Topics in Fusion and Plasma Studies 2
(459.667, 3 Credits)

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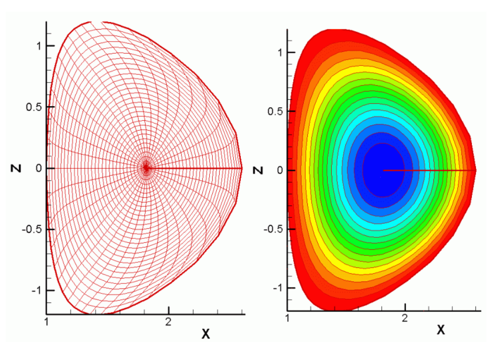
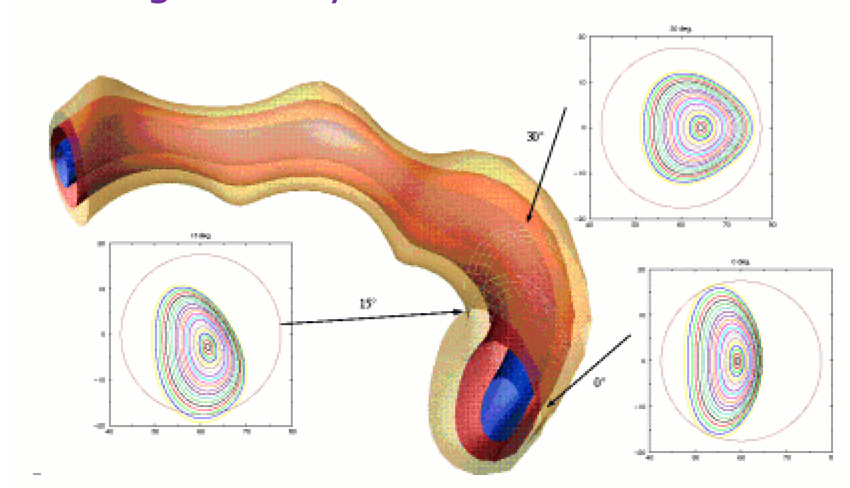
Equilibrium: General Considerations

- **Introduction**

- Focusing on the common features of the configurations possessing attractive MHD activities
- A description of the basic equilibrium equations
- The need for toroidicity
- The concept of magnetic flux surfaces
- The definition of the basic plasma parameters and figures of merit describing an MHD equilibrium and stability
- Fundamental conflict between the requirements for equilibrium and stability in toroidal geometry





Equilibrium: General Considerations

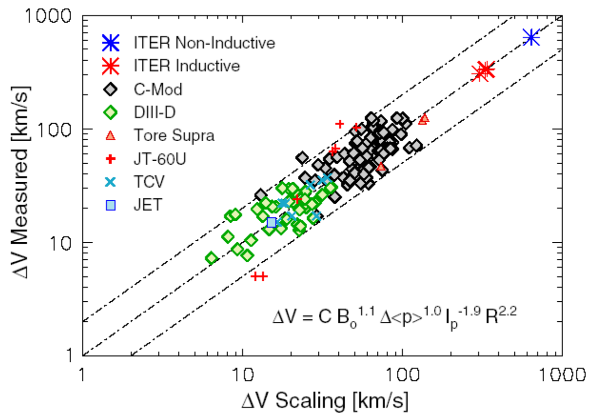
- **Basic Equations**
  - MHD equilibrium equations:
    time-independent with v=0 (static)

$$\vec{J}\times\vec{B}=\nabla p$$

$$\nabla\times\vec{B}=\mu_0\vec{J}$$

$$\nabla\cdot\vec{B}=0$$

cf) stationary equilibrium with nonzero flows:
v ~ V_{Ti} within the context of ideal MHD



J. E. Rice *et al*, *Nucl. Fusion* **47** 1618 (2007)

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\vec{v}=0$$

$$\rho\frac{d\vec{v}}{dt}=\vec{J}\times\vec{B}-\nabla p$$

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right)=0$$

$$\vec{E}+\vec{v}\times\vec{B}=0$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

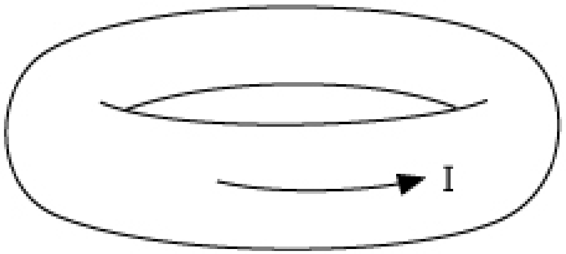
$$\nabla\times\vec{B}=\mu_0\vec{J}$$

$$\nabla\cdot\vec{B}=0$$

Equilibrium: General Considerations

- **The Virial Theorem (Shafranov, 1966)**

- Question: Can the following configuration be held in equilibrium only by its own currents?
  If so, we could build a fusion reactor with modest, or no coils.
- Any MHD equilibrium must be supported by externally supplied currents.
- It is not possible to create a configuration confined solely by the currents flowing within the plasma itself.



$$V(r) = ar^n \rightarrow \overline{T} = \frac{n+1}{2}\overline{V}$$

RJE. Clausius, "On a Mechanical Theorem Applicable to Heat"
Philosophical Magazine, Ser. 4 **40**: 122–127 (1870)

Equilibrium: General Considerations

- **The Virial Theorem (Shafranov, 1966)**

- Question: Can the following configuration be held in equilibrium only by its own currents?
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- Any MHD equilibrium must be supported by externally supplied currents
- It is not possible to create a configuration confined solely by the currents flowing within the plasma itself

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot \overleftrightarrow{T} = 0 \longrightarrow \nabla \cdot \overleftrightarrow{T} = 0$$

$$\overleftrightarrow{T} = \rho \vec{v}\vec{v} + \left(p + \frac{B^2}{2\mu_0} \right) \overleftrightarrow{I} - \frac{\vec{B}\vec{B}}{\mu_0}$$ stress tensor

General Properties of Ideal MHD

- **Local Conservation Relations**

$$\overleftrightarrow{T} = \rho\vec{v}\vec{v} + \left(p + \frac{B^2}{2\mu_0}\right)\overleftrightarrow{I} - \frac{\vec{B}\vec{B}}{\mu_0}$$

Reynolds stress

$$\overleftrightarrow{T}_R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho v^2 \end{pmatrix}$$

$$\overleftrightarrow{T}_B = \begin{pmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_{||} \end{pmatrix}$$

Important with large fluid flow, but usually not in studies of plasma stability where the flows are either small or zero

$$p_\perp = p + \frac{B^2}{2\mu_0}, \quad p_{||} = p - \frac{B^2}{2\mu_0}$$

isotropic plasma pressure

Total pressure parallel to the field: negative magnetic pressure correspond to a tension along the field lines

Equilibrium: General Considerations

- **The Virial Theorem**

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot \overleftrightarrow{T} = 0 \longrightarrow \nabla \cdot \overleftrightarrow{T} = 0$$

$$\overleftrightarrow{T} = \cancel{\rho \vec{v}\vec{v}} + \left(p + \frac{B^2}{2\mu_0} \right) \overleftrightarrow{I} - \frac{\vec{B}\vec{B}}{\mu_0}$$

stress tensor

$$= p_\perp (1 - \vec{b}\vec{b}) + p_{||} \vec{b}\vec{b}$$

$$\overleftrightarrow{T} = \begin{pmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_{||} \end{pmatrix} \qquad p_\perp = p + \frac{B^2}{2\mu_0}, \quad p_{||} = p - \frac{B^2}{2\mu_0}, \quad \vec{b} = \vec{B}/B$$

