Topics in Fusion and Plasma Studies 2
(459.667, 3 Credits)

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Equilibrium: 2-D Configurations

- **Introduction**

- Consider the axisymmetric torus, the simplest, multi-dimensional configuration
- We shall derive the Grand–Shafranov equation for axisymmetric equilibria.
- This provides a complete description of toroidal equilibrium:

  radial pressure balance, toroidal force balance,

  β limits, q-profiles, magnetic well, etc.
- It applies to the following configurations (circular and noncircular);

  RFP, ohmic tokamak, highβ tokamak, flux conserving tokamak,

  spherical tokamak, spheromak, toroidal multipole

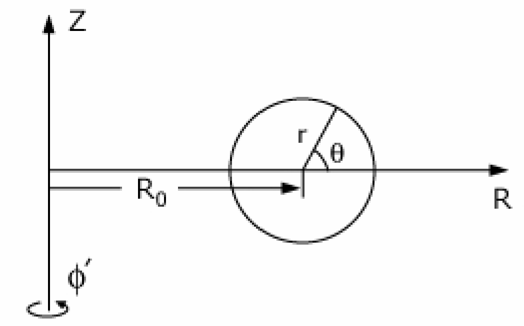
Equilibrium: 2-D Configurations

- **The Grad-Shafranov Equation**
  - obtained from the reduction of the ideal MHD equations
  - exact (no expansion)
  - Toroidal axisymmetric $\partial/\partial\phi=0$
  - 2 dimensional
  - nonlinear
  - partial differential equation
  - elliptic characteristics
  - Grad and Rubin (1958), Shafranov (1960)

$$\vec{J}\times\vec{B}=\nabla p$$

$$\nabla\times\vec{B}=\mu_0\vec{J}$$

$$\nabla\cdot\vec{B}=0$$



- Plan of action

1. Derive the exact Grad–Shafranov equation
2. Solve by means of an asymptotic expansion in a/R
3. Zero order: $(a/R)^0 \rightarrow$ 1-D screw pinch radial pressure balance
4. First order: $(a/R)^1 \rightarrow$ toroidal force balance

Equilibrium: 2-D Configurations

- **The Grad-Shafranov Equation**
  - The $\nabla \cdot B$ Equation

$$\nabla \cdot \vec{B} = 0 \qquad \frac{1}{R}\frac{\partial (RB_R)}{\partial R} + \frac{1}{R}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = 0$$

$$B_R = -\frac{1}{R}\frac{\partial \psi}{\partial Z}, \quad B_z = \frac{1}{R}\frac{\partial \psi}{\partial R} \qquad \psi = RA_\phi$$

Stream function for the poloidal magnetic field

Toroidal component of vector potential

$$\vec{B} = B_\phi \vec{e}_\phi + \vec{B}_p$$

$$\vec{B}_p = \frac{1}{R}\nabla \psi \times \vec{e}_\phi$$

$$\vec{B}_p = \nabla \times \vec{A} = \nabla \times (A_\phi \vec{e}_\phi) = \frac{1}{R}\frac{\partial}{\partial R} RA_\phi \vec{e}_Z - \frac{\partial A_\phi}{\partial Z}\vec{e}_R$$

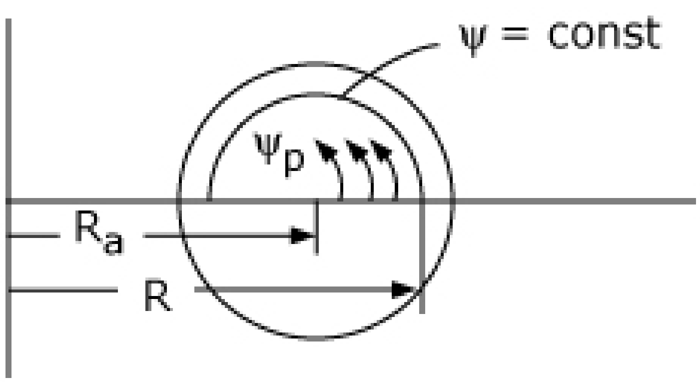
Equilibrium: 2-D Configurations

- **The Grad-Shafranov Equation**

  - The stream function ψ is closely related to the poloidal flux in the plasma.

