Lecture 11

Monte Carlo Simulation

Statistics for
Civil & Environmental Engineers

Definition

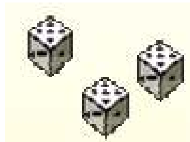
- Simulation
  - An imitation of some real thing, state of affairs, or process (Wikipedia)
  - The process of replicating the real world based on a set assumptions and conceived models of reality (Text)
- Monte Carlo Simulation
  - Simulation when dealing with random variables
  - The procedure is usually repeated to generate a different set of values of the variables in accordance with a specified probability distribution

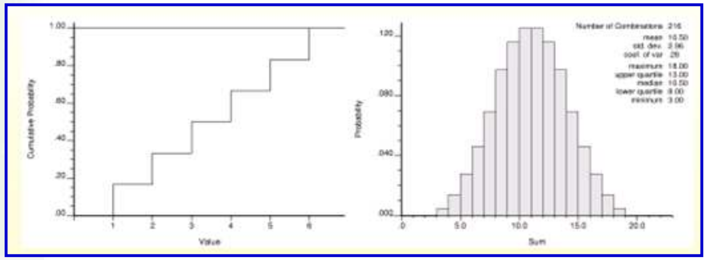
Why Monte Carlo Simulation?

- Provides a Model Uncertainty Before and After a Transfer Function
  - To reproduce a model of variability
  - To assess the uncertainty of the output after a transfer function

A Simple Example

- Experiment: Throw 3 dices 100 times and compute the sum of the values obtained
- Transfer Function= Σx_i
- 3 Independent Variables

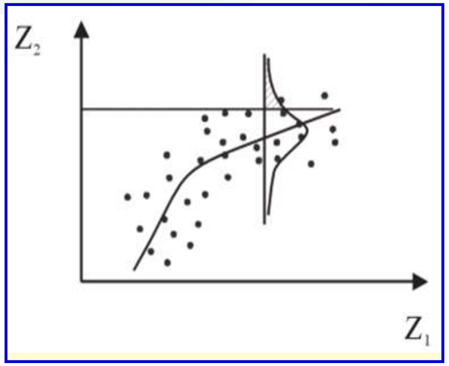




The Real World is Much More Complicated!

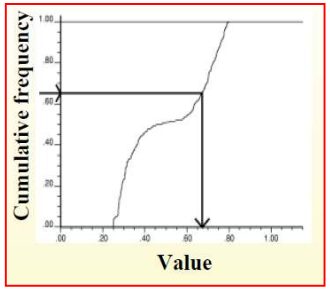
- **Continuous Distribution**
- **Large Number of Variables**
- **Nonlinear Average**
- **Average above a threshold**

→ Impossible to deduce the total number of combination



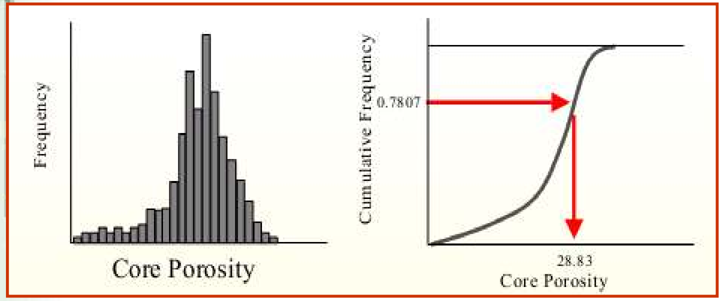
MC Procedure

- **Unknown to be Known: The output distribution F_Y**
- **Given: The input distribution F_X and the transfer function (or model) *TR***
- **Quantile Transformation**
  - **Equation: $q(p) = F^{-1}(p) = x_p$**
  - **Draw a uniform random number in (0, 1) → "p" (a probability)**
  - **Read the quantile function q(p)**
  - **to get the simulation value x_p**



- **Input x_p to the transfer function to obtain y_p**
- **Repeat the above procedure to generate as many of y_p's as you want**

MCS Example



- **Monte Carlo Simulation / Stochastic Simulation / Random Drawing proceed by reading quantiles from a cumulative distribution**

MCS Example

- **The procedure**
  - **Generate a random number between 0 and 1 (calculator, table, program, …)**
  - **Read the quantile associated to that random number**
- **For example**

Random Number	Simulated Number	
0.7807	28.83	
0.1562		
0.6587		
0.8934		

…

MCS Text Examples

Example 8.1. Computation of π. Consider a horizontal floor on which parallel lines are drawn at equal distances a. A needle of length $b < a$ is dropped at random on the floor. The problem is to find the probability that the needle will intersect a line. Let X be a random variable that gives the distance of the midpoint of the needle to the nearest line, with $0 < x \le a/2$, and let Y be the variable that gives the acute angle between the needle (or its extension) and the line.

