Numerical Solutions of Tokamak Equilibria-(1/2)

- 핵융합 플라즈마 특강 2 -

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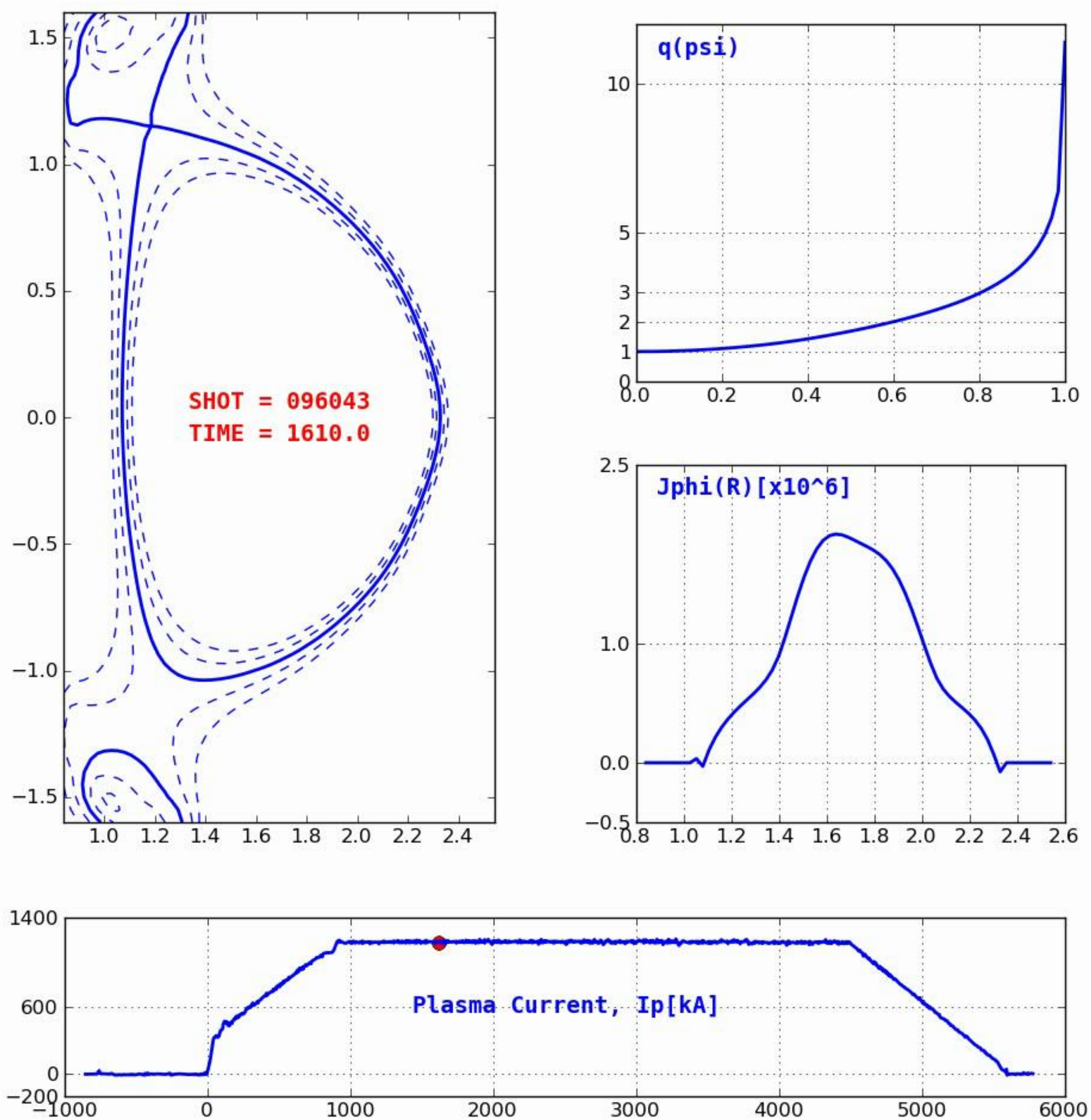
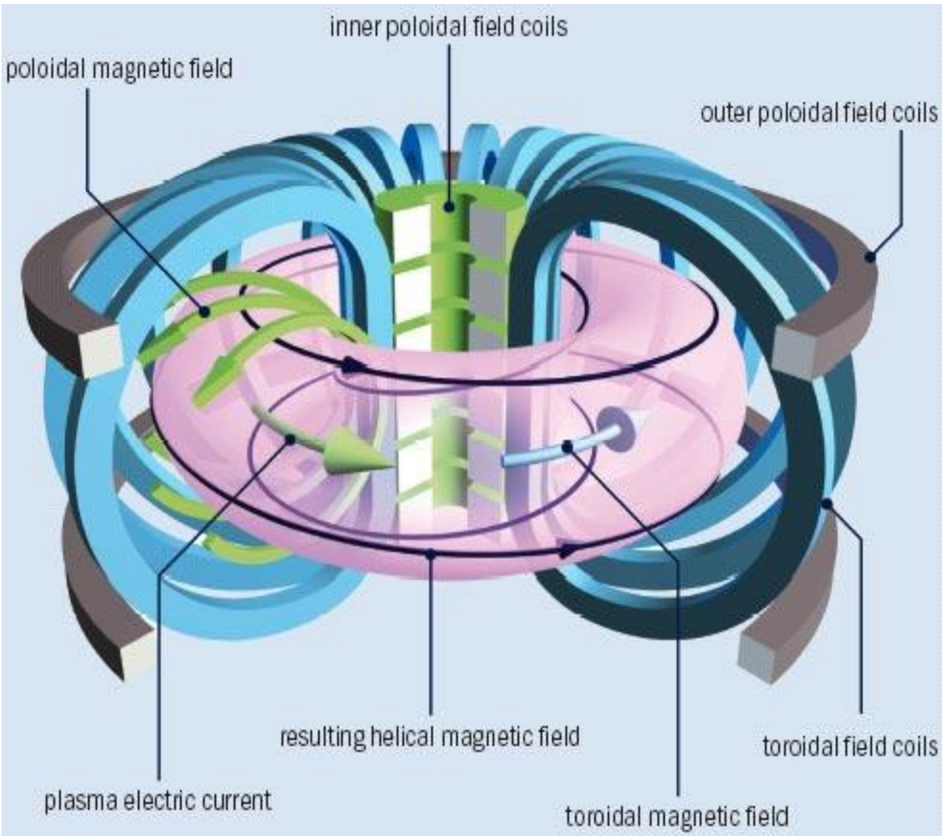
Example of Tokamak Equilibrium Analysis