$$
\begin{aligned}
\psi_p &= \int B_p \cdot dA \\
&= \int_0^{2\pi} d\phi \int_{R_a}^{R} dRRB_Z(R, Z=0) \\
&= \int_{R_a}^{R} 2\pi R \frac{1}{R}\frac{\partial \psi}{\partial R} dR \\
&= 2\pi\{\psi(R,0) - \psi(R_a,0)\} \\
&= 2\pi\psi
\end{aligned}
$$

Poloidal flux on axis is zero



Equilibrium: 2-D Configurations

- **The Grad-Shafranov Equation**
  - Ampere's law

$$\mu_0 \vec{J} = \nabla \times \vec{B}$$

$$\mu_0 \vec{J} = \nabla \times \left(RB_\phi \frac{\vec{e}_\phi}{R} + \frac{1}{R} \nabla \psi \times \vec{e}_\phi \right)$$

$$= \nabla (RB_\phi) \times \frac{\vec{e}_\phi}{R} + RB_\phi \nabla \times \frac{\vec{e}_\phi}{R} + \frac{\vec{e}_\phi}{R} \cdot \nabla(\nabla \psi) - \nabla \psi \cdot \nabla \frac{\vec{e}_\phi}{R} - \frac{\vec{e}_\phi}{R} \nabla \cdot \nabla \psi + \nabla \psi \nabla \cdot \frac{\vec{e}_\phi}{R}$$

$$= \nabla (RB_\phi) \times \frac{\vec{e}_\phi}{R} - \frac{\vec{e}_\phi}{R} \nabla^2 \psi + \frac{1}{R^2} \frac{\partial}{\partial \phi} \left(\frac{\partial \psi}{\partial R} \vec{e}_\phi + \frac{\partial \psi}{\partial Z} \vec{e}_Z \right) - \left(\frac{\partial \psi}{\partial R} \frac{\partial}{\partial R} + \frac{\partial \psi}{\partial Z} \frac{\partial}{\partial Z} \right) \frac{\vec{e}_\phi}{R}$$

$$= \nabla (RB_\phi) \times \frac{\vec{e}_\phi}{R} - \frac{\vec{e}_\phi}{R} \left(\frac{\partial^2 \psi}{\partial R^2} + \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial Z^2} - \frac{2}{R} \frac{\partial \psi}{\partial R} \right)$$

$$= \nabla (RB_\phi) \times \frac{\vec{e}_\phi}{R} - \frac{\vec{e}_\phi}{R} \left(R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial Z^2} \right) \longleftarrow$$

$$\frac{1}{R^2} \frac{\partial}{\partial \phi} \left(\frac{\partial \psi}{\partial R} \vec{e}_\phi + \frac{\partial \psi}{\partial Z} \vec{e}_Z \right) = \frac{1}{R^2} \frac{\partial \psi}{\partial R} \vec{e}_\phi$$

$$-\left(\frac{\partial \psi}{\partial R} \frac{\partial}{\partial R} + \frac{\partial \psi}{\partial Z} \frac{\partial}{\partial Z} \right) \frac{\vec{e}_\phi}{R} = \frac{1}{R^2} \frac{\partial \psi}{\partial R} \vec{e}_\phi$$

Equilibrium: 2-D Configurations

- **The Grad-Shafranov Equation**
  - Ampere's law

$$\mu_0 \vec{J} = \mu_0 J_\phi \vec{e}_\phi + \mu_0 \vec{J}_p$$

$$\mu_0 \vec{J}_p = \frac{1}{R} \nabla (R B_\phi) \times \vec{e}_\phi$$

$$\mu_0 \vec{J}_\phi = -\frac{1}{R} \Delta^* \psi$$

$$\Delta^* \psi \equiv R^2 \nabla \cdot \left(\frac{\nabla \psi}{R^2} \right) = R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial Z^2}$$

elliptic operator

Equilibrium: 2-D Configurations

- **The Grad-Shafranov Equation**
  - Momentum equation $\vec{J}\times\vec{B}=\nabla p$

$$\vec{B}\cdot\nabla p=0 \qquad \vec{e}_\phi\cdot\nabla\psi\times\nabla p=0$$

$$\frac{B_\phi}{R}\frac{\partial p}{\partial\phi}+\frac{\nabla\psi\times\vec{e}_\phi}{R}\cdot\nabla p=0$$