The outcomes of X and Y are bounded as $0 < x \le a/2$ and $0 < y \le \pi/2$. Since $\Pr[x < X \le x + dx] = (2/a)\,dx$ and $\Pr[y < Y \le y + dy] = (2/\pi)\,dy$, one obtains $f_X(x) = 2/a$ and $f_Y(y) = 2/\pi$. Noting that X and Y are independent, the joint pdf is the product of the marginals; that is, $f_{X,Y}(x, y) = 4/(a\pi)$. From Figure 8.1.1*a*, it is seen that the needle actually crosses a line when $X \le (b/2)\sin Y$.

MCS Text Examples

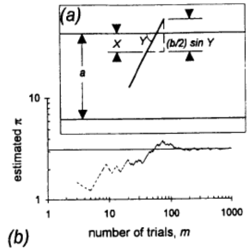


FIGURE 8.1.1
Buffon's needle problem: (*a*) sketch of the experiment and (*b*) results of simulation.

The probability of this event is given by

$$p = \frac{4}{a\pi}\int_0^{\pi/2}\int_0^{(b/2)\sin y} dx\,dy = \frac{2b}{a\pi}.$$

When this expression is equated to the frequency of hits (or crossings) observed in actual (physical) experiments, accurate values of π can be obtained. First, we specify a and $b < a$ and assume an appropriate value of π. Then m independent pairs of X and Y that follow the foregoing uniform distributions are generated numerically. Second, p is estimated as the ratio between the random number N of those pairs (x, y) that satisfy $x \le (b/2)\sin y$ and the number of trials m. Finally, π is computed as $2b/(ap)$ or $2bm/(aN)$. The accuracy increases with the number m of trials, as shown in Fig. 8.1.1*b*. Such experiments are called *urn extractions* and are used for generation.

MCS Text Examples

Example 8.2. Monte Carlo integration. The definite integral of a function $g(u) > 0$ from a to b—that is,

$$I = \int_a^b g(u)\,du,$$

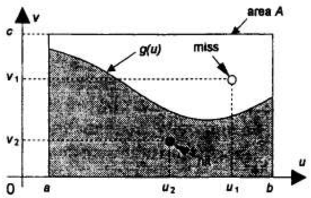


FIGURE 8.1.2
Monte Carlo method of integration. Random points are chosen within the area A. The integral of the function $g(\cdot)$ is estimated as the area of A rescaled by the fraction of random points falling below the curve g.

is the area bounded by the curve $g(u)$ within the interval $[a, b]$, as shown in Figure 8.1.2. Consider a rectangle embedding this area, and suppose that one were to throw darts at the rectangle of area $A = c(b - a)$, where $c \ge g(u)$ for $a \le u \le b$. Let N denote the (large) number of darts thrown uniformly against this target. If N is the number of darts falling below the curve $g(u)$, the integral may be estimated as the area A multiplied by the fraction N/n of random points that fall below $g(u)$; that is,

$$I = \int_a^b g(u)\,du \approx c(b - a)\frac{N}{n},$$

MCS Text Examples

where $p = N/n$ is the probability of a hit. Instead of throwing darts, one might generate n pairs u and v of two independent uniformly distributed variates U and V, with $a \le u \le b$ and $0 \le v \le c$, respectively, and count the number N of pairs with $v \le g(u)$. For an increasing number of generated pairs, one expects that the value of the estimated integral approaches its theoretical value. This method can also be used in the multidimensional case by picking n random points, say, $x_1, \ldots, x_n$, uniformly distributed in a multidimensional volume Ω. Then, the basic theorem of Monte Carlo integration estimates the integral of a function g over Ω as

$$I = \int_\Omega g\,d\Omega = \Omega\langle g\rangle \pm Z,$$

where Z is a random variable representing the error in the estimated integral, with zero mean and standard deviation

$$\sigma_Z = \sqrt{\frac{\langle g^2\rangle - \langle g\rangle^2}{n}}.$$

Here, the angle brackets denote taking the arithmetic mean over the n sample points; that is,

$$\langle g\rangle = \frac{1}{n}\sum_{i=1}^{n} g(x_i) \qquad \text{and} \qquad \langle g^2\rangle = \frac{1}{n}\sum_{i=1}^{n}[g(x_i)]^2.$$