Physical Understanding of Tokamak Equilibrium



* Image fetched from IOP-physicsweb

$$\nabla p = \mathbf{J}\times\mathbf{B}, \quad \nabla\times\mathbf{B} = \mu_0\mathbf{J}, \quad \nabla\cdot\mathbf{B} = \mathbf{0}$$

$$\Downarrow$$

$$\underbrace{\nabla\cdot\left(r^{-2}\nabla\psi\right)}_{\mathbf{B_p}} = \underbrace{\mu_0 p'(\psi)}_{\nabla \mathbf{p}} + \underbrace{\frac{F(\psi)F'(\psi)}{r^2}}_{\mathbf{B_\phi}}$$

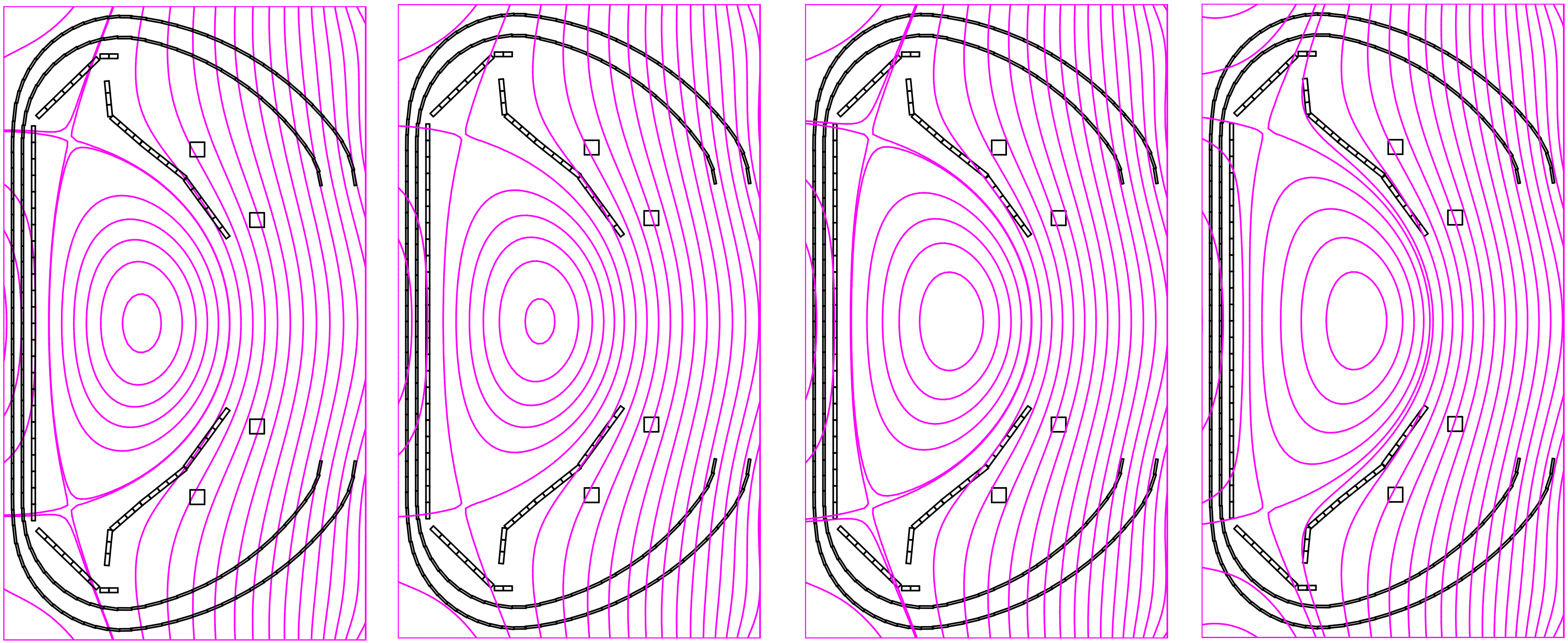
How much plasma can be supported by B_ϕ and B_p ?

Physical Understanding of Tokamak Equilibrium

"how much plasma" ➔ **plasma beta** $\beta \equiv \dfrac{\langle p \rangle}{\langle B^2 \rangle / 2\mu_0}$

High beta plasma ➔ **large shafranov shift & $J_\phi(\psi)$ profile change**

$J_\phi(\psi)$ profile change ➔ **safety factor** $q \equiv 2\pi/\iota$ **& inductance** $l_i(\psi)$



β_p=0.088, β_N[%]=0.185 β_p=0.484, β_N[%]=0.944 β_p=1.129, β_N[%]=1.991 β_p=2.098, β_N[%]=3.113

Numerical/Mathematical Understanding of Tokamak Equilibrium

Magnetic Flux

Current Sources

$$\boxed{\Delta^* \psi = -\mu_0 r J_\phi(\psi(r,z))}$$

where $\Delta^* \psi \equiv r^2 \nabla \cdot \left(r^{-2} \nabla \psi\right)$, $\quad J_\phi(\psi) = -rp'(\psi) - \dfrac{F(\psi)F'(\psi)}{\mu_0 r}$

- **Non-linear, elliptic PDE (partial differential equation)**
- **A sort of non-linear Poisson equation in toroidal geometry**
- **Unknown: $\psi(r,z)$, Known(given): $J_\phi(\psi(r,z))$**

➔ Solve mag. flux, $\psi(r,z)$, with given source, $J_\phi(\psi(r,z))$

(Ref.) Green Function Method In General

- **A linear differential equation in a general form can be expressed as follows**

$$L(x)u(x) = f(x)$$

$$\begin{cases} L(x): \text{a linear, self-adjoint differential operator} \\ u(x): \text{unknown function} \\ f(x): \text{known non-homogeneous term} \end{cases}$$

- **Its solution can be written as**

$$u(x) = L^{-1}(x) f(x) \qquad \text{, where } LL^{-1} = L^{-1}L = I = \text{Identity operator}$$

- **More specifically, define the inverse operator as**

$$L^{-1} f \equiv \int G(x;x') f(x') dx'$$

where the kernel $G(x;x')$ is the Green's Function of operator L

- **Recall the properties of the Dirac delta function** $\delta(x)$

$$\int_{-\infty}^{\infty} \delta(x - x') f(x') dx' = f(x), \qquad \int_{-\infty}^{\infty} \delta(x') dx' = 1$$

*** Reference : http://www.boulder.nist.gov/div853/greenfn/tutorial.html**

(Ref.) Green Function Method – Cont.

