Naval Architecture & Ocean Engineering

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Engineering Mathematics 2

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Elementary Row Operations. Row-Equivalent System

Elementary *Row* Operations for Matrices :

Elementary Operations for Equations :

Elementary Row Operations.
Row-Equivalent System

Elementary *Row* Operations for Matrices :

1) ***Interchange of two rows***

Elementary Operations for Equations :

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Elementary Row Operations. Row-Equivalent System

Elementary *Row* Operations for Matrices :

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2) *Addition of a constant multiple of one row to another row*

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Elementary Row Operations. Row-Equivalent System

Elementary *Row* Operations for Matrices :

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2) ***Addition of a constant multiple of one row to another row***

3) ***Multiplication of a row by a nonzero constant*** c

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Elementary Row Operations. Row-Equivalent (행동치) System

Clearly, the interchange of two equations does not alter the solution set. Neither does that addition because we can undo it by a corresponding subtraction. Similarly for that multiplication, which we can undo by multiplying the new equation by $1/c$ (since $c\neq 0$), producing the original equation.

We now call a linear system S_1 row-equivalent to a linear system S_2 if S_1 can be obtained from S_2 by (finitely many!) row operations. Thus we have proved the following result, which also justifies the Gauss elimination.

Because of this theorem, systems having the same solution sets are often called equivalent systems. But note well that we are dealing with row operations.

No column operations on the augmented matrix are permitted in this context because they would generally alter the solution set.

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↓ Raw Operations

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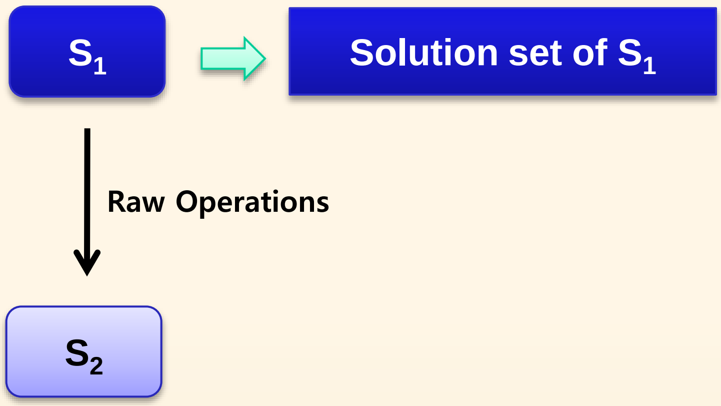

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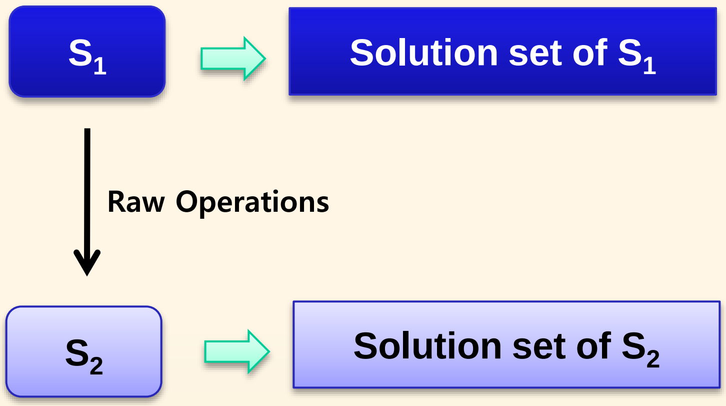

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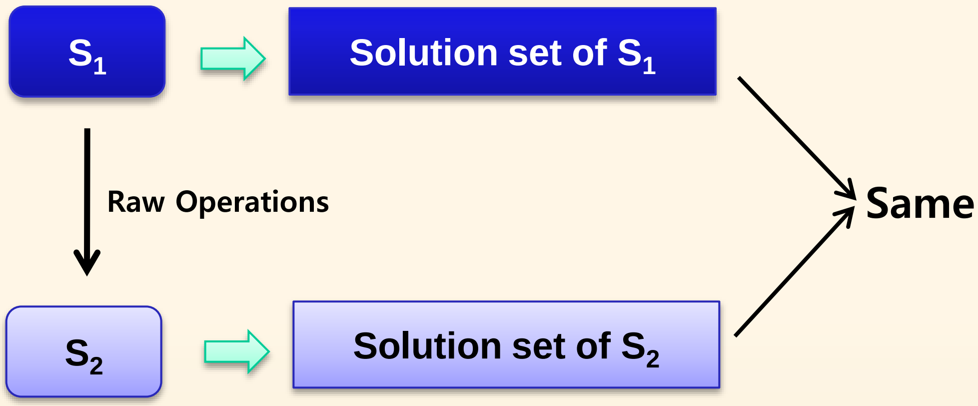

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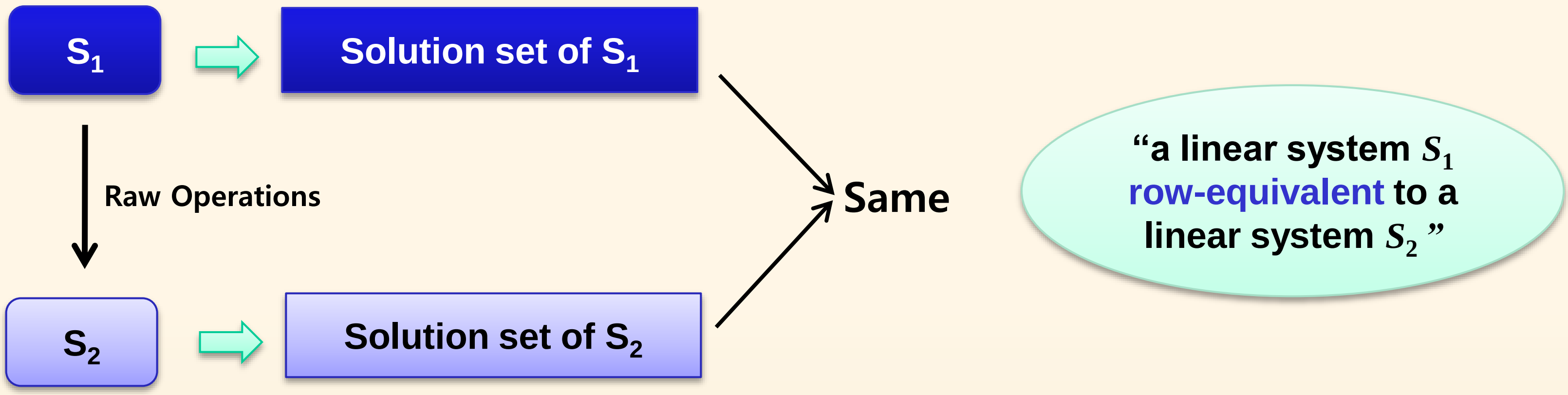