There is no guarantee that the error is distributed as normal, so the error term should be taken only as a rough indication of probable error. Note that the implementation of this method requires the generation of uniform random numbers in a specified domain, say, the rectangle A or the hypervolume Ω.

MCS Text Examples

Example 8.4. Seepage under a dike wall. Figure 8.1.3 shows a river valley resting on a homogeneous alluvial layer of porous material bounded by impervious rock. Supposing the level of the groundwater table BL is constant at h_g, the total river head is constant at h_r and there is a head loss of $h_l = h_r - h_g$. Also, the boundaries $ABCDEFGHI$ and LMN are impervious and there are no sinks in the flow. One superimposes a square grid, or lattice, on the aquifer to study the movement of a particle of water at an interior point under a two-dimensional random walk; in other words, this particle is made to move horizontally and vertically from one point to another through the lattice.[2] Whenever the particle meets one of the aforementioned impervious boundaries on its random walk in the limited zone, it is reflected back. On the other hand, when it reaches the groundwater table or the riverbed, the path is terminated and values of h_g and h_r, respectively, are given to it. After n such random walks from an interior point S (where n is large, say, greater than 400), suppose that n_g paths are assigned h_g and that n_r paths are assigned h_r, where $n = n_g + n_r$. We estimate the head at point S:

$$h_S = \frac{n_g h_g + n_r h_r}{n_g + n_r}.$$

Consider an interior node 0 and a particle at any of its neighboring grid points, 1, 2, 3, or 4 in Figure 8.1.3. The probability of a particle arriving at point 0 is obtained by weighting the probabilities that it arrives at the four neighboring points. For the random walk, we assign equal weights of 1/4, as shown. If the point 0 is on a boundary, such as point Q, then the weighting is adjusted to 1/4, 1/2, and 1/4, as shown. A Monte Carlo simulation of the random walks from point 0 in Fig. 8.1.3 produced 423 paths that terminated at the groundwater table boundary and 577 paths that terminated at the riverbed boundary. The groundwater table boundary represents a total head of $h_g = 0$ m, and the total head at the riverbed boundary is $h_r = 3$ m. Hence, the pressure

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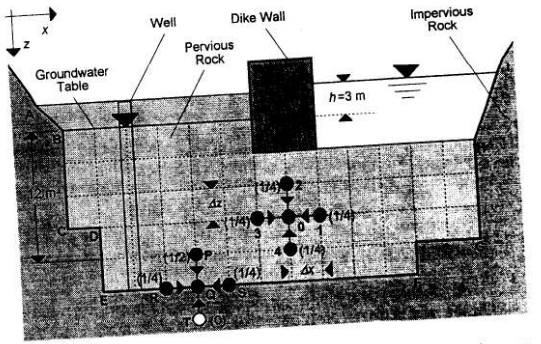

FIGURE 8.1.3
Groundwater head under a dike wall in a river valley transect. The values in parentheses are the next-step probabilities in the traced direction.

head at 0 is

$$h_0 = \frac{n_g h_g + n_r h_r}{n} = \frac{423 \times 0 + 577 \times 3}{423 + 577} = 1.73 \text{ m}.$$

The elevation head at point 0, relative to the groundwater table, is 9 m. Hence, the pressure in the water at point 0 is $9.81 \times (9 + 1.73) \approx 105$ kPa.