Equilibrium: General Considerations

- **The Virial Theorem**

Integrate the following identity over the plasma volume

$$\nabla \cdot (\vec{r} \cdot \overleftrightarrow{T}) = \vec{r} \cdot (\nabla \cdot \overleftrightarrow{T}) + \mathrm{Trace}(\overleftrightarrow{T})$$

$$\mathrm{Trace}(\overleftrightarrow{T}) = 2p_{\perp} + p_{||} = 3p + \frac{B^2}{2\mu_0}$$

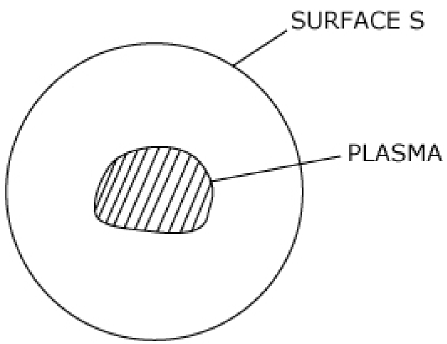
$$\int \left(3p + \frac{B^2}{2\mu_0} \right) d\vec{V} = \int \nabla \cdot \left(\vec{r} \cdot \overleftrightarrow{T} \right) d\vec{V} = \int \vec{n} \cdot (\vec{r} \cdot \overleftrightarrow{T}) dS$$

$$= \int \left[\left(p + \frac{B^2}{2\mu_0} \right) (\vec{n} \cdot \vec{r}) - \frac{B^2}{2\mu_0} (\vec{r} \cdot \vec{b})(\vec{n} \cdot \vec{b}) \right] dS \longleftarrow d\vec{S} = \vec{n} dS$$

Now assume the plasma can be confined by its own currents. Show that this leads to a contradiction.

Equilibrium: General Considerations

- **The Virial Theorem**



$$B(S) \le K / r^3$$

$$B^2 r dS \propto \frac{1}{r^3}$$

S $\to \infty$

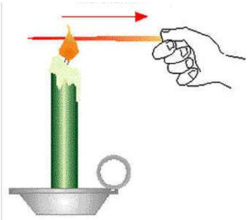
$$\int \left(3p + \frac{B^2}{2\mu_0}\right) d\vec{V} = \int \left[\left(p + \frac{B^2}{2\mu_0}\right)(\vec{n}\cdot\vec{r}) - \frac{B^2}{2\mu_0}(\vec{r}\cdot\vec{b})(\vec{n}\cdot\vec{b})\right] dS$$

finite

This is a contradiction.
There must be current carrying coils outside the plasma.
The RHS must be evaluated over the surface of the conductors.

Equilibrium: General Considerations

- **Toroidicity**

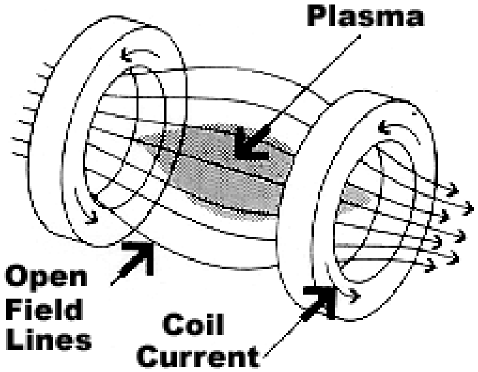


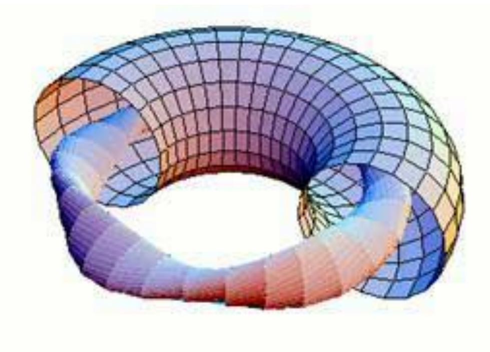
- Question: Why are most fusion configurations toroidal?

- Answer: Avoid parallel end losses
- Dominant loss mechanism is heat loss via thermal conduction.
- Heat loss is more severe along B than ⊥ to B
  because charged particles move freely along magnetic field lines.
  The magnetic field confines particles in the ⊥ direction.

$$\frac{\kappa_{\|e}}{\kappa_{\perp i}} = 1.12\left(\frac{m_i}{m_e}\right)^{1/2}(\omega_{ci}\tau_{ii})^2 \approx 6.2\times10^{10}\left(\frac{B^2T_i^3}{n^2}\right) \longleftarrow$$

$$Z=1,\quad m_i = 2m_{proton}$$
$$T_e = T_i$$
$$\ln\Lambda = 15$$

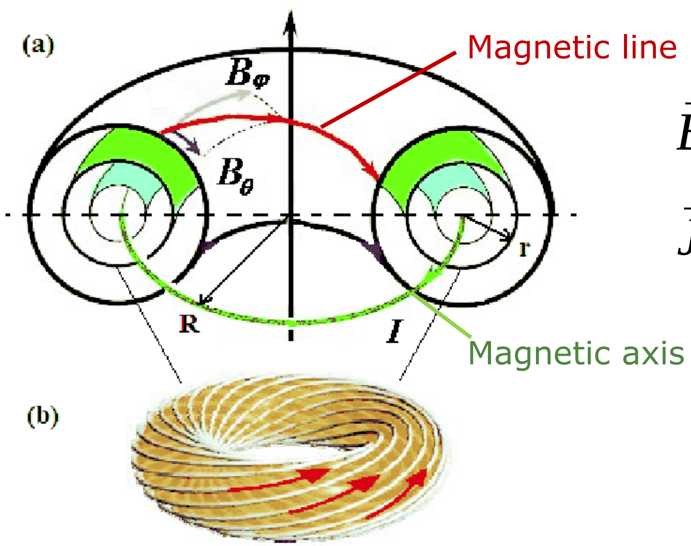




$$\kappa_{\|e}/\kappa_{\perp i} = 3.1\times10^{12}$$
$$T_i = 2\text{keV},\quad B = 5\text{T},\quad n = 2\times10^{20}\,\text{m}^{-3}$$

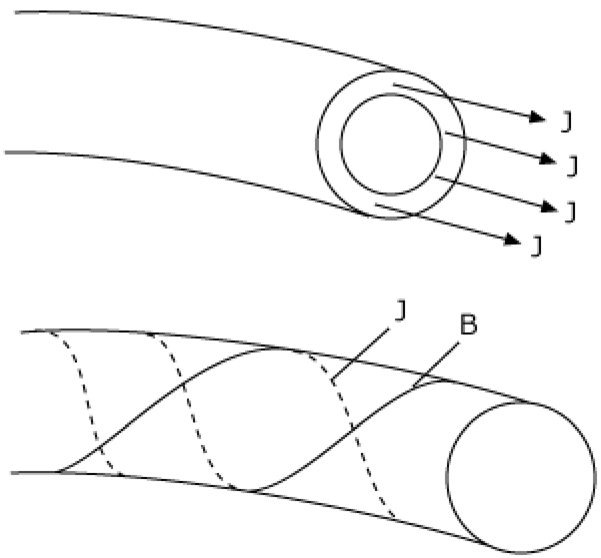
Equilibrium: General Considerations

- **Magnetic Flux Surfaces**
  - In fusion configurations with confined plasmas the magnetic lines lie on a set of nested toroidal surfaces called flux surfaces.
  - Pressure is constant along a magnetic field line.
  - Magnetic lines lie in surfaces of constant pressure.
  - Flux surfaces are surfaces of constant pressure.
  - The current lines lie on surfaces of constant pressure.
  - The current flows between flux surfaces and not across them.
  - The angle between J and B is arbitrary.



