Lecture 12

Risk and Reliability

: Measures

Statistics for
Civil & Environmental Engineers

Risk and Reliability

- **Definition**
  - The ***risk*** that a system is incapable of meeting the demand is defined as the probability of failure p_f over the specified system lifetime under specified operating conditions. System ***reliability***, denoted by r, is the (complementary) probability of nonfailure,

$$r = 1 - p_f$$

- **Capacity (X) vs. Demand (Y)**
  - Strength vs. Load, or Resistance vs. Force
  - e.g.

    Landing capacity vs. the flight arrival rate of an airport

    spillway capacity vs. flood discharge

- ***Capacity*** **and** ***Demand*** **are both** ***Uncertain*****!**

Factor of Safety

- **Definition**
  - The ***safety factor*** of a system, treated as a random variable and defined as $Z = X/Y$, is the ratio between capacity X and demand Y for the system

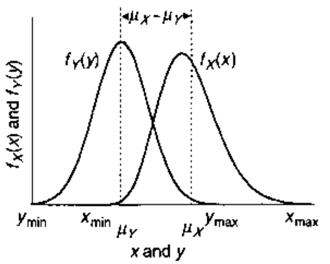

FIGURE 9.1.2
Probability density functions of capacity X and demand Y.

Factor of Safety Example(1-1)

Example 9.1. Structural safety factor for independent lognormally distributed load and strength. Consider a structure whose load-carrying capacity, or strength X, and load Y are independent lognormal variates, with means and standard deviations μ_X, μ_Y and σ_X, σ_Y, respectively. In this case, the safety factor, $Z = X/Y$, is also a lognormal variate. As shown in Eqs. (4.2.28),

$$\mu_{\ln(Z)} = \mu_{\ln(X)} - \mu_{\ln(Y)} = \ln \mu_X - \tfrac{1}{2}\ln(1 + V_X^2) - \ln \mu_Y + \tfrac{1}{2}\ln(1 + V_Y^2),$$

where $V_X = \sigma_X/\mu_X$ and $V_Y = \sigma_Y/\mu_Y$ are the coefficients of variation of X and Y, respectively, and

$$\sigma_{\ln(Z)}^2 = \sigma_{\ln(X)}^2 + \sigma_{\ln(Y)}^2 = \ln(1 + V_X^2) + \ln(1 + V_Y^2).$$

In terms of the medians, m_X and m_Y, it follows from Eq. (4.2.28*d*) that

$$\mu_{\ln(Z)} = \ln(m_X) - \ln(m_Y) = \ln(m_X/m_Y),$$

where the ratio (m_X/m_Y) represents the median safety factor. Since $\ln(Z)$ is normally distributed with mean $\mu_{\ln(Z)}$ and standard deviation $\sigma_{\ln(Z)}$, the random variable $[\ln(Z) - \mu_{\ln(Z)}]/\sigma_{\ln(Z)}$ is a standard normal variate. Therefore, the probability of failure is found using Eq. (9.1.2) as

$$p_f = F_Z(1) = \Phi\left(\frac{\ln 1 - \mu_{\ln(z)}}{\sigma_{\ln(z)}}\right) = \Phi\left(-\frac{\mu_{\ln(z)}}{\sigma_{\ln(z)}}\right)$$

$$= 1 - \Phi\left(\frac{\ln(m_X/m_Y)}{\sqrt{\ln(1 + V_X^2) + \ln(1 + V_Y^2)}}\right),$$

where $\Phi(\cdot)$ denotes the cdf of the standard normal distribution. Accordingly, the reliability of the structure, $r = \Pr[Z > 1] = 1 - F_Z(1) = 1 - p_f$, is

Factor of Safety Example(1-2)

Thus, if X and Y are independent and lognormally distributed, the reliability is a function of the median safety factor and the standard deviation $\sigma_{\ln(X/Y)}$.

Consider, for example, a rigid timber beam (see Fig. 2.1.3) with an estimated average strength of 39.1 N/mm² and coefficient of variation of 25 percent. If the beam is designed to carry a load of 24.0 N/mm², with a coefficient of variation of 15 percent, one can compute the failure probability as follows. Since the means and coefficients of variation of strength X and load Y are $\mu_X = 39.1$ N/mm², $V_X = .25$, $\mu_Y = 24.0$ N/mm², and $V_Y = .15$, respectively, assuming X and Y are independent and lognormally distributed,

$$\sigma_{\ln(Z)} = [\ln(1 + .25^2) + \ln(1 + .15^2)]^{1/2} = .288,$$

$$\mu_{\ln(Z)} = \ln(39.1) - \tfrac{1}{2}\ln(1 + .25^2) - \ln(24.0) + \tfrac{1}{2}\ln(1 + .15^2) = 0.469.$$

The probability of failure is thus

$$p_f = F_Z(1) = \Phi(-.469/.288) = \Phi(-1.628) = .052,$$

which indicates that the beam has a reliability of 94.8 percent. The pdf's of X and Y are shown in Fig. 9.1.4a, and the corresponding cdf's in Fig. 9.1.4b. The pdf and cdf of the safety factor Z are shown in Fig. 9.1.5.

