Lecture 7

Model Estimation Techniques

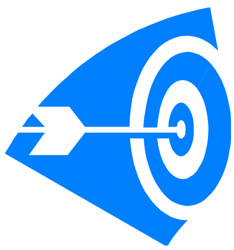
Statistics for
Civil & Environmental Engineers

Precision and Accuracy

- **Precision**
  - the ability of an estimator to provide repeated estimates that are close together

    ((note))

    1. due to random error
    2. measured with the variance of a estimator



- **Accuracy**
  - precision + unbiasedness

    ((note))

    1. measured with the MSE = variance + bias^2

Properties of Estimators: Unbiasedness

- **Definition**
  - **A point estimator $\hat{\theta}$ is an unbiased estimator of the population parameter θ if $E(\hat{\theta}) = \theta$.**
  - **If the estimator is biased, the bias= $E(\hat{\theta}) = \theta$**

Unbiasedness Example

Example 5.1. Mean and variance of the sample mean. It can be shown that the sample mean $\overline{X}$ and the sample variance

$$\hat{S}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

are unbiased estimators of μ and σ^2.

The first result follows immediately by taking expectations of a random sample of size n,

$$\overline{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n),$$

which yields

$$E(\overline{X}) = \frac{1}{n}(nE[X_i]) = \frac{1}{n}(n\mu) = \mu.$$

For the variance [as in Eqs. (1.2.6) and (1.2.7)]

$$E[\hat{S}^2] = \frac{1}{n-1}E\left[\sum_{i=1}^{n}(X_i - \overline{X})^2\right] = \frac{1}{n-1}E\left[\sum_{i=1}^{n}(X_i - \mu)^2 - n(\overline{X} - \mu)^2\right]$$

$$= \frac{1}{n-1}\left\{\sum_{i=1}^{n}E\left[(X_i - \mu)^2\right] - nE\left[(\overline{X} - \mu)^2\right]\right\} = \frac{1}{n-1}\left\{\sum_{i=1}^{n}\sigma^2 - n\,\mathrm{Var}\left[\overline{X}\right]\right\}$$

$$= \frac{1}{n-1}\left(n\sigma^2 - n\frac{\sigma^2}{n}\right) = \sigma^2.$$

Properties of Estimators: Efficiency

- **Definition**
  - An estimator that has minimum mean square error among all possible unbiased estimator is called an ***efficient estimator***. The mean square error of an estimator, which is equivalent to the sum of its variance and the square of its bias, can be used as a relative measure of efficiency when comparing two or more estimator

$$E[(A-\theta)^2] = E[\{(A-E[A])-(\theta-E[A])\}^2]$$
$$= E[(A-E[A])^2] + (\theta-E[A])^2$$
$$= Var[A] + (bias)^2$$

Efficiency Example

Example 5.4. Relative efficiencies of the estimators of the mean of concrete densities. From Tables 1.2.1 and 1.2.2, the mean of the densities of 40 concrete test cubes is 2445 kg/m^3. However, if we had only the first five test cubes, the estimated mean would be 2431 kg/m^3. Both estimators are unbiased as seen in Example 5.1. Hence the relative efficiencies, as given by the ratio of the mse values, are equivalent to the ratio of the variances; that is,

$$\frac{\sigma^2/40}{\sigma^2/5} = \frac{1}{8}.$$

This result merely confirms what we already know; that is, the large-sample estimator for the mean is more efficient than that based on a smaller sample. Also, efficiency is inversely proportional to the sample size n.

