Chapter 3 Kinematics of Fluid Motion

3.1 Steady and Unsteady Flow, Streamlines, and Streamtubes

3.2 One-, Two-, and Three-Dimensional Flows

3.3 Velocity and Acceleration

3.4 Circulation, Vorticity, and Rotation

Objectives:

- treat kinematics of idealized fluid motion along streamlines and flowfields
- learn how to describe motion in terms of displacement, velocities, and accelerations without regard to the forces that cause the motion
- distinguish between rotational and irrotational regions of flow based on the flow property vorticity

사랑하는 사람이 있다면
새벽 강으로 데리고 오세요
모든 강들이 한 다발의 꽃으로
기다리고 있는
먼저 고백하기 좋은 하얀 새벽 강으로
데리고 오세요

젖은 모래에 서로의 발자국을 찍어가며
잠 덜 깬 손을 잡아보세요
당신 가슴에 강물처럼 흐르고 싶다고 얘기해도
하얀 안개가 상기된 볼을 숨겨줄 테니
사랑하는 사람이 있다면
새벽 강으로 데리고 오세요

박은화의 《새벽강》중에서

3.1 Steady and Unsteady Flow, Streamlines, and Streamtubes

- Two basic means of describing fluid motion

1) Lagrangian views

Joseph Louis Lagrange (1736-1813), Italian mathematician

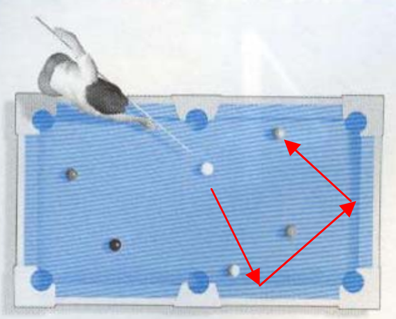


FIGURE 4–1
With a small number of objects, such as billiard balls on a pool table, individual objects can be tracked.

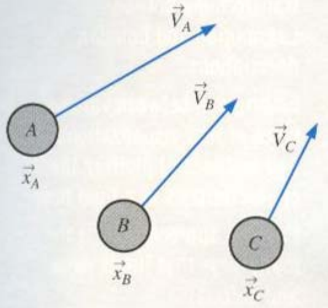


FIGURE 4–2
In the Lagrangian description, one must keep track of the position and velocity of individual particles.

~ Each fluid particle is labeled by its spatial coordinates at some initial time.

~ Then fluid variables (path, density, velocity, and others) of each individual particle are traced as time passes.

~ used in the dynamic analyses of solid particles

▪ Difficulties of Lagrangian description for fluid motion

- We cannot easily define and identify particles of fluid as they move around.
- A fluid is a continuum, so interactions between parcels of fluid are not easy to describe as are interactions between distinct objects in solid mechanics.
- The fluid parcels continually deform as they move in the flow.

▪Path line

~ the position is plotted as a function of time = trajectory of the particle → path line

~ since path line is <u>tangent to the instantaneous velocity</u> at each point along the path, changes in the particle location over an infinitesimally small time are given by

$$dx = udt;\ dy = vdt;\ dz = wdt$$

This means that

$$u = \frac{dx}{dt};\ v = \frac{dy}{dt}\ \ w = \frac{dz}{dt} \tag{3.1a}$$

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{dt}{1} \tag{3.1b}$$

The acceleration components are

$$a_x = \frac{du}{dt};\ a_y = \frac{dv}{dt};\ a_z = \frac{dw}{dt} \tag{3.1c}$$

2) Eulerian view

Leonhard Euler (1707-1783), Swiss mathematician

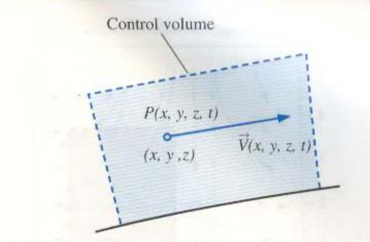


FIGURE 4–3
In the Eulerian description, one defines field variables, such as the pressure field and the velocity field, at any location and instant in time.

~ attention is focused on particular points in the space filled by the fluid

~ motion of individual particles is no longer traced

→ A finite volume called a <u>control volume</u> (flow domain) is defined, through which fluid flows in and out.

~The values and variations of the velocity, density, and other fluid variables are determined <u>as a function of space and time</u> within the control volume. → flow field

Define velocity field as a vector field variable in Cartesian coordinates

$$\vec{v} = \vec{v}(x, y, z, t) \qquad \text{(E1)}$$

Acceleration field

$$\vec{a} = \vec{a}(x, y, z, t) \qquad \text{(E2)}$$

Define the pressure field as a scalar field variable

$$p = p(x,\ y,\ z,\ t) \tag{E3}$$

where

$$\vec{v} = u\overrightarrow{e_x} + v\overrightarrow{e_y} + w\overrightarrow{e_z} \tag{E4}$$

$\overrightarrow{e_x},\ \overrightarrow{e_y},\ \overrightarrow{e_z}$ = unit vectors

Substitute (E4) into (E1) to expand the velocity field

$$\vec{v} = (u, v, w) = u(x,\ y,\ z,\ t)\overrightarrow{e_x} + v(x,\ y,\ z,\ t)\ \overrightarrow{e_y} + w(x,\ y,\ z,\ t)\ \overrightarrow{e_z} \tag{E5}$$

▪Difference between two descriptions

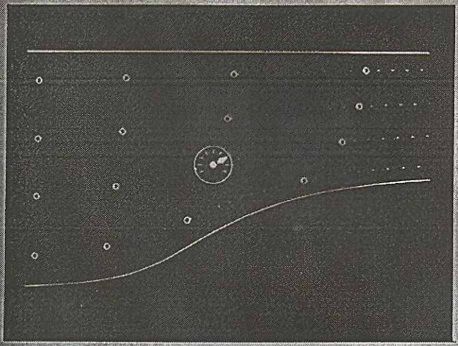
Imagine a person standing beside a river, measuring its properties.

- Lagrangian approach: he throws in a prove that moves downstream with the river flow
- Eulerian approach: he anchors the probe at a fixed location in the river

~ Eulerian approach is practical for most fluid engineering problems (experiments).

motion, we would have to give the velocity of all the pieces of matter in the flow as a function of time and initial position.

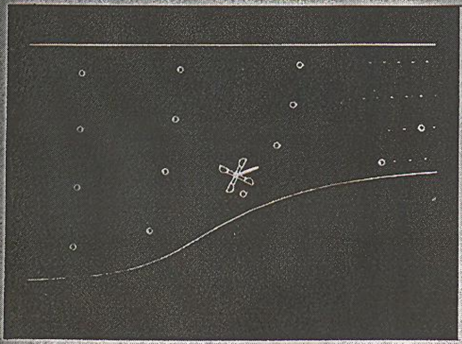
Such a description, in terms of material points, is called a *Lagrangian* description of the flow. The identifying co-ordinates are called *Lagrangian*, or sometimes *material*, co-ordinates. Given the Lagrangian velocity field, we can easily calculate the Lagrangian displacement by integration in time, and the acceleration field by partial differentiation with respect to time.



7. A pressure gauge is attached to one of the moving points.

(Lagrangian)

To make what we might call a Lagrangian measurement, we can imagine attaching an instrument like a pressure gauge to a fluid material point (Fig. 7). This sort of measurement is attempted in the atmosphere with balloons of neutral buoyancy. If the balloon does indeed move faithfully with the air, it gives the Lagrangian displacement, i.e. the displacement of an identified fluid "element." Such Lagrangian measurements are actually very difficult, particularly in the laboratory. We usually prefer to make measurements at points fixed in laboratory co-ordinates; it is relatively easy to hold an instrument at a fixed location.