Factor of Safety Example(1-3)

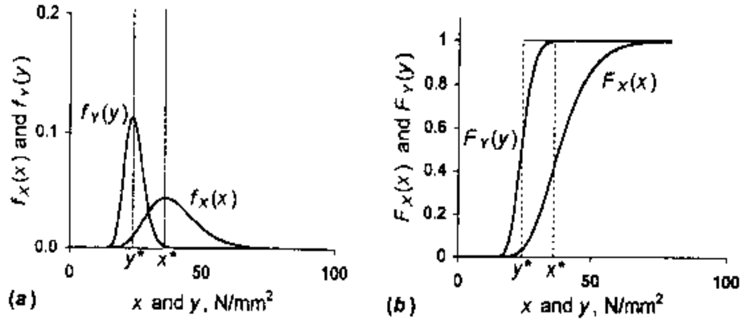

FIGURE 9.1.4
Timber strength and load illustration: pdf's (a) and cdf's (b) for independent lognormally distributed X and Y.

Factor of Safety Example(1-4)

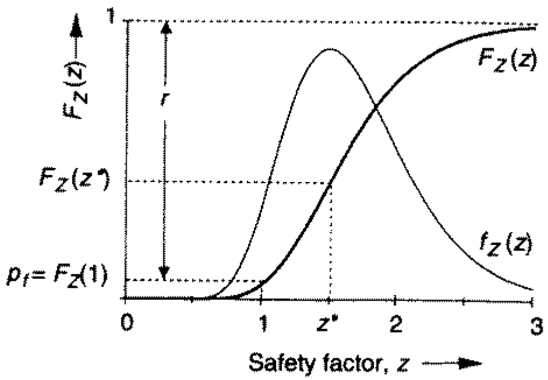

FIGURE 9.1.5
Timber strength and load illustration; pdf and cdf of safety factor $Z = X/Y$ for independent lognormally distributed X and Y.

Factor of Safety

- **Central Safety Factor**
  - ζ =

  where μ_X: expected capacity, μ_Y: demand

- **Nominal Safety Factor**
  - **Nominal value of capacity: $x^* = \mu_X - h_X\sigma_X$**
  - **Nominal value for the demand: $y^* = \mu_Y + h_Y\sigma_Y$**
  - **Safety factor**

  z^* =

  where h_X and h_Y are sigma units of their respective functions

Factor of Safety Example(2-1)

Example 9.2. Central safety factor for a pumping station. A pumping station was designed using a safety factor ζ^* of 1.8, or 9/5. An engineer has the task of assessing the reliability of the system without any knowledge of possible fluctuations of capacity and demand. Therefore, the coefficients of variation of capacity and demand are assumed to be equal, $(V_X = V_Y = V)$, as are the sigma bounds $(h_X = h_Y = h)$. From Eq. (9.1.6),

$$\zeta^* = \frac{\mu_X - h_X V_X \mu_X}{\mu_Y + h_Y V_Y \mu_Y} = \frac{\mu_X}{\mu_Y}\frac{1 - hV}{1 + hV} = \zeta\frac{1 - hV}{1 + hV}.$$

The engineer further assumes that the possible range of V is $.1 \le V \le .5$, and $0 \le h \le 1$, so that the possible range of hV is $0 \le hV \le 0.5$. Since no other information is available regarding the moments of hV, the principle of maximum entropy suggests that hV can be modeled as a uniformly distributed variate with $E[hV] = 1/4$, which yields $\zeta/\zeta^* = 5/3$. Therefore, to improve system reliability in order to achieve a safety factor of ζ, the engineer must increase the nominal capacity x^* of the pumping station from $(9/5)y^*$ to $(5/3)(9/5)y^* = 3y^*$.