- The Green's function *G(x;x')* then satisfies

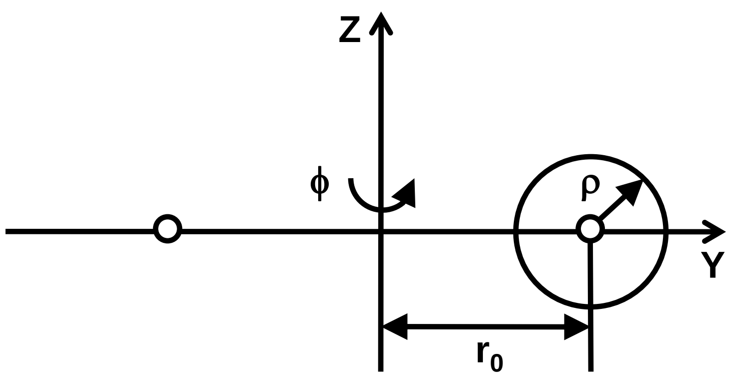
$$L(x)G(x;x^{'}) = \delta(x - x^{'})$$

- The solution can then be written directly in terms of the Green's function as

$$u(x) = \int_{-\infty}^{\infty} G(x;x^{'}) f(x^{'}) dx^{'}$$

- From basic physics, the Green's function gives the potential at the point x due to a point charge at the point x' the source point. And the Green's function only depends on the distance between the source and field points.

Green Function for Toroidal Ring Current



In cylindrical (r, ϕ, z) coordinates

$$\left[\Delta^* \psi = -\mu r J_\phi\right] + \left[\Delta^* G(\mathbf{r},\mathbf{r}_0) = -\mu r \delta(\mathbf{r}-\mathbf{r}_0)\right]$$

$$\psi(\mathbf{r}) = \begin{cases} \int_\Omega J_\phi G(\mathbf{r},\mathbf{r}_0)\, dS & \text{from plasma} \\ I_c G(\mathbf{r},\mathbf{r}_c) & \text{from coils} \end{cases}$$

$$G(\mathbf{r},\mathbf{r}_0) = \frac{\mu}{2\pi k}(r r_0)^{1/2}\left[(2-k^2)K(k) - 2E(k)\right]$$

$$\begin{cases} k^2 \equiv \dfrac{4 r r_0}{(r+r_0)^2 + (z-z_0)^2} \\ K(k) = \text{complete elliptic integral of the first kind} \\ E(k) = \text{complete elliptic integral of the second kind} \end{cases}$$

- Howard S. Cohl et. al., "A Compact Cylindrical Green's Function Expansion for the Solution of Potential Problems", *The Astrophysical Jornal*, Vol. 527, p86-101 (1999)
- B.J. Braams, "The interpretation of tokamak magnetic diagonostics", *Plasma Physics and Controlled Fusion*, Vol. 33, No. 7, p715~748 (1991)

Remark On Use Of Green Function

- **If there is a toroidal ring current source such as PF coil, then one can calculate the magnetic flux at the position, (r,z), from the source, (r_0,z_0), directly using Green function.**

$$\psi(r,z) = I_c G(r,z;r_0,z_0)$$

- **In addition, one can calculate the magnetic field strength easily as follows with *B_ϕ=0***

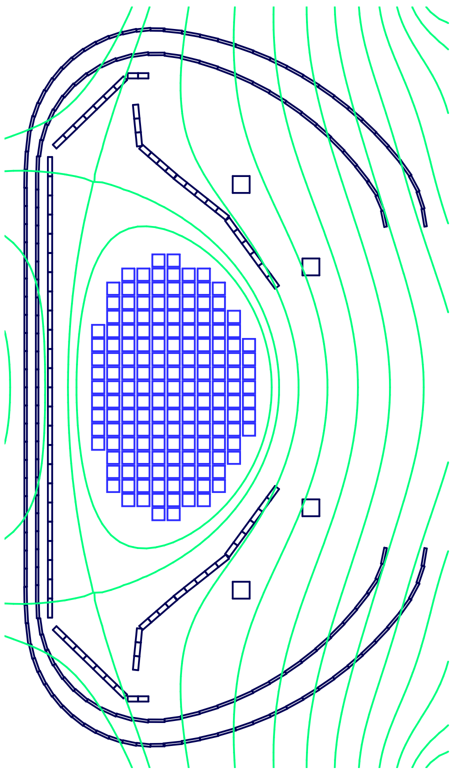
$$B_r = -\frac{1}{r}\frac{\partial\psi}{\partial z} = -\frac{I_c}{r}\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial z}$$

$$B_z = +\frac{1}{r}\frac{\partial\psi}{\partial r} = +\frac{I_c}{r}\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial r}$$

