환)**

$$f(x) = \int_0^\infty A(w)\cos wx\,dw, \quad A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv$$

$$A(w) = \sqrt{2/\pi}\ \hat{f}_c(w)$$

$$\Rightarrow \textbf{Fourier Cosine Transform}: \mathcal{F}_c(f) = \hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx,$$

$$\textbf{Inverse Fourier Cosine Transform}: f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(w)\cos wx\,dw$$

11.8 Fourier Cosine and Sine Transforms (푸리에 코사인 및 사인 변환)

- **Fourier Sine Transform (푸리에 사인 변환)**

$$f(x)=\int_0^\infty B(w)\sin wx\,dw,\quad B(w)=\frac{2}{\pi}\int_0^\infty f(v)\sin wv\,dv$$

$$B(w)=\sqrt{2/\pi}\;\hat{f}_s(w)$$

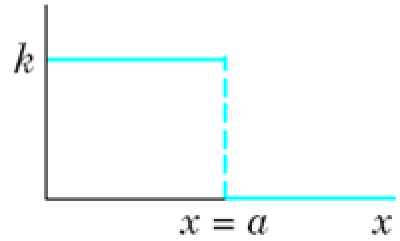
$$\Rightarrow \textbf{Fourier Sine Transform}:\ \mathcal{F}_s(f)=\hat{f}_s(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx\,dx,$$

$$\textbf{Inverse Fourier Sine Transform}:\ f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(w)\sin wx\,dw$$

11.8 Fourier Cosine and Sine Transforms (푸리에 코사인 및 사인 변환)

■ Ex. 1 **Fourier Cosine and Fourier Sine Transforms**

$$f(x) = \begin{cases} k & (0 < x < a) \\ 0 & (x > a) \end{cases}$$



11.8 Fourier Cosine and Sine Transforms (푸리에 코사인 및 사인 변환)

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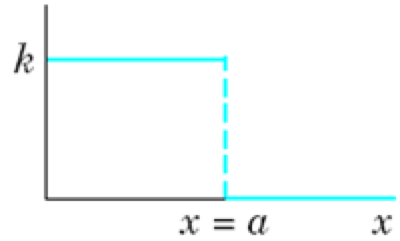

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} k \int_0^a \cos wx\, dx = \sqrt{\frac{2}{\pi}} k \left(\frac{\sin aw}{w} \right)$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} k \int_0^a \sin wx\, dx = \sqrt{\frac{2}{\pi}} k \left(\frac{1 - \cos aw}{w} \right)$$

11.8 Fourier Cosine and Sine Transforms (푸리에 코사인 및 사인 변환)

- **Linearity**

$$\mathcal{F}_c(af+bg)=a\mathcal{F}_c(f)+b\mathcal{F}_c(g)$$

$$\mathcal{F}_s(af+bg)=a\mathcal{F}_s(f)+b\mathcal{F}_s(g)$$

- **Cosine and Sine Transforms of Derivatives**

$f(x)$가 연속이고, x축 상에서 절대적분 가능

$f'(x)$가 모든 유한구간에서 구분 연속

$$x\to\infty,\quad f(x)\to 0$$

$$\Rightarrow\quad \mathcal{F}_c\{f'(x)\}=w\mathcal{F}_s\{f(x)\}-\sqrt{\frac{2}{\pi}}f(0),\quad \mathcal{F}_s\{f'(x)\}=-w\mathcal{F}_c\{f(x)\}$$

Prove!

$$\mathcal{F}_c\{f''(x)\}=-w^2\mathcal{F}_c\{f(x)\}-\sqrt{\frac{2}{\pi}}f'(0),\quad \mathcal{F}_s\{f''(x)\}=-w^2\mathcal{F}_s\{f(x)\}+\sqrt{\frac{2}{\pi}}wf(0)$$

11.8 Fourier Cosine and Sine Transforms (푸리에 코사인 및 사인 변환)

- Ex. 3 **An Application of the Operational Formula (9)**

  Find the Fourier cosine transform $f(x) = e^{-ax} \left(a > 0 \right)$ **in two ways**

11.8 Fourier Cosine and Sine Transforms (푸리에 코사인 및 사인 변환)

- Ex. 3 **An Application of the Operational Formula (9)**

  Find the Fourier cosine transform $f(x)=e^{-ax}(a>0)$ **in two ways**

미분에 의하여, $(e^{-ax})''=a^2e^{-ax}$이므로 $a^2 f(x)=f''(x)$

$$\Rightarrow \quad a^2\mathcal{F}_c(f)=\mathcal{F}_c(f'')=-w^2\mathcal{F}_c(f)-\sqrt{\frac{2}{\pi}}f'(0)=-w^2\mathcal{F}_c(f)+a\sqrt{\frac{2}{\pi}}$$

$$\Rightarrow \quad (a^2+w^2)\mathcal{F}_c(f)=a\sqrt{\frac{2}{\pi}}$$

$$\therefore \quad \mathcal{F}_c(e^{-ax})=\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+w^2}\right) \qquad (a>0)$$

11.8 Fourier Cosine and Sine Transforms (푸리에 코사인 및 사인 변환)

PROBLEM SET 11.8

HW: 4, 18

11.9 Fourier Transform. Discrete and Fast Fourier Transforms (푸리에 변환. 이산 및 고속 푸리에 변환)

- **Complex Form of the Fourier Integral (푸리에 적분의 복소형식)**

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]\,dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$$

11.9 Fourier Transform. Discrete and Fast Fourier Transforms (푸리에 변환. 이산 및 고속 푸리에 변환)

- **Complex Form of the Fourier Integral (푸리에 적분의 복소형식)**

$$f(x) = \int_0^{\infty} [A(w)\cos wx + B(w)\sin wx]\, dw$$

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv\, dv$$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv\, dv$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) \cos(wx - wv)\, dv \right] dw$$

Prove!