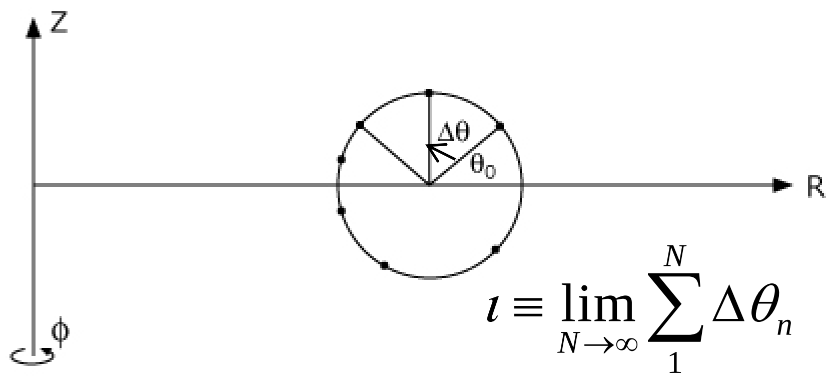
$$\vec{B}\cdot\nabla p = 0$$

$$\vec{J}\cdot\nabla p = 0$$



Equilibrium: General Considerations

- **Magnetic Flux Surfaces**
  - Three classes of magnetic field line trajectories
  - Rational
  - Ergodic
  - Stochastic



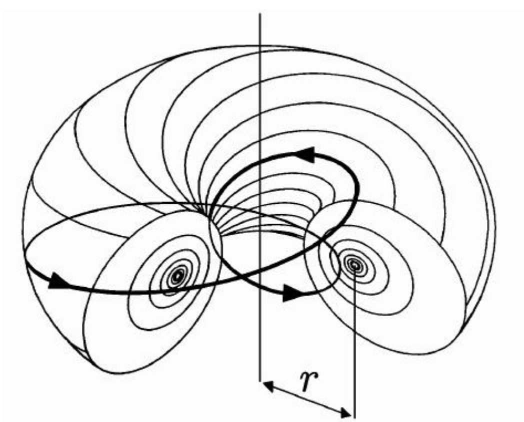
$$\iota \equiv \lim_{N\to\infty} \sum_{1}^{N} \Delta\theta_n$$

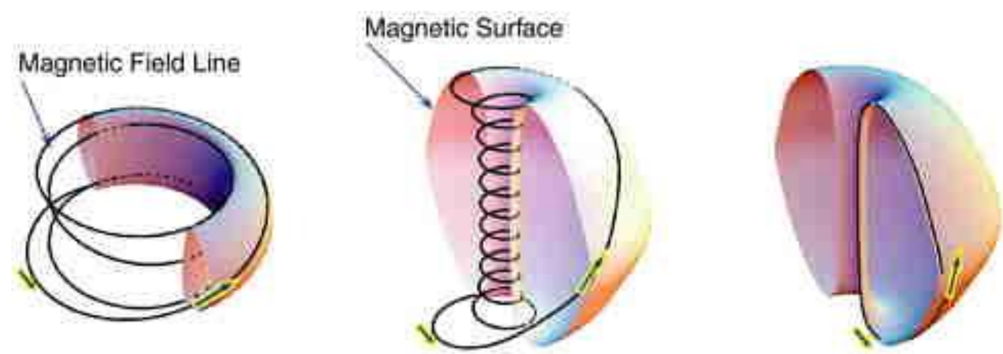
Rotational transform:
the average change in poloidal angle per single transit
in the toroidal direction

Equilibrium: General Considerations

• Magnetic Flux Surfaces

- Rational surface
- $\iota/2\pi$ is rational fraction.
- Magnetic lines close on themselves after a finite number of transits, $\iota/2\pi = n/m$.
- Ergodic surface
- $\iota/2\pi$ is not rational.
- Magnetic lines eventually cover an entire surface.
- For ergodic surfaces, the flux surfaces can be traced out by plotting contours of constant *p* or by following magnetic field line trajectories.
- For rational surfaces, we must plot contours of constant *p.*

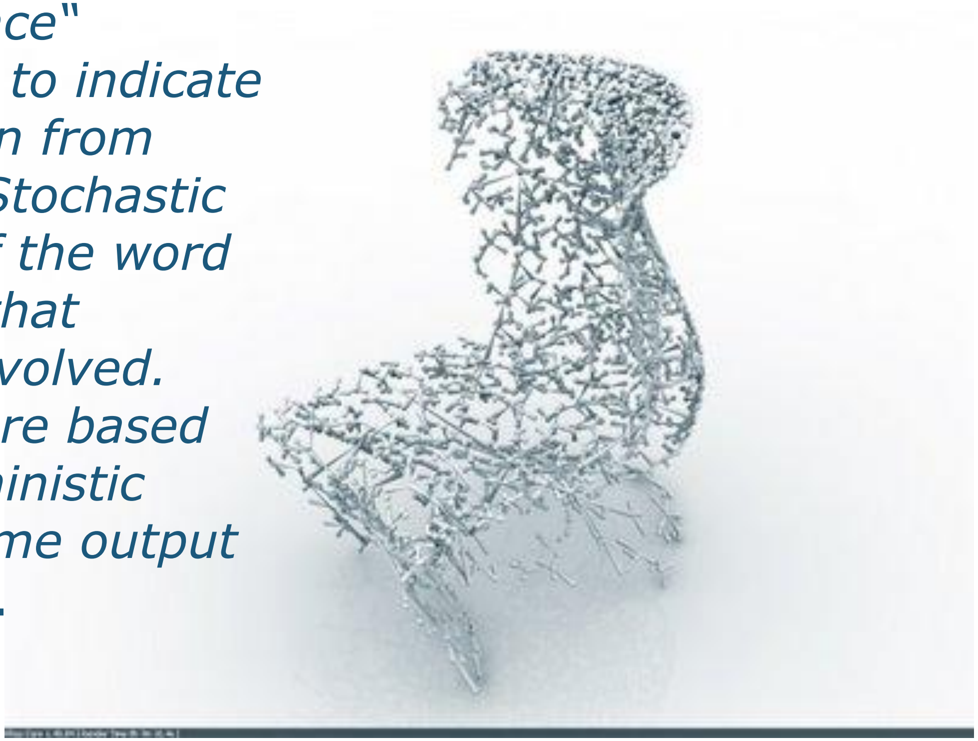




Tokamak Plasma (safety factor q = 4)   Spherical Torus Plasma (safety factor q = 12)   Spheromak Plasma (safety factor q = 0.03)

Equilibrium: General Considerations

STOCHASTIC *is synonymous with "random." The word is of Greek origin and means "pertaining to chance" (Parzen 1962, p. 7). It is used to indicate that a particular subject is seen from point of view of randomness. Stochastic is often used as counterpart of the word "deterministic," which means that random phenomena are not involved. Therefore, stochastic models are based on random trials, while deterministic models always produce the same output for a given starting condition. .*
(ie mathworld.wolfram.com)