- **Basically one could solve the GS equation using Green function by assuming the plasma as a set of toroidal ring current elements. However it requires rather large computational costs.**

Direct Solution By Green Function

By assuming the plasma as a set of current elements, the magnetic flux can be calculated by Green function directly as follows



$$\Delta^{*}\psi = -\mu_0 r J_{\phi,\text{total}}\left(\psi(r,z)\right)$$

$$\begin{aligned}
\psi(r,z) &= \int_{\Omega} J_{\phi,\text{total}}(r',z')G(r,z;r',z')dS(r',z') \\
&= \int_{\Omega} \left(J_{\phi,\text{plasma}}(\mathbf{r}') + J_{\phi,\text{external}}(\mathbf{r}')\right)G(\mathbf{r};\mathbf{r}')dS(\mathbf{r}') \\
&= \sum_{l} \left((\Delta S_l)J_{\phi,\text{plasma}}(\mathbf{r}_l')\right)G(\mathbf{r};\mathbf{r}_l') + \sum_{c} I_c G(\mathbf{r};\mathbf{r}_c')
\end{aligned}$$

Drawbacks →

- **Slow computation speed**
- **Not enough smoothness of solution**

Various Numerical Approaches For Tokamak Equilibria

Numerical Methods For Tokamak Equilibria

Real space solver

- Iterative method
  - SOR, ADI, or MGM with FDM/FEM
- Direct method
  - DCR (double cyclic reduction) or FACR (fourier analysis cyclic reduction)

Inverse equilibrium solver

- Iterative metric method

$$J = \nabla\psi \times \nabla\theta \cdot \nabla\phi = \mu\left(\frac{r}{R_0}\right)^m \psi^n$$

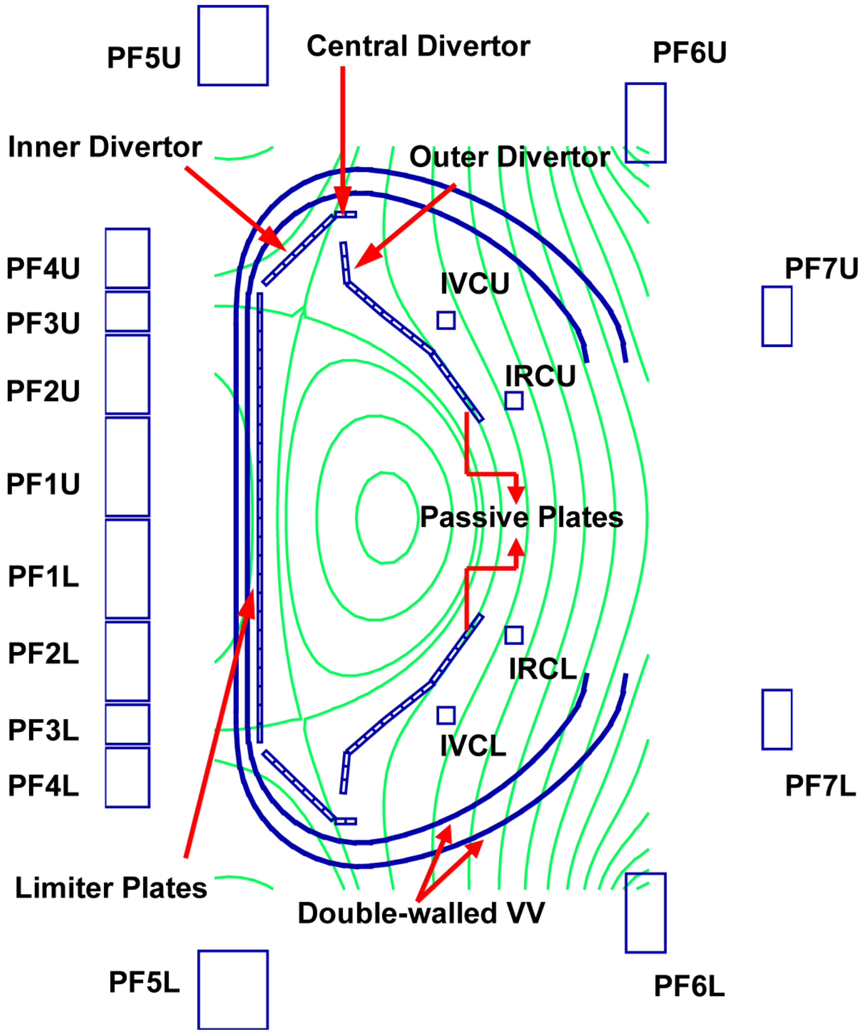
- Direct inverse solution method
- Poloidal angle expansion method

