Topics in Fusion and Plasma Studies 2
(459.667, 3 Credits)

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Stability: General Considerations

- **Variational Formulation**

$$\omega^2 = \frac{\delta W(\xi^*,\xi)}{K(\xi^*,\xi)}$$

$$\delta W(\xi^*,\xi) = -\frac{1}{2}\int \xi^* \cdot \vec{F}(\xi) d\vec{r}$$

$$K(\xi^*,\xi) = \frac{1}{2}\int \rho|\xi|^2 d\vec{r}$$

Advantages

- Allowing use of trial functions to estimate ω^2
- Can be applied to multidimensional systems efficiently

Disadvantages

- Still somewhat complicated
- Giving more information than minimum required

Stability: General Considerations

- It is often of primary interest to determine whether a given system is stable or unstable.
- No great need to know the specific values of growth rates (ω^2).
- Growth times ($\leq$ 50 μs) are typically much shorter than experimental times ($\geq$ 1 ms).
- Since MHD instabilities are very strong, it is more important to know whether the system is stable or not, rather than know the precise growth rates (which can be easily estimated).
- In such cases, the variational formulation can be further simplified, leading to a powerful minimizing principle which exactly determines stability boundaries.

• The Energy Principle

- Representing the most efficient and often the most intuitive method of determining plasma stability.
- Can be applied in a direct manner to systems in which the plasma is surrounded by a conducting wall.
- If a vacuum region is present, difficulties are resolved by introducing a natural BC into the Extended Energy Principle.

Stability: General Considerations

- **The Energy Principle**

Variational principle

$$\omega^2 = \frac{\delta W}{K} \geq 0 \qquad \text{stable}$$

Energy principle

$$\delta W \geq 0 \qquad \text{stable}$$

- If the minimum value of potential energy is positive for all displacements, the system is stable.
- It it is negative for any displacement, the system is unstable.

Stability: General Considerations

- **The Energy Principle**

Proof

$$-\omega_n^2 \rho \xi_n = \vec{F}(\xi_n)$$ complete set of normal modes, orthonormal

$$\xi(r) = \sum a_n \xi_n(r)$$ arbitrary trial function

$$\int \rho \xi_n^* \cdot \xi_m d\vec{r} = \delta_{nm}$$

$$\delta W(\xi^*, \xi) = -\frac{1}{2}\int \xi^* \cdot \vec{F}(\xi) d\vec{r} = -\frac{1}{2}\sum a_n^* a_m \int \xi_n^* \cdot \vec{F}(\xi_m) d\vec{r}$$

$$= -\frac{1}{2}\sum a_n^* a_m \int \xi_n^* \cdot (-\omega_m^2 \rho \xi_m) d\vec{r}$$

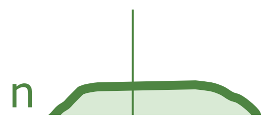
$$= \frac{1}{2}\sum |a_m|^2 \omega_m^2$$

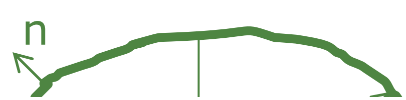
Not completely valid for the general case because of the existence of continua.

- If a trial function can be found which makes $\delta W < 0$, then at least one $\omega_m^2 < 0 \rightarrow$ instability
- If all trial function make $\delta W > 0$, then all $\omega_m^2 > 0 \rightarrow$ stability

Stability: General Considerations

- **The Extended Energy Principle**
  - The Energy principle is valid if the plasma is either directly surrounded by a conducting wall or isolated from the wall by a vacuum region.
  - When a vacuum region is present the situation is more complicated because the vacuum fields do not appear explicitly in δW but enter only through the BCs at the plasma surface.





$$\vec{n}\cdot\hat{B}_1\big|_{r_w} = 0$$

$$\vec{B}]]_{r_p} = 0$$

  - Difficulty of generating trial functions satisfying the pressure balance jump condition.
  - δW not convenient because of complicated BC with wall, and no explicit appearance of vacuum energy.