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S_1 ⇨ Solution set of S_1

Raw Operations ①

S_2 ⇨ Solution set of S_2

Same

"a linear system S_1 **row-equivalent** to a linear system S_2 "

① : "elimination"

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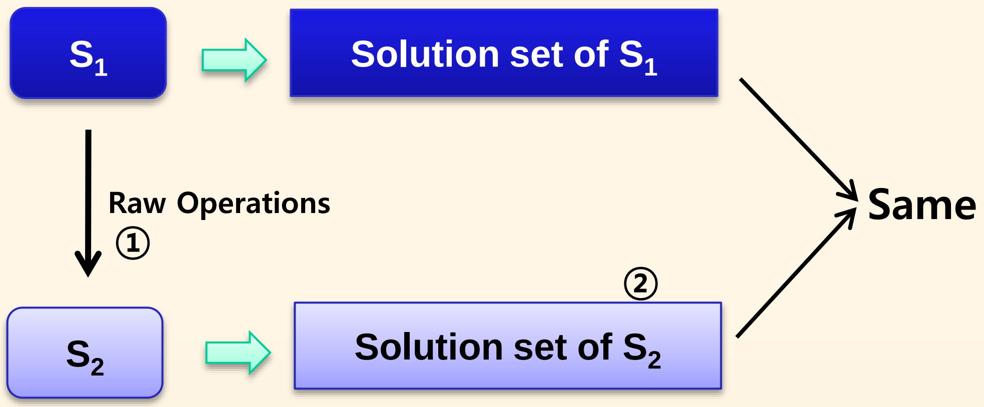

"a linear system S_1 row-equivalent to a linear system S_2 "

① : "elimination" →② : Solution of S_2 found

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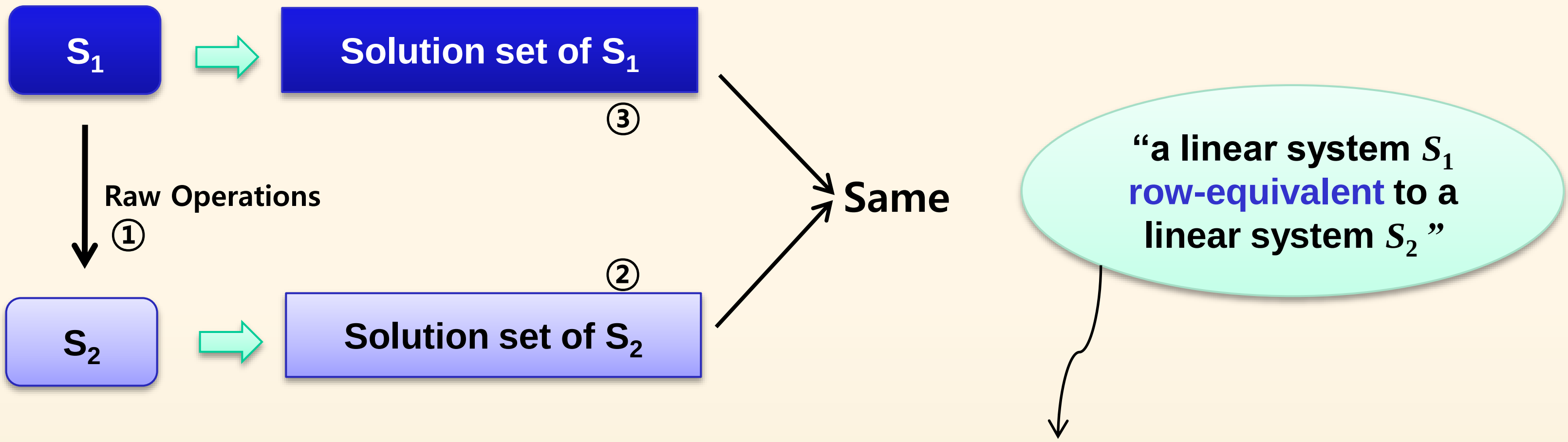

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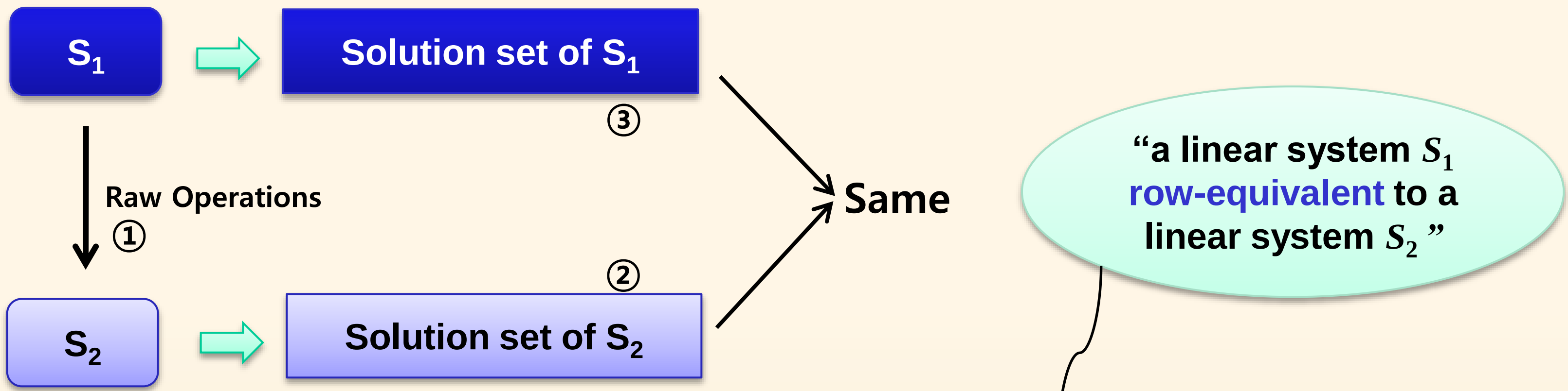

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Gauss Elimination

Gauss Elimination and Back Substitution

This is a standard elimination method for solving linear systems that proceeds systematically irrespective of particular features of the coefficients.

If a system is in "triangular form" we can solve it by "back substitution".