Generation of Random Numbers

❖ **Uniform Distribution**

$$k_{i+1} = (ak_i + c)\ (\text{modulo } m)$$

$$k_{i+1} = (ak_i + c) - m\eta_i$$

$$u_{i+1} = k_{i+1}/m = ak_i/m + c/m - \text{Int}[(ak_i + c)/m]$$

where, $u_i \sim U(0,1)$, a= multiplier, c= increment, m= modulo

$$\eta_i = \text{Int}[(ak_i + c)/m]$$

Generation of Random Number

❖ **Continuous Distribution**

TABLE 8.2.2
Summary of independent random numbers x generated from selected distributions; $u_1, u_2, \ldots, u_i, \ldots$ denote uniform (0, 1) random numbers

Distribution	Parameters	x
Standard normal		$\sqrt{2\ln u_1}\sin(2\pi u_2)$ and $\sqrt{-2\ln u_1}\cos(2\pi u_2)$
Standard beta	α, β	$u_1^{1/\alpha} \Big/ \left[u_1^{1/\alpha} + u_1^{1/\beta}\right]$, provided that $u_1^{1/\alpha} + u_2^{1/\beta} \le 1$*
Standard gamma	r	for $r > 1$: $\ln\left(\prod_{i=1}^{r} u_i\right) = -\sum_{i=1}^{r} \ln u_i$, for integer r; and, in general, $-\sum_{i=4}^{r'+3} \ln u_i + (-\ln u_3)u_1^{r} \Big/ \left[u_2^{1/r} + u_3^{1/(1-r)}\right]$, with $r' = \text{Int}(r)$
		for $r < 1$: if $u_1(e + r)/e \le 1$ and $u_2 \le e^{-x}$, then $x = [u_1(e + r)/e]^{1/r}$; if $u_2 \le x^{r-1}$, then $x = -\ln[(e + r)(1 - u_1)/(er)]$; otherwise reject and repeat until accept†
Binomial	n, p	$\sum_{i=1}^{n} k_i$, with $k_i = 1$, if $u_i < p$; and $k_i = 1$, if $u_i \ge p$
Poisson	ν	x such that $\sum_{i=1}^{x} -\nu^{-1}\ln(u_i) \le 1$, and $\sum_{i=1}^{x+1} -\nu^{-1}\ln(u_i) > 1$
Geometric	p	$\ln u/\ln(1 - p) - 1$, rounded to the next integer
Negative binomial	ξ, p	$\sum_{i=1}^{m} h_i$, with $h_i = 1$, if $u_i \ge p$; $h_i = 1$, if $u_i < p$; and m such that $\xi = \sum_{i=1}^{m}(1 - h_i)$

Text Example

Example 8.8. Linear congruential algorithm. Suppose we assume low values for the constants in Eq. (8.2.1): $a = 5$, $c = 1$, and $m = 8$. Let $k_0 = 1$ be the seed for generating a sequence of random integers k_i, $i = 1, 2, 3, \ldots$ For $i = 1$, one has

$$k_1 = ak_0 + c - m\,\mathrm{Int}[(ak_0 + c)/m] = 5 \times 1 + 1 - 8 \times \mathrm{Int}[(5 \times 1 + 1)/8]$$
$$= 5 + 1 - 8 \times \mathrm{Int}(0.75) = 5 + 1 - 8 \times 0 = 6,$$

and, from Eq. (8.2.3),

$$u_1 = k_1/m = 6/8 = 0.75.$$

The second iteration yields

$$k_2 = ak_1 + c - m\,\mathrm{Int}[(ak_1 + c)/m] = 5 \times 6 + 1 - 8 \times \mathrm{Int}[(5 \times 6 + 1)/8]$$
$$= 30 + 1 - 8 \times \mathrm{Int}(3.875) = 30 + 1 - 8 \times 3 = 7,$$
$$u_2 = k_2/m = 7/8 = 0.875.$$

The subsequent iterations yield the following sequence

0.5, 0.625, 0.25, 0.375, 0, 0.125, <u>0.75, 0.875, 0.5, 0.625, 0.25, 0.375, 0, 0.125,</u> 0.75, 0.875, 0.5, 0.625, 0.25, 0.375, 0, 0.125, <u>0.75, 0.875, 0.5, 0.625, 0.25, 0.375, 0, 0.125,</u> ...,

Text Example

which is seen to be cyclic with a period of 8, because the underlined sequence of 8 values is repeated indefinitely. This is clearly shown by plotting u_{i+1} against u_i in Fig. 8.2.1.