$$\vec{n}\cdot\xi\big|_{r_w} = 0$$

$$[[\,p + B^2/2\mu_0\,]]_{r_p} = 0$$

Stability: General Considerations

- **The Extended Energy Principle**
  - The vacuum contribution explicitly appears and natural BCs are introduced.

$$\delta W(\xi^*,\xi) = -\frac{1}{2}\int \xi^* \cdot \vec{F}(\xi)\,d\vec{r} \qquad \vec{Q} \equiv \vec{B}_1 = \nabla \times (\xi \times \vec{B})$$

$$= -\frac{1}{2}\int \xi^* \cdot [\frac{1}{\mu_0}(\nabla \times \vec{Q}) \times \vec{B} + \frac{1}{\mu_0}(\nabla \times \vec{B}) \times \vec{Q} + \nabla(\gamma p \nabla \cdot \xi + \xi \cdot \nabla p)]\,d\vec{r}$$

$$\delta W = \frac{1}{2}\int d\vec{r}\left\{\frac{\left|\vec{Q}\right|^2}{\mu_0} + \gamma p\left|\nabla \cdot \xi\right|^2 - \xi^* \cdot [\vec{J} \times \vec{Q} + \nabla(\xi \cdot \nabla p)]\right\} - \frac{1}{2}\int d\vec{S}(\vec{n} \cdot \xi^*)\left(\gamma p \nabla \cdot \xi - \frac{\vec{B} \cdot \vec{B}_1}{\mu_0}\right)$$

$$\xi = \xi_\perp + \xi_{||}\vec{b}$$

$$\xi_{||}^*\vec{b} \cdot [\vec{J} \times \vec{Q} + \nabla(\xi \cdot \nabla p)] = 0$$

$$\delta W = \delta W_F + \mathrm{B.T.}$$

Stability: General Considerations

- **The Extended Energy Principle**
  - The vacuum contribution explicitly appears and natural BCs are introduced.

$$\delta W = \delta W_F + \text{B.T.}$$

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left[\frac{\left|\vec{Q}\right|^2}{\mu_0} - \xi_\perp^* \cdot \vec{J} \times \vec{Q} + \gamma p \left|\nabla \cdot \xi\right|^2 + (\xi_\perp \cdot \nabla p)\nabla \cdot \xi_\perp^*\right]$$

standard form of the fluid energy

$$\text{B.T.} = \frac{1}{2}\int_S d\vec{S}(\vec{n} \cdot \xi_\perp^*)\left(\frac{\vec{B} \cdot \vec{B}_1}{\mu_0} - \gamma p \nabla \cdot \xi - \xi_\perp \cdot \nabla p\right)$$

boundary term

Stability: General Considerations

- **The Extended Energy Principle**
  - Complicated pressure balance jump relation introduced into the Energy Principle as a natural BC

$$[[\, p + B^2/2\mu_0]]_{r_p} = 0$$

$$\left[p_1 + \frac{\vec{B}\cdot\vec{B}_1}{\mu_0} + \xi\cdot\nabla\left(p + \frac{B^2}{2\mu_0}\right)\right]_{r_p} = \left[\frac{\hat{B}\cdot\hat{B}_1}{\mu_0} + \xi\cdot\nabla\frac{\hat{B}^2}{2\mu_0}\right]_{r_p}$$

$$\mathrm{B.T.} = \delta W_S + \frac{1}{2}\int_S d\vec{S}(\vec{n}\cdot\xi_\perp^*)\left(\frac{\hat{B}\cdot\hat{B}_1}{\mu_0}\right)$$

$$\delta W_S = \frac{1}{2}\int_S d\vec{S}(\vec{n}\cdot\xi_\perp^*)[[\xi_\perp\cdot\nabla\left(p + B^2/2\mu_0\right)]]$$

$$= \frac{1}{2}\int_S d\vec{S}\left|\vec{n}\cdot\xi_\perp\right|^2 \vec{n}\cdot[[\nabla\left(p + B^2/2\mu_0\right)]]$$

Surface contribution: non-zoro only if surface currents flow on the plasma-vacuum boundary