Factor of Safety Example(2-2)

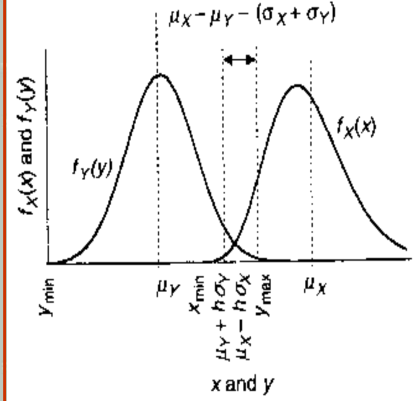

FIGURE 9.1.6
Sigma bounds of capacity X and demand Y.

Safety of Margin

- **Definition**
  - **The *safety margin* of a system is the random difference $S = X - Y$ between capacity X and demand Y of the system**

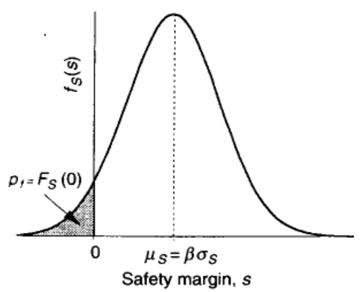

FIGURE 9.1.7
pdf of safety margin S.

Safety of Margin Example(1-1)

Example 9.3. Structural margin of safety for independent normally distributed load and strength. Consider a structure whose load-carrying capacity or strength X and load Y are independent normal variates, with means and standard deviations μ_X, μ_Y and σ_X, σ_Y respectively. In this case, the safety margin, $S = X - Y$, is shown in Example 3.60 to be also a normal variate with

$$\mu_S = \mu_X - \mu_Y, \qquad \sigma_S^2 = \sigma_X^2 + \sigma_Y^2.$$

Since S is normally distributed with mean μ_S and standard deviation σ_S, the random variable $(S - \mu_S)/\sigma_S$ is a standard normal variate, and the reliability of the structure is, from Eq. (9.1.9),

$$r = 1 - F_S(0) = 1 - \Phi\left(\frac{0 - \mu_S}{\sigma_S}\right) = 1 - \left[1 - \Phi\left(\frac{\mu_S}{\sigma_S}\right)\right] = \Phi\left(\frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right),$$

where $\Phi(\cdot)$ denotes the cdf of the standard normal distribution.

For example, consider again the rigid timber beam of Example 9.1, and assume normal and independent strength X and load Y. The probability of failure can be computed as follows. Since

$$\mu_X = 39.1 \text{ N/mm}^2,\ V_X = .25,\ \mu_Y = 24.0 \text{ N/mm}^2,\ V_Y = .15,$$

one has

$$\sigma_X = 39.1 \times .25 = 9.775 \text{ N/mm}^2, \sigma_Y = 24.0 \times .15 = 3.6 \text{ N/mm}^2,$$

so that

$$\mu_S = 39.1 - 24.0 = 15.1 \text{ N/mm}^2, \qquad \sigma_S = (9.775^2 + 3.6^2)^{1/2} = 10.41 \text{ N/mm}^2.$$

Safety of Margin Example(1-2)

The required probability of failure is

$$p_f = F_S(0) = \Phi(-15.1 / 10.41) = \Phi(-1.450) = .074,$$

which indicates a beam reliability of 92.6 percent. The pdf's of X, Y, and S are shown in Fig. 9.1.8*a*, and the corresponding cdf's are given in Fig. 9.1.8*b*. The estimated reliability from the independent normal model differs from that obtained from the independent lognormal model by only about 2 percent.

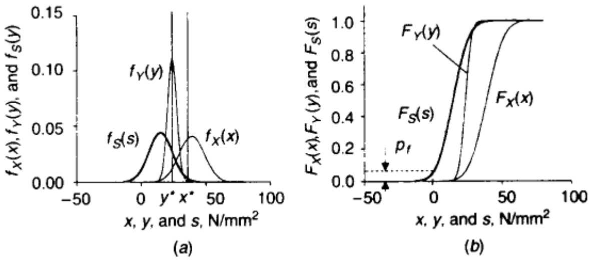

FIGURE 9.1.8
Timber strength and load illustration: pdf (a) and cdf (b) of safety margin S for independent normally distributed X and Y.