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) \sin(wx - wv)\, dv \right] dw = 0$$

$$e^{ix} = \cos x + i \sin x$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) e^{-iw(x-v)}\, dv\, dw$$

Prove!

11.9 Fourier Transform. Discrete and Fast Fourier Transforms (푸리에 변환. 이산 및 고속 푸리에 변환)

- **Complex Form of the Fourier Integral (푸리에 적분의 복소형식)**

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-iwv} dv] e^{iwx} dw$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw$$

$$F(s) = \mathcal{L}(f) = \int_{0}^{\infty} e^{-st} f(t) dt$$

11.9 Fourier Transform. Discrete and Fast Fourier Transforms (푸리에 변환. 이산 및 고속 푸리에 변환)

- **Complex Form of the Fourier Integral (푸리에 적분의 복소형식)**
  - **Complex Fourier Integral:** $f(x)=\dfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)e^{-iw(x-v)}dv\,dw$
  - **Fourier Transform:** $\mathcal{F}(f)=\hat{f}(w)=\dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx$
  - **Inverse Fourier Transform:** $f(x)=\dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}dw$
- **Existence of the Fourier Transform (푸리에 변환의 존재)**

$f(x)$가 x축 상에서 절대적분가능이고 모든 유한구간에서 구분연속

$\Rightarrow$ $f(x)$의 푸리에 변환 $\hat{f}(w)$는 존재하며, $\hat{f}(w)=\dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx$

11.9 Fourier Transform. Discrete and Fast Fourier Transforms (푸리에 변환. 이산 및 고속 푸리에 변환)

- Ex. 1 **Fourier Transform**

$$f(x) = 1 \text{ if } |x| < 1 \text{ and } \; f(x) = 0 \text{ otherwise}$$

11.9 Fourier Transform. Discrete and Fast Fourier Transforms (푸리에 변환. 이산 및 고속 푸리에 변환)

- Ex. 1 **Fourier Transform**

$$f(x) = 1 \text{ if } |x| < 1 \text{ and } \ f(x) = 0 \text{ otherwise}$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \cdot \left.\frac{e^{-iwx}}{-iw}\right|_{-1}^{1} = \frac{1}{-iw\sqrt{2\pi}} \left(e^{-iw} - e^{iw}\right)$$

$$= \frac{-2i \sin w}{-iw\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}$$

11.9 Fourier Transform. Discrete and Fast Fourier Transforms (푸리에 변환. 이산 및 고속 푸리에 변환)

- **Linearity. Fourier Transform of Derivatives (도함수의 푸리에 변환)**
- **Linearity of the Fourier Transform**

$$\mathcal{F}(af+bg)=a\mathcal{F}(f)+b\mathcal{F}(g)$$

- **Fourier Transform of Derivatives**

함수 $f(x)$가 x축 상에서 연속

$f'(x)$가 x축 상에서 절대적분 가능

$$|x|\to\infty,\quad f(x)\to 0$$

$$\Rightarrow\quad \mathcal{F}\{f'(x)\}=iw\mathcal{F}\{f(x)\},\quad \mathcal{F}\{f''(x)\}=-w^2\mathcal{F}\{f(x)\}$$ **Prove!**

11.9 Fourier Transform. Discrete and Fast Fourier Transforms (푸리에 변환. 이산 및 고속 푸리에 변환)

- Ex. 3 **Application of the Operational Formula (9)**

  Find the Fourier transform of xe^{-x^2}

$$\mathcal{F}\left(e^{-ax^2}\right)=\frac{1}{\sqrt{2a}}e^{-w^2/4a} \quad (a>0)$$

11.9 Fourier Transform. Discrete and Fast Fourier Transforms (푸리에 변환. 이산 및 고속 푸리에 변환)

- Ex. 3 **Application of the Operational Formula (9)**

  Find the Fourier transform of xe^{-x^2}

$$\mathcal{F}\left(e^{-ax^2}\right)=\frac{1}{\sqrt{2a}}e^{-w^2/4a} \quad (a>0)$$

$$\mathcal{F}\left(xe^{-x^2}\right)=\mathcal{F}\left\{-\frac{1}{2}\left(e^{-x^2}\right)'\right\}=-\frac{1}{2}\mathcal{F}\left\{\left(e^{-x^2}\right)'\right\}=-\frac{1}{2}iw\mathcal{F}\left(e^{-x^2}\right)$$

$$=-\frac{1}{2}iw\frac{1}{\sqrt{2}}e^{-w^2/4}=-\frac{iw}{2\sqrt{2}}e^{-w^2/4}$$

11.9 Fourier Transform. Discrete and Fast Fourier Transforms (푸리에 변환. 이산 및 고속 푸리에 변환)

$$\mathcal{F}(fg) \neq \mathcal{F}(f)\mathcal{F}(g)$$

- **Convolution (합성곱)**

- Convolution: $(f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)dp = \int_{-\infty}^{\infty} f(x-p)g(p)dp$

- Convolution Theorem (합성곱 정리):

  함수 $f(x)$와 $g(x)$가 구분연속이고, 유한하며, x축 상에서 절대적분가능

$$\mathcal{F}(f * g) = \sqrt{2\pi}\,\mathcal{F}(f)\mathcal{F}(g)$$ **Prove using *x*-*p*=*q***

11.9 Fourier Transform. Discrete and Fast Fourier Transforms (푸리에 변환. 이산 및 고속 푸리에 변환)

PROBLEM SET 11.9

HW: 14