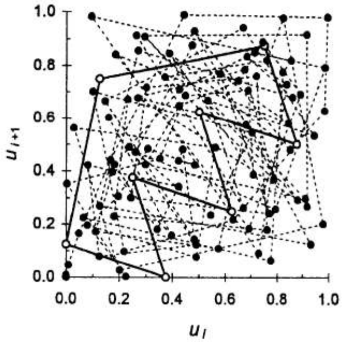

FIGURE 8.2.1
Trajectory of 100 sequentially generated standard uniform random numbers with $a = 5$, $c = 1$, and $m = 8$ (solid line), and with $a = 2^7 + 1$, $c = 1$, and $m = 2^{35}$ (dotted line).

Also shown in Fig. 8.2.1 are results from the generator $a = 2^7 + 1$, $c = 1$, and $m = 2^{35}$, which yields a much larger period of cyclicity. This choice gives satisfactory results for binary computers; and $a = 101$, $c = 1$, and $m = 2^b$ for a decimal computer with a word length b.

Sample Size and Accuracy of MC Experiment

In Monte Carlo integration it is seen that choosing n points uniformly and randomly distributed in a multidimensional space leads to an error term that decreases as $n^{-1/2}$, because each new point sampled adds linearly to an accumulated sum of squares that will become the variance, and the estimated error comes from the square root of the variance. In designing a Monte Carlo experiment, one must determine how many simulations are required to assess the system behavior. When simulation is used to evaluate the probability p that some event occurs, such as unsatisfactory system performance, one must search for the sample size required to obtain a specified accuracy of the estimated p. If N denotes the observed number of occurrences of the event in a sample of size n, the obvious estimator of p is the proportion N/n. When sequential simulations are independent of each other, N is a binomial variate with parameters n and p. From Eq. (5.3.7), the standard error of the estimated proportion is

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}, \tag{8.1.2}$$

and for large n (say, $n > 30$ and $np > 5$), the sampling distribution is very nearly normal with mean np and variance $np(1-p)$. In practice, the sample estimate $\hat{p}$ is

Sample Size and Accuracy of MC Experiment

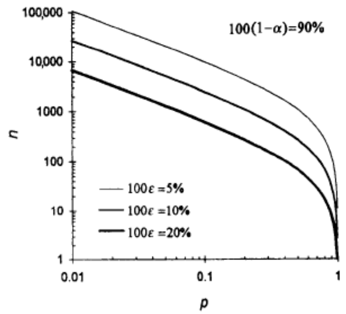

FIGURE 8.1.5
Simulation sample size n required to estimate the probability p of the design event within $100\varepsilon\%$ of its true value with $100(1-\alpha)\%$ confidence.

substituted for p, and the $100(1-\alpha)$ percent two-sided confidence limits on the true value p given n and an observed value of the estimator $\hat{p}$ are determined as

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \quad \text{and} \quad \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \tag{8.1.3}$$

(where $z_{\alpha/2}$ denotes a standard normal variate that is exceeded with probability $\alpha/2$).

Sample Size and Accuracy of MC Experiment

The necessary sample size n to ensure that the $100(1-\alpha)$ percent confidence limits are within 100ε percent of the true value of p, where $0 \le \varepsilon \le 1$; that is,

$$z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}} \le \varepsilon p,$$

is given by

$$n \ge \frac{z_{\alpha/2}^2(1-p)}{\varepsilon^2 p}. \qquad (8.1.4)$$

Since n is a function of p, which is unknown before the experiment is performed, one must estimate the value of p before the experiment. Figure 8.1.5 shows the increase of n for decreasing p and different values of acceptable tolerance ε.