Safety of Margin

- **For Normal Capacity and Demand**
  - **Mean and variance**

    $\mu_S = \mu_X - \mu_Y$

    $\sigma_S^2 = \sigma_X^2 - 2\rho_{XY}\sigma_X\sigma_Y + \sigma_Y^2$
  - **Probability of failure**
  - **Reliability cdf of the standard normal distribution**

Safety of Margin Example(2-1)

Example 9.4. Irrigation water supply. During the growing season the expected demand Y from an irrigation scheme is 10 units with a coefficient of variation of 50 percent, which accounts for fluctuations associated with weather variability. The mean available water X, which is diverted from a river barrage, is 20 units, with a coefficient of variation of 20 percent, which accounts for fluctuations associated with hydrologic variability in that season. Because of the relationship between hydrology and climate, the natural water availability often tends to decrease when the demand increases, so that the correlation coefficient between X and Y is negative. The estimated value of ρ_{XY} is −.5. An irrigation engineer needs to estimate the reliability of the system assuming that both capacity X and demand Y are normally distributed variates.

The standard deviations of capacity and demand are

$$\sigma_X = V_X\mu_X = .2 \times 20 = 4 \text{ units},$$

$$\sigma_Y = V_Y\mu_Y = .5 \times 10 = 5 \text{ units},$$

respectively. The safety margin, $S = X - Y$, is normally distributed with mean

$$\mu_S = \mu_X - \mu_Y = 20 - 10 = 10 \text{ units},$$

and standard deviation

$$\sigma_S = (\sigma_X^2 - 2\rho_{XY}\sigma_X\sigma_Y + \sigma_Y^2)^{1/2} = (4^2 + 2 \times .5 \times 4 \times 5 + 5^2)^{1/2} = 7.81 \text{ units}.$$

Safety of Margin Example(2-2)

The required risk of failure is

$$p_f = F_S(0) = \Phi(-10/7.81) = 1 - \Phi(1.28) = 1 - .9 = .1,$$

and the associated reliability is 90 percent. In order to increase the reliability of the system to 95 percent, the diversion of a neighboring stream is considered for the purpose of increasing the mean capacity. Assuming that both V_X and ρ_{XY} do not change, the mean capacity μ_X must be increased by a factor of a so that

$$r = 1 - F_S(0) = \Phi\left(\frac{a\mu_X - \mu_Y}{\sqrt{a^2\mu_X^2V_X^2 - 2\rho_{XY}aV_X\mu_X\sigma_X + \sigma_Y^2}}\right) = .95;$$

that is,

$$\frac{a\mu_X - \mu_Y}{\sqrt{a^2\mu_X^2V_X^2 - 2\rho_{XY}aV_X\mu_X\sigma_X + \sigma_Y^2}} \doteq 1.65.$$

Hence,

$$\frac{20a - 10}{\sqrt{4^2a^2 + 2 \times 0.5 \times 4 \times 5 \times a + 5^2}} = \frac{20a - 10}{\sqrt{16a^2 + 20a + 25}} = 1.65,$$

which yields $a = 1.20$. This means that the new source must provide a 20 percent increase in the average water availability if the goal is to increase the reliability of the irrigation system to 95 percent.

Reliability Index

❖ **Definition**

- **The *reliability index* of a system, denoted by β, is defined as the ratio between the mean and standard deviation of the safety margin of the system**
- **Reliability index**

$$\beta = \mu_S / \sigma_S$$

- **Reliability index in terms of the first two moments of the capacity and the demand functions**

Reliability Index Example(1-1)

Example 9.5. Structural reliability index for normally distributed safety margin. Consider again a structure whose load-carrying capacity, or strength, X and load Y are independent normal variates (see Example 9.3). Since $r = \Phi(\mu_S/\sigma_S)$, r is a function of the ratio μ_S/σ_S, which is the safety margin expressed in units of σ_S, that is, the reliability index β. Therefore, system reliability can be written as $r = \Phi(\beta)$, and the corresponding probability of failure is given by $p_f = 1 - r = 1 - \Phi(\beta)$. For normal S, a value of $\beta = 0$ corresponds to $r = .5$ (50% reliability). Similarly, $\beta = 1.28$ with 90% reliability, $\beta = 1.65$ with 95%, $\beta = 2.33$ with 99%, $\beta = 3.10$ with 99.9%, and $\beta = 3.72$ with 99.99%. This illustrates that the level of reliability is a function of both the relative position of $f_X(x)$ and $f_Y(y)$, as measured by the mean safety margin $\mu_S = \mu_X - \mu_Y$, and the degree of dispersion, as measured in terms of the standard deviation $\sigma_S = (\sigma_X^2 + \sigma_Y^2)^{1/2}$. The reliability index β reflects the combined effect of both these factors. A useful approximation of the failure probability is given by

$$p_f \cong 2 \times 10^{-\beta},$$

which can be used for reliability analysis with β taking values from 1 to 2.7, as shown in Fig. 9.1.9.