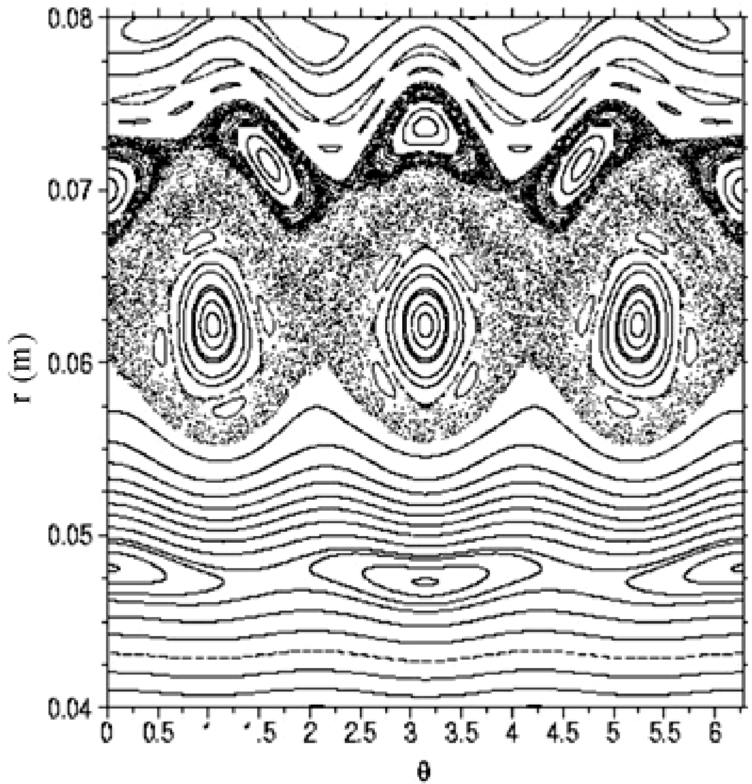


http://www.theverymany.net/labels/rhinoscript.html

Equilibrium: General Considerations

- **Magnetic Flux Surfaces**

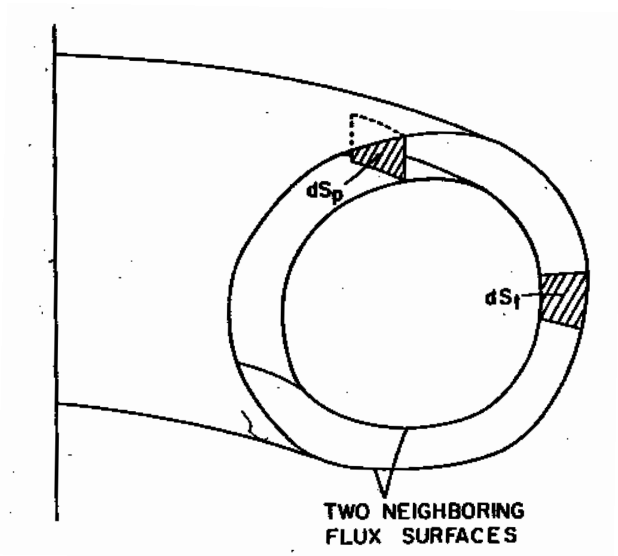
- Stochastic volume: lines fill up a volume → ultra-poor confinement



Equilibrium: General Considerations

- **Surface Quantities: Basic Plasma Parameters and Figures of Merit**

- By carrying out appropriate integrals over the flux surfaces, one can define a number of quantities that are of important to the application of MHD theory.
- These "surface quantities" describe the global properties of the equilibria and as such are essential to distinguish different configurations and to interpret experimental data.
- One dimensional functions depending only upon the flux surface label.



$$\psi_p \equiv \int \vec{B} \cdot d\vec{S}_p \qquad \psi_p = \psi_p(p)$$ poloidal flux

$$\psi_t = \int \vec{B} \cdot d\vec{S}_t$$ toroidal flux

$$I_t = \int \vec{J} \cdot d\vec{S}_t$$ toroidal current

$$I_p = \int \vec{J} \cdot d\vec{S}_p$$ poloidal current

$$B_p = B_p(p, \theta, \phi) \neq B_p(p)$$

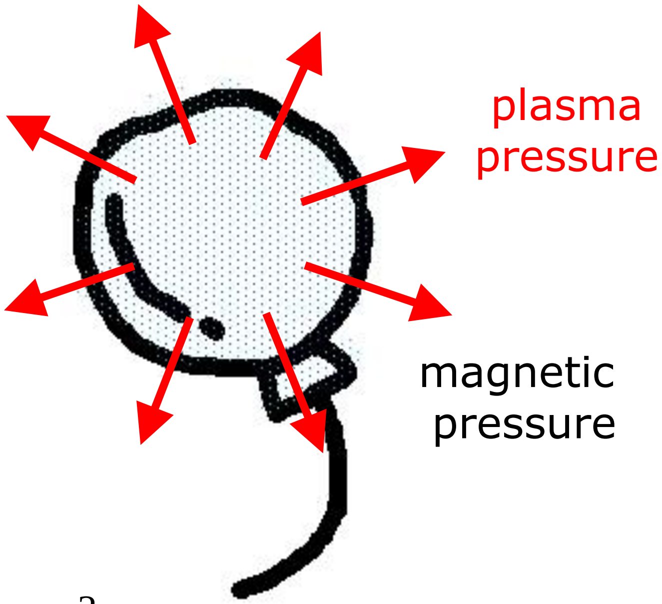
Equilibrium: General Considerations

• Surface Quantities: Basic Plasma Parameters and Figures of Merit

- Requirements of surface quantities
- Must relatively easy to evaluate but still accurately reflect the physics issue under consideration
- Must be expressible in terms of simple physical parameters, easily related to experiment
- Must be defined in a sufficiently general manner so as to apply to all configurations of interest:
  using the volume V contained within a flux surface as the flux surface label.

Equilibrium: General Considerations

- **Surface Quantities: Basic Plasma Parameters and Figures of Merit**
  - Plasma β

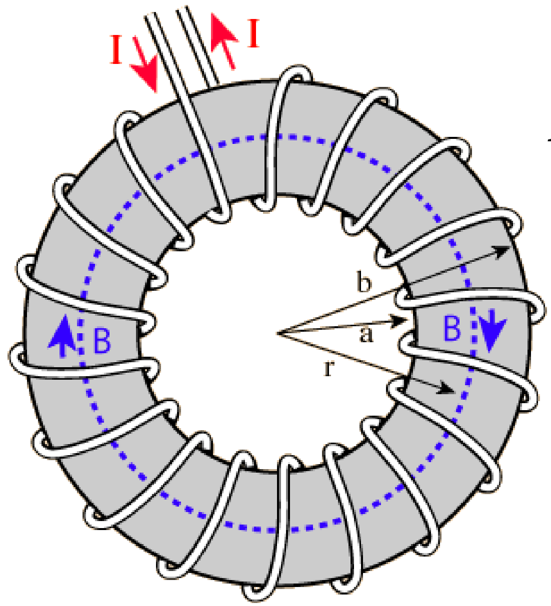


$$\beta = 2\mu_0 p / B^2$$

- The ratio of the plasma pressure to the magnetic field pressure
- A measure of the degree to which the magnetic field is holding a non-uniform plasma in equilibrium.

Equilibrium: General Considerations

- **Surface Quantities: Basic Plasma Parameters and Figures of Merit**
  - Plasma β



$$B = \frac{\mu N I}{2\pi r}$$

$$\beta = 2\mu_0 p / B^2$$

- The power output for a given magnetic field and plasma assembly is proportional to the square of beta.
- In a reactor it should exceed 0.1: economic constraint

Equilibrium: General Considerations

• Surface Quantities: Basic Plasma Parameters and Figures of Merit

- Plasma β

$$\overline{\beta} \equiv \frac{2\mu_0 \langle p \rangle}{\langle B_t^2 + B_p^2 \rangle} \qquad \langle p \rangle = \frac{1}{V_0} \int_0^{V_0} p(V)\, dV$$

Replace with simpler but similar expressions more easily related to experiments

$$\langle B_t^2 \rangle \to B_0^2 \qquad \langle B_p^2 \rangle \to \overline{B}_p^2 \qquad \longleftarrow \qquad \overline{B}_p^2 = \frac{\mu_0 I_0}{2\pi a \kappa} \qquad \kappa = A / \pi a^2$$

plasma elongation

$$\overline{\beta} \equiv \frac{2\mu_0 \langle p \rangle}{B_0^2 + \overline{B}_p^2} \qquad \beta_t \equiv \frac{2\mu_0 \langle p \rangle}{B_0^2} \qquad \beta_p = \frac{2\mu_0 \langle p \rangle}{\overline{B}_p^2} = \frac{8\pi^2 a^2 \kappa^2 \langle p \rangle}{\mu_0 I_0^2}$$

- β is related with fusion reactor economics and technology.
- Maximum allowable value is set by MHD equilibrium requirements and instabilities driven by the pressure gradient.