Sample Size and Accuracy of MC Experiment

Number of Samples to Generate. Enough samples should be generated so that the required output statistics are estimated accurately. To a large extent, the number of samples depends on what statistic of the Monte Carlo simulation output is of interest for the problem at hand. For example, to determine the mean of the output O with a given accuracy, one can use the normal approximation to establish the $1-\alpha$ confidence limits on the population mean $\mu(O)$, from which one can write $P[-u_{1-\alpha/2}\ \sigma(O)/\sqrt{m} < \bar{O} - \mu(O) < u_{1-\alpha/2}\ \sigma(O)/\sqrt{m}] = 1-\alpha$ where $\bar{O}$ = sample mean of the output, $\sigma(O)$ = population variance, $u_{1-\alpha/2} = 1-\alpha/2$ quantile of the normal distribution with mean zero and variance one, $1-\alpha$ = confidence level, and m = sample size. Thus, if $\bar{O}$ must be within $0.1\ \sigma(O)$ of $\mu(O)$ with a probability $1-\alpha = 0.95$, then $u_{0.975} = 1.96$ and the sample size required is given by $1.96\ \sigma(O)/\sqrt{m} = 0.1\sigma(O)$, which gives $m = 384$. Likewise, for an accuracy of $0.2\sigma(O)$, $m = 96$. A better approximation may be generally obtained by using instead the confidence limits based on the t distribution. In this case, the number of samples is obtained, for instance, by solving $t_{1-\alpha/2,m-1} = 0.1\sqrt{m}$ for m [for $0.1\sigma(O)$ accuracy], where $t_{1-\alpha/2,m-1}$ is the $1-\alpha/2$ quantile of the t distribution with $m-1$ degrees of freedom.

Sample Size and Accuracy of MC Experiment

Likewise, to determine the standard deviation of the output O with a given accuracy, one can establish the confidence limits on the population variance $\sigma^2(O)$. Thus, one can write

$$P\left(\frac{\sqrt{m-1}}{\sqrt{\chi^2_{1-\alpha/2,m-1}}} < \frac{\sigma(O)}{S(O)} < \frac{\sqrt{m-1}}{\sqrt{\chi^2_{\alpha/2,m-1}}}\right) = 1-\alpha$$

in which $S(O)$ = sample standard deviation and $\chi^2_{\beta,m-1}$ = β-quantile of the chi-square distribution with $m-1$ degrees of freedom. For example, for $m = 384$ and $1-\alpha = 0.95$, $P[0.934 < \sigma(O)/S(O) < 1.077] = 0.95$, which means that with a sample size of 384 one can determine the sample standard deviation of the output such that its ratio with the population standard deviation is within about 15 percent.

Example 1-1

Example 8.16. Pier scour. Pier foundations of bridges over water can be undermined by local scour. The best-fit scour model for bridge piers proposed by Johnson (1992) gives the scour depth X measured from the average channel bed to the bottom of the scour hole as

$$X = 2.02Y(b/Y)^{0.98}F_r^{0.21}W^{-0.24},$$

where Y is the depth of flow just upstream of the pier, F_r is the upstream Froude number ($F_r = V/(gY)^{1/2}$, V and g denote the approach flow velocity and acceleration due to gravity, respectively), W is sediment gradation (equal to $d_{84\%}/d_{50\%}$, the ratio between the 84% quantile to the median sediment diameter), and b is the pier width. All these quantities are measured in metric units. Using the Manning formula to compute the velocity for a wide rectangular channel cross section,

$$V = (1/n)S^{1/2}Y^{2/3},$$

where n is the roughness coefficient and S is the slope. Hence, the Froude number is

$$F_r = V/(gY)^{1/2} = S^{1/2}Y^{1/6}n^{-1}g^{-1/2},$$

thus,

$$X = 2.02Y(b/Y)^{0.98}(S^{1/2}Y^{1/6}n^{-1}g^{-1/2})^{0.21}W^{-0.24},$$

which, after substituting 9.81 m/s^2 for g, can be written as

$$X = 1.59b^{0.980}Y^{0.055}S^{0.105}n^{-0.210}W^{-0.240}.$$

Example 1-2

The estimation of Y, S, n, and W is affected by uncertainties. We propose to model all these quantities as random variables. One can thus determine the probability distribution of X by simulation if the probability distributions of Y, S, n, and W are known. For a pier width of 2.5 m, suppose that sediment gradation $W \sim \text{lognormal}(4, 1.6^2)$, the slope $S \sim N(0.002, 0.0004^2)$, the depth $Y \sim N(4.75 \text{ m}, 1.2^2 \text{ m}^2)$, and the roughness coefficient $n \sim \text{uniform}(0.02, 0.04)$. Also, one can reasonably assume that Y, S, n, and W are independent of each other. To perform each simulation, one will generate a standard uniform random number, u_i, and three independent standard normal numbers, z_{1i}, z_{2i}, and z_{3i}. The ith outcome of the roughness coefficient n is found by rescaling u_i as