- $p=p(\psi)$, p is an arbitrary free function of ψ.
- There is no way to determine $p(\psi)$ from ideal MHD.
  We need transport theory or some other simple physical model.

$$\vec{J}\cdot\nabla p=0 \qquad \vec{e}_\phi\cdot\nabla\psi\times\nabla(RB_\phi)=0$$

$$\frac{J_\phi}{R}\frac{\partial p}{\partial\phi}+\vec{J}_p\cdot\nabla p=0$$

$$\frac{1}{R}\nabla RB_\phi\times\vec{e}_\phi\cdot\nabla\psi\frac{dp}{d\psi}=0 \qquad \frac{1}{R}\frac{dp}{d\psi}\left(\vec{e}_\phi\cdot\nabla\psi\times RB_\phi\right)=0 \qquad RB_\phi=F(\psi)$$

Equilibrium: 2-D Configurations

- **The Grad-Shafranov Equation**

Interpretation of F(ψ)

$$I_p = \int \vec{J}_p \cdot dA$$

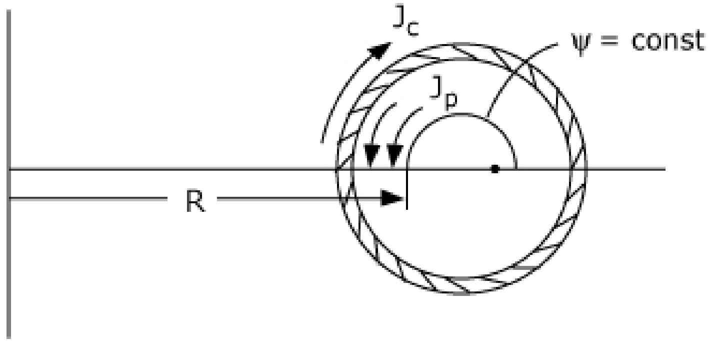
$$= -\int_0^{2\pi} d\phi \int_0^R dR R J_Z (R, Z=0)$$

$$= -\int_0^{2\pi} d\phi \int_0^R dR R \frac{1}{R} \nabla (RB_\phi) \times \vec{e}_\phi$$

$$= -\int_0^R 2\pi R \frac{1}{R} \frac{\partial F}{\partial R} dR$$

$$= -2\pi \left\{ F(R,0) - F(0,0) \right\}$$

$$= -2\pi F(\psi)$$



$I_p(\psi)$ is the total poloidal current passing through the circle $\psi(R, 0)$ = const.

Equilibrium: 2-D Configurations

- **The Grad-Shafranov Equation**

$$\nabla\psi\cdot(\vec{J}\times\vec{B}-\nabla p)=0$$

$$T_1=-\nabla\psi\cdot\nabla P=-\frac{dp}{d\psi}(\nabla\psi)^2$$

$$T_2=(\vec{J}_p+J_\phi\vec{e}_\phi)\times(\vec{B}_p+B_\phi\vec{e}_\phi)\cdot\nabla\psi$$
$$=\left\{\vec{J}_p\times\vec{B}_p+\vec{J}_p\times B_\phi\vec{e}_\phi+\vec{e}_\phi\times\vec{B}_pJ_\phi+(\vec{e}_\phi\times\vec{e}_\phi)J_\phi B_\phi\right\}\cdot\nabla\psi$$

$$T_a=\frac{1}{\mu_0}\left(\frac{1}{R}\nabla F\times\vec{e}_\phi\times\frac{1}{R}\nabla\psi\times\vec{e}_\phi\right)\cdot\nabla\psi$$
$$=\frac{1}{\mu_0R^2}\frac{dF}{d\psi}\left\{(\nabla\psi\times\vec{e}_\phi)\times(\nabla\psi\times\vec{e}_\phi)\right\}\cdot\nabla\psi=0$$

$$T_b=\frac{1}{\mu_0}\left\{\left(\frac{1}{R}\nabla F\times\vec{e}_\phi\right)\times\vec{e}_\phi\right\}\cdot\nabla\psi B_\phi$$
$$=\frac{F}{\mu_0R^2}\frac{dF}{d\psi}(\nabla\psi\times\vec{e}_\phi)\times\vec{e}_\phi\cdot\nabla\psi=-\frac{F}{\mu_0R^2}\frac{dF}{d\psi}(\nabla\psi)^2$$

$$T_c=\vec{e}_\phi\times\frac{\nabla\psi\times\vec{e}_\phi}{R}\left(-\frac{1}{\mu_0R}\Delta^*\psi\right)\cdot\nabla\psi$$
$$=-\frac{1}{\mu_0R^2}\Delta^*\psi(\nabla\psi)^2$$