$$\frac{1}{\overline{\beta}} = \frac{1}{\beta_t} + \frac{1}{\beta_p}$$

Equilibrium: General Considerations

• Surface Quantities: Basic Plasma Parameters and Figures of Merit

- Kink safety factor: measuring stability against long-wavelength modes driven by the toroidal plasma current

$$q_* \equiv \frac{aB_0}{R_0\overline{B}_p} = \frac{2\pi a^2 \kappa B_0}{\mu_0 R_0 I_0}$$

Plasma current limit set by kink instabilities

- Rotational transform

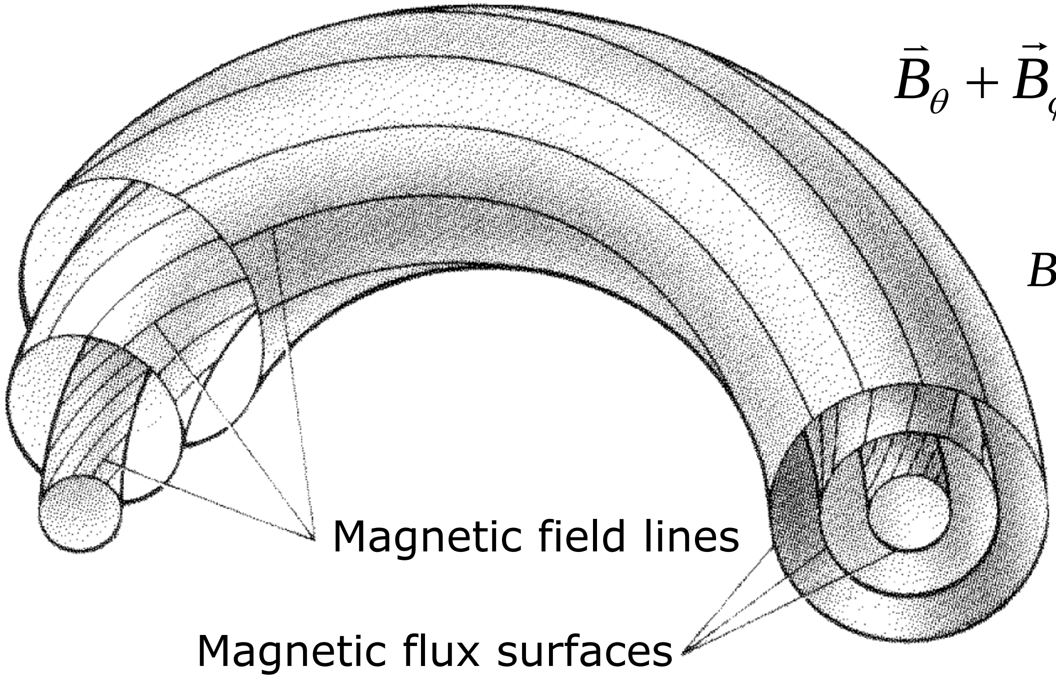
$$\psi_t = \psi_t(V)$$

$$\psi_p = \psi_p(V)$$

$$\iota(V) \equiv 2\pi\left(\frac{d\psi_p / dV}{d\psi_t / dV}\right)$$

Equilibrium: General Considerations

- MHD Safety factor q = number of toroidal orbits per poloidal orbit



$$\vec{B}_\theta + \vec{B}_\varphi$$

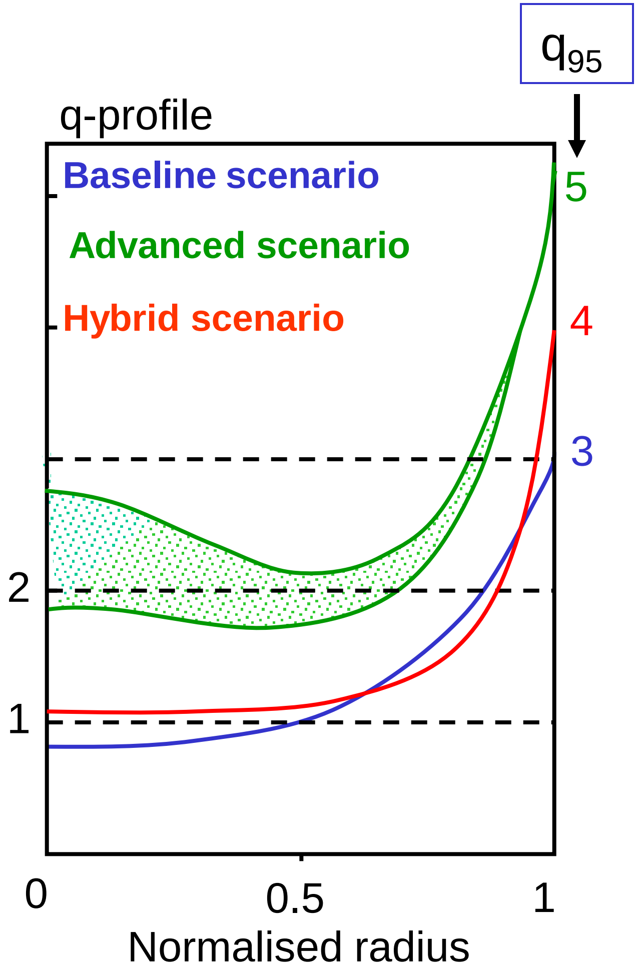
$$B_\theta = \frac{\mu_0}{r}\int_0^r j_\varphi(r')r'dr'$$

$$q = \frac{\text{number of toroidal windings}}{\text{number of poloidal windings}}$$

$$q(V) \equiv \frac{2\pi}{\iota} = \frac{d\psi_t / dV}{d\psi_p / dV}$$

Equilibrium: General Considerations

- MHD Safety factor q = number of toroidal orbits per poloidal orbit

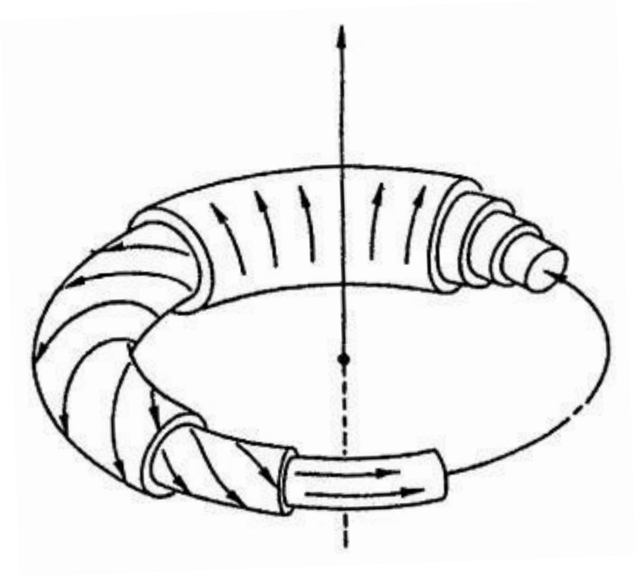


Equilibrium: General Considerations

• Surface Quantities: Basic Plasma Parameters and Figures of Merit

- Magnetic shear
- measuring the change in pitch angle of a magnetic field line from one flux surface to the next
- Playing an important role in stabilizing MHD instabilities, particularly those driven by the pressure gradient

$$s(V) \equiv 2\frac{V}{q}\frac{dq}{dV}$$



Equilibrium: General Considerations

• Surface Quantities: Basic Plasma Parameters and Figures of Merit

- Magnetic well
  - Measuring plasma stability against short perpendicular wavelength modes driven by the plasma pressure gradient
  - Closely related to the average curvature of a magnetic field line
  - A configuration has favourable stability properties if *W*(*V*) is large and positive. Such systems tend to confine plasma in regions of low *B*, thus, instabilities driven by the pressure gradient are suppressed since the plasma has difficulty expanding into a high-B region.

$$\widehat{W} = 2\frac{V}{\langle B^2 \rangle}\frac{d}{dV}\left\langle \frac{B^2}{2} \right\rangle \qquad \langle Q \rangle \equiv \int_0^L \frac{Q\,dl}{B} \Big/ \int_0^L \frac{dl}{B}$$