Others

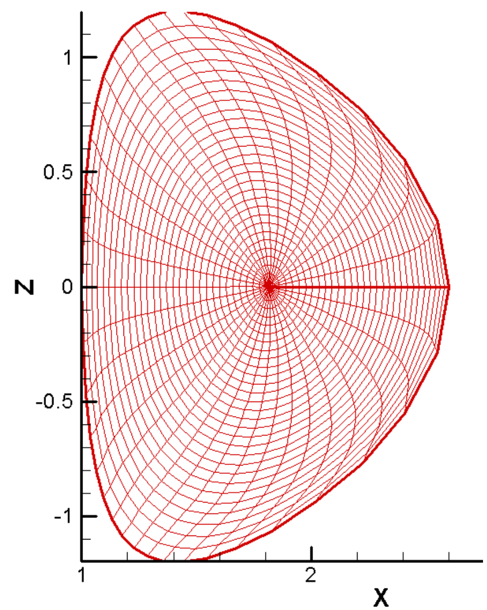
- Green function method
- Orthogonal function expansion method
- Conformal mapping method

Types Of Equilibrium Solutions

Real space equilibrium With free boundary



Inverse equilibrium With fixed boundary



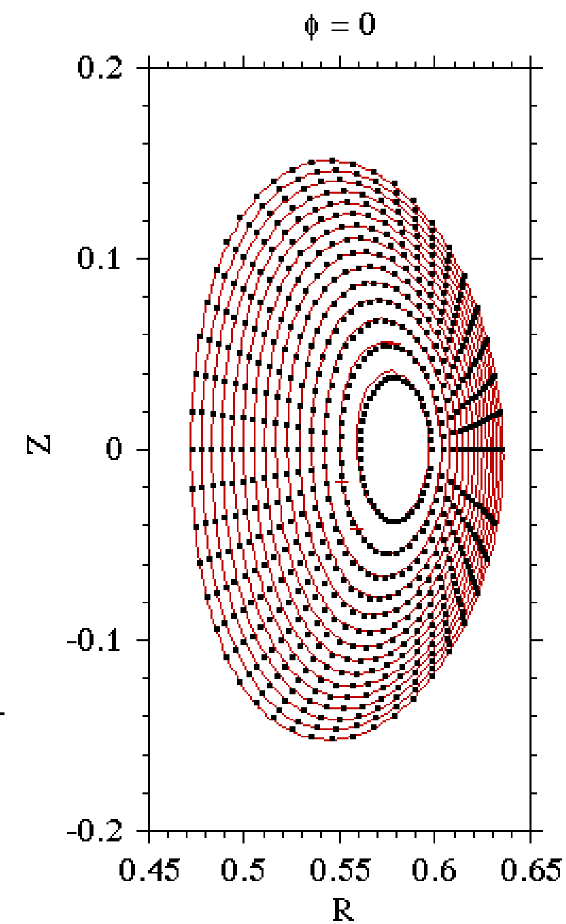


Image fetched from http://www-fusion.ciemat.es

Approximation Using Finite Difference Method

Let's consider a simple discretization based on rectangular grids

Using Centered Difference Scheme

$$\begin{cases} \left.\dfrac{\partial \psi}{\partial X}\right|_{X=X_k} = \dfrac{\psi_{k+1/2} - \psi_{k-1/2}}{h_k} = \dfrac{\psi_{k+1} - \psi_{k-1}}{2h_k} \\ \left.\dfrac{\partial^2 \psi}{\partial X^2}\right|_{X=X_k} = \dfrac{1}{h_k}\left(\psi'_{k+1/2} - \psi'_{k-1/2}\right) = \dfrac{1}{h_k^2}\left(\psi_{k+1} - 2\psi_k + \psi_{k-1}\right) \end{cases}$$