Reliability Index Example(1-2)

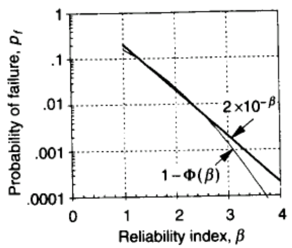

FIGURE 9.1.9
Power approximation of the probability of failure as a function of the reliability index.

For example, in the case of the rigid timber beam of Example 9.3, where strength X and load Y are normal and independent variates,

$$\beta = \mu_S/\sigma_S = 15.1/10.41 = 1.451,$$

so that the reliability index is 1.451 sigma units. The power approximation of the corresponding probability of failure is

$$p_f \cong 2 \times 10^{-1.451} = .071.$$

The previously computed value of $p_f = 1 - \Phi(1.451) = .073$, so the error in the power approximation is about 3%.

Reliability Index

❖ **Influence of correlation**

$$\beta V_Y = \frac{\zeta - 1}{\sqrt{\nu^2\zeta^2 - 2\rho\nu\zeta + 1}}$$

where $\nu = V_X / V_Y$, $\rho = \rho_{XY}$

- **For $V_X = V_Y = V$**
- **To measure the correlation coefficient between capacity and demand is not an easy task, because it depends on many factors**
- ***What would happen if we neglect ρ?***

Reliability Index

❖ **Influence of correlation**

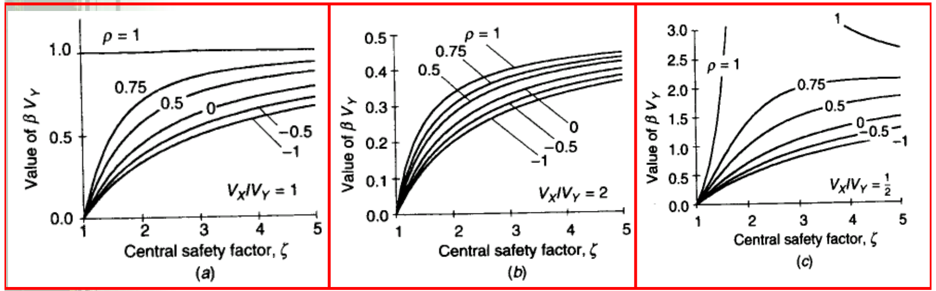

FIGURE 9.1.10
βV_Y versus ζ for different correlation coefficients between capacity X and demand Y for (a) $V_X/V_Y = 1$, (b) $V_X/V_Y = 2$, and (c) $V_X/V_Y = \frac{1}{2}$.

Reliability Index Example(2)

Example 9.7. Irrigation water supply. Consider again the irrigation problem of Example 9.4, and assume that correlation between capacity and demand can be neglected. Assuming that $\rho_{XY} = 0$ yields

$$\sigma_S = (\sigma_X^2 + \sigma_Y^2)^{1/2} = (4^2 + 5^2)^{1/2} = 6.40 \text{ units}$$

and

$$\beta = \mu_S/\sigma_S = 10/6.40 = 1.56;$$

thus, the estimated reliability of the system is $r = \Phi(1.56) = .94$. If compared with the original estimate of 90 percent, this result illustrates that an engineer who disregards the correlation between capacity and demand can come to the misleading conclusion that the goal of 95 percent reliability can be reached.

One can also use Eqs. (9.1.15) and (9.1.16) to compute the failure and nonfailure probabilities if either X or Y or both are nonnormal. This is a straightforward exercise for two independent lognormal variates, as shown in the following example.

Reliability Index

❖ **For Lognormal Capacity and Demand**

➢ **For $V_X = V_Y = V$**

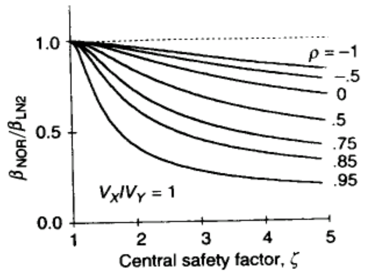

FIGURE 9.1.11
β_{NOR} β_{LN2} versus ζ for different correlation coefficient between capacity X and demand Y.

Reliability Index Example(3-1)

Example 9.9. Thermal pollution in a river. The discharge Y from the cooling system of a thermal power plant flows into a river. To prevent thermal pollution in the river, it is desirable that Y does not exceed a fraction of the natural flow Q in the river. say, $X = Q/a$, where a denotes a constant that depends on the difference in temperature between the two flows. An engineer wishes to evaluate the risk that thermal pollution occurs in the river. Assume that Y is normally distributed with mean 2 m^3/s and coefficient of variation 20 percent, as shown in Fig. 9.1.12.