Equilibrium: 2-D Configurations

- **The Grad-Shafranov Equation**

$$\nabla\psi\cdot(\vec{J}\times\vec{B}-\nabla p)=0$$

combine terms

$$(\nabla\psi)^2\left(-\frac{dp}{d\psi}-\frac{1}{\mu_0 R^2}\frac{d}{d\psi}\frac{F^2}{2}-\frac{1}{\mu_0 R^2}\Delta^*\psi\right)=0$$

$$\Delta^*\psi=-\mu_0 R^2\frac{dp}{d\psi}-F\frac{dF}{d\psi}$$

$$p=p(\psi),\quad F=F(\psi)$$

$$\vec{B}=\frac{1}{R}\nabla\psi\times\vec{e}_\phi+\frac{F}{R}\vec{e}_\phi$$

$$\mu_0\vec{J}=\frac{1}{R}\frac{dF}{d\psi}\nabla\psi\times\vec{e}_\phi-\frac{1}{R}\Delta^*\psi\vec{e}_\phi$$

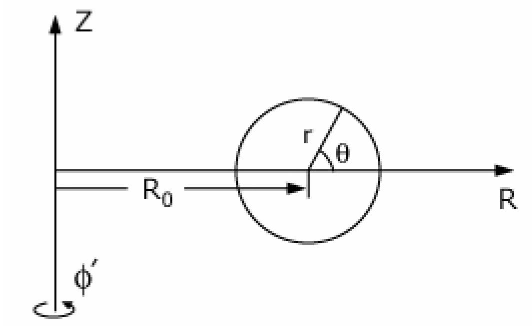
$$\psi_p=2\pi\psi,\quad I_p=2\pi F$$

Equilibrium: 2-D Configurations

- **The Grad-Shafranov Equation**
  - Plasma Parameters and Figures of Merit

$$R = R_0 + r\cos\theta, \quad Z = r\sin\theta$$

$$r = \hat{r}(\theta,\psi) \longleftarrow \psi = \psi(r,\theta)$$



The Volume V

$$V(\psi) = \int_0^{2\pi}\int_0^{2\pi}\int_0^{\hat{r}} Rr\,dr\,d\theta\,d\phi$$

$$V(\psi) = \pi R_0 \int_0^{2\pi} d\theta\,\hat{r}^2\left[1 + \frac{2}{3}\left(\frac{\hat{r}}{R_0}\right)\cos\theta\right]$$

Equilibrium: 2-D Configurations

- **The Grad-Shafranov Equation**
  - Plasma Parameters and Figures of Merit

Safety Factor *q*

$$\frac{Rd\phi}{B_\phi} = \frac{dr}{B_r} = \frac{rd\theta}{B_\theta} = \frac{dl_p}{B_p}$$

$$dl_p = [(dr)^2 + (rdr)^2]^{1/2}$$ poloidal arc length

$$B_p = (B_r^2 + B_\theta^2)^{1/2} = |\nabla \psi| / R$$ poloidal magnetic field

$$\Delta\phi = \int_0^{\Delta\phi} d\phi = \int_0^{2\pi} \left(\frac{rB_\phi}{RB_\theta} \right)_S d\theta$$

$$\frac{\iota}{2\pi} = \frac{\Delta\theta}{\Delta\phi} = \frac{2\pi}{\Delta\phi}$$

$$q(\psi) = \frac{2\pi}{\iota} = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{rB_\phi}{RB_\theta} \right)_S d\theta = \frac{F(\psi)}{2\pi} \oint \frac{dl_p}{R^2 B_p} \longleftarrow RB_\phi = F(\psi)$$

