Lecture 2-1

Probability Examples

Statistics for
Civil & Environmental Engineers

Example 2.19: Dam Reliability

Example 2.19. Dam reliability. Consider again the problem of the failure of a dam (see Example 2.14) caused by the occurrence of either a flood exceeding the design capacity of the spillway (event A) or a destructive earthquake producing the structural collapse of the dam (event B). Let $\Pr[A] = a$ and $\Pr[B] = b$ denote the respective probabilities of occurrence in a year. If the two events are statistically independent, their joint probability equals the product of their marginal probabilities, that is,

$$\Pr[AB] = \Pr[A]\,\Pr[B] = ab.$$

Accordingly, the risk of failure of the dam in a year is given by

$$\Pr[A + B] = \Pr[A] + \Pr[B] - \Pr[AB] = a + b - ab.$$

If we assume typical values for a and b for small dams of .02 and .01, respectively, the risk of failure in a year is

$$\Pr[A + B] = a + b - ab = .02 + .01 - .02 \times .01 = .0298.$$

Note that this result is very close to that (.03) obtained by using Boole's inequality (see Example 2.13). When rare events with small marginal probabilities are considered, their joint probability generally plays a minor role, so that its effect can often be neglected in risk assessment.

Example 2.19: Dam Reliability

One can see from Fig. 2.1.6*g* that the sample space for this experiment is given by four elementary events, namely,

$$\Omega = \{AB, AB^c, A^cB, A^cB^c\},$$

whereas the failure event is represented by $A + B = (AB) \cup (AB^c) \cup (A^cB)$. Therefore, the probability of survival of the dam in any year can be measured as

$$\Pr[A^cB^c] = 1 - \Pr[A + B] = 1 - (a + b - ab),$$

which is defined as the reliability in a year. For $a = .02$ and $b = .01$,

$$\Pr[A^cB^c] = 1 - (a + b - ab) = 1 - (.02 + .01 - .01 \times .02) = .9702,$$

which means a chance of more than 97 percent that the dam will survive in a year. However, the designer is mainly interested in evaluating the system reliability during its lifetime, say, m years, and must therefore consider failure probability after $i = 1, 2, \ldots, m$ years from the construction of the dam. The survival probability after the first year is again $\Pr[A^cB^c]$. The experiment is repeated in subsequent years. The survival probability after the second year is given by $\Pr[(A^cB^c)_1 \cap (A^cB^c)_2]$, which can be written as

$$\Pr[(A^cB^c)_1(A^cB^c)_2] = \Pr[(A^cB^c)_1]\ \Pr[(A^cB^c)_2 \mid (A^cB^c)_1].$$

Example 2.19: Dam Reliability

where subscripts denote years. Because floods and earthquakes occurring in a year can be assumed to be independent of those occurring in another year, the conditional probability on the right-hand side can be simply written as $\Pr[(A^cB^c)_2 \mid (A^cB^c)_1] = \Pr[(A^cB^c)_2]$. We also assume that the survival probability in any one year is the same as that in any other year. The survival probability after the second year is therefore

$$\Pr[(A^cB^c)_1(A^cB^c)_2] = \Pr[(A^cB^c)_1]\ \Pr[(A^cB^c)_2] = \Pr[A^cB^c]^2$$
$$= [1 - (a + b - ab)]^2.$$

By using the same procedure, the m-year survival probability can be evaluated as

$$\Pr[(A^cB^c)_1(A^cB^c)_2 \cdots (A^cB^c)_m] = \{\Pr[A^cB^c]\}^m = [1 - (a + b - ab)]^m,$$

which can be taken as a reliability measure of the system, assuming constant probabilities of failure. For a design lifetime of $m = 50$ years,

$$\Pr[(A^cB^c)_1(A^cB^c)_2 \cdots (A^cB^c)_{50}] = .9702^{50} = .2203,$$

which means a design reliability of 22 percent. The design risk will be given by the complementary probability, which means that there is a design risk of 78 percent of dam failure within the design lifetime of 50 years.

Example 2.19: Dam Reliability

The risk of failure in the ith year of dam operation is given by the probability that either an overtopping flood or a destructive earthquake or both will occur exactly in that year without any previous occurrence. This is given by

$$\begin{aligned}\Pr[(A^cB^c)_1(A^cB^c)_2\cdots(A^cB^c)_{i-1}(A+B)_i] &= (\Pr[A^cB^c])^{i-1}\Pr[(A+B)_i \mid (A^cB^c)_{i-1}] \\ &= (\Pr[A^cB^c])^{i-1}\Pr[A+B] \\ &= [1-(a+b-ab)]^{i-1}(a+b-ab),\end{aligned}$$

which is simply the design reliability rescaled by the elementary risk of failure. (Details of this geometric distribution, with parameter $(a+b-ab)$, are given in Section 4.1.) For instance, the risk of dam failure during the tenth year of operation will be

$$\Pr[(A^cB^c)_1(A^cB^c)_2\cdots(A^cB^c)_9(AB)_{10}] = .9702^9 \times .0298 = .0227.$$

Figure 2.2.4 gives reliabilities for varying design periods, say, m, and probabilities of first-time failure during the mth year of dam operation.