Triangular Matrices

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ -8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}$$

Gauss Elimination and Back Substitution

$$x_1 + 2x_2 + x_3 = 1$$
$$3x_1 - x_2 - x_3 = 2$$
$$2x_1 + 3x_2 - x_3 = -3$$

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row2 +
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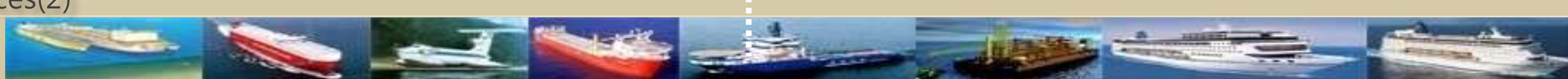

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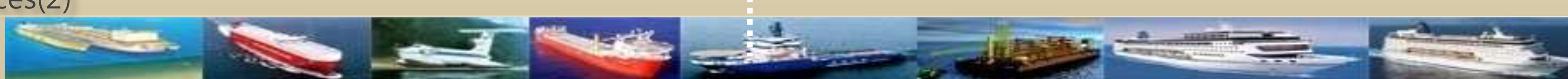

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Gauss Elimination and Back Substitution

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - \ \ x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ -7x_2 - 4x_3 &= -1 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$

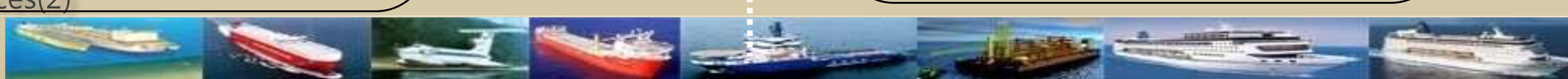
row3 + row1x(−2)

row3 + row1x(−2)

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= -3 \\ +)\ -2x_1 - 4x_2 - 2x_3 &= -2 \\ \hline -x_2 - 3x_3 &= -5 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ -7x_2 - 4x_3 &= -1 \\ -x_2 - 3x_3 &= -5 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 0 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix}$$



Gauss Elimination and Back Substitution

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row 2 ↔ row 3

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ -x_2 - 3x_3 &= -5 \\ -7x_2 - 4x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & -7 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -1 \end{bmatrix}$$

row 2x(−1)

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_2 + 3x_3 &= 5 \\ -7x_2 - 4x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -7 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

Gauss Elimination and Back Substitution

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

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Gauss Elimination and Back Substitution

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\begin{aligned}
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\end{aligned}
$$

$$
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$$

row 3+ row 2x7

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 1 \\
x_2 + 3x_3 &= 5 \\
17x_3 &= 34
\end{aligned}
$$

Gauss Elimination and Back Substitution

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row 3+ row 2x7

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_2 + 3x_3 &= 5 \\ 17x_3 &= 34 \end{aligned}$$

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Gauss Elimination and Back Substitution

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row 3+ row 2x7

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$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}$$

The last equations and matrix are equal to given equations.

Gauss Elimination and Back Substitution

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 1 \\
x_2 + 3x_3 &= 5 \\
17x_3 &= 34
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}
$$

We can solve this by "Back substitution", that is, solve the last equation for the variable, and then work backward, substituting the value of the variable into the above equation and solve it for another variable.

Gauss Elimination and Back Substitution

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 1 \\
x_2 + 3x_3 &= 5 \\
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$$
\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}
$$

$$
x_3 = \frac{34}{17} = 2
$$

We can solve this by "Back substitution", that is, solve the last equation for the variable, and then work backward, substituting the value of the variable into the above equation and solve it for another variable.

Gauss Elimination and Back Substitution

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 1 \\
x_2 + 3x_3 &= 5 \\
17x_3 &= 34
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}
$$

$$x_3 = \frac{34}{17} = 2$$

$$x_2 + 3x_3 = x_2 + 3 \cdot 2 = 5$$

We can solve this by "Back substitution", that is, solve the last equation for the variable, and then work backward, substituting the value of the variable into the above equation and solve it for another variable.

Gauss Elimination and Back Substitution

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_2 + 3x_3 &= 5 \\ 17x_3 &= 34 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}$$

$$x_3 = \frac{34}{17} = 2$$

$$x_2 + 3x_3 = x_2 + 3 \cdot 2 = 5$$

$$\therefore x_2 = -1$$

We can solve this by "Back substitution", that is, solve the last equation for the variable, and then work backward, substituting the value of the variable into the above equation and solve it for another variable.

Gauss Elimination and Back Substitution

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_2 + 3x_3 &= 5 \\ 17x_3 &= 34 \end{aligned}$$

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$$x_1 + 2x_2 + x_3 = x_1 + 2 \cdot (-1) + 2 = 1$$

Gauss Elimination and Back Substitution

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_2 + 3x_3 &= 5 \\ 17x_3 &= 34 \end{aligned}$$

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$$x_3 = \frac{34}{17} = 2$$

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Gauss Elimination and Back Substitution

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 1 \\
x_2 + 3x_3 &= 5 \\
17x_3 &= 34
\end{aligned}
\qquad x_3
$$

$$
\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}
$$

$$x_3 = \frac{34}{17} = 2$$

We can solve this by "Back substitution", that is, solve the last equation for the variable, and then work backward, substituting the value of the variable into the above equation and solve it for another variable.

$$x_2 + 3x_3 = x_2 + 3\cdot 2 = 5$$

$$\therefore x_2 = -1$$

$$x_1 + 2x_2 + x_3 = x_1 + 2\cdot(-1) + 2 = 1$$

$$\therefore x_1 = 1$$

Gauss Elimination and Back Substitution

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_2 + 3x_3 &= 5 \\ 17x_3 &= 34 \end{aligned}$$

x_2

x_3

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}$$

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Gauss Elimination and Back Substitution

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_2 + 3x_3 &= 5 \\ 17x_3 &= 34 \end{aligned}$$

x_1

x_2

x_3

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}$$

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Gauss Elimination and Back Substitution

Since a linear system is completely determined by its augmented matrix, Gauss elimination can be done by merely considering the matrices.

Gauss Elimination and Back Substitution

Since a linear system **is completely determined by its** augmented matrix**, Gauss elimination can be done by merely considering the matrices.**

augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & -1 & -1 & 2 \\ 2 & 3 & -1 & -3 \end{array}\right]$$

Gauss Elimination and Back Substitution

Since a linear system is completely determined by its augmented matrix, Gauss elimination can be done by merely considering the matrices.