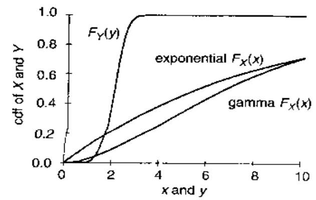

FIGURE 9.1.12
cdf's of capacity X and demand Y for Example 9.9.

For the period in which the river receives the discharge Y, Q can be approximated by an exponential distribution with mean 40 m^3/s, and $a = 5$. Inflow Y and streamflow Q are further assumed to be independent variates. The problem is approached by using the univariate Rosenblatt transformation in order to determine the equivalent normal distribution for the exponential capacity X. Since X is exponentially distributed with mean $40/5 = 8$ m^3/s,

$$f_X(x) = (1/8)\exp(-x/8) = .125\exp(-.125x)$$

and

$$F_X(x) = 1 - \exp(-.125x).$$

Reliability Index Example(3-2)

The mean μ_{X^*} and the standard deviation σ_{X^*} of the equivalent normal distribution for the exponential capacity X are found from the assumption that, at failure point x^*,

$$\Phi[(x^* - \mu_{X^*})/\sigma_{X^*}] = F_X(x^*),$$

where $\Phi(\cdot)$ denotes the cdf of the standard normal variate. Thus,

$$\mu_{X^*} = x^* - \sigma_{X^*}\Phi^{-1}[F_X(x^*)],$$

where $\Phi^{-1}(\xi)$ denotes the ξth quantile of the standard normal distribution. It also follows from the previous assumption at the failure point that, by equating the corresponding probability densities at the failure point,

$$(1/\sigma_{X^*})\phi[(x^* - \mu_{X^*})/\sigma_{X^*}] = f_X(x^*),$$

where $\phi(\cdot)$ denotes the pdf of the standard normal variate. Hence,

$$\sigma_{X^*} = \phi\{\Phi^{-1}[F_X(x^*)]\}/f_X(x^*).$$

By substitution,

$$\sigma_{X^*} = \phi\{\Phi^{-1}[1 - \exp(-.125x^*)]\}/[.125\exp(-.125x^*)]$$

and

$$\mu_{X^*} = x^* - \sigma_{X^*}\Phi^{-1}[1 - \exp(-.125x^*)],$$

whereas $\mu_Y = 2$ m^3/s and $\sigma_Y = .2 \times 2 = .4$ m^3/s.

Reliability Index Example(3-3)

Because the failure point is unknown, the problem is solved by iteration. If $x^* = 1$ m^3/s is taken as the initial value,

$$\sigma_{X^*} = \phi\{\Phi^{-1}[1 - \exp(-.125 \times 1)]\}/[.125\exp(-.125 \times 1)]$$
$$= \phi[\Phi^{-1}(.118)]/.110$$
$$= \phi(-1.188)/.110 = 1.79$$

and

$$u_{X^*} = x - \sigma_{X^*}\Phi^{-1}[1 - \exp(-0.125 \times 1)] = 1 - 1.79\Phi^{-1}(0.118) = 3.12;$$

these are used in Eq. (9.1.14) to obtain, for independent capacity and demand,

$$\beta = (\mu_{X^*} - \mu_Y)/(\sigma_{X^*}^2 + \sigma_Y^2)^{1/2} = (3.12 - 2)/(1.79^2 + 0.4^2)^{1/2} = 0.61.$$

For the second iteration, one takes $x^* = 1.5$, which yields $\beta = 0.74$. As shown in Table 9.1.1, this procedure is then followed until the difference between two subsequent estimates of β is negligible. Accordingly, one obtains $\beta = 0.74$; that is, the reliability of the system $r = \Phi(.74) = .77$.