Example 2.19: Dam Reliability

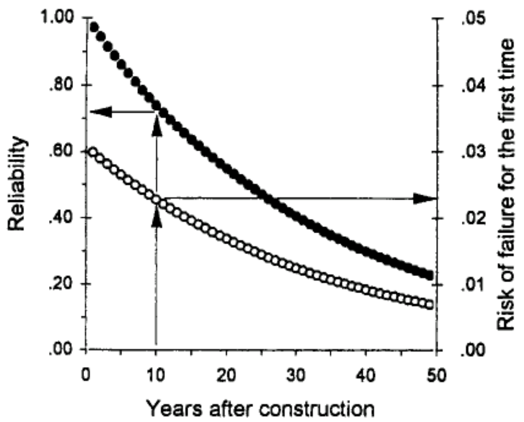


FIGURE 2.2.4 Dam survival probability (m-year design reliability) and probability of failure in the mth year of operation (risk of failure for the first time).

Example 2.23: Water Quality

Example 2.23. Water quality. Consider the concurrent data on DO and BOD recorded at 38 sites on the Blackwater River, England, in Table E.1.3. Owing to similarities in water uses, one can assume that the observations are from the same population. The means of the data are 7.5 mg/L and 3.2 mg/L, respectively. Define the following mutually exclusive and collectively exhaustive events:

$$B_1 = \{DO \le 7.5 \text{ mg/L}, BOD > 3.2 \text{ mg/L}\},$$
$$B_2 = \{DO > 7.5 \text{ mg/L}, BOD > 3.2 \text{ mg/L}\},$$
$$B_3 = \{DO > 7.5 \text{ mg/L}, BOD \le 3.2 \text{ mg/L}\},$$
$$B_4 = \{DO \le 7.5 \text{ mg/L}, BOD \le 3.2 \text{ mg/L}\}.$$

These are given in Fig. 2.2.8. By using relative frequencies,

$$\Pr[B_1] = 17/38 = .45,$$
$$\Pr[B_2] = 0/38 = .00,$$
$$\Pr[B_3] = 19/38 = .50,$$
$$\Pr[B_4] = 2/38 = .05.$$

Example 2.23: Water Quality

The standard deviations are 1.0 and 0.5 mg/L, respectively. Let A be the event defined by concurrent values of DO and BOD within the range (mean − standard deviation) to (mean + standard deviation), that is,

$$A = \{6.5 < DO < 8.5 \text{ mg/L}, 2.7 < BOD < 3.7 \text{ mg/L}\}.$$

The conditional probabilities of event A given that B_i occurs are

$$\Pr[A \mid B_1] = 7/17 = .41,$$
$\Pr[A \mid B_2]$ is undefined because $\Pr[B_2] = 0$,
$$\Pr[A \mid B_3] = 11/19 = .58,$$
$$\Pr[A \mid B_4] = 1/2 = .50.$$

From the theorem of total probability [Eq. (2.2.15)],

$$\Pr[A] = \Pr[A \mid B_1]\Pr[B_1] + \Pr[A \mid B_2]\Pr[B_2] + \Pr[A \mid B_3]\Pr[B_3] + \Pr[A \mid B_4]\Pr[B_4],$$

that is,

$$\Pr[A] = \frac{7}{17}\frac{17}{38} + (\text{undefined})(0) + \frac{11}{19}\frac{19}{38} + \frac{1}{2}\frac{2}{38} = \frac{7}{38} + \frac{11}{38} + \frac{1}{38} = \frac{19}{38} = .50,$$

which means that the monitored values of DO and BOD have a 50 percent chance of lying in the previously defined range.

Example 2.23: Water Quality

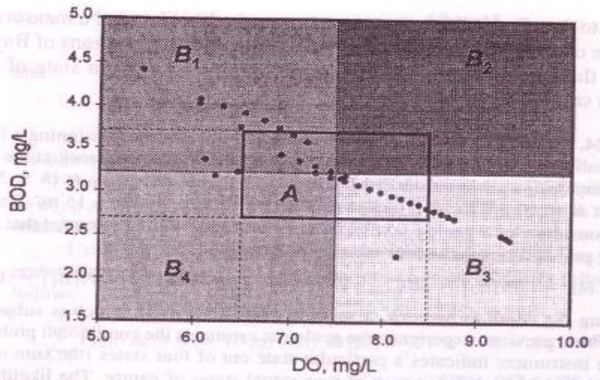


FIGURE 2.2.8
Scatter diagram of water quality data and events in application of Bayes' theorem.