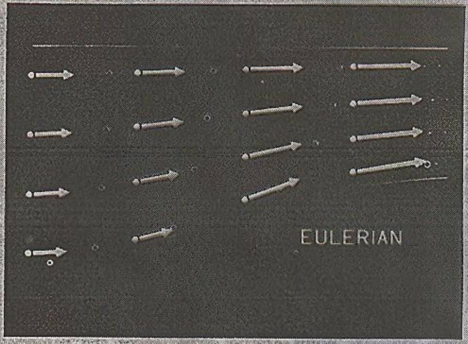


8. An anemometer is placed at a fixed position in the flow field.

(Eulerian)

Eulerian Description

Classically, the idea of a field, such as an electric, magnetic, or temperature field, is defined by how the response of a test body or probe, like the anemometer in Fig. 8, varies with time at each point in some spatial co-ordinate system. In Fig. 8 the fixed anemometer probes in laboratory co-ordinates. We will always use solid points and solid arrows to indicate such probing positions, fixed in our laboratory, and the velocities measured there.

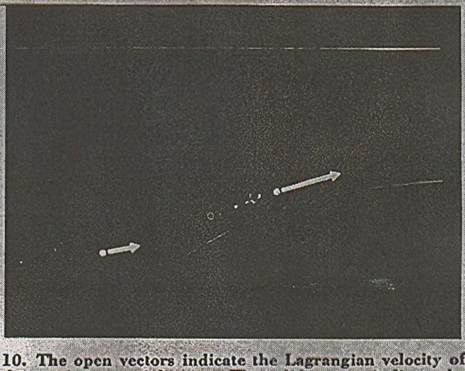


9. The solid vectors indicate the velocities at fixed points in the contraction.

In Fig. 9 we have a grid of points fixed in space with an arrow at each to indicate the velocity at each point. A description like this which gives the spatial velocity distribution in laboratory co-ordinates is called an *Eulerian* description of the flow.

Relation Between Eulerian and Lagrangian Frames

Although the physical field is the same, the Eulerian and Lagrangian representations are not the same, be-



10. The open vectors indicate the Lagrangian velocity of the moving material point. The solid vectors indicate the Eulerian velocities at two fixed points. When the moving point coincides with a fixed point, the Eulerian and Lagrangian velocities coincide.

- Two types of flow

→ In Eulerian view, two types of flow can be identified.

1) Steady flow

~ The fluid variables at any point do not vary (change) with time.

~ The fluid variables may be a function of a position in the space. → non-uniform flow

→ In Eulerian view, steady flow still can have accelerations (advective acceleration).

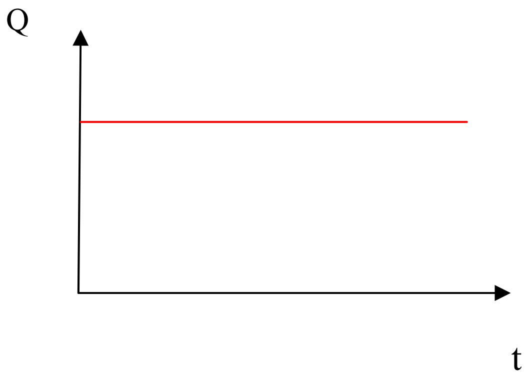
2) Unsteady flow

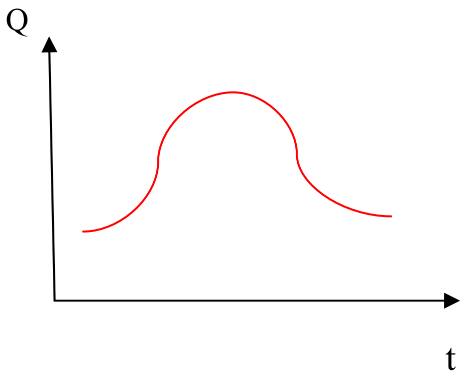
~ The fluid variables will vary with time at the spatial points in the flow.

- Fig. 3.1

when valve is being opened or closed → unsteady flow

when valve opening is fixed → steady flow





[Re] Mathematical expressions

Let fluid variables (pressure, velocity, density, discharge, depth) = F

$$F = f(x, y, z, t)$$

$$\frac{\partial F}{\partial t} = 0 \rightarrow \text{steady flow}$$

$$\frac{\partial F}{\partial t} \neq 0 \rightarrow \text{unsteady flow}$$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0 \rightarrow \text{uniform flow}$$

$$\frac{\partial F}{\partial x} \neq 0 \rightarrow \text{non-uniform (varied) flow}$$

- Flow lines

1) Streamline

~ The curves drawn at an instant of time in such a way that the tangent at any point is in the direction of the velocity vector at that point are called instantaneous streamlines, and they continually evolve in time in an unsteady flow.

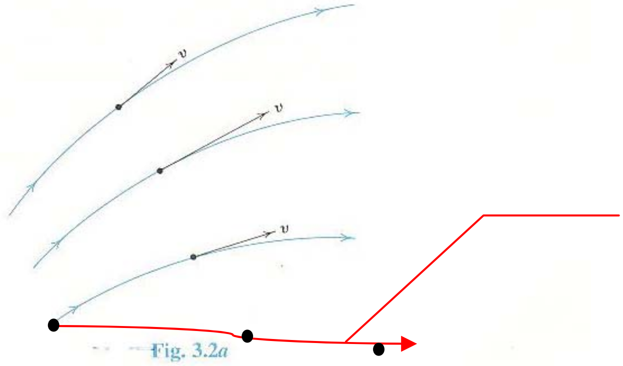


Fig. 3.2*a*

Path line for particle *a* in an unsteady flow

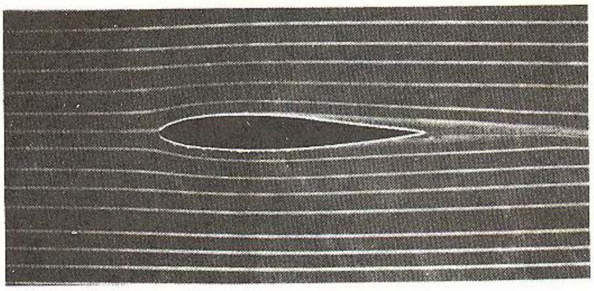


Fig. 3.2*b* Steady flow past an airfoil in a smoke tunnel (streamlines are created by introducing small jets of smoke at a number of upstream points in the flow).

Individual fluid particles must travel on paths whose tangent is always in the direction of the fluid velocity at any point.

→ In an unsteady (turbulent) flow, path lines are not coincident with the instantaneous streamlines.

2) Path line

~ trajectory of the particle

3) Streak line

~ current location of all particles which have passed through a fixed point in space

~ The streak lines can be used to trace the travel of a pollutant downstream from a smoke stack or other discharge.

~ In steady flow, Lagrangian path lines are the same as the Eulerian streamlines, and both are the same as the streak lines, because the streamlines are then fixed in space and path lines, streak lines and streamlines are tangent to the steady velocities.

→ In a steady flow, all the particles on a streamlines that passes through a point in space also passed through or will pass through that point as well.

~ In an unsteady flow, the path lines, the streak lines, and the instantaneous streamlines are not coincident.

• In a steady flow, streamlines can be defined integrating Eq. (3.1) in space.

[Ex] A fluid flow has the following velocity components; $u = 1$ m/s, $v = 2x$ m/s. Find an equation for the streamlines of this flow.