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$$\vdots \qquad\qquad \vdots$$

Gauss Elimination and Back Substitution

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$$\vdots \qquad\qquad \vdots$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 17 & 34 \end{array}\right]$$

Row-echelon form

Gauss Elimination : The Three Possible Cases of Systems

case 1 : Gauss Elimination if Infinitely Many Solutions Exist

three equations < four unknowns

$$\left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 2.7 \\ 1.2x_1 - 0.3x_2 + 0.3x_3 + 2.4x_4 &= 2.1 \end{aligned}$$

Row2-0.2*Row1

$\Downarrow$

Row3-0.4*Row1

$\Downarrow$

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Row2-0.2*Row1

Row3-0.4*Row1

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right]$$

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ -1.1x_2 - 1.1x_3 + 4.4x_4 &= -1.1 \end{aligned}$$

$$\Downarrow$$

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$\Downarrow$

Row2-0.2*Row1

Row3-0.4*Row1

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right]$$

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$\Downarrow$

Row3+Row2

Gauss Elimination : The Three Possible Cases of Systems

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$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

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Row2-0.2*Row1

⇩

Row3-0.4*Row1

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right]$$

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ -1.1x_2 - 1.1x_3 + 4.4x_4 &= -1.1 \end{aligned}$$

⇩

Row3+Row2

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ 0 &= 0 \end{aligned}$$

Gauss Elimination : The Three Possible Cases of Systems

case 1 : Gauss Elimination if Infinitely Many Solutions Exist

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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Back substitution.

Gauss Elimination : The Three Possible Cases of Systems

case 1 : Gauss Elimination if Infinitely Many Solutions Exist

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Back substitution.

From the second equation : $x_2 = 1 - x_3 + 4x_4$

Gauss Elimination : The Three Possible Cases of Systems

case 1 : Gauss Elimination if Infinitely Many Solutions Exist

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Gauss Elimination : The Three Possible Cases of Systems

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Back substitution.

From the second equation : $x_2 = 1 - x_3 + 4x_4$

From the first equation : $x_1 = 1 - x_4$

Since x_3 and x_4 remain arbitrary, we have infinitely many solutions.

Gauss Elimination : The Three Possible Cases of Systems

case 1 : Gauss Elimination if Infinitely Many Solutions Exist

$$\left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ 0 &= 0 \end{aligned}$$

Back substitution.

From the second equation : $x_2 = 1 - x_3 + 4x_4$

From the first equation : $x_1 = 1 - x_4$

Since x_3 and x_4 remain arbitrary, we have infinitely many solutions.

If we choose a value of x_3 and a value of x_4, then the corresponding values of x_1 and x_2 are uniquely determined.

Gauss Elimination : The Three Possible Cases of Systems

case 2 : Gauss Elimination if no Solution Exists

$$\begin{bmatrix} 3 & 2 & 1 & \vdots & 3 \\ 2 & 1 & 1 & \vdots & 0 \\ 6 & 2 & 4 & \vdots & 6 \end{bmatrix} \qquad \begin{cases} 3x_1 + 2x_2 + \ \ x_3 = 3 \\ 2x_1 + \ \ x_2 + \ \ x_3 = 0 \\ 6x_1 + 2x_2 + 4x_3 = 6 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow$$

Row2-2/3*Row1

Row3-2*Row1

$$\Downarrow \qquad\qquad \Downarrow$$

Gauss Elimination : The Three Possible Cases of Systems

case 2 : Gauss Elimination if no Solution Exists

$$\begin{bmatrix} 3 & 2 & 1 & \vdots & 3 \\ 2 & 1 & 1 & \vdots & 0 \\ 6 & 2 & 4 & \vdots & 6 \end{bmatrix} \qquad \begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ 2x_1 + x_2 + x_3 = 0 \\ 6x_1 + 2x_2 + 4x_3 = 6 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\begin{bmatrix} 3 & 2 & 1 & \vdots & 3 \\ 0 & -1/3 & 1/3 & \vdots & -2 \\ 0 & -2 & 2 & \vdots & 0 \end{bmatrix} \qquad \begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ -1/3x_2 + 1/3x_3 = -2 \\ - 2x_2 + 4x_3 = 0 \end{cases}$$

Row2-2/3*Row1

Row3-2*Row1

$$\Downarrow \qquad\qquad \Downarrow$$

Row3-6*Row3

Gauss Elimination : The Three Possible Cases of Systems

case 2 : Gauss Elimination if no Solution Exists

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \qquad \begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ 2x_1 + x_2 + x_3 = 0 \\ 6x_1 + 2x_2 + 4x_3 = 6 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow$$

Row2-2/3*Row1

Row3-2*Row1

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \qquad \begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ -1/3x_2 + 1/3x_3 = -2 \\ -2x_2 + 4x_3 = 0 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow$$

Row3-6*Row3

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] \qquad \begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ -1/3x_2 + 1/3x_3 = -2 \\ 0 = 12 \end{cases}$$

Gauss Elimination : The Three Possible Cases of Systems

case 2 : Gauss Elimination if no Solution Exists

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \qquad \begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ 2x_1 + x_2 + x_3 = 0 \\ 6x_1 + 2x_2 + 4x_3 = 6 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \qquad \begin{array}{l} \text{Row2-2/3*Row1} \\ \text{Row3-2*Row1} \end{array} \qquad \begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ -1/3x_2 + 1/3x_3 = -2 \\ -2x_2 + 4x_3 = 0 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow$$

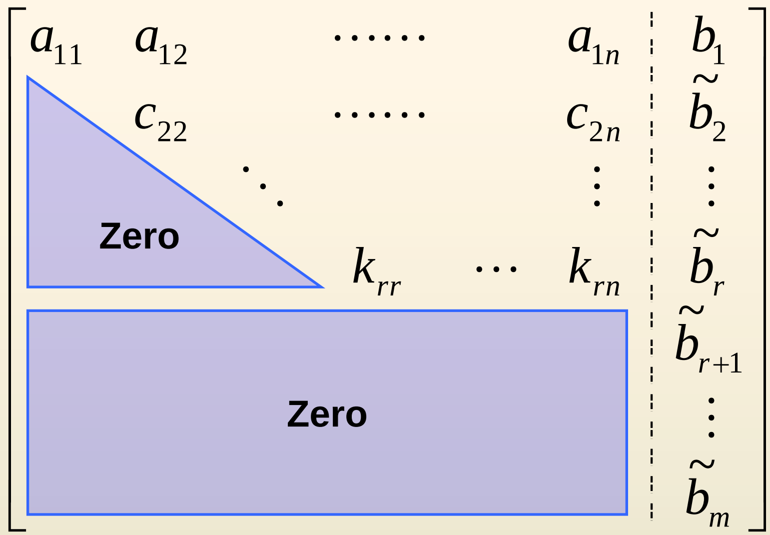
$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] \qquad \text{Row3-6*Row3} \qquad \begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ -1/3x_2 + 1/3x_3 = -2 \\ 0 = 12 \end{cases}$$

The false statement 0=12 show that the system has no solution.