Reliability Index Example(3-4)

TABLE 9.1.1
Risk evaluation for thermal pollution in a river with exponentially distributed streamflow

Exponential capacity, X						
Mean of X =	**8**					
λ =	.125					
Iteration process						
Point of failure, x^* =	1.0	1.5	2.0	2.5	2.1	**1.9**
$F(x^*)$ =	0.1175	0.1710	0.2212	0.2684	0.2309	0.2114
$f(x^*)$ =	0.1103	0.1036	0.0974	0.0915	0.0961	0.0986
$\Phi^{-1}[F(x^*)]$ =	−1.188	−0.950	−0.768	−0.618	−0.736	−0.802
$\phi\{\Phi^{-1}[F(x^*)]\}$ =	0.197	0.254	0.297	0.330	0.304	0.289
Mean of X^* =	3.12	3.83	4.34	4.73	4.43	4.25
Standard deviation of X^* =	1.79	2.45	3.05	3.60	3.17	2.94
Normal demand, Y						
Mean of Y =	**2**					
Standard Deviation of Y =	**0.4**					
Evaluation of reliability index, β						
β =	**.61**	**.74**	**.76**	**.75**	**.76**	**.76**
Reliability: $\Phi(\beta)$ = .777						
Risk: $1 - \Phi(\beta)$ = .223						

Reliability Index Example(3-5)

One can also use the same approach for a capacity distribution different from the exponential. For example, if X is gamma distributed with mean 8 m^3/s and its coefficient of variation is $1/\sqrt{2}$ (see Fig. 9.1.12), the parameters of the gamma pdf are found to be, by the method of moments,

$$r = (1/V_X)^2 = (1/\sqrt{2})^2 = 2, \qquad \lambda = r/\mu_X = 2/8 = .25 \text{ m}^{-3}\text{s}.$$

Thus, from Eq. (4.2.7),

$$f_X(x) = [\lambda^r/\Gamma(r)]x^{r-1}\exp(-\lambda x) = .25^2 x\exp(-.25x),$$

and, for $r = 2$,

$$F_X(x) = \int_0^x \frac{\lambda^2}{\Gamma(2)} z^{2-1}\exp(-\lambda z)\,dz = 1 - (1 + \lambda x)e^{-\lambda x} = 1 - (1 + .25x)e^{-.25x}.$$

Using this procedure, one gets, for the initial value of $x^* = 1$ m^3/s,

$$\sigma_{X^*} = \phi\{\Phi^{-1}[1 - (1 + 0.25 \times 1) \times \exp(-0.25 \times 1)]\}/[(0.25^2 \times 1) \times \exp(-0.25 \times 1)]$$
$$= 1.26,$$
$$\mu_{X^*} = 1 - 1.26 \times \Phi^{-1}[1 - (1 + 0.25 \times 1) \times \exp(-0.25 \times 1)] = 3.44;$$

and, using these values in Eq. (9.1.14),

$$\beta = (\mu_{X^*} - \mu_Y)/(\sigma_{X^*}^2 + \sigma_Y^2)^{1/2} = (3.44 - 2)/(1.26^2 + 0.4^2)^{1/2} = 1.09.$$

After some iterations, the reliability index is found to be 1.32. Hence, from Eq. (9.1.16) reliability is about 91%. The procedure is detailed in Table 9.1.2.

Reliability Index Example(3-6)

TABLE 9.1.2
Risk evaluation for thermal pollution in a river with gamma distributed streamflow

Gamma capacity, X						
Mean of X =	**8**					
Coefficient of variation of X =	**.707**					
r =	2					
λ =	.25					
Iteration process						
Point, x^* =	1.0	1.5	2.0	2.5	1.9	2.1
$F(x^*)$ =	0.0265	0.0550	0.0902	0.1302	0.0827	0.0979
$f(x^*)$ =	0.0487	0.0644	0.0758	0.0836	0.0738	0.0776
$\Phi^{-1}[F(x^*)]$ =	−1.935	−1.598	−1.339	−1.125	−1.387	−1.294
$\phi\{\Phi^{-1}[F(x^*)]\}$ =	0.061	0.111	0.163	0.212	0.152	0.173
Mean of X^* =	3.44	4.26	4.87	5.35	4.76	4.98
Standard deviation of X^* =	1.26	1.73	2.15	2.53	2.06	2.23
Normal demand, Y						
Mean of Y =	**2**					
Standard deviation of Y =	**0.4**					
Evaluation of reliability index, β						
β =	**1.09**	**1.27**	**1.32**	**1.31**	**1.31**	**1.32**
Reliability: $\Phi(\beta)$ =	0.906					
Risk: $1-\Phi(\beta)$ =	0.094					

Performance Function

- **Definition**
  - **The *performance function* of a system is the random function $g(X, Y)$ of capacity X and demand Y describing system performance, related to its possible failure, or limiting state of interest, given by $g(X, Y) = 0$**
- **For reduced variables**

$$X' = (X - \mu_X) / \sigma_X, \quad Y' = (Y - \mu_Y) / \sigma_Y$$

$$g(X', Y') = \sigma_X X' - \sigma_Y Y' + \mu_X - \mu_Y = 0$$

- **Reliability index for uncorrelated variables**

Performance Function

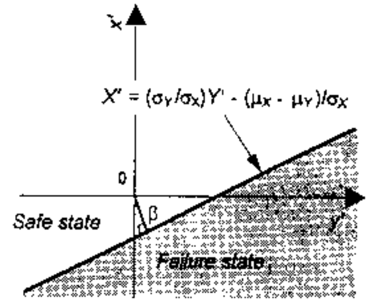

FIGURE 9.1.14
Failure state, safe state, and limiting state in a reduced coordinate system.