Equilibrium: 2-D Configurations

- **The Grad-Shafranov Equation**
  - Plasma Parameters and Figures of Merit

The Magnetic Well and Shear

$$\frac{dV}{d\psi} = 2\pi\int_0^{2\pi} d\theta R\hat{r}\,\frac{\partial \hat{r}}{\partial \psi} = 2\pi\int_0^{2\pi}\left(\frac{\hat{r}}{B_\theta}\right)_S d\theta = 2\pi\oint\frac{dl_p}{B_p} \leftarrow \frac{\partial \hat{r}}{\partial \psi} = \left(\frac{\partial \psi}{\partial \hat{r}}\right)^{-1} = \left(\frac{1}{RB_\theta}\right)_S$$

$$s(\psi) = 2\left(\frac{V}{V'}\right)\left(\frac{q'}{q}\right)$$

$$W(\psi) = 2\left(\frac{V}{V'}\right)\left(\frac{\left\langle B^2/2+\mu_0 p\right\rangle'}{\left\langle B^2\right\rangle}\right)$$

$$\left\langle Q\right\rangle = \frac{\int (rQ/B_\theta)_S\, d\theta}{\int (r/B_\theta)_S\, d\theta} = \frac{\oint Q dl_p / B_p}{\oint dl_p / B_p}$$

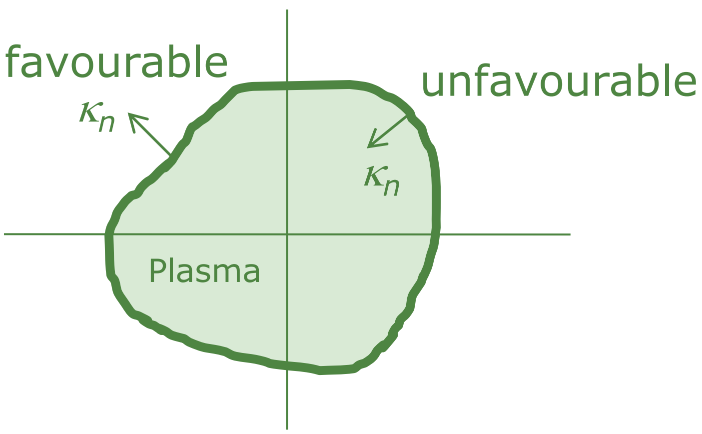
$$\left\langle B^2\right\rangle = \left\langle [F^2 + (\nabla\psi)^2]/R^2\right\rangle$$

Equilibrium: 2-D Configurations

- **The Grad-Shafranov Equation**
  - Plasma Parameters and Figures of Merit

The Magnetic Well and Shear

$$W(\psi) = 2\left(\frac{V}{V'}\right)\left(\frac{\left\langle B^2/2+\mu_0 p\right\rangle'}{\left\langle B^2\right\rangle}\right)$$



Equilibrium: 2-D Configurations

- **The Grad-Shafranov Equation**
  - Plasma Parameters and Figures of Merit

Plasma Beta and Kink Safety Factor

$$\beta_t = \frac{2\mu_0 \langle p \rangle}{B_0^2}, \quad \beta_p = \frac{2\mu_0 \langle p \rangle}{\overline{B}_p^2}$$

$$q_* = \frac{aB_0}{R_0 \overline{B}_p}$$

$$\langle p \rangle = \frac{1}{V_0} \int_0^{V_0} p(V) dV = \frac{1}{V_0} \int_0^{\psi_b} pV' d\psi$$

$$\overline{B}_p = \frac{\mu_0 I_0}{2\pi a \kappa} = \frac{1}{2\pi a \kappa} \oint B_p dl_p$$

$$\kappa = \frac{1}{2\pi} \int_0^{2\pi} d\theta \left[\frac{\hat{r}(\psi_b \theta)}{a} \right]^2$$ field line curvature

Equilibrium: 2-D Configurations

- **The Grad-Shafranov Equation**
  - For systems with helical symmetry

$$B_r = -\frac{1}{r}\frac{\partial \psi}{\partial \alpha}, \quad \alpha \equiv l\theta + hz$$

l: order of the poloidal periodicity
$2\pi/h$: helical period

$$lB_\theta + hrB_z = \frac{\partial \psi}{\partial r}$$

$$lB_z - hrB_\theta = F(\psi)$$

$$\Delta^* \psi = -\mu_0 \frac{dp}{d\psi} - \frac{F}{l_0^2}\frac{dF}{d\psi} + \frac{2hlF}{l_0^4}$$

$$\Delta^* \psi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{l_0^2}\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \alpha^2}$$

$$l_0^2 \equiv l^2 + h^2 r^2$$