Stability: General Considerations

- **The Extended Energy Principle**

$$\mathrm{B.T.} = \delta W_S + \frac{1}{2}\int_S d\vec{S}(\vec{n}\cdot\xi_\perp^*)\left(\frac{\hat{B}\cdot\hat{B}_1}{\mu_0}\right)$$

$\downarrow$

perturbed magnetic energy in the vacuum region

$$\delta W_V = \frac{1}{2}\int_V d\vec{r}\,\frac{\left|\hat{B}_1\right|^2}{\mu_0} = \frac{1}{2}\int_V d\vec{r}\,\frac{\left|\nabla\times\hat{A}_1\right|^2}{\mu_0}$$

$$= \frac{1}{2}\int_V d\vec{r}\,\frac{\nabla\cdot(\hat{A}_1^*\times\nabla\times\hat{A}_1) - \hat{A}_1^*\cdot\nabla\times\nabla\times\hat{A}_1}{\mu_0}$$

$$= -\frac{1}{2}\int_S d\vec{S}\,\frac{\vec{n}\cdot(\hat{A}_1^*\times\hat{B}_1)}{\mu_0}$$

$\hat{A}_1 = \xi_\perp\times B + \nabla\phi$ $\qquad$ $\hat{B}_1\cdot(\vec{n}\times\nabla\phi) = 0$ choose as gauge

Stability: General Considerations

- **The Extended Energy Principle**

$$\hat{A}_1 = \xi_\perp \times B + \nabla \phi \qquad \hat{B}_1 \cdot (\vec{n} \times \nabla \phi) = 0$$ choose as gauge

$$\begin{aligned}\delta W_V &= -\frac{1}{2}\int_S d\vec{S}\,\frac{\vec{n}\cdot(\xi_\perp^* \times B)\times \hat{B}_1}{\mu_0} \\ &= \frac{1}{2}\int_S d\vec{S}(\vec{n}\cdot\xi_\perp^*)\left(\frac{\hat{B}\cdot\hat{B}_1}{\mu_0}\right)\end{aligned}$$

Stability: General Considerations

- **The Extended Energy Principle**

$$\delta W = \delta W_F + \delta W_S + \delta W_V$$

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left[\frac{\left|\vec{Q}\right|^2}{\mu_0} - \xi_\perp^* \cdot (\vec{J}\times\vec{Q}) + \gamma p\left|\nabla\cdot\xi\right|^2 + (\xi_\perp\cdot\nabla p)\nabla\cdot\xi_\perp^*\right]$$

$$\delta W_S = \frac{1}{2}\int_S d\vec{S}\left|\vec{n}\cdot\xi_\perp\right|^2 \vec{n}\cdot[[\nabla\left(p + B^2/2\mu_0\right)]]$$

$$\delta W_V = \frac{1}{2}\int_V d\vec{r}\,\frac{\left|\hat{B}_1\right|^2}{\mu_0}$$

Boundary conditions on trial functions

$$\vec{n}\cdot\hat{B}_1\Big|_{r_w} = 0$$

$$\vec{n}\cdot\hat{B}_1\Big|_{r_p} = \vec{n}\cdot\nabla\times(\xi_\perp\times\hat{B})\Big|_{r_p} = \hat{B}_1\cdot\nabla(\vec{n}\cdot\xi_\perp) - (\vec{n}\cdot\xi_\perp)[\vec{n}\cdot(\vec{n}\cdot\nabla)\hat{B}_1]\Big|_{r_p}$$

Stability: General Considerations

- **The Intuitive Form of δW_F**
  - Derive an alternate form that provides additional physical insight into the behaviour of MHD instabilities

