Topics in Fusion and Plasma Studies 2
(459.667, 3 Credits)

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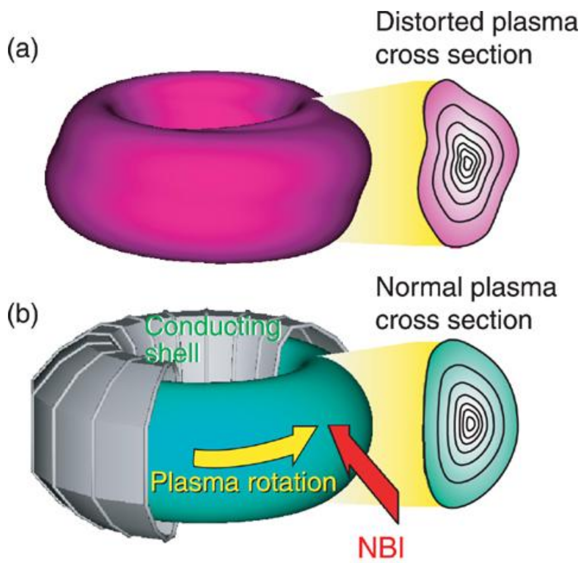
Contents

Contents

General Properties of Ideal MHD

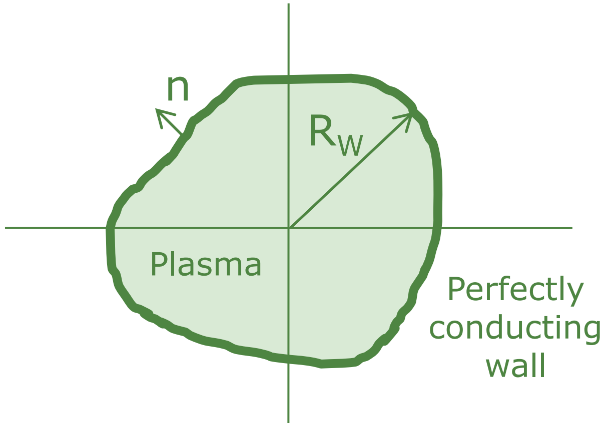
- **Introduction**

- Short description of the three most common classes of boundary conditions
- Conservation of mass, momentum, and energy, both locally and globally despite the significant number of approximations made in ideal MHD
- Consequence of the perfect conductivity assumption



General Properties of Ideal MHD

- **Boundary Conditions**
  - Perfectly conducting wall:
    tangential electric field and normal magnetic field vanish on the conducting wall



$$\vec{n} \times \vec{E}\Big|_{R_W} = 0$$

$$\vec{n} \cdot \vec{B}\Big|_{R_W} = 0$$

$$\vec{n} \cdot \vec{v}\Big|_{R_W} = 0 \quad \longleftarrow \quad \vec{n} \times \vec{E} + (\vec{n} \cdot \vec{B})\vec{v} - (\vec{n} \cdot \vec{v})\vec{B} = 0$$

ideal Ohm's law

General Properties of Ideal MHD

- **Boundary Conditions**
  - Insulating vacuum region:

  assume that the equations can be solved in each region

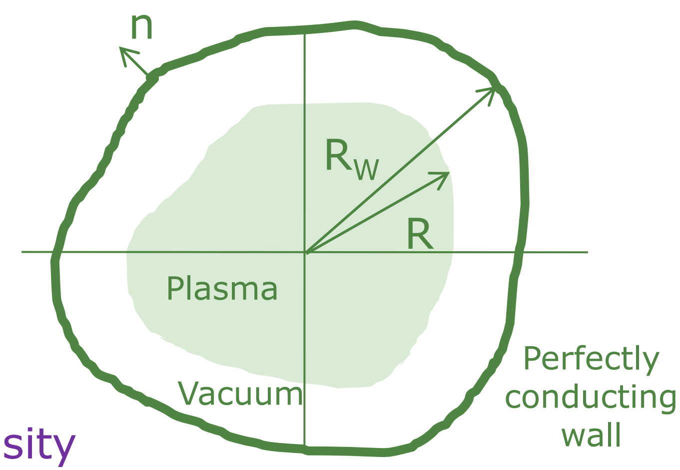
$$\nabla \times \hat{B} = 0$$

$$\nabla \cdot \hat{B} = 0$$

vacuum

$$\hat{\mathrm{n}} \cdot \vec{B}\Big|_{R_W} = 0$$

wall



$$[[\mathrm{n} \cdot \vec{B}]]_R = 0 \qquad [[Q]] = \hat{Q} - Q$$

$$[[\mathrm{n} \times \vec{B}]]_R = \mu_0 \vec{K}$$

Surface current density

$$[[p + B^2 / 2\mu_0]]_R = 0$$

$\vec{n} \cdot$

Free boundary problem

.VS.

Fixed boundary problem

$\vec{n}$

B

$B^2|_R = B^2|_R$

and the magnetic pressure are continuous

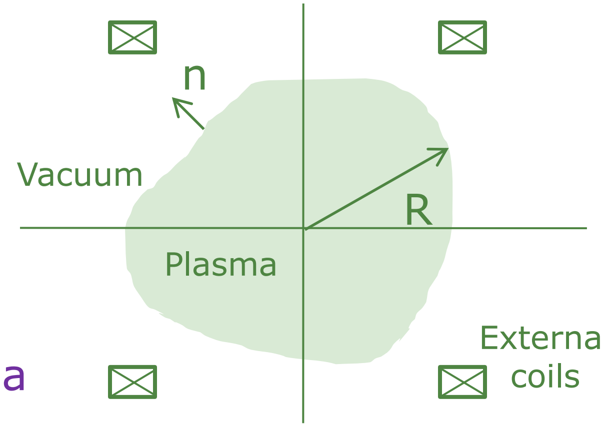
General Properties of Ideal MHD

- **Boundary Conditions**
  - Plasma surrounded by external coils:
    difficult but realistic situation where the plasma is confined by the magnetic fields created by a fixed set of external conductors

$$[[\mathrm{n} \cdot \vec{B}]]_R = 0$$

$$[[\mathrm{n} \times \vec{B}]]_R = \mu_0 \vec{K}$$

$$[[p + B^2 / 2\mu_0]]_R = 0$$



$$\hat{B} = \vec{B}_a + \tilde{B}$$ applied field by coils and induced field due to the plasma