Gauss Elimination : The Three Possible Cases of Systems

Row Echelon Form

At the end of the Gauss elimination (before the back substitution) the row-echelon form(행 사다리꼴) of the augmented matrix will be



$\cdots(8)$

Here, $r \le m$ and $a_{11} \ne 0,\ c_{22} \ne 0,\ \cdots,\ k_{rr} \ne 0$, and all the entries in the blue triangle as well as in the blue rectangle are zero. From this we see that with respect to solutions of the system with augmented matrix (8) (and thus with respect to the originally given system) there are three possible cases:

Gauss Elimination : The Three Possible Cases of Systems

Row Echelon Form

(a) Exactly one solution

if $r = n$ and $\tilde{b}_{r+1},\cdots\tilde{b}_m$, if present, are zero. To get the solution, solve the nth equation corresponding to (8) (which is $k_{nn}x_n=b_n$) for x_n, then the (n-1)st equation for x_{n-1}, and so on up the line.

(b) Infinitely many solutions

if $r < n$ and $\tilde{b}_{r+1},\cdots\tilde{b}_m$, if present, are zero. To obtain any of these solutions, choose values of x_{r-1}, ⋯, x_n arbitrary . Then solve the rth equation for x_r, then the (r-1)st equation for x_{r-1}, and so on up the line.

(c) No solution

if $r < m$ and one of the entries $\tilde{b}_{r+1},\cdots\tilde{b}_m$ is not zero.

$$
r=3\left\{\begin{array}{l} \\ \\ \\ \end{array}\right.
\overbrace{\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]}^{n=3}
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} m=4
\qquad
r=2\left\{\begin{array}{l} \\ \\ \end{array}\right.
\overbrace{\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}^{n=4}
\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} m=3
\qquad
r=2\left\{\begin{array}{l} \\ \\ \end{array}\right.
\overbrace{\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]}^{n=3}
\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} m=3
$$

$$
\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + 11x_2 + 24x_3 = 90 \\ 3x_1 - 3x_2 - 92x_3 = -190 \\ 2x_1 - 2x_2 + 2x_3 = 0 \end{cases}
\Rightarrow
\begin{cases} x_1 - x_2 + x_3 = 0 \\ 0\cdot x_1 + 10x_2 + 25x_3 = 90 \\ 0\cdot x_1 + 0\cdot x_2 - 95x_3 = -190 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = 0 \end{cases}
$$

Gauss Elimination : The Three Possible Cases of Systems

Row Echelon Form

(a) Exactly one solution

if $r = n$ and $\tilde{b}_{r+1},\cdots\tilde{b}_m$, if present, are zero. To get the solution, solve the nth equation corresponding to (8) (which is $k_{nn}x_n=b_n$) for x_n, then the $(n$-1)st equation for x_{n-1}, and so on up the line.

(b) Infinitely many solutions

if $r < n$ and $\tilde{b}_{r+1},\cdots\tilde{b}_m$, if present, are zero. To obtain any of these solutions, choose values of x_{r-1}, ⋯, x_n arbitrary . Then solve the rth equation for x_r, then the $(r$-1)st equation for x_{r-1}, and so on up the line.

(c) No solution **(no. of equations > no. of unknowns)**

if $r < m$ and one of the entries $\tilde{b}_{r+1},\cdots\tilde{b}_m$ is not zero.

$$r=3\left\{\begin{array}{c}\\ \\ \\ \end{array}\right.\overbrace{\begin{bmatrix} 1 & -1 & 1 & \vdots & 0 \\ 0 & 10 & 25 & \vdots & 90 \\ 0 & 0 & -95 & \vdots & -190 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}}^{n=3}\left.\begin{array}{c}\\ \\ \\ \\ \end{array}\right\} m=4
\qquad
r=2\left\{\begin{array}{c}\\ \\ \end{array}\right.\overbrace{\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & \vdots & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & \vdots & 1.1 \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}}^{n=4}\left.\begin{array}{c}\\ \\ \\ \end{array}\right\} m=3
\qquad
r=2\left\{\begin{array}{c}\\ \\ \end{array}\right.\overbrace{\begin{bmatrix} 3 & 2 & 1 & \vdots & 3 \\ 0 & -1/3 & 1/3 & \vdots & -2 \\ 0 & 0 & 0 & \vdots & \boxed{12} \end{bmatrix}}^{n=3}\left.\begin{array}{c}\\ \\ \\ \end{array}\right\} m=3$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + 11x_2 + 24x_3 = 90 \\ 3x_1 - 3x_2 - 92x_3 = -190 \\ 2x_1 - 2x_2 + 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 - x_2 + x_3 = 0 \\ 0\cdot x_1 + 10x_2 + 25x_3 = 90 \\ 0\cdot x_1 + 0\cdot x_2 - 95x_3 = -190 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = 0 \end{cases}$$

Gauss Elimination : The Three Possible Cases of Systems

Row Echelon Form

(a) Exactly one solution (no. of equations 'r' = no. of unknowns 'n')

if $r = n$ and $\tilde{b}_{r+1}, \cdots \tilde{b}_m$, if present, are zero. To get the solution, solve the nth equation corresponding to (8) (which is $k_{nn}x_n = b_n$) for x_n, then the $(n-1)$st equation for x_{n-1}, and so on up the line.

(b) Infinitely many solutions

if $r < n$ and $\tilde{b}_{r+1}, \cdots \tilde{b}_m$, if present, are zero. To obtain any of these solutions, choose values of $x_{r-1}, \cdots, x_n$ arbitrary . Then solve the rth equation for x_r, then the $(r-1)$st equation for x_{r-1}, and so on up the line.