Sol.: $\frac{dy}{dx} = \frac{v}{u} = \frac{2x}{1} \quad (a)$

Integrating (a) gives

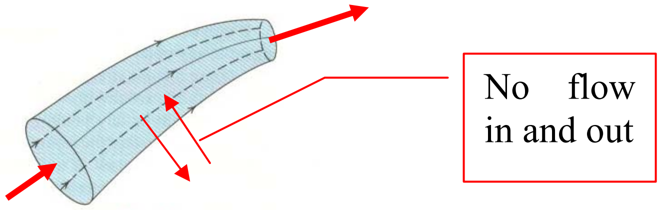
$$y = x^2 + c$$

- Stream tube

~ aggregation of streamlines drawn through a closed curve in a steady flow forming a boundary across which fluid particles cannot pass because the velocity is always tangent to the boundary

~ may be treated as if isolated from the adjacent fluid

~ many of the equations developed for a small streamtube will apply equally well to a streamline.



No flow in and out

Fig. 3.3

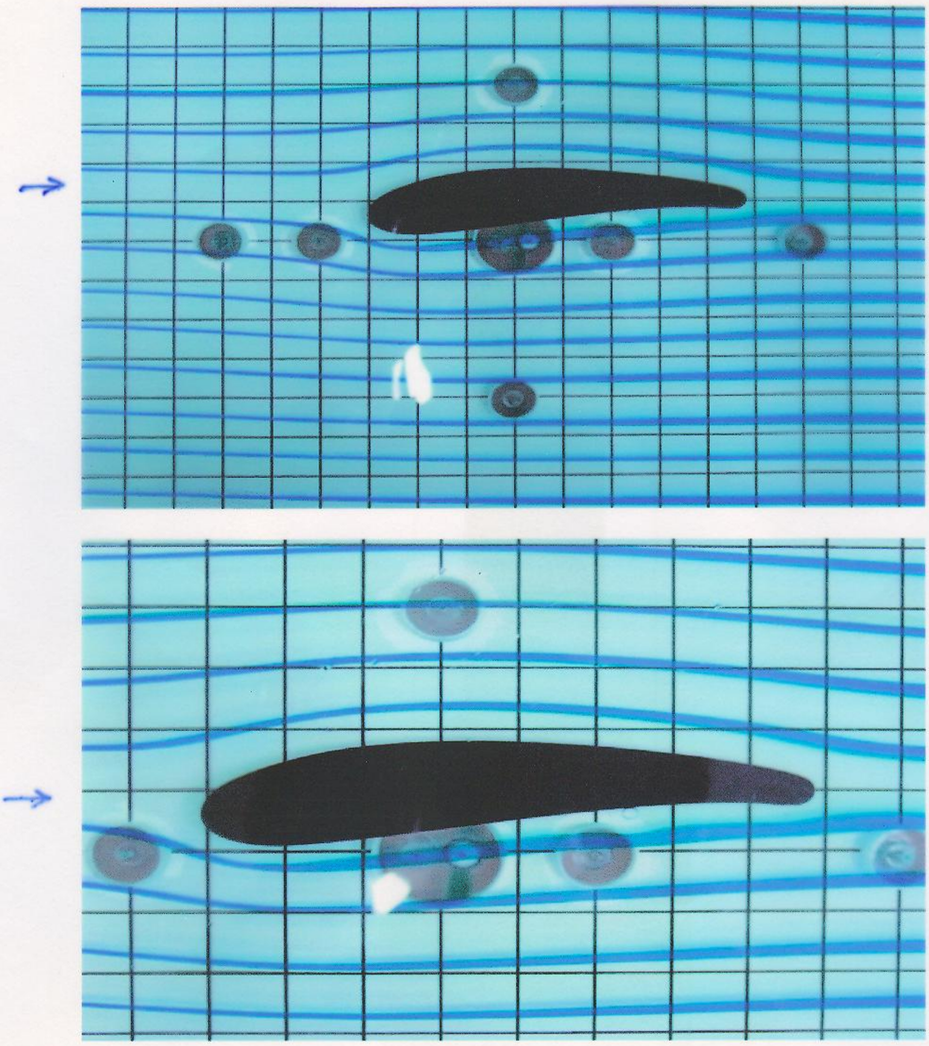
[Re] How to shoot flow lines?

1) streamline: shoot bunch of reflectors instantly $(\Delta t \rightarrow 0)$

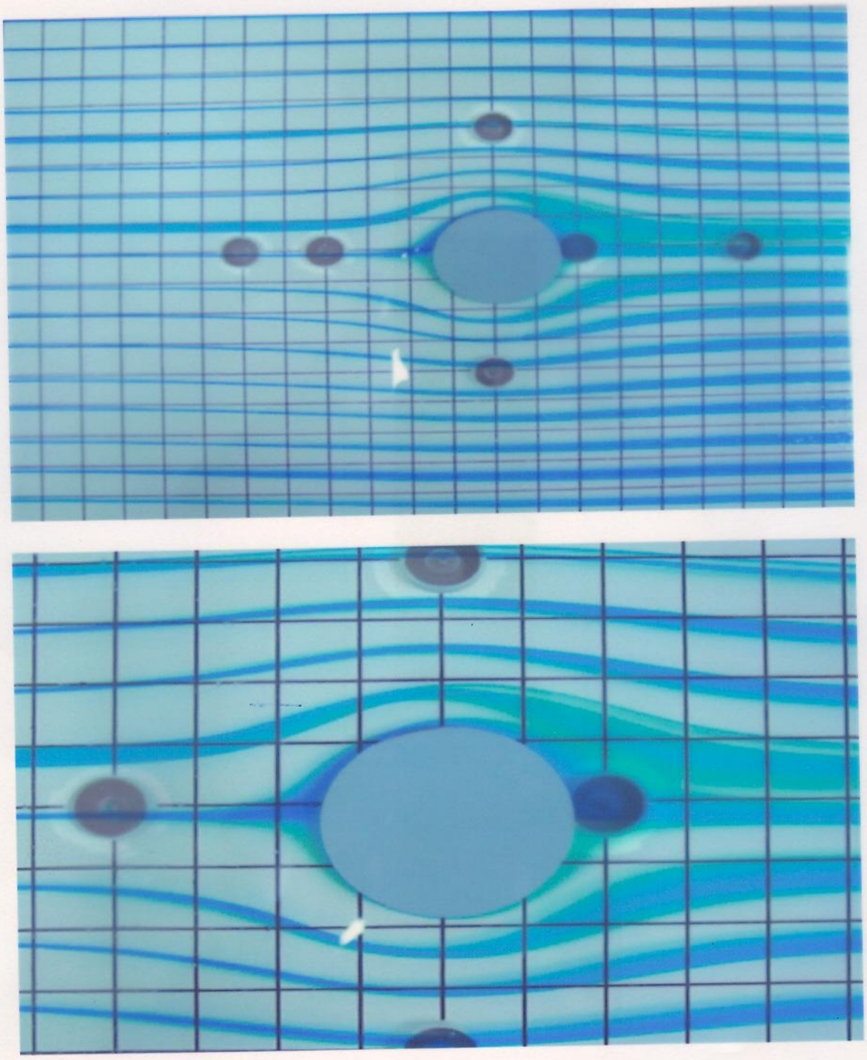
2) path line: shoot only one reflector with long time exposure