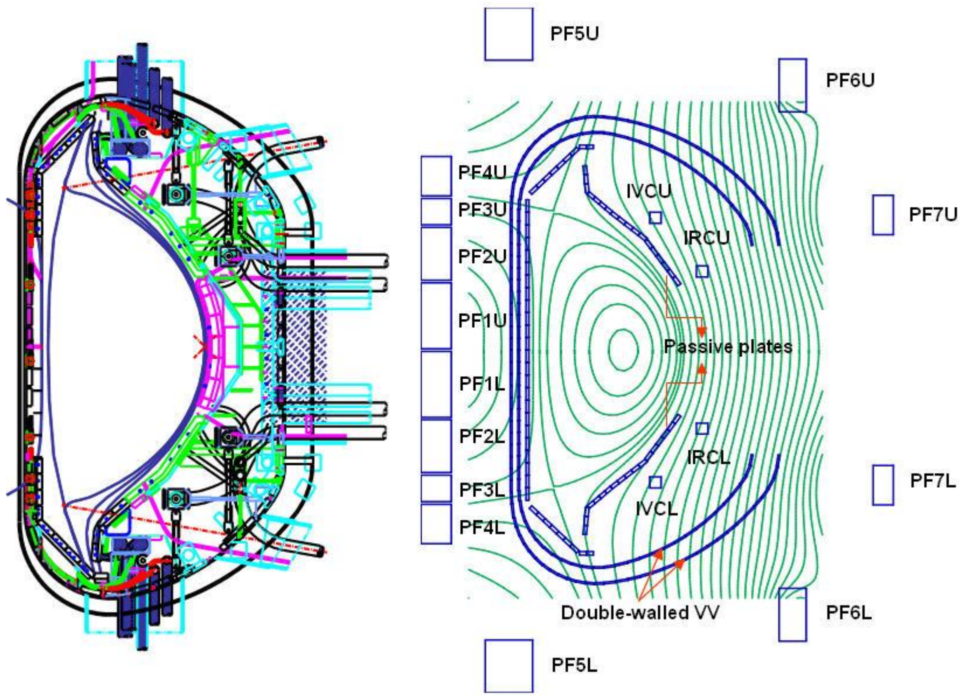
$$\vec{B}_a = \nabla \times \vec{A}$$

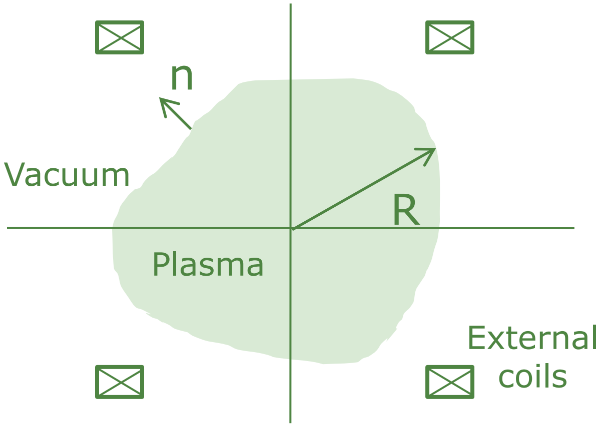
$$\vec{A} = \frac{\mu_0}{4\pi} \sum_i \int \frac{\vec{J}_i}{|\vec{r} - \vec{r}\,'|} d\vec{r}\,'$$ Biot-Savart law

$$\nabla \times \tilde{B} = \nabla \cdot \tilde{B} = 0$$ induced field in the vacuum region

General Properties of Ideal MHD

- **Boundary Conditions**
  - Plasma surrounded by external coils:
  difficult but realistic situation where the plasma is confined by the magnetic fields created by a fixed set of external conductors





General Properties of Ideal MHD

- **Local Conservation Relations**
  - Since a considerable number of assumptions were made in the derivation of the MHD equations it is important to investigate whether the resulting model still satisfies the basic conservation laws.

$$\frac{\partial}{\partial t}(\quad)+\nabla\cdot(\quad)=0$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0 \qquad \overleftrightarrow{T}=\rho\vec{v}\vec{v}+\left(p+\frac{B^2}{2\mu_0}\right)\overleftrightarrow{I}-\frac{\vec{B}\vec{B}}{\mu_0}$$

stress tensor

$$\frac{\partial \rho\vec{v}}{\partial t}+\nabla\cdot\overleftrightarrow{T}=0 \qquad w=\frac{1}{2}\rho v^2+\frac{B^2}{2\mu_0}+\frac{p}{\gamma-1}$$

energy

$$\frac{\partial w}{\partial t}+\nabla\cdot\vec{s}=0 \qquad \vec{s}=\left(\frac{1}{2}\rho v^2+\frac{\gamma}{\gamma-1}p\right)\vec{v}+\frac{1}{\mu_0}\vec{E}\times\vec{B}$$

energy flux

General Properties of Ideal MHD

- **Local Conservation Relations**

$$\overleftrightarrow{T} = \rho\vec{v}\vec{v} + \left(p + \frac{B^2}{2\mu_0}\right)\overleftrightarrow{I} - \frac{\vec{B}\vec{B}}{\mu_0}$$

Reynolds stress

$$\overleftrightarrow{T}_R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho v^2 \end{pmatrix}$$

Important with large fluid flow, but usually not in studies of plasma stability where the flows are either small or zero

$$\overleftrightarrow{T}_B = \begin{pmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_{||} \end{pmatrix}$$

$$p_\perp = p + \frac{B^2}{2\mu_0}, \quad p_{||} = p - \frac{B^2}{2\mu_0}$$

isotropic plasma pressure

Total pressure parallel to the field: negative magnetic pressure correspond to a tension along the field lines