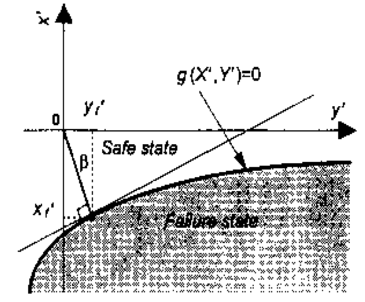

FIGURE 9.1.15
Failure state, safe state, and nonlinear limiting state in a reduced coordinate system.

Performance Function

- **In General,**

$$g(X_1, X_2, \ldots, X_n) = a_0 + \sum_{i=1}^{n} a_i X_i$$

$$\beta = \frac{a_0 + \sum_{i=1}^{n} a_i \mu_i}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \rho_{ij} \sigma_i \sigma_j}}$$

- **For Mutually Independent Variates**

Performance Function Example(1)

Example 9.10. Lake phytoplankton. Climate and water quality are among the factors influencing the quantity of phytoplankton in shallow lakes. Assume that the rate of increase of phytoplankton can be expressed as a linear function $g(X_1, X_2, X_3)$ of three variables: the temperature of water X_1, global radiation X_2, and the concentration of nutrients X_3. The equilibrium corresponds to the limiting state of interest, $g(X_1, X_2, X_3) = 0$, and positive growth rates must be avoided to prevent eutrophication. Field observations indicate that X_1, X_2, and X_3 can be modeled as normal variates with the following means and coefficients of variation:

$$\mu_1 = 16\,°\text{C}, \qquad \mu_2 = 150\ \text{W/m}^2, \qquad \mu_3 = 100\ \text{mg/m}^3$$

and

$$V_1 = .5, \qquad V_2 = .3, \qquad V_3 = .7.$$

Thus,

$$\sigma_1 = 8\,°\text{C}, \qquad \sigma_2 = 45\ \text{W/m}^2, \qquad \sigma_3 = 70\ \text{mg/m}^3.$$

Performance Function Example(2)

Although it is observed that temperature and radiation have no effect on the concentration of nutrients, so that $\rho_{13} = \rho_{23} = 0$, mutually they are highly correlated, with $\rho_{12} = .8$. The equilibrium function is

$$g(X_1, X_2, X_3) = a_0 + a_1X_1 + a_2X_2 + a_3X_3,$$

with $a_0 = -1.5$ mg/m^3, $a_1 = 0.08$ mg/(m$^3\cdot$°C), $a_2 = 0.01$ mg/m W, and $a_3 = 0.05$. Other variables should be incorporated, such as those accounting for predation and natural wastage; these are included in the constant a_0 because of difficulties in estimating them separately. The reliability index is then computed using Eq.(9.1.29). The numerator is given by

$$a_0 + a_1\mu_1 + a_2\mu_2 + a_3\mu_3 = -1.5 + .08 \times 16 + .01 \times 150 + .05 \times 100 = 6.28,$$

and the argument in the square root of the denominator is

$$a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + a_3^2\sigma_3^2 + 2a_1a_2\rho_{12}\sigma_1\sigma_2$$
$$+ 2a_1a_3\rho_{13}\sigma_1\sigma_3 + 2a_2a_3\rho_{23}\sigma_2\sigma_3 = .08^2 \times 8^2 + .01^2 \times 45^2 + .05^2 \times 70^2 + 2$$
$$\times (-.08) \times (-.01) \times .8 \times 8 \times 45 + 0 + 0 = 13.32.$$

Thus,

$$\beta = 6.28/\sqrt{13.32} = 6.28/3.65 = 1.72,$$

and, from Eq. (9.1.16),

$$r = \Phi(1.72) = .957.$$

This means there is a 96 percent chance that the equilibrium situation is preserved; that is, the risk that the algal biomass will increase is only 4%.

Assignment 6

- **Chapter 9**

Problems

9.3 / 9.5 / 9.7 / 9.12 / 9.18

Due date: December 1st, 2009