3) streak line: shoot dye injecting from on slot with instant exposure

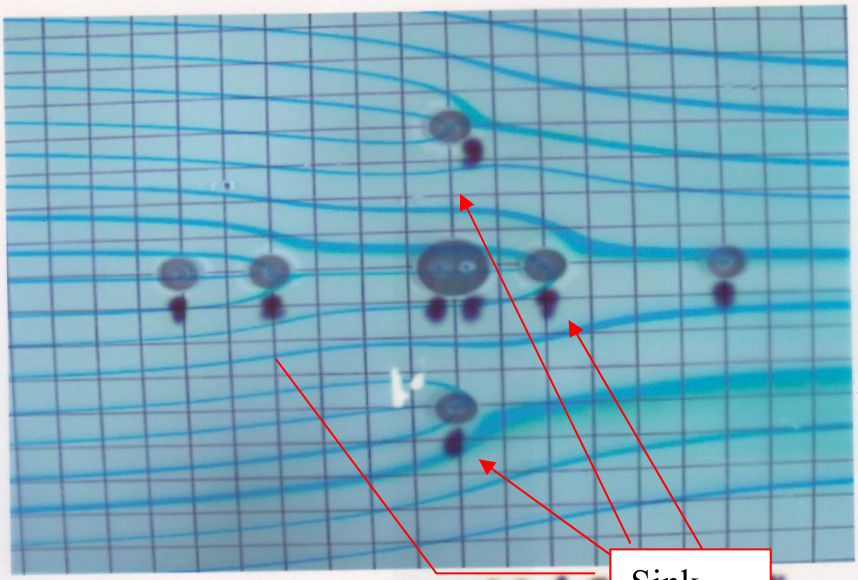
YF15w 20wf 36w 2 πx u



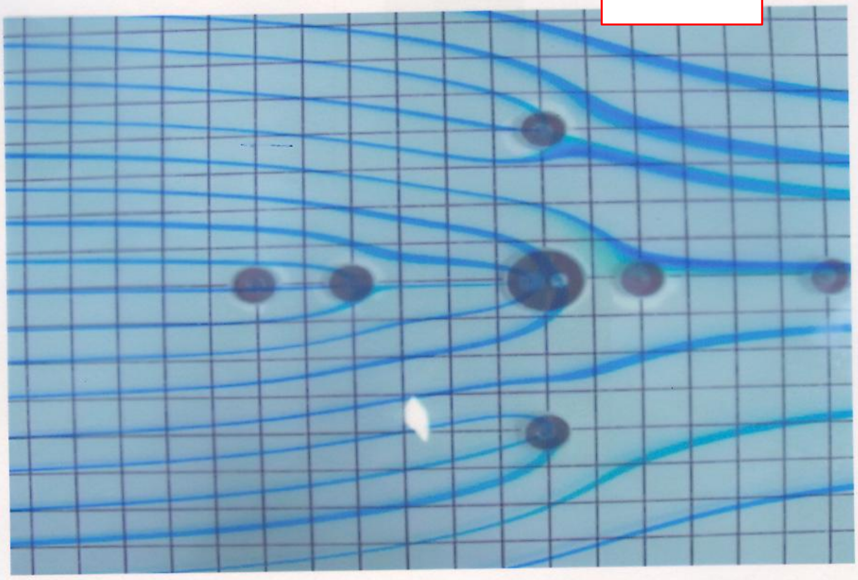
Streamlined object



Circular cylinder



2,3,6,7 k



2,3,4,5,6 제거시 sink

[Re] Substantial (total) derivative

$$\underbrace{\frac{dF}{dt}}_{\text{total derivative}}=\frac{\partial F}{\partial t}+\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial z}\frac{dz}{dt}$$

$$=\underbrace{\frac{\partial F}{\partial t}}_{\text{local derivative}}+\underbrace{u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}}_{\text{advective (convective) derivative}}$$

- steady flow: $\frac{\partial F}{\partial t}=0$
- uniform flow: $u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0$

3.2 One-, Two-, and Three-Dimensional Flows

- One-dimensional flow

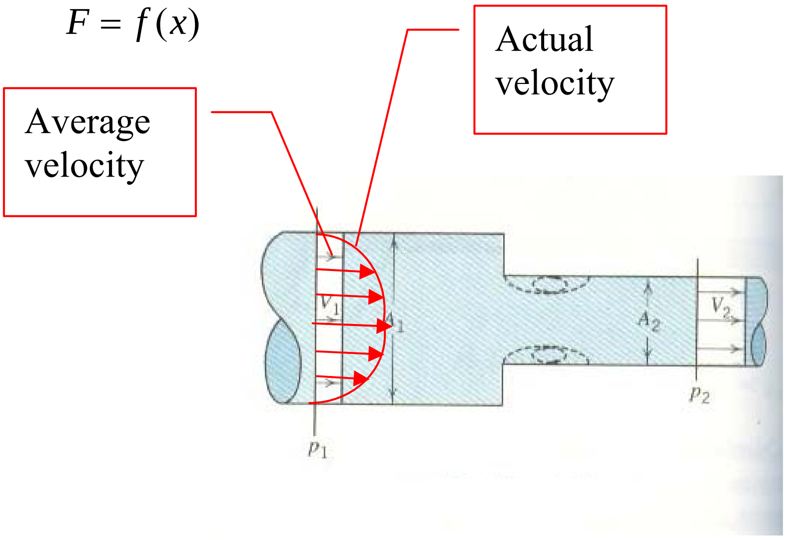
  ~ All fluid particles are assumed uniform over any cross section.

  ~ The change of fluid variables perpendicular to (across) a streamline is negligible compared to the change along the streamline.

  ~ powerful, simple

  ~ pipe flow, flow in a stream tube - average fluid properties are used at each section

$$F = f(x)$$



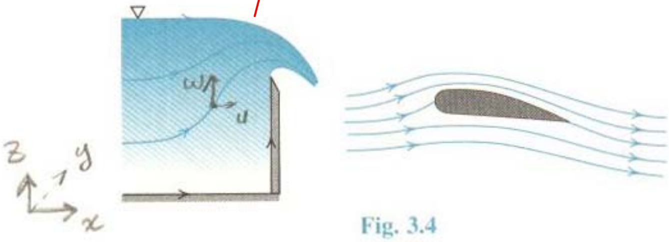
- Two-dimensional flow

  ~ flow fields defined by streamlines in a single plane (unit width)

  ~ flows over weir and about wing - Fig. 3.4

  ~ assume end effects on weir and wing is negligible

$$v = f(x,z)$$



flow over the weir

Fig. 3.4

- Three-dimensional flow

~ The flow fields defined by streamlines in space.

~ axisymmetric three-dim. flow - Fig. 3.5

→ streamlines = stream surfaces

$$v = f(x,y,z)$$

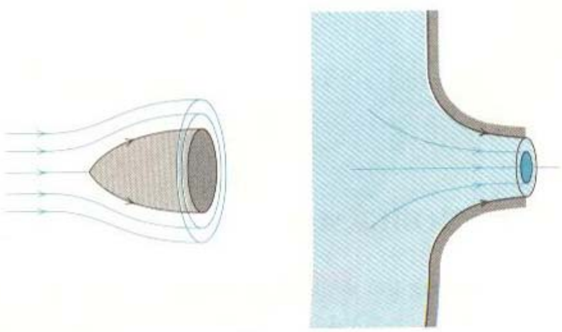


Fig. 3.5

3.3 Velocity and Acceleration

velocity, $\vec{v}$

acceleration, $\vec{a}$

~ vector: magnitude

direction - known or assumed

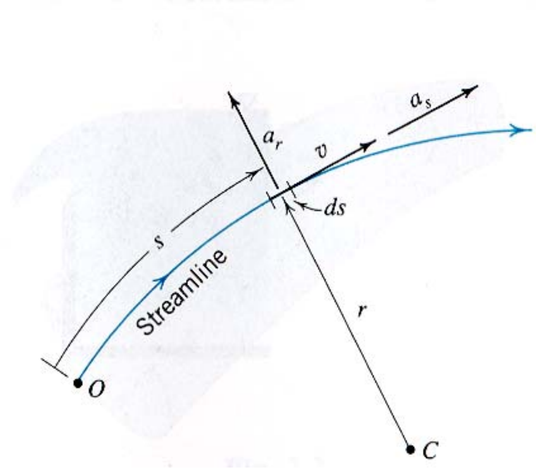


Fig. 3.6

◈ <u>One-dimensional flow</u> along a streamline

Select a fixed point 0 as a reference point and define the displacement s of a fluid particle along the streamline in the direction of motion.