General Properties of Ideal MHD

- **Local Conservation Relations**

$$w = \frac{1}{2}\rho v^2 + \frac{B^2}{2\mu_0} + \frac{p}{\gamma - 1} = w_K + w_P$$

$$w_K = \frac{1}{2}\rho v^2$$ plasma kinetic energy

$$w_P = \frac{B^2}{2\mu_0} + \frac{p}{\gamma - 1}$$ Potential energy

Magnetic energy+
internal energy of the plasma

$$\vec{s} = \left(\frac{1}{2}\rho v^2 + \frac{p}{\gamma - 1}\right)\vec{v} + p\vec{v} + \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

Flow of plasma
kinetic+internal energy

Poynting flux:
flow of electromagnetic
energy

Mechanical work
done on or by the
plasma as it moves

General Properties of Ideal MHD

- **Global Conservation Laws**

  Obtained by integrating the local conservation laws over the volumes appropriate to each of the three sets of boundary conditions.

  - Perfectly conducting wall

$$\frac{dM}{dt} = 0$$

$$\frac{d\vec{P}}{dt} = -\int \left(p + \frac{B^2}{2\mu_0} \right) \vec{n}\, dS$$

$$\frac{dW}{dt} = 0$$

Total force exerted by the walls on the plasma. If the system remains in place, zero.

$$M = \int \rho\, d\vec{r}$$ total mass of plasma

$$\vec{P} = \int \rho \vec{v}\, d\vec{r}$$ mechanical momentum of plasma

$$W = \int \left(\frac{1}{2}\rho v^2 + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0} \right) d\vec{r}$$

total kinetic, potential energy of plasma and magnetic field: energy can be transferred each other

General Properties of Ideal MHD

- **Global Conservation Laws**
  - Insulating vacuum region

  - More complicated since the plasma is allowed to move.
  - The combined plasma-vacuum energy is conserved.

$$Z(t) = \int z(\vec{r},t)d\vec{r}$$ global quantity

$$\frac{dZ(t)}{dt} = \int_V \frac{\partial z}{\partial t} d\vec{r} + \int_S z\vec{n}\cdot\vec{u}dS$$

Total time derivative in a volume whose boundary is moving with u

plasma energy →

$$\frac{dW}{dt} = -\int \left(p + \frac{B^2}{2\mu_0} \right) \vec{n}\cdot\vec{v}dS$$

$$W = \int \left(\frac{1}{2}\rho v^2 + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0} \right) d\vec{r} \qquad w = \frac{1}{2}\rho v^2 + \frac{B^2}{2\mu_0} + \frac{p}{\gamma - 1}$$

General Properties of Ideal MHD

- **Global Conservation Laws**

$$\frac{dZ(t)}{dt} = \int_V \frac{\partial z}{\partial t} d\vec{r} + \int_S z\vec{n}\cdot\vec{u}\,dS$$

vacuum energy $\longrightarrow$

$$\frac{d\hat{W}}{dt} = \int \frac{1}{\mu_0}\left(\hat{B}\cdot\frac{\partial \hat{B}}{\partial t}\right)d\vec{r} - \int \frac{\hat{B}^2}{2\mu_0}\vec{n}\cdot\vec{v}\,dS$$

$$\hat{W} = \int \frac{\hat{B}^2}{2\mu_0} d\vec{r}$$

$$\frac{d\hat{W}}{dt} = \int \frac{\hat{B}^2}{2\mu_0}\vec{n}\cdot\vec{v}\,dS$$

$\longleftarrow$ Faraday's law, Divergence theorem

$$\hat{E} = -\vec{v}\times\hat{B}, \quad \vec{n}\cdot\hat{B}\big|_S = 0$$

General Properties of Ideal MHD

- **Global Conservation Laws**

$$\frac{dW}{dt} = -\int \left(p + \frac{B^2}{2\mu_0} \right) \vec{n} \cdot \vec{v} dS \qquad \frac{d\hat{W}}{dt} = \int \frac{\hat{B}^2}{2\mu_0} \vec{n} \cdot \vec{v} dS$$

$$\frac{d}{dt}(W + \hat{W}) = 0 \quad \longleftarrow \quad [[\, p + B^2 / 2\mu_0 \,]]_R = 0$$

- When an ideal MHD plasma is isolated from a conducting wall by a vacuum region, the combined energy of the plasma-vacuum system is conserved.