(c) No solution (no. of equations > no. of unknowns)

if $r < m$ and one of the entries $\tilde{b}_{r+1}, \cdots \tilde{b}_m$ is not zero.

$$\begin{array}{c} n=3 \\ r=3 \left\{ \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right] \right\} m=4 \end{array}$$

$$\begin{array}{c} n=4 \\ r=2 \left\{ \left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \right\} m=3 \end{array}$$

$$\begin{array}{c} n=3 \\ r=2 \left\{ \left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & \boxed{12} \end{array}\right] \right\} m=3 \end{array}$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + 11x_2 + 24x_3 = 90 \\ 3x_1 - 3x_2 - 92x_3 = -190 \\ 2x_1 - 2x_2 + 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 - x_2 + x_3 = 0 \\ 0 \cdot x_1 + 10x_2 + 25x_3 = 90 \\ 0 \cdot x_1 + 0 \cdot x_2 - 95x_3 = -190 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \end{cases}$$

Gauss Elimination : The Three Possible Cases of Systems

Row Echelon Form

(a) Exactly one solution (no. of equations 'r' = no. of unknowns 'n')

if $r = n$ and $\tilde{b}_{r+1}, \cdots \tilde{b}_m$, if present, are zero. To get the solution, solve the nth equation corresponding to (8) (which is $k_{nn}x_n = b_n$) for x_n, then the $(n\text{-}1)$st equation for $x_{n\text{-}1}$, and so on up the line.

(b) Infinitely many solutions (no. of equations < no. of unknowns)

if $r < n$ and $\tilde{b}_{r+1}, \cdots \tilde{b}_m$, if present, are zero. To obtain any of these solutions, choose values of $x_{r\text{-}1}, \cdots, x_n$ arbitrary . Then solve the rth equation for x_r, then the $(r\text{-}1)$st equation for $x_{r\text{-}1}$, and so on up the line.

(c) No solution (no. of equations > no. of unknowns)

if $r < m$ and one of the entries $\tilde{b}_{r+1}, \cdots \tilde{b}_m$ is not zero.

$$r=3\left\{\begin{array}{l}\\ \\ \\ \end{array}\right. \overbrace{\quad\quad\quad\quad}^{n=3} \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right] \left.\begin{array}{l}\\ \\ \\ \\ \end{array}\right\} m=4$$

$$\left\{\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + 11x_2 + 24x_3 &= 90 \\ 3x_1 - 3x_2 - 92x_3 &= -190 \\ 2x_1 - 2x_2 + 2x_3 &= 0 \end{aligned}\right. \Rightarrow \left\{\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 0 \cdot x_1 + 10x_2 + 25x_3 &= 90 \\ 0 \cdot x_1 + 0 \cdot x_2 - 95x_3 &= -190 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 &= 0 \end{aligned}\right.$$

$$r=2\left\{\begin{array}{l}\\ \\ \end{array}\right. \overbrace{\quad\quad\quad\quad\quad}^{n=4} \left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \left.\begin{array}{l}\\ \\ \\ \end{array}\right\} m=3$$

$$\left\{\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ -x_1 - 0.3x_2 - 0.3x_3 + 0.2x_4 &= -2.3 \\ 1.5x_1 + 1.0x_2 + 1.0x_3 - 2.5x_4 &= 4.0 \end{aligned}\right.$$

$$\Downarrow$$

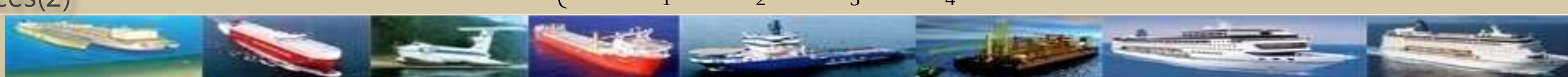
$$\left\{\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 0 \cdot x_1 + 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 &= 0.0 \end{aligned}\right.$$

$$r=2\left\{\begin{array}{l}\\ \\ \end{array}\right. \overbrace{\quad\quad\quad}^{n=3} \left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & \boxed{12} \end{array}\right] \left.\begin{array}{l}\\ \\ \\ \end{array}\right\} m=3$$

$$\left\{\begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ 9x_1 + 7x_2 + 2x_3 &= 15 \\ 3x_1 + 2x_2 + x_3 &= -9 \end{aligned}\right.$$

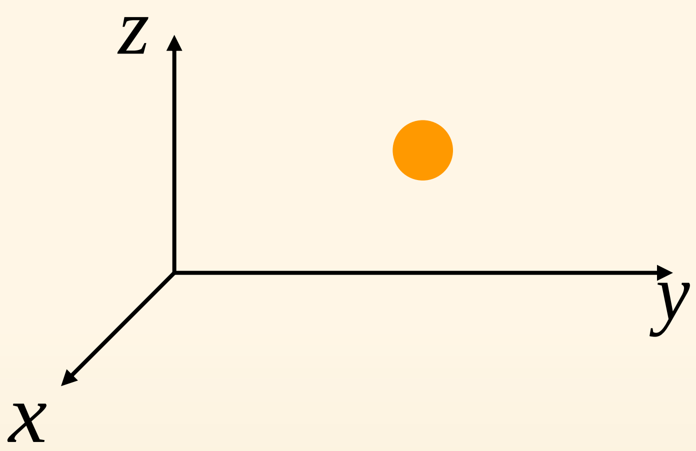
$$\Downarrow$$

$$\left\{\begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ -1/3x_2 + 1/3x_3 &= -2 \\ 0 &= 12 \end{aligned}\right.$$

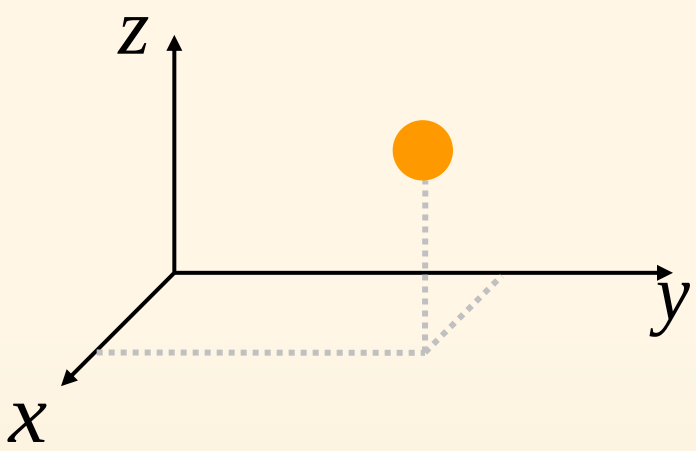


Rank of a Matrix.
Linear Independence.