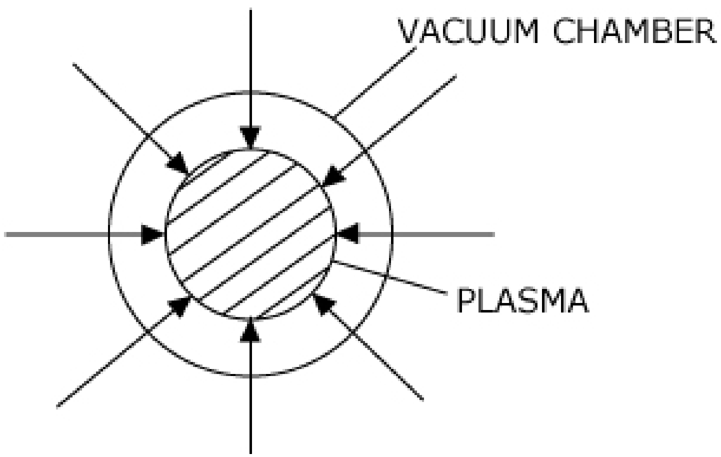
$$W = 2\frac{V}{\langle B^2 \rangle}\frac{d}{dV}\left\langle \mu_0 p + \frac{B^2}{2} \right\rangle$$ for finite J

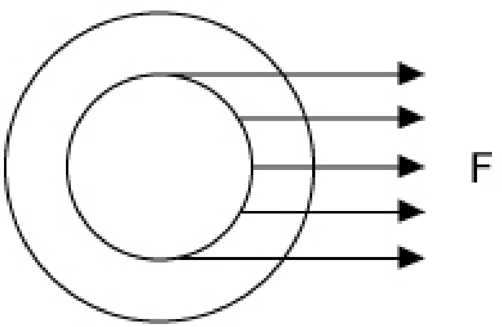
Equilibrium: General Considerations

- **The Basic Problem of Toroidal Equilibrium**
  - Radial pressure balance

amplitude **V** difficulty **Λ**

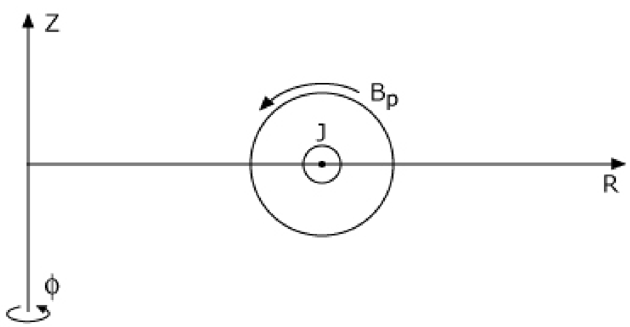
  - Toroidal force balance

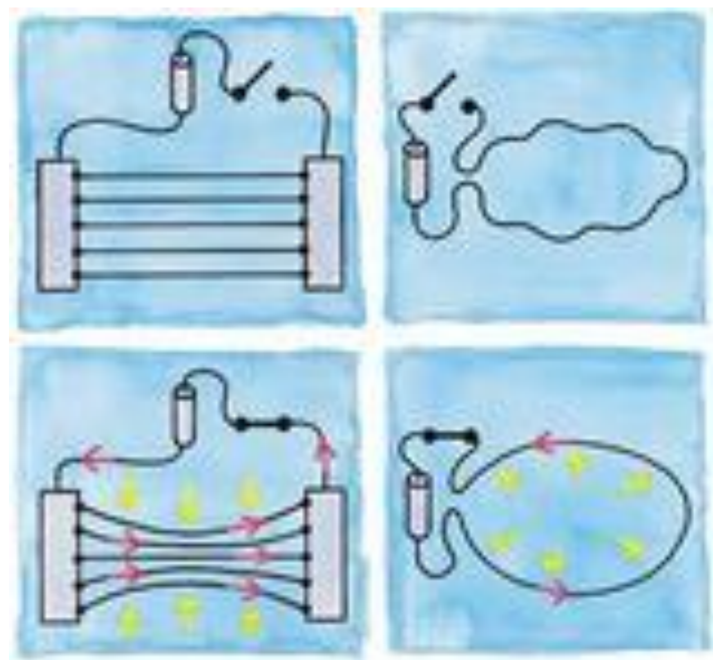




Equilibrium: General Considerations

- **The Basic Problem of Toroidal Equilibrium**
  - Consider a configuration with a purely poloidal field
    - Hoop force





Equilibrium: General Considerations

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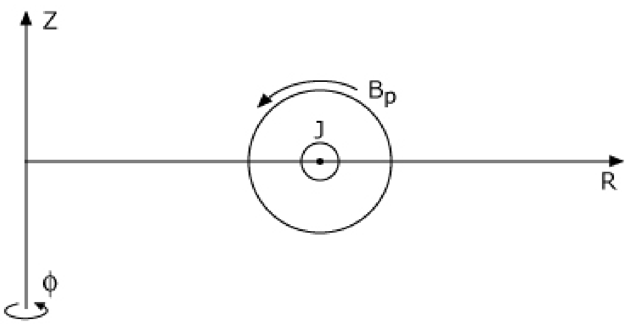
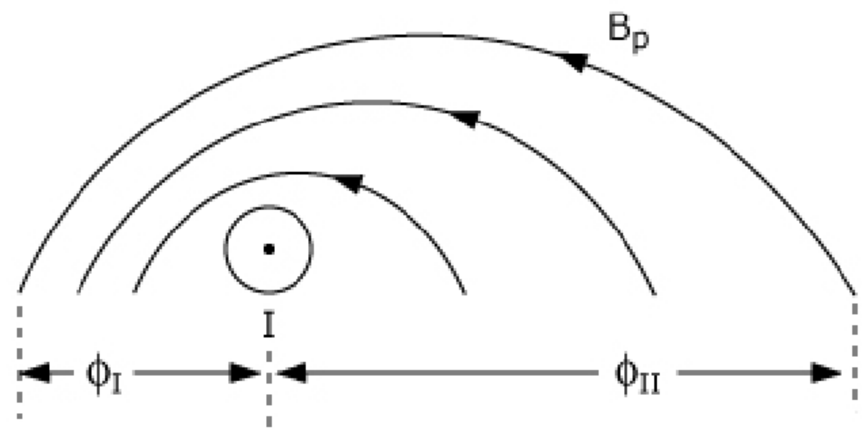
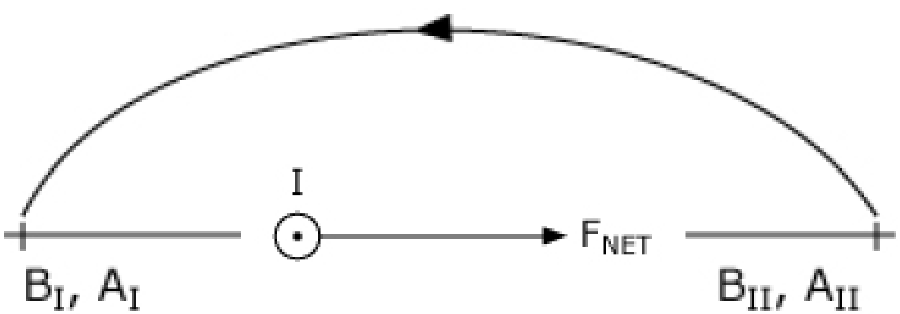




$$\phi_I = \phi_{II}$$

$$B_I > B_{II}, \quad A_I < A_{II}$$

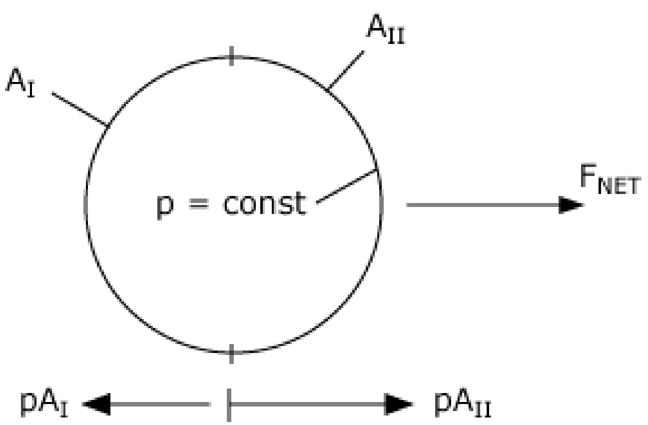
$$B_I^2 A_I > B_{II}^2 A_{II}$$



$$F_{NET} \sim e_R (B_I^2 A_I - B_{II}^2 A_{II}) / 2\mu_0$$

Equilibrium: General Considerations

- **The Basic Problem of Toroidal Equilibrium**
  - Consider a configuration with a purely poloidal field
  - Tire tube force