→ In time *dt* the particle will cover a differential distance *ds* along the streamline.

1) Velocity

- magnitude of velocity $v = \frac{ds}{dt}$

where s = displacement

- direction of velocity = tangent to the streamline

2) Acceleration

- acceleration along (tangent to) the streamline = a_s
- acceleration (normal to) the streamline = a_r

$$a_s = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds} \tag{3.2}$$

$$a_r = -\frac{v^2}{r} \leftarrow \text{particle mechanics} \tag{3.3}$$

where r = radius of curvature of the streamline at s

[Re] Uniform circular motion

→ Particle moves in a circle with a constant speed.

- direction of $\Delta\vec{v}$: pointing inward, approximately toward the center of circle

Apply similar triangles OPP' and $P'QQ'$

$$\therefore \quad \frac{\overline{PP'}}{r} = \frac{\Delta v}{v}$$

Now, approximate $\overline{PP'}$ (chord length) as $\overset{\frown}{PP'}$ (arc length) when θ is small.

$$\therefore \quad \frac{\Delta v}{v} = \frac{v\Delta t}{r}$$

$$\frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

$a_r = -a = -\frac{v^2}{r}$ (along a radius inward toward the center of the circle)

$a_r =$ radial (centripetal) acceleration= constant in magnitude directed radially inward

- Angular velocity, ω

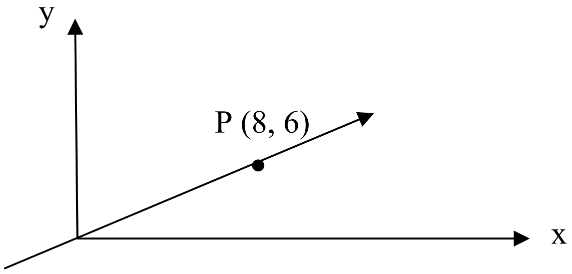
$$v = r\omega$$

$$a_r = \frac{(r\omega)^2}{r} = r\omega^2$$

[IP 3.1] p. 96

Along a straight streamline, $v = 3\sqrt{x^2 + y^2}$ m/s

Calculate velocity and acceleration at the point (8.6)



$$a_s = v\frac{dv}{ds}, \qquad a_r = -v^2 / r$$

We observe that

1) The streamline is straight. → $a_r = 0$

2) The displacement s is give as

→ $s = \sqrt{x^2 + y^2}$

Therefore, $v = 3s$

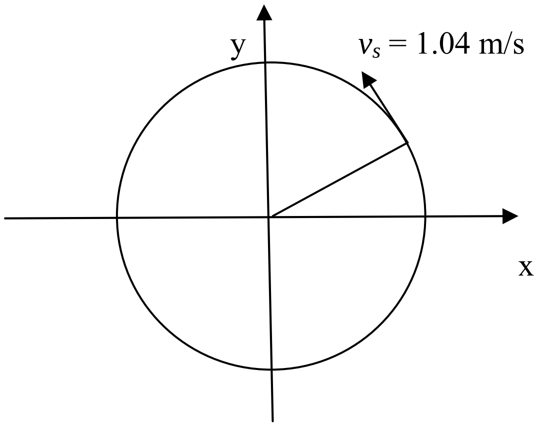
At (8.6) → $s = 10$ → $v = 3(10) = 30$ m/s

$$a_s = v\frac{dv}{ds} = 3s(3s)' = 9s = 90 \text{ m/s}^2$$

[IP 3.2]

The fluid at the wall of the tank moves along the circular streamline with a constant tangential velocity component, $v_s = 1.04\ m/s$.

Calculate the tangential and radial components of acceleration at any point on the streamline.



1) a_s

$$\boxed{\frac{dv}{ds} = 0}$$

→ Because tangential velocity is constant, $a_s = v\dfrac{dv}{ds} = 0$

2) a_r

$$a_r = -\frac{v^2}{r} = -\frac{(1.04)^2}{2} = -0.541\ \mathrm{m/sec^2}$$

→ directed toward the center of circle

◈ Flowfield

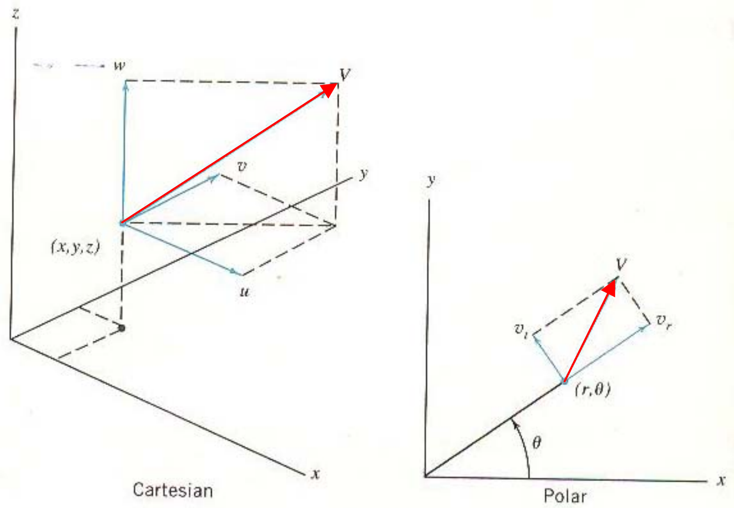


Fig. 3.7

The velocities are everywhere different in magnitude and direction at different points in the flowfield and at different times. → three-dimensional flow

At each point, each velocity has components *u, v, w* which are parallel to the *x*-, *y*-, and *z*-axes.

In Eulerian view,

$$u = u(x,\ y,\ z,\ t),\ v = v(x,\ y,\ z,\ t),\ w = w(x,\ y,\ z,\ t)$$

In Lagrangian view, velocities can be described in terms of displacement and time as

$$u = \frac{dx}{dt},\ v = \frac{dv}{dt},\ w = \frac{dz}{dt} \qquad (3.4)$$

where *x, y,* and *z* are the actual coordinates of a fluid particle that is being tracked

→ The velocity at a point is the same in both the Eulerian and the Lagrangian view.

$$a_x = \frac{du}{dt},\ a_y = \frac{dv}{dt},\ a_z = \frac{dw}{dt} \tag{3.5}$$

- Total (substantial, material) derivatives (App. 6)

$$du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$$dv = \frac{\partial v}{\partial t} dt + \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz$$

$$dw = \frac{\partial w}{\partial t} dt + \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

- Accelerations

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

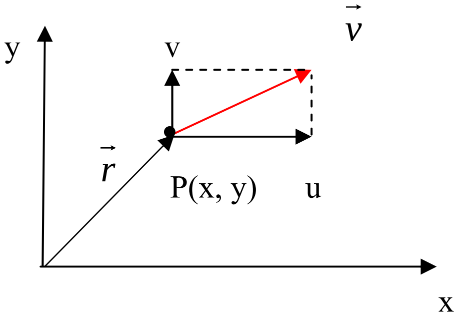
$$a_y = \frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \tag{3.6}$$

$$a_z = \frac{dw}{dt} = \underbrace{\frac{\partial w}{\partial t}}_{\text{local acceleration}} + \underbrace{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}}_{\text{convective acceleration}}$$

For steady flow, $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0$, however convective acceleration is not zero.