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left[\frac{\left|\vec{Q}\right|^2}{\mu_0} - \xi_\perp^* \cdot (\vec{J}\times\vec{Q}) + \gamma p\left|\nabla\cdot\xi\right|^2 + (\xi_\perp\cdot\nabla p)\nabla\cdot\xi_\perp^*\right]$$

$$\longleftarrow \quad \left|\vec{Q}\right|^2 = \left|\vec{Q}_\perp\right|^2 + \left|\vec{Q}_{||}\right|^2$$

$$\xi_\perp^* \cdot (\vec{J}\times\vec{Q}) = J_{||}(\xi_\perp^*\times\vec{b})\cdot\vec{Q}_\perp + Q_{||}\xi_\perp^*\cdot\vec{J}_\perp\times\vec{b}$$

$$\longleftarrow \quad \vec{J}_\perp = \frac{\vec{b}\times\nabla p}{B}$$

$$\vec{Q}_{||} = \vec{b}\cdot\nabla\times(\xi_\perp\times\vec{B})$$

$$= \vec{b}\cdot(\vec{B}\cdot\nabla\xi_\perp - \xi_\perp\cdot\nabla\vec{B} - \vec{B}\nabla\cdot\xi_\perp)$$

$$= -B(\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\vec{\kappa}) + \frac{\mu_0}{B}\xi_\perp\cdot\nabla p$$

Stability: General Considerations

- **The Intuitive Form of δW_F**

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left[\underbrace{\frac{\left|\vec{Q}_\perp\right|^2}{\mu_0} + \frac{B^2}{\mu_0}\left|\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\vec{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\xi\right|^2}_{\text{stabilising}} \overbrace{- 2(\xi_\perp\cdot\nabla p)(\vec{\kappa}\cdot\xi_\perp^*) - J_{||}(\xi_\perp^*\times\vec{b})\cdot\vec{Q}_\perp}^{\text{destabilising}}\right]$$

Pressure-driven modes (+ or -)

current-driven (kinks) modes (+ or -)

Energy required to bend magnetic field lines: dominant potential energy contribution to the shear Alfvén wave

Energy necessary to compress the magnetic field: major potential energy contribution to the compressional Alfvén wave

Energy required to compress the plasma: main source of potential energy for the sound wave

Stability: General Considerations

- **Incompressibility**
  - Because of the simple way in which $\xi_{||}$ appears in δW, it is possible to minimize once for all with respect to $\xi_{||}$ and eliminate it from the calculation.

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left[\frac{\left|\vec{Q}_\perp\right|^2}{\mu_0} + \frac{B^2}{\mu_0}\left|\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\vec{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\xi\right|^2 - 2(\xi_\perp\cdot\nabla p)(\vec{\kappa}\cdot\xi_\perp^*) - J_{||}(\xi_\perp^*\times\vec{b})\cdot\vec{Q}_\perp\right]$$

$$\xi_{||} \rightarrow \xi_{||} + \delta\xi_{||}$$

$$\delta(\delta W)_{\xi_{||}} = \int_P d\vec{r}\,\gamma p(\nabla\cdot\xi)\nabla\cdot\left(\delta\xi_{||}\frac{\vec{B}}{B}\right)$$

$$= -\int_P d\vec{r}\,\frac{\delta\xi_{||}}{B}\vec{B}\cdot\nabla(\gamma p\nabla\cdot\xi)$$

$$= -\int_P d\vec{r}\,\frac{\delta\xi_{||}}{B}\gamma p\vec{B}\cdot\nabla(\nabla\cdot\xi) = 0$$

Stability: General Considerations

- **Incompressibility**

Several minimising condition

$$\vec{B}\cdot\nabla(\nabla\cdot\xi)=0$$

If non-singular $\nabla\cdot\xi=0$

$$\delta W_F=\frac{1}{2}\int_P d\vec{r}\left[\frac{\left|\vec{Q}_\perp\right|^2}{\mu_0}+\frac{B^2}{\mu_0}\left|\nabla\cdot\xi_\perp+2\xi_\perp\cdot\vec{\kappa}\right|^2+\cancel{\gamma p\left|\nabla\cdot\xi\right|^2}-2(\xi_\perp\cdot\nabla p)(\vec{\kappa}\cdot\xi_\perp^*)-J_{||}(\xi_\perp^*\times\vec{b})\cdot\vec{Q}_\perp\right]$$