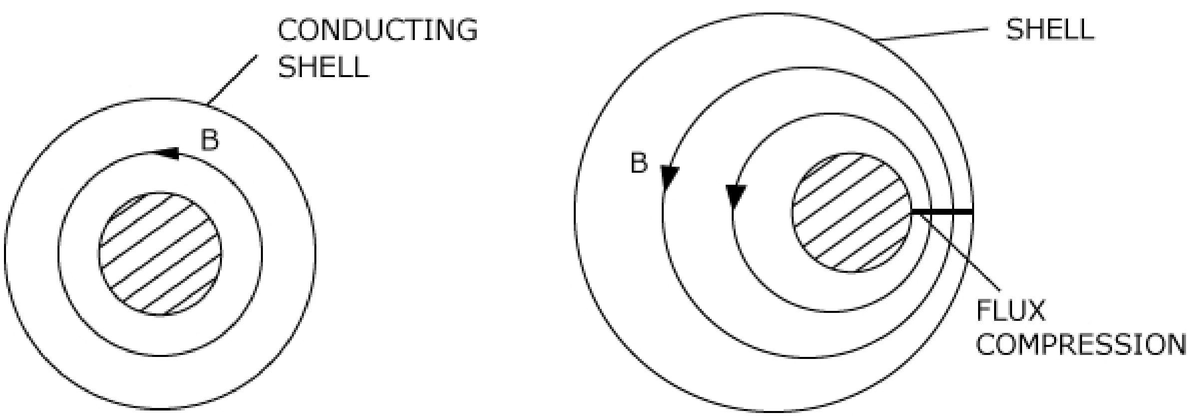
$$F \sim -e_R(pS_1 - pS_2)$$

How to compensate the outward forces?

Equilibrium: General Considerations

- **The Basic Problem of Toroidal Equilibrium**

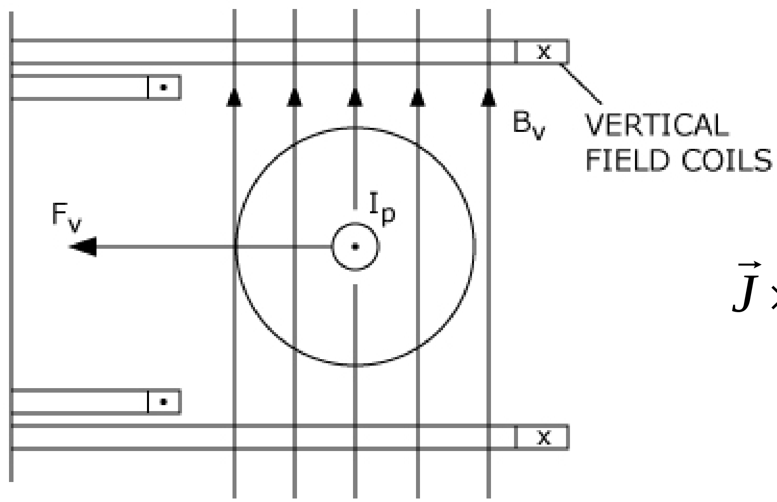
  1. Perfectly conducting shell around the plasma



Equilibrium: General Considerations

- **The Basic Problem of Toroidal Equilibrium**

2. Vertical field coils



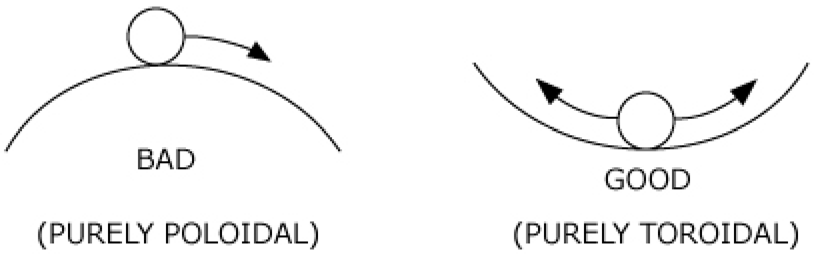
$$\vec{J} \times \vec{B}_v$$

$$F_v = BIL = 2\pi R_0 I_p B_v$$

Equilibrium: General Considerations

- **The Basic Problem of Toroidal Equilibrium**
  - Consider a configuration with a purely poloidal field

- Conclusion
- It is easy to provide toroidal equilibrium using purely poloidal magnetic fields.
- But, we shall see that such systems are very unstable to macroscopic MHD modes.
- However, we shall also see that systems with large toroidal fields have much better stability properties.



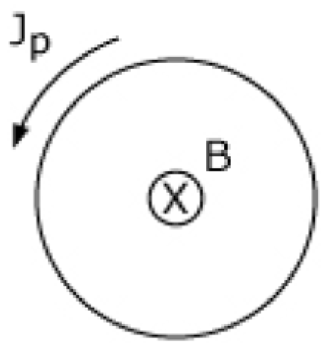
Equilibrium: General Considerations

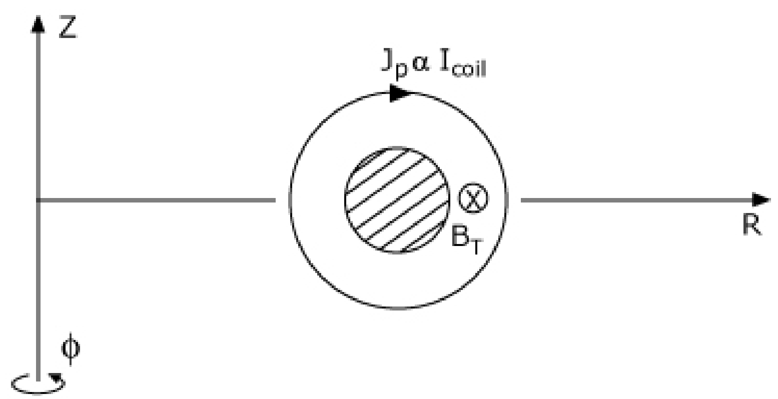
- **The Basic Problem of Toroidal Equilibrium**
  - Consider a configuration with a purely toroidal field

- Assumption
- All the current flows solely in a thin layer at the plasma surface.
- The pressure is a constant in the plasma and zero outside.
- The plasma is completely diamagnetic so that the magnetic field in the plasma is zero.