- 2-D steady flow

(i) Cartesian coordinate; P(x, y)



$$\vec{r} = \vec{i}x + \vec{j}y$$

$$\vec{v} = \vec{i}u + \vec{j}v$$

$$u = u(x, y)$$ - horizontal

$$v = v(x, y)$$ - vertical

$$u = \frac{dx}{dt}; \quad v = \frac{dy}{dt}$$

$$\vec{a} = \frac{\overrightarrow{dv}}{dt} = \vec{i}a_x + \vec{j}a_y$$

$$a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \qquad \boxed{\frac{\partial u}{\partial t} = 0}$$

$$a_y = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \qquad \boxed{\frac{\partial v}{\partial t} = 0}$$

(ii) Polar coordinate $P(r,\theta)$

$$\theta = \text{radian} = \frac{s}{r}$$

where $s =$ arc length, $r =$ radius

$$v_r = \frac{dr}{dt} \qquad \text{- radial} \tag{3.7a}$$

$$v_t = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega \qquad \text{- tangential} \tag{3.7b}$$

where $ds = rd\theta,\ \omega = \dfrac{d\theta}{dt} = \dfrac{v_t}{r}$

[Re] Conversion

$$x = r\cos\theta,\ y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2},\ \theta = \arctan\frac{y}{x}$$

- Total derivative in Polar coordinates

$$dv_r = \frac{\partial v_r}{\partial r} dr + \frac{\partial v_r}{\partial \theta} d\theta$$

$$dv_t = \frac{\partial v_t}{\partial r} dr + \frac{\partial v_t}{\partial \theta} d\theta$$

$$\frac{dv_r}{dt} = \frac{\partial v_r}{\partial r}\frac{dr}{dt} + \frac{\partial v_r}{\partial \theta}\frac{d\theta}{dt} = v_r \frac{\partial v_r}{\partial r} + \frac{\partial v_r}{\partial \theta}\frac{1}{r}\frac{\partial s}{\partial t} = v_r \frac{\partial v_r}{\partial r} + \frac{\partial v_r}{\partial \theta}\frac{1}{r} v_t$$

$$\frac{dv_t}{dt} = \frac{\partial v_t}{\partial r}\frac{dr}{dt} + \frac{\partial v_t}{\partial \theta}\frac{d\theta}{dt} = \frac{\partial v_t}{\partial r} v_r + \frac{\partial v_t}{\partial \theta}\frac{1}{r} v_t$$

- Acceleration in Polar coordinates → Hydrodynamics (Lamb, 1959)

$$a_r = a_{rr} + a_{rt}$$

where a_{rr} = acceleration due to variation of v_r in $r-$direction;

a_{rt} = acceleration due to variation of v_t in $r-$direction

$$\boxed{\omega = \frac{v_t}{r}}$$

$$a_r = \underset{\substack{\downarrow \\ a_{rr}}}{\frac{dv_r}{dt}} - \underset{\substack{\downarrow \\ a_{rt}}}{\omega v_t} = \underline{v_r \frac{\partial v_r}{\partial r} + v_t \frac{1}{r}\frac{\partial v_r}{\partial \theta}} - \frac{d\theta}{dt} v_t = v_r \frac{\partial v_r}{\partial r} + v_t \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_t^2}{r}$$

$$\boxed{\frac{dv_r}{dt} = v_r \frac{\partial v_r}{\partial r} + \frac{\partial v_r}{\partial \theta}\frac{1}{r} v_t}$$

$a_t = a_{tt} + a_{tr}$

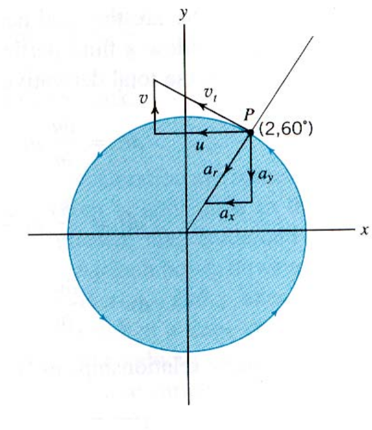
$$a_t = \underset{\overset{\downarrow}{a_{tt}}}{\frac{dv_t}{dt}} + \underset{\overset{\downarrow}{a_{tr}}}{\omega v_r} = v_r \frac{\partial v_t}{\partial r} + v_t \frac{1}{r}\frac{\partial v_t}{\partial \theta} + \frac{v_r v_t}{r}$$

[IP 3.3] p. 99

For circular streamline along which $v_t = 1.04 \text{ m/s}$, $r = 2 \text{ m}$ (radius of curvatures)

Calculate $\begin{pmatrix} u,\ v,\ v_t,\ v_r \\ a_x,\ a_y,\ a_t,\ a_r \end{pmatrix}$ at $P(2 \text{ m}, 60^\circ)$

[Sol]



Determine $P(x, y)$

$$x = 2\cos 60^\circ = 1$$

$$y = 2\sin 60^\circ = \sqrt{3}$$

1) Velocity

- Polar coordinate

$$v_t = 1.04 \text{ m/s}$$

$$v_r = \frac{dr}{dt} = 0$$

- Cartesian coordinate

Apply similar triangles

$$v_t : u = r : y\,,\;\; r = \sqrt{x^2 + y^2} = 2$$

$$u = \frac{-v_t y}{r} = -v_t \sin 60^\circ u = -(1.04)\sin 60^\circ = -0.90 \text{ m/s}$$

$$v = \frac{v_t x}{r} = v_t \cos 60^\circ = (1.04)\cos 60^\circ = 0.52 \text{ m/s}$$

2) Accelerations

- Cartesian coordinate

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1.04y}{r}\underline{\cancel{\frac{\partial}{\partial x}\left(-\frac{1.04y}{r}\right)}} + \frac{1.04x}{r}\frac{\partial}{\partial y}\left(-\frac{1.04y}{r}\right)$$

$$= \frac{-(1.04)^2}{r^2}x = -\frac{1.082}{4}x$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1.04y}{r}\frac{\partial}{\partial x}\left(\frac{1.04x}{r}\right) + \underline{\cancel{\frac{1.04x}{r}\frac{\partial}{\partial y}\left(\frac{1.04x}{r}\right)}}$$

$$= -\frac{1.082}{4}y$$

At , $P(1,\sqrt{3})$, $$a_x = -\frac{1.082}{4}(1) = -0.27 \text{ m/s}^2$$

$$a_y = -\frac{1.082}{4}\sqrt{3} = -0.47 \text{ m/s}^2$$

- Polar coordinate

$$a_r = \cancel{v_r}\frac{\partial v_r}{\partial r} + v_t \cancel{\frac{\partial v_r}{r\partial \theta}} - \frac{v_t^2}{r} = -\frac{(1.04)^2}{2} = -0.54 \text{ m/s}^2$$

→ direction toward the center of circle

$$a_t = \cancel{v_r}\frac{\partial v_t}{\partial r} + v_t \frac{\partial v_t}{r\partial \theta} + \cancel{\frac{v_r v_t}{r}} = 1.04\frac{\partial}{r\partial \theta}(1.04) = 0 \text{ m/s}^2$$