- The fact that only the total is conserved indicates that, in general, energy will flow from one region to the other as the plasma moves.

General Properties of Ideal MHD

- **Global Conservation Laws**
  - Plasma surrounded by external coils:
  - The energy of the system is no longer conserved.
  - With external sources present, energy can be supplied to or extracted from the system.

$$\frac{d}{dt}(W+\hat{W}) = -\int \frac{1}{\mu_0}(\vec{E}\times\vec{B})\cdot\vec{n}_W\, dS_W$$

The rate of increase of the total energy in the combined plasma-vacuum system is equal to the electromagnetic power flowing into the region.

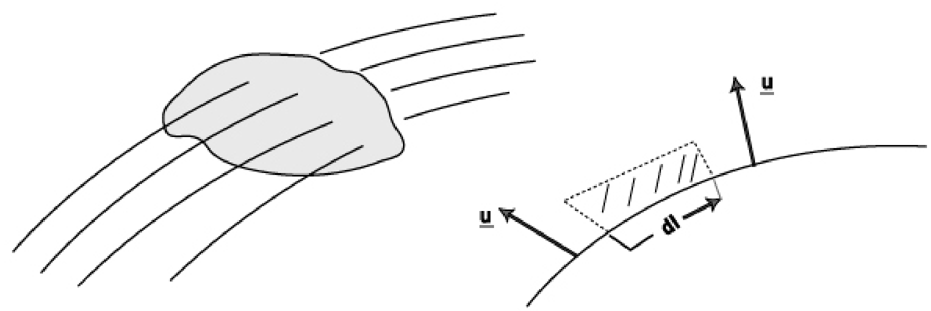
General Properties of Ideal MHD

- **Conservation of Flux: "Frozen" Field Line Picture**
  - A consequence of the perfect conductivity Ohm's law, is that the magnetic flux passing through any arbitrary open surface area moving with the plasma is constant.
  - Flux is conserved locally as well as globally.

$$\frac{d\psi}{dt} = \int \frac{\partial \hat{B}}{\partial t} \cdot \vec{n} dS - \oint \vec{u} \times \vec{b} \cdot d\vec{l}$$

time rate of change of the flux passing through any moving surface, S

$$\psi = \int \vec{B} \cdot \vec{n} dS$$



$$\frac{d\psi}{dt} = -\oint (\vec{E} + \vec{u} \times \vec{B}) \cdot d\vec{l}$$

⟵ Faraday's law, Stokes theorem

$$\frac{d\psi}{dt} = 0$$

⟵ u=v (plasma velocity), ideal Ohm's law

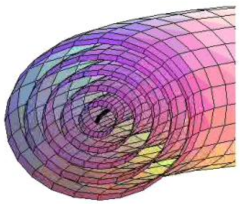
General Properties of Ideal MHD

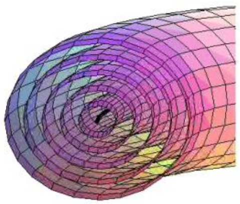
- **Conservation of Flux: "Frozen" Field Line Picture**

$$\frac{d\psi}{dt} = 0$$

- The total flux in an ideal MHD plasma is conserved.
- Magnetic lines move with the plasma; they are "frozen" into the fluid.
- Any allowable physical motion of the plasma requires that neighboring fluid elements remain adjacent to one another; fluid elements are not allowed to tear or break into separate pieces.
- Since the magnetic lines move with the plasma, the field line topology must thus be preserved during any physically allowable MHD motion.

Ideal MHD:
$\eta = 0$





Resistive MHD:
$\eta \neq 0$