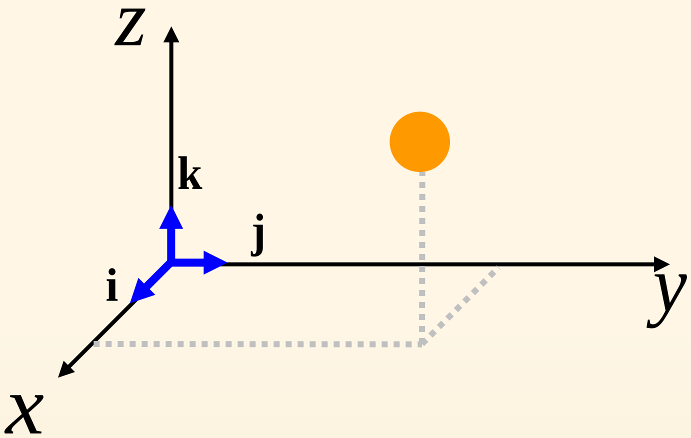
Linearly independent vectors



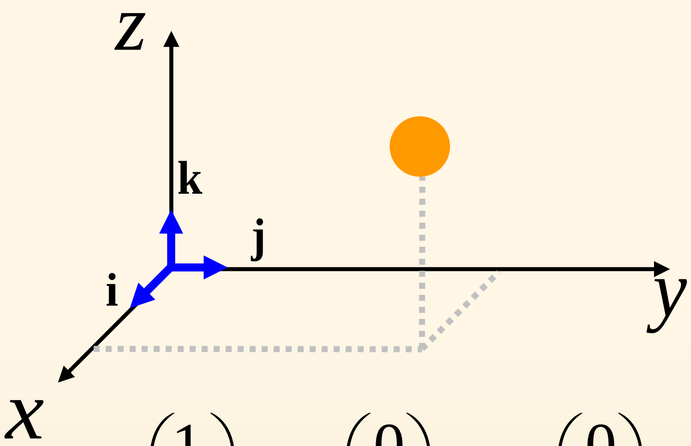
Linearly independent vectors



Linearly independent vectors

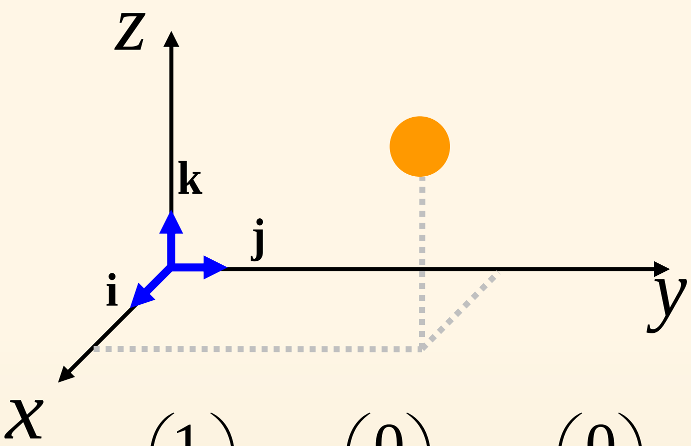


Linearly independent vectors



$$\mathbf{i}=\begin{pmatrix}1\\0\\0\end{pmatrix},\mathbf{j}=\begin{pmatrix}0\\1\\0\end{pmatrix},\mathbf{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

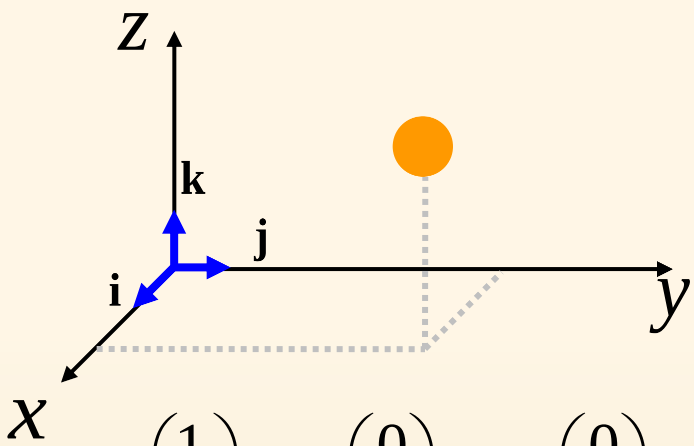
Linearly independent vectors



$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We can express the location of the point with i, j, k.

Linearly independent vectors

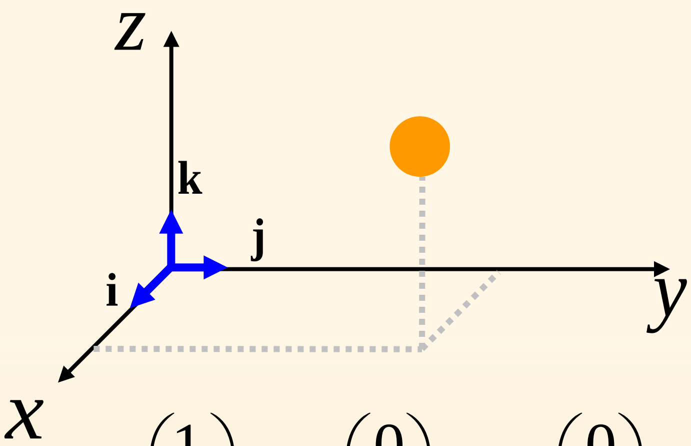


$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We can express the location of the point with i, j, k.

$$a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Linearly independent vectors



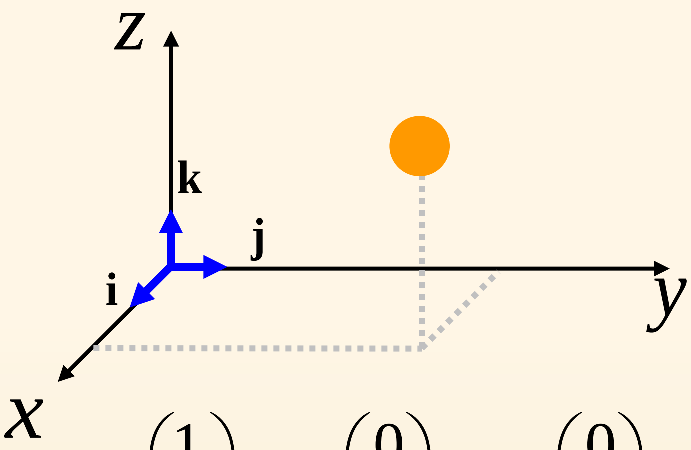
$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We can express the location of the point with i, j, k.

$$a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

If the point is at the origin, the equation becomes

Linearly independent vectors



$$\mathbf{i}=\begin{pmatrix}1\\0\\0\end{pmatrix},\mathbf{j}=\begin{pmatrix}0\\1\\0\end{pmatrix},\mathbf{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