Example 2.23: Water Quality

From Bayes' theorem [Eq. (2.2.16)],

$$\Pr[B_1 \mid A] = \frac{\Pr[A \mid B_1]\ \Pr[B_1]}{\sum_{i=1}^{n} \Pr[A \mid B_i]\ \Pr[B_i]} = \frac{(7/17)(17/38)}{19/38} = \frac{7}{19} = .37$$

which means that, if the monitored values of DO and BOD lie in the previously defined range, there is a 37 percent chance that DO does not exceed its sample mean and BOD does exceed its sample mean. Using Bayes' theorem, one also obtains

$$\Pr[B_2 \mid A] = 0,$$

$$\Pr[B_3 \mid A] = .05,$$

$$\Pr[B_2 \mid A] = .58.$$

Example 2.22: Water Shortage

Example 2.22. Water shortage. The water supply of a city located in the northern Mediterranean area relies on both groundwater and surface water. The collection of surface water is provided by run-of-river draft (directly from rivers) and also reservoir storage in the Apennine mountains surrounding the city; in addition, wells collect groundwater from the coastal aquifer (see Fig. 2.2.7).

The operation of the system is influenced by regional droughts, which occur randomly with an annual frequency of 20 percent. When a regional drought occurs, there is a 40 percent chance that extremely low flows will occur in mountain streams. The drought and low flows can cause a water shortage in the city with a probability of .3 due to lack of available water from either of the sources (event A). When extremely low flows occur in mountain streams without affecting water availability, there is also a 25 percent chance that groundwater in the coastal aquifer will be polluted by brackish water (event B). When extremely low flows in mountain streams do not occur in a dry year, there is also a 10 percent chance that reservoir storage will not be sufficient to meet target release (event C).

Assume that the occurrence of either A, or B, or C yields a water shortage in the city (events). The problem is to estimate the probability that a water shortage occurs in a year. For this purpose, one defines the following events:

Example 2.22: Water Shortage

$S \equiv$ {occurrence of a water shortage in the city},

$D \equiv$ {occurrence of a regional drought}, $\Pr(D) = .2$,

$L \equiv$ {occurrence of extremely low flows in mountain streams}, $\Pr(L \mid D) = .4$,

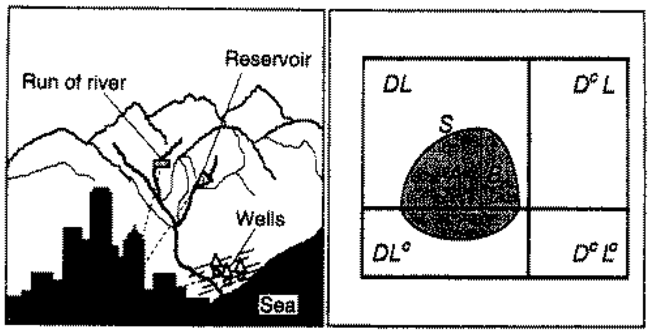
$A \equiv$ {water shortage caused by occurrence of extremely low flows in mountain streams under drought conditions}, $\Pr(A \mid DL) = .3$,

$B \equiv$ {water shortage caused by brackish groundwater when drought and extremely low flows occur in mountains but do not cause water shortage}, $\Pr(S \mid DLA^c) = .25$,

and

$C \equiv$ {water shortage caused by inadequate reservoir storage when drought conditions occur but extremely low flows do not occur in mountain streams}, $\Pr(S \mid DL^c) = .10$.

Example 2.22: Water Shortage



(a) (b)

FIGURE 2.2.7
(a) Layout of the water supply system; (b) Venn diagram showing the events involved in the occurrence of a water shortage (event S, shaded).

Example 2.22: Water Shortage

The events DL, DL^c, D^cL, and D^cL^c, shown in Fig. 2.2.7, are mutually exclusive and collectively exhaustive. From Eq. (2.2.11), then,

$$\Pr[DL] = \Pr[L \mid D]\Pr[D] = .40 \times .20 = .08$$

and

$$\Pr[DL^c] = \Pr[L^c \mid D]\Pr[D] = (1 - .40) \times .20 = .12.$$

Also one can assume that

$$\Pr[D^cL] = 0 \qquad \text{and} \qquad \Pr[L^c \mid D^c] = 1.$$

Hence

$$\Pr[D^cL^c] = \Pr[L^c \mid D^c]\Pr[D^c] = 1 \times (1 - .20) = .80.$$

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Example 2.22: Water Shortage

By using the theorem of total probability [Eq. (2.2.15)] one can estimate the probability of failure as follows:

$$\Pr[S] = \Pr[S \mid DL]\ \Pr[DL] + \Pr[S \mid DL^c]\ \Pr[DL^c] + \Pr[S \mid D^cL]\ \Pr[D^cL] + \Pr[S \mid D^cL^c]\ \Pr[D^cL^c],$$

where

$$\Pr[S \mid DL] = \Pr[S \mid DLA]\ \Pr[A \mid DL] + \Pr[S \mid DLA^c]\ \Pr[A^c \mid DL] = 1 \times .3 + .25 \times (1 - .3) = .475,$$

which can also be obtained from Eq. (2.2.15) by making S conditional to DL throughout. It is assumed that

$$\Pr[S \mid D^cL] = \Pr[S \mid D^cL^c] = 0,$$

because a water-shortage occurs only when a regional drought occurs. Thus,

$$\Pr[S] = .475 \times .08 + .10 \times .12 + 0 \times 0 + 0 \times .80 = .05.$$

That is, there is a 5 percent chance that a water shortage occurs in the city in any year.