$$\vec{B} = B_T \vec{e}_T$$

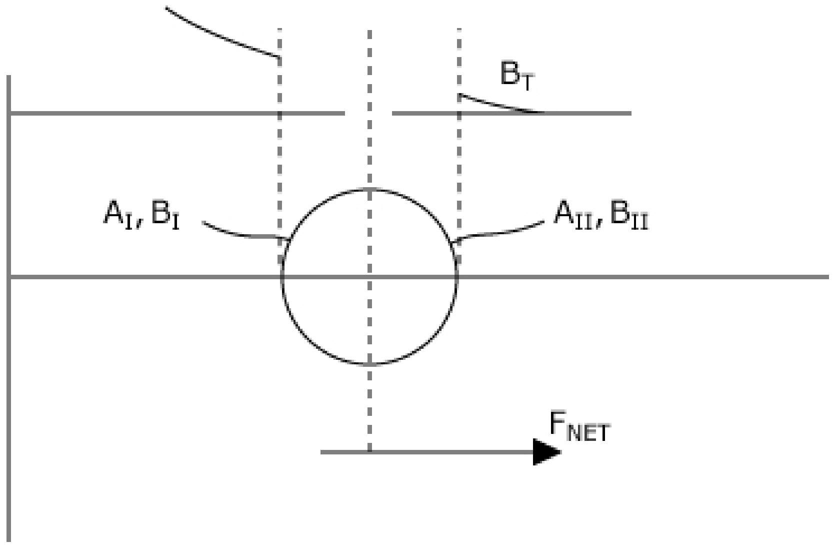
$$B_T = B_0 (R_0 / R), \quad B_0 = \mu_0 I_{coil} / 2\pi R_0$$





Equilibrium: General Considerations

- **The Basic Problem of Toroidal Equilibrium**
  - Consider a configuration with a purely toroidal field
  - ~~Hoop force~~
  - Tire tube force
  - $1/R$ force



$$\phi_I = \phi_{II}$$
$$B_I > B_{II}, \quad A_I < A_{II}$$
$$B_I^2 A_I > B_{II}^2 A_{II}$$

Equilibrium: General Considerations

- **The Basic Problem of Toroidal Equilibrium**
  - Consider a configuration with a purely toroidal field

Combined outward force

$$F_R = -\int [[\, p + B^2 / 2\mu_0 \,]] \vec{e}_r \cdot \vec{e}_R dS$$

$$[[\, p + B^2 / 2\mu_0 \,]] = \frac{B_0^2}{2\mu_0}\left(\frac{R_0}{R}\right)^2 - p$$

$$e_r \cdot e_R = \cos\theta$$

$$dS = 2\pi a R d\theta$$

$$R = R_0 + a\cos\theta, \quad R_0 / a >> 1$$

$$F_R = 2\pi^2 a^2 \left(p + \frac{B^2}{2\mu_0} \right)$$

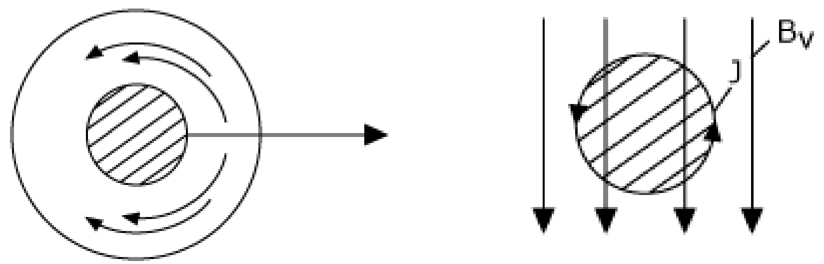
tire tube force

1/*R* force

Equilibrium: General Considerations

- **The Basic Problem of Toroidal Equilibrium**
  - Consider a configuration with a purely toroidal field

- Can a conducting shell balance outward force in purely toroidal case?
- No! Magnet flux is not trapped. Lines are free to slide around plasma.
- Can a vertical field balance outward force in purely toroidal case?
- No! There is no net inward force because of the basic field directions.



A purely toroidal field cannot hold a plasma in toroidal equilibrium. The toroidal force cannot be balanced.

Equilibrium: General Considerations

- **The Basic Problem of Toroidal Equilibrium**
  - Consider a configuration with a purely toroidal field

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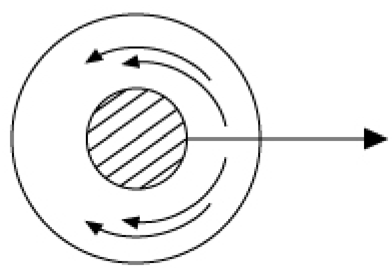
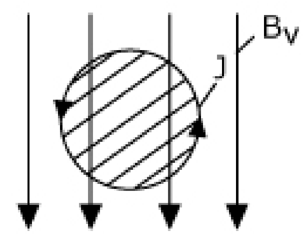




- Time scale for loss of equilibrium is in the μ sec range.

$$\tau = \left(\frac{2b}{F_R / M}\right)^{1/2} = \left(\frac{2bR_0\rho_0}{p + B^2/2\mu_0}\right)^{1/2}$$

$$M = \rho_0 V_0$$

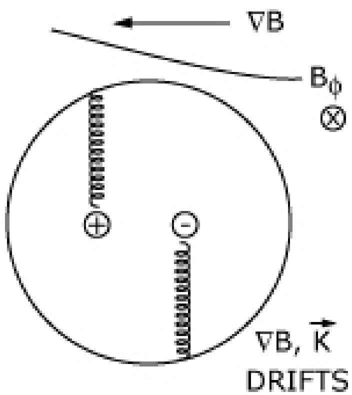
$$b = 1m, \quad R_0 = 1\text{km}, \quad B_0 = 5\text{T},$$

$$n_0 = 10^{21}\text{m}^{-3}, \quad p = B_0^2/2\mu_0, \quad \text{D}$$

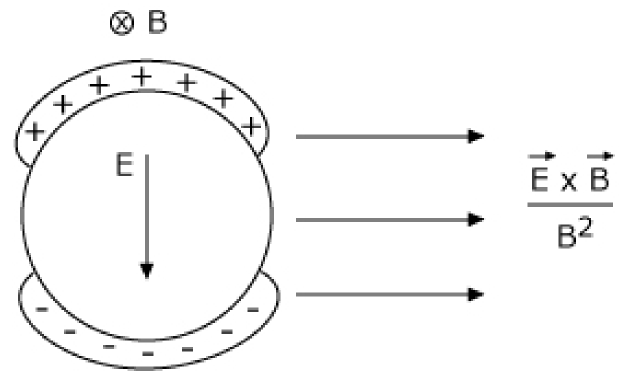
$$\rightarrow \tau = 18\mu\text{s}$$

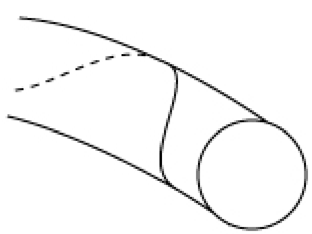
Equilibrium: General Considerations

- **The Basic Problem of Toroidal Equilibrium**
  - Effect of the poloidal field – single particle picture

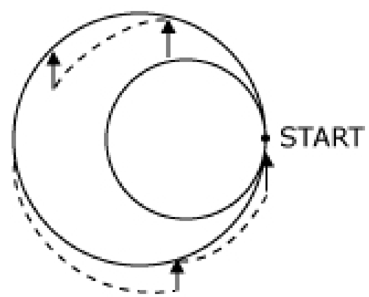


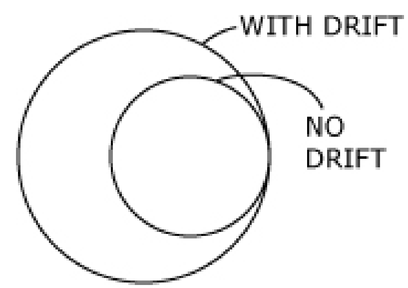
$$\vec{V}_{g\perp} = \frac{1}{\omega_{c0} R_0}\left(\frac{u_\perp^2}{2} + u_{||}^2\right)\vec{e}_Z$$





poloidal field applied





Equilibrium: General Considerations

- **The Basic Problem of Toroidal Equilibrium**

- Conclusion

- Poloidal fields: Good equilibrium poor stability
- Toroidal fields: Poor equilibrium good stability
- Our goal is to optimize the advantages and minimize the disadvantages.

  This is the challenge of creating desirable fusion geometries.