[Re]

$$a_x^2 + a_y^2 = -(0.27)^2 + (0.47)^2 = 0.073 + 0.22 = 0.29$$

$$a_r^2 = (-0.54)^2 = 0.29$$

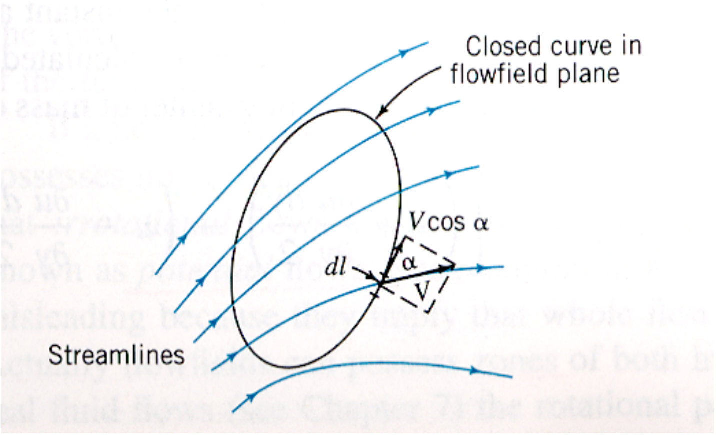
$$\therefore a_r^2 + a_t^2 = a_x^2 + a_y^2$$

3.4 Circulation, Vorticity, and Rotation

3.4.1 Circulation

As shown in IP 3.3, tangential components of the velocity cause the fluid in a flow a swirl.

→ A measure of swirl can be defined as circulation.



- Circulation, Γ

= line integral of the tangential component of velocity around a closed curve fixed in the flow (circle and squares)

$$d\Gamma = (V\cos a)dl$$

$$\Gamma = \oint_C d\Gamma = \oint (V\cos\alpha)dl = \oint \vec{V}\cdot\vec{dl} \qquad (3.9)$$

where $\vec{dl}$ = elemental vector of size dl and direction tangent to the control surface at each point

[Re] vector dot product

$$\vec{a}\cdot\vec{b}=ab\cos\alpha$$

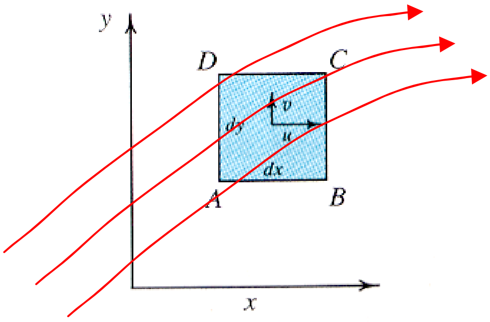
[Cf] Integral of <u>normal component of velocity</u> → Continuity equation

$$\oint_{C.S.}\rho\vec{V}\cdot\vec{n}\,dA=0\quad\rightarrow \text{Ch. 4}$$

$\vec{n}$ = unit normal vector

• Point value of the circulation in a flow for square of differential size

→ proceed from A counterclockwise around the boundary of the element



$$d\Gamma\cong\begin{bmatrix}\text{mean velocity}\\ \text{along } AB\end{bmatrix}dx\cos 0^\circ+\begin{bmatrix}\text{mean velocity}\\ \text{along } BC\end{bmatrix}dy\ \cos 0^\circ$$

$$+\begin{bmatrix}\text{mean velocity}\\ \text{along } CD\end{bmatrix}dx\cos 180^\circ+\begin{bmatrix}\text{mean velocity}\\ \text{along } DA\end{bmatrix}dy\cos 180^\circ$$

$$\begin{bmatrix}\text{mean velocity}\\ \text{along } AB\end{bmatrix} dx\cos 0^\circ = \left[u - \frac{\partial u}{\partial y}\frac{dy}{2}\right]dx$$

$$\begin{bmatrix}\text{mean velocity}\\ \text{along } CD\end{bmatrix} dx\cos 180^\circ = \left[u + \frac{\partial u}{\partial y}\frac{dy}{2}\right](-dx)$$

$$d\Gamma \cong \left[u - \frac{\partial u}{\partial y}\frac{dy}{2}\right]dx + \left[v + \frac{\partial v}{\partial x}\frac{dx}{2}\right]dy - \left[u + \frac{\partial u}{\partial y}\frac{dy}{2}\right]dx - \left[v - \frac{\partial v}{\partial x}\frac{dx}{2}\right]dy$$

Expanding the products and retaining only the terms of lowest order (largest magnitude) gives

$$d\Gamma = udx - \frac{\partial u}{\partial y}\frac{dxdy}{2} + vdy + \frac{\partial v}{\partial x}\frac{dxdy}{2} - udx - \frac{\partial u}{\partial y}\frac{dxdy}{2} - vdy + \frac{\partial v}{\partial x}\frac{dxdy}{2}$$

$$\boxed{d\Gamma = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy}$$

where *dx dy* is the area inside the control surface

3.4.2 Vorticity, ξ

~ measure of the rotational movement

~ differential circulation per unit area enclosed

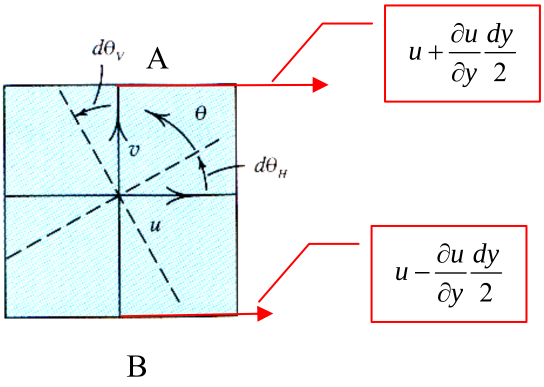
$$\boxed{\xi = \frac{d\Gamma}{dxdy} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}} \tag{3.10}$$

[Cf] continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

For polar coordinates

$$\xi = \frac{\partial v_t}{\partial r} + \frac{v_t}{r} - \frac{\partial v_r}{r\partial\theta} \tag{3.11}$$

3.4.3 Angular rotation



If the fluid element tends to rotate, two lines will tend to rotate also.

→ For the instant their average angular velocity can be calculated.

For vertical line AB

$$\widehat{d\theta_V} = \frac{ds}{dy} = \frac{dudt}{dy} = du\frac{dt}{dy}$$

$$= -\left[\left(\cancel{u} + \frac{\partial u}{\partial y}\frac{dy}{2}\right) - \left(\cancel{u} - \frac{\partial u}{\partial y}\frac{dy}{2}\right)\right]\frac{dt}{dy} = -\frac{\partial u}{\partial y}dt$$

$$\therefore \omega_V = \frac{d\theta_V}{dt} = -\frac{\partial u}{\partial y}$$

where $\omega =$ rate of rotation

For horizontal line

$$\widehat{d\theta_H} = +\left[\left(v + \frac{\partial v}{\partial x}\frac{dx}{2}\right) - \left(v - \frac{\partial v}{\partial x}\frac{dx}{2}\right)\right]\frac{dt}{dx} = +\frac{\partial v}{\partial x}dt$$

$$\omega_H = \frac{d\theta_H}{dt} = \frac{\partial v}{\partial x}$$

Consider average rotation

$$\boxed{\omega = \frac{1}{2}(\omega_V + \omega_H) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)} \qquad (3.24)$$

$$\boxed{\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2\omega}$$

rotational flow ~ flow possesses vorticity → $\xi \neq 0$

irrotational flow ~ flow possesses no vorticity, no net rotation → $\xi = 0$

= <u>potential flow</u> (velocity potential exists) ← Ch. 5

- Actually flow fields can possess zones of both irrotational and rotational flows.

free vortex flow → irrotational flow, bath tub, hurricane, morning glory spillway

forced vortex flow → rotational flow, rotating cylinder

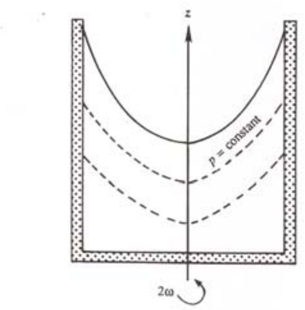


Figure 5.2 Constant pressure surfaces in a solid-body rotation generated in a rotating tank containing liquid.