By applying it to GS eq.

$$\frac{1}{h_Z^2}\psi_{j-1,l} + \left(\frac{1}{h_R^2} + \frac{1}{R_l}\frac{1}{2h_R}\right)\psi_{j,l-1} - 2\left(\frac{1}{h_R^2} + \frac{1}{h_Z^2}\right)\psi_{j,l} + \left(\frac{1}{h_R^2} - \frac{1}{R_l}\frac{1}{2h_R}\right)\psi_{j,l+1} + \frac{1}{h_Z^2}\psi_{j+1,l} = -\mu_0 R_l J_{\phi\, j,l}$$

$$\psi_{j,l} = \left\{2\left(\frac{1}{h_R^2} + \frac{1}{h_Z^2}\right)\right\}^{-1} \times \left[\frac{1}{h_Z^2}\psi_{j-1,l} + \left(\frac{1}{h_R^2} + \frac{1}{R_l}\frac{1}{2h_R}\right)\psi_{j,l-1} + \left(\frac{1}{h_R^2} - \frac{1}{R_l}\frac{1}{2h_R}\right)\psi_{j,l+1} + \frac{1}{h_Z^2}\psi_{j+1,l} + \mu_0 R_l J_{\phi\, j,l}\right]$$

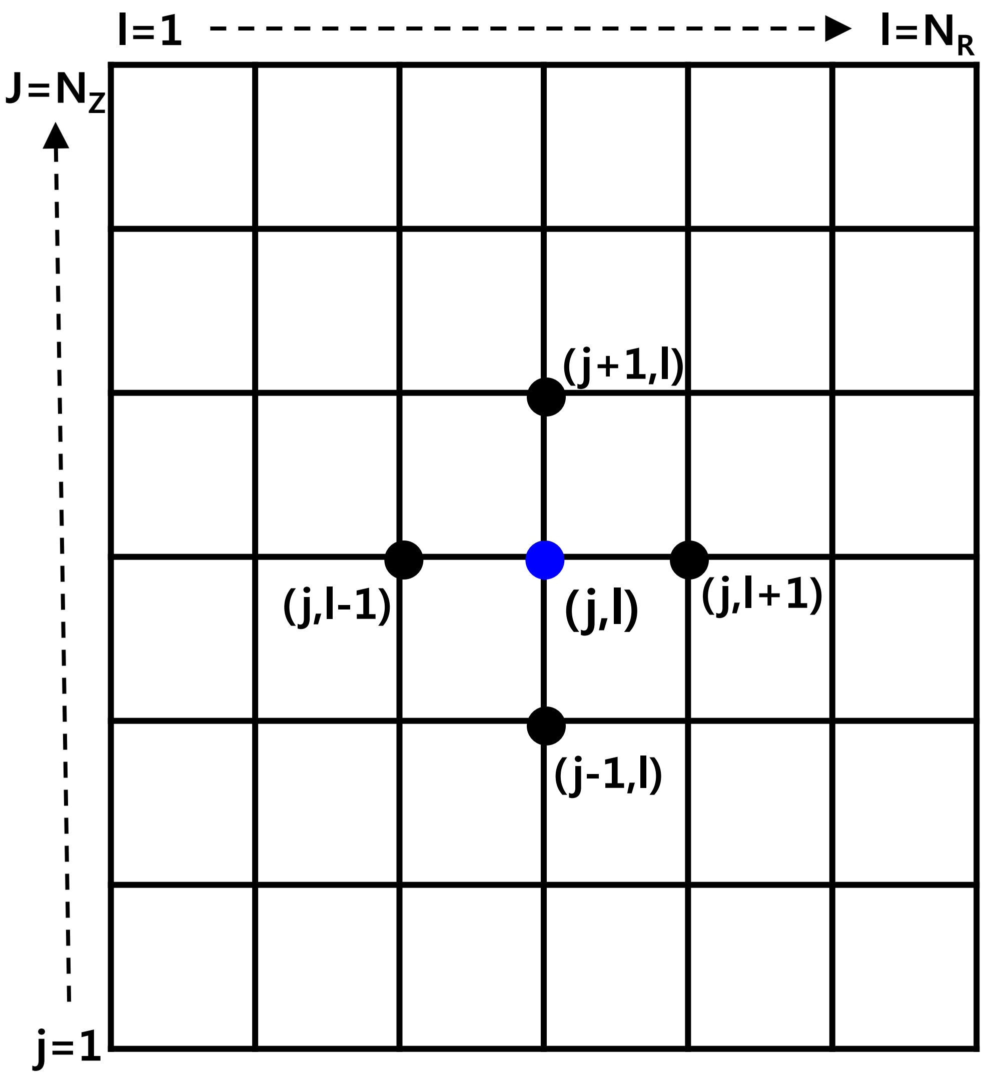
Approximation Using Finite Difference Method