$$n_i = 0.02 + (0.04 - 0.02)u_{1i},$$

and those of W, S, and Y are computed as

$$w_i = \exp(1.312 + 0.385 z_{1i}),$$

$$s_i = 0.002 + 0.0004 z_{2i}, \text{ and}$$

$$y_i = 4.75 + 1.2 z_{3i}.$$

Example 1-3

Figure 8.3.1a shows the sampling cdf $F_X(x)$ of the scour depth resulting from the first 10, 100, and 1000 simulation cycles. It is seen that the size-10 sample provides a rough approximation to the size-1000 sampling cdf, which is better approximated by the sampling cdf's obtained from the size-100 sample. The estimated means and standard deviations of the specified variates are shown in Figs. 8.3.1b and c for an increasing number of simulation cycles. Note that the sampling means and standard deviations of Y, S, n, and W estimated from 1000 simulation cycles practically overlap with those used as inputs to the simulation procedure.

The 1000-cycle simulated mean and standard deviation of X are 3.39 m and 0.36 m, respectively. Note that the estimated mean is very close to the nominal value of 3.32 m determined by substituting the mean values for the corresponding variates in the pier scour model. These results can also be compared with the approximated mean and standard deviation, which are computed by using Taylor's series expansion about the means of independent variates (see Section 3.4). The first and second partial derivatives of X with respect to each independent variate are as follows:

Example 1-4

$$(\partial X/\partial y)_\mu = 1.59 \times 0.0551 b^{0.980} \mu_Y^{-0.945} \mu_S^{0.105} \mu_n^{-0.210} \mu_W^{-0.240} = 0.0325,$$

$$(\partial X/\partial s)_\mu = 1.59 \times 0.1051 b^{0.980} \mu_Y^{0.055} \mu_S^{-0.895} \mu_n^{-0.210} \mu_W^{-0.240} = 0.0006,$$

$$(\partial X/\partial n)_\mu = -1.59 \times 0.2100 b^{0.980} \mu_Y^{0.055} \mu_S^{0.105} \mu_n^{-1.210} \mu_W^{-0.240} = -23.21,$$

$$(\partial X/\partial w)_\mu = -1.59 \times 0.2400 b^{0.980} \mu_Y^{0.055} \mu_S^{0.105} \mu_n^{-0.210} \mu_W^{-1.240} = -0.1989,$$

$$(\partial^2 X/\partial y^2)_\mu = -1.59 \times 0.0520 b^{0.980} \mu_Y^{-1.945} \mu_S^{0.105} \mu_n^{-0.210} \mu_W^{-0.240} = -0.0065,$$

$$(\partial^2 X/\partial s^2)_\mu = -1.59 \times 0.0940 b^{0.980} \mu_Y^{0.055} \mu_S^{-1.895} \mu_n^{-0.210} \mu_W^{-0.240} = -17863.8,$$

$$(\partial^2 X/\partial n^2)_\mu = 1.59 \times 0.2542 b^{0.980} \mu_Y^{0.055} \mu_S^{0.105} \mu_n^{-2.210} \mu_W^{-0.240} = 936.4,$$

$$(\partial^2 X/\partial w^2)_\mu = 1.59 \times 0.2975 b^{0.980} \mu_Y^{0.055} \mu_S^{0.105} \mu_n^{-0.210} \mu_W^{-2.240} = 0.0616,$$

where μ_Y, μ_S, μ_n, and μ_W denote the means of Y, S, n, and W. From Eq. (3.4.36),

$$E[X] \approx 3.32 + 0.5(-0.0065 \times 1.2^2 - 17863.8 \times 0.0004^2 + 936.4 \times 0.0058^2 + 0.0616 \times 1.6^2) = 3.40 \text{ m},$$

and, from Eq. (3.4.37),

$$\text{Var}[X] \approx 0.0325^2 \times 1.2^2 + 0.0006^2 \times 0.0004^2 + (-23.21)^2 \times 0.0058^2 + (-0.1989)^2 \times 1.6^2 = 0.1208 \text{ m}^2,$$

which yields an approximated standard deviation of 0.35 m. These approximations provide accurate estimates of the mean and standard deviation of scour depth as determined from simulation.