If $\nabla\cdot\xi\neq 0$

- Existing sufficient equilibrium symmetry

$\vec{B}=B_\theta(r)\vec{e}_\theta,\quad \xi\sim\xi(r)\exp[i(m\theta+kz)]$ *Z* pinch

$\vec{B}\cdot\nabla\dfrac{\xi_{||}}{B}=\dfrac{B_\theta}{r}\dfrac{\partial}{\partial\theta}\dfrac{\xi_{||}}{B_\theta}=\dfrac{im\xi_{||}}{r}$ = 0 for *m* = 0

Stability: General Considerations

- **Incompressibility**

Several minimising condition

If $\nabla \cdot \xi \neq 0$

- Existing sufficient equilibrium symmetry

$$\vec{B} \cdot \nabla \frac{\xi_{||}}{B} = \frac{B_\theta}{r} \frac{\partial}{\partial \theta} \frac{\xi_{||}}{B_\theta} = \frac{im\xi_{||}}{r} \quad = 0 \text{ for } m = 0$$

$$\nabla \cdot \xi = \nabla \cdot \xi_\perp + \nabla \cdot \frac{\xi_{||}}{B} \vec{B} = \nabla \cdot \xi_\perp + \vec{B} \cdot \nabla \frac{\xi_{||}}{B}$$

$$= \nabla \cdot \xi_\perp$$

$\xi_{||}$ does not appear.
The term must be maintained for the rest of the minimisation.

$$\int_P \gamma p |\nabla \cdot \xi|^2 dr = \int_P \gamma p |\nabla \cdot \xi_\perp|^2 dr$$

Stability: General Considerations

- **Incompressibility**

Several minimising condition

If $\nabla\cdot\xi \neq 0$

- Existing sufficient equilibrium symmetry
- Existing closed-line symmetry (periodicity constraints) $\xi_{||}(l)=\xi_{||}(l+L)$

$$\vec{B}\cdot\nabla(\nabla\cdot\xi)=0 \;\rightarrow\; \nabla\cdot\xi = F(p)$$ homogeneous solution

$$\vec{B}\cdot\nabla\frac{\xi_{||}}{B} = -\nabla\cdot\xi_{\perp} + F(p)$$

$$\frac{\xi_{||}}{B} = -\int_0^l \frac{\nabla\cdot\xi_{\perp}}{B}dl + \int_0^l \frac{F(p)}{B}dl = -\int_0^l \frac{\nabla\cdot\xi_{\perp}}{B}dl + F(p)\int_0^l \frac{dl}{B}$$

$$F(p) = \langle\nabla\cdot\xi_{\perp}\rangle = \frac{\oint \frac{dl}{B}\nabla\cdot\xi_{\perp}}{\oint \frac{dl}{B}}$$

$$\int_P \gamma p|\nabla\cdot\xi|^2 d\vec{r} = \int_P \gamma p F(p)^2 d\vec{r}$$
$$= \int_P \gamma p|\langle\nabla\cdot\xi_{\perp}\rangle|^2 d\vec{r}$$

In periodicity choose

$\xi_{||}$ does not appear.

Stability: General Considerations

- **Incompressibility**

For the Energy Principle

$$\delta W_F + \delta W_S + \delta W_V \geq 0 \quad \text{ideal MHD}$$

$$\delta W_\perp + \delta W_S + \delta W_V \geq 0 \quad \text{collisionless MHD}$$

For the corresponding variational principles

$$\omega^2 = (\delta W_F + \delta W_S + \delta W_V)/K \geq 0 \quad \text{ideal MHD}$$

$$\omega^2 = (\delta W_\perp + \delta W_S + \delta W_V)/K_\perp \geq 0 \quad \text{collisionless MHD}$$

- As expected, the quantity $\xi_{||}$ never appears in the collisionless MHD model.
- The only difference is the plasma compressibility contribution.
- In the general case where $\nabla \cdot \xi = 0$, δW (ideal MHD) = δW (collisionless MHD), indicating that both models predict the identical stability boundaries.
- The collisionless MHD model is more pessimistic with regard to growth rate.

$$\omega^2(\text{coll.MHD}) \leq \omega^2(\text{ideal MHD})$$