$$\psi_{j,l} = \frac{1}{2}\left[\left(\frac{h_R^2}{h_R^2+h_Z^2}\right)\psi_{j-1,l} + \left(\frac{h_R^2}{h_R^2+h_Z^2}\right)\psi_{j+1,l} + \left(\frac{h_Z^2}{h_R^2+h_Z^2}\right)\left(1+\frac{h_R}{2R_l}\right)\psi_{j,l-1} + \left(\frac{h_Z^2}{h_R^2+h_Z^2}\right)\left(1-\frac{h_R}{2R_l}\right)\psi_{j,l+1} + \left(\frac{h_R^2 h_Z^2}{h_R^2+h_Z^2}\right)\mu_0 R_l J_{\phi\, j,l} \right]$$

Discretized Grad-Shafranov Eq.

- **$(j,l)^{th}$ element can be updated/adjusted with adjacent 4 elements and its current source on the same grid point**
- **Now it's a boundary value problem (B.C. required).**
- **This linearized equation can be solved using various numerical algorithms such as SOR and MGM so on.**

Boundary Condition



- $(N_R-2)x(N_Z-2)$ eqs. Obtained from the previous page
  - $(j,l)_{th}$ element is calculated with adjacent 4 elements
- $2(N_R-2) + 2(N_Z-2) – 4$ eqs. required to close the linear equations→ boundary condition

- For free boundary problems, B.C. can be given from external current sources and plasma currents using '**free-space Green function**'

- For fixed boundary problems, the B.C. can be specified by user. Normally zero flux.

How To Handle Non-linearity ??
"Picard Iteration"

Existence/Uniqueness Of Solution : Picard-Lindelof Theorem

- **In the initial value problem as follows**

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0, \quad t \in [t_0 - \alpha, t_0 + \alpha]$$

- **Suppose f is Lipschitz continuous in y and continuous in t . Then, for some value $\varepsilon > 0$, there exists a unique solution $y(t)$ to the initial value problem within the range $[t_0 - \varepsilon, t_0 + \varepsilon]$.**

Successive Approximation By Picard Iteration

$$(1)\ \text{guess } y_0(t)$$

$$(2)\ \text{solve } y_{k+1}(t) \text{ from } y'_{k+1}(t) = f_k(t, y_k(t))$$

$$(3)\ \text{iterate}(2) \text{ with } k = 1, \cdots \text{ until } y_{k+1}(t) \text{ is converged}$$

- **Refer to the chapter 2 of 'First-order differential equations' in 'Advanced Engineering Mathematics'**

Homework

- Make two contour plots of $\psi(r,z)$ with a computational region ($1.1\text{m} \leq r \leq 2.4\text{m}$, $-1.5\text{m} \leq z \leq 1.5$). Here $\psi(r,z)$ is solved with following information. One contour is without PF0 and the other one is with PF0. Compare these two plots and understand the difference. Probably 3D contour might be better than 2D contour.

Current Sources	R [m]	Z [m]	# of turn	Current [kA]
PF1U/PF1L	0.57	±0.25	180	-8.85
PF2U/PF2L	0.57	±0.70	144	-1.43
PF3U/PF3L	0.57	±1.00	72	9.49
PF4U/PF4L	0.57	±1.26	108	10.17
PF5U/PF5L	1.09	±2.30	208	10.66
PF6U/PF6L	3.09	±1.92	128	-2.07
PF7U/PF7L	3.73	±0.98	72	-17.20
PF0	1.80	0.00	1	2000

Tips

- Use Green function formula as a direct method
- One can solve this in an iterative way studied above if one want.