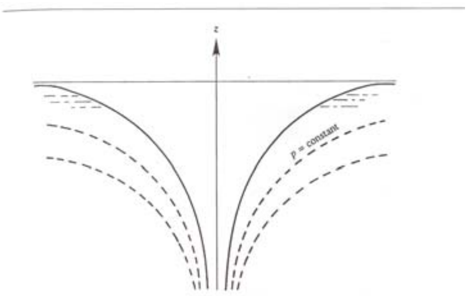
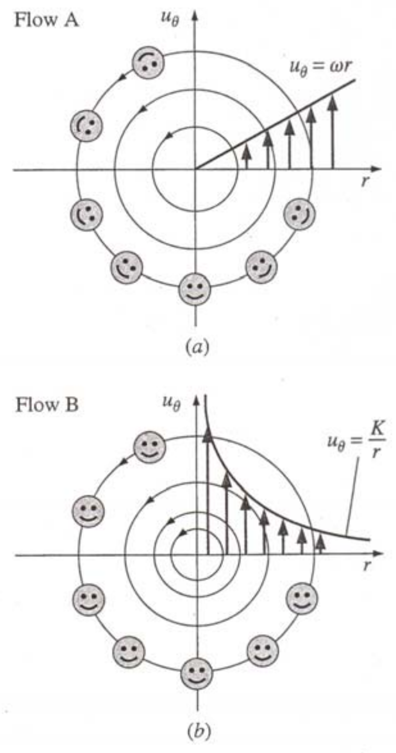


Figure 5.3 Irrotational vortex in a liquid.

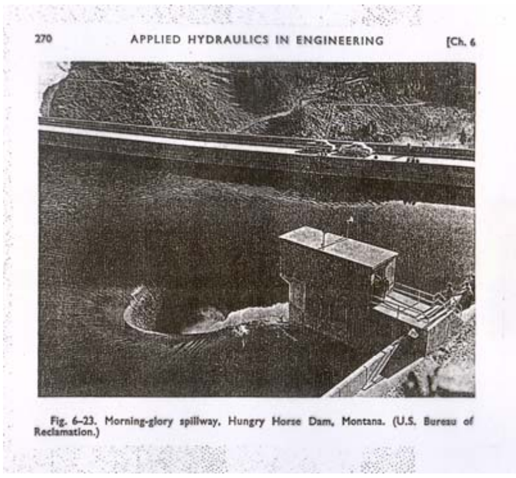


Rotational

Irrotational

FIGURE 4–28

Streamlines and velocity profiles for (*a*) flow A, solid-body rotation and (*b*) flow B, a line vortex. Flow A is rotational, but flow B is irrotational everywhere except at the origin.



270 APPLIED HYDRAULICS IN ENGINEERING [Ch. 6

Fig. 6-23. Morning-glory spillway, Hungry Horse Dam, Montana. (U.S. Bureau of Reclamation.)

(*a*) (*b*)

FIGURE 4–29

A simple analogy: (*a*) *rotational* circular flow is analogous to a roundabout, while (*b*) *irrotational* circular flow is analogous to a Ferris wheel.

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와류실험기를 통한 강제와류와 자유와류 실험

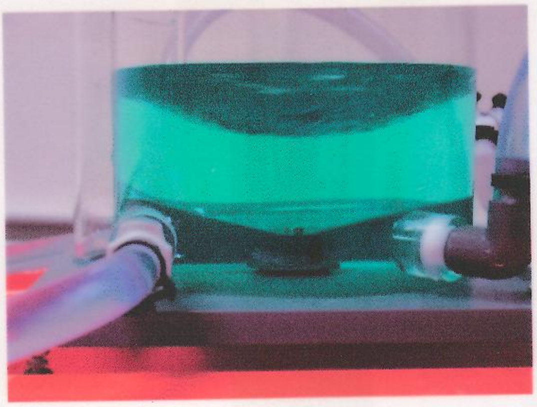


Figure. 1 Forced Vortex

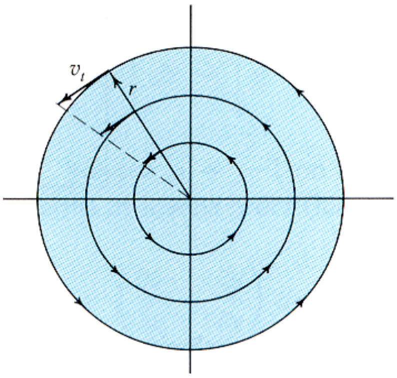
Figure. 2 Free Vortex

[IP 3.4] Calculate the vorticity of two-dimensional flowfield described by the equations

$v_t = \omega r$ and $v_r = 0$

→ Forced vortex

→ A cylindrical container is rotating at an angular velocity ω.



[Sol]

$$\xi = \frac{\partial v_t}{\partial r} + \frac{v_t}{r} - \frac{\partial v_r}{r \partial \theta}$$

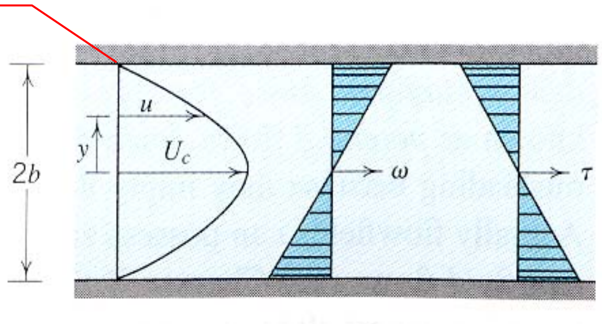
$$\xi = \frac{\partial}{\partial r}(\omega r) + \frac{\omega r}{r} - \frac{\partial}{r \partial \theta}(0) = \omega + \omega - 0 = 2\omega \neq 0$$

→ rotational flow (forced vortex) possessing a constant vorticity over the whole flow field

→ streamlines are concentric circles

[IP 3.5] When a viscous, incompressible fluid flow between two plates and the flow is laminar and two-dimensional, the velocity profile is parabolic, $u = U_c\left(1-\frac{y^2}{b^2}\right)$.

Calculate τ and ω (rotation)



No slip condition for real fluid

[Sol]

$$\frac{\partial v}{\partial x}=0; \qquad \frac{\partial u}{\partial y}=-U_c\frac{2y}{b^2}$$

1) $\omega=\frac{1}{2}(\omega_V+\omega_H)=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$

$$=-\frac{1}{2}\left(-U_c\frac{2y}{b^2}\right)=\left(\frac{U_c}{b^2}\right)y$$

2) $\tau=\mu\frac{du}{dy}$

$$\tau=\mu\frac{du}{dy}=\left(-\frac{2\mu U_c}{b^2}\right)y$$

$$\tau=-2\mu\omega=-\mu\xi$$

→ rotation and vorticity are large where shear stress is large.

Homework Assignment # 3

Due: 1 week from today

Prob. 3.3

Prob. 3.5

Prob. 3.6

Prob. 3.10

Prob. 3.12

Prob. 3.15