Method of Moment Estimation

- **Definition**
  - estimates population parameters using the moments of samples
  - equates the first *m* parameters of the distribution to the first *m* sample moments
  - computationally simple

Method of Moment Estimation Example(1)

Example 3.20. Timber strength. The mean $\bar{x}$ and the standard deviation $\hat{s}$ of the sample data of timber strength are given in Table E.1.1. It was assumed in Example 3.19 that these data are distributed according to the following gamma distribution, with pdf

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)} e^{-\lambda x} x^{r-1}.$$

Also,

$$E[X] = \frac{r}{\lambda} \quad \text{and} \quad \mathrm{Var}[X] = \frac{r}{\lambda^2}$$

Therefore, substituting sample estimates $\bar{x}$ and $\hat{s}^2$ for the mean and variance, respectively, one can obtain the following estimates of the parameters of the gamma pdf:

$$\hat{r} = \frac{\bar{x}^2}{\hat{s}^2} \quad \text{and} \quad \hat{\lambda} = \frac{\bar{x}}{\hat{s}^2}$$

Method of Moment Estimation Example(2)

If the timber strength data sample without the zero value is modeled using this distribution, the statistics of Table 1.2.2 give a mean of 39.09 N/mm^2 and standard deviation of 9.92 N/mm^2, so that we obtain $\hat{r} = 15.5$ and $\hat{\lambda} = 0.40$ N/mm^2. The curve of the theoretical cdf of timber strength is compared with the cumulative relative frequency curve in Fig. 3.2.4.

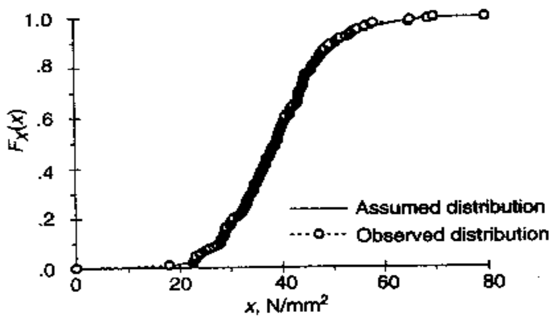


FIGURE 3.2.4 Gamma cdf of modulus of rupture, X, of timber compared with the cumulative relative frequency curve of observed data.

Maximum Likelihood Estimation

- **Definition**
  - **The parameter set $\tilde{\theta}$ for which $L(\tilde{\theta})$ takes a maximum given a sample $x_1, x_2, \ldots, x_n$ is a maximum likelihood estimator set**
- **Likelihood Function**

$$L(\theta) = \prod_{i=1}^{n} f_X(x_i|\theta)$$

- **Log Likelihood Function**

$$\ln L = \sum_{i=1}^{n} f_X(x_i)$$

$$\ln L / \partial\theta_1 = 0, \ \ldots, \ \ln L / \partial\theta_n = 0$$

Maximum Likelihood Estimation Example(1)

Example 3.22. Flood occurrence. Consider the following Bernoulli pmf specified in Example 3.7, where x is a discrete variable and p is a parameter, to model flood occurrences.

$$P_X(x_j) = \Pr[X = x_j] = p^{x_j}(1-p)^{1-x_j} \qquad \text{for } x_j = 0, 1.$$

If there are n trials or outcomes,

$$L(p) = \prod_{j=1}^{n} p^{x_j}(1-p)^{1-x_j}, \qquad \text{where } x_j = 0, 1.$$

We find that $L(p)$ is positive and that the value of p that maximizes it is the same value that maximizes $\ln L(p)$. The latter is more convenient in such cases and also when applied to a pdf with an exponential term. Thus,

$$\ln L(p) = \sum_{j=1}^{n} x_j \ln p + \left(n - \sum_{j=1}^{n} x_j\right) \ln(1-p)$$

Maximum Likelihood Estimation Example(2)

and

$$\frac{d \ln L(p)}{dp}s = \frac{\sum_{j=1}^{n} x_j}{p} - \frac{n - \sum_{j=1}^{n} x_j}{(1-p)}.$$

Hence by equating the foregoing to zero, the estimate of the parameter is obtained as

$$\hat{p} = \frac{\sum_{j=1}^{n} x_j}{n}$$

MOM & MLE Example: Uniform Distribution

Example 6.1. Use the method of moments to estimate the parameters of the uniform distribution based on the following sample: 1,4,3,4,5,6,7,6,9,5. What are the maximum likelihood estimators for this sample?

By method of moments

By maximum likelihood