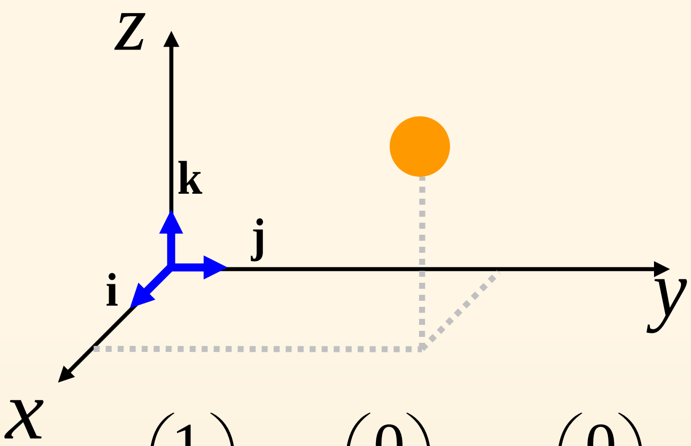
We can express the location of the point with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

$$a\mathbf{i}+b\mathbf{j}+c\mathbf{k}=a\begin{pmatrix}1\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\0\end{pmatrix}+c\begin{pmatrix}0\\0\\1\end{pmatrix}$$

If the point is at the origin, the equation becomes

$$a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\1\\0\end{bmatrix}+c\begin{bmatrix}0\\0\\1\end{bmatrix}=\mathbf{0}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Linearly independent vectors



$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We can express the location of the point with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

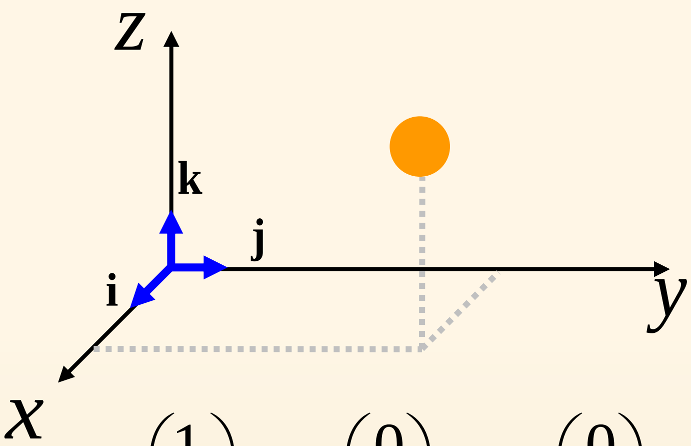
$$a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

If the point is at the origin, the equation becomes

$$a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The equation above is satisfied if and only if $a=b=c=0$.

Linearly independent vectors



$$\mathbf{i}=\begin{pmatrix}1\\0\\0\end{pmatrix},\mathbf{j}=\begin{pmatrix}0\\1\\0\end{pmatrix},\mathbf{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

We can express the location of the point with i, j, k.

$$a\mathbf{i}+b\mathbf{j}+c\mathbf{k}=a\begin{pmatrix}1\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\0\end{pmatrix}+c\begin{pmatrix}0\\0\\1\end{pmatrix}$$

If the point is at the origin, the equation becomes

$$a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\1\\0\end{bmatrix}+c\begin{bmatrix}0\\0\\1\end{bmatrix}=\mathbf{0}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The equation above is satisfied if and only if $a=b=c=0$.

Then, i, j, k are linearly independent.

Linear Independence and Dependence of Vectors

Given any set of m vectors $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(m)}$ (with the same number of components), a linear combination of these vectors is an expression of the form

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)}$$

$c_1, c_2, \cdots, c_m$ **are any scalars. Now consider the equation.**

$$\boxed{c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0} \quad \cdots\cdots(1)}$$

$$c_1 = c_2 = \cdots = c_m = \mathbf{0}$$

$$\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \cdots, \mathbf{a}_{(m)}$$

비교

Definition 3.1 **Linear Dependence / Independence**

A set of functions $f_1(x),\ f_2(x), \ldots,\ f_n(x)$ is said to be **'linearly dependent'** on an interval I if there exist constant $c_1, c_2, \ldots c_n$, not all zero such that $c_1 f_1(x) + c_2 f_2(x) + \cdots c_n f_n(x) = 0$ for every x in the interval.

If the set of functions is not linearly dependent on the interval, it is said to be **'linearly independent'**

In other words, a set of functions is 'linearly independent' if the only constants for

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

are $c_1 = c_2 = \cdots = c_n = 0$

Linear Independence and Dependence of Vectors

Given any set of m vectors $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(m)}$ (with the same number of components), a linear combination of these vectors is an expression of the form

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)}$$

$c_1, c_2, \cdots, c_m$ **are any scalars. Now consider the equation.**

$$\boxed{c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0} \quad \cdots\cdots(1)}$$

$$c_1 = c_2 = \cdots = c_m = \mathbf{0}$$

$$\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \cdots, \mathbf{a}_{(m)}$$

vectors linearly independent set or linearly independent.

비교

Definition 3.1 **Linear Dependence / Independence**

A set of functions $f_1(x),\ f_2(x), \ldots,\ f_n(x)$ is said to be 'linearly dependent' on an interval I if there exist constant $c_1, c_2, \ldots c_n,$ not all zero such that $c_1 f_1(x) + c_2 f_2(x) + \cdots c_n f_n(x) = 0$ for every x in the interval.
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비교

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When $c_1 = c_2 = \cdots = c_m = \mathbf{0}$

Vector $\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \cdots, \mathbf{a}_{(m)}$ **vectors linearly independent set or linearly independent.**

비교

Function

Definition 3.1 **Linear Dependence / Independence**

A set of functions $f_1(x),\ f_2(x), \ldots,\ f_n(x)$ is said to be 'linearly dependent' on an interval I if there exist constant $c_1, c_2, \ldots c_n,$ not all zero such that $c_1 f_1(x) + c_2 f_2(x) + \cdots c_n f_n(x) = 0$ for every x in the interval.
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Linear Independence and Dependence of Vectors

$$\boxed{c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0} \quad \cdots\cdots(1)}$$

$$\mathbf{a}_{(1)} = k_2\mathbf{a}_{(2)} + \cdots + k_m\mathbf{a}_{(m)}, \qquad (\text{where } \; k_j = -c_j / c_1)$$

Linear Independence and Dependence of Vectors

$$\boxed{c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0} \quad \cdots\cdots(1)}$$

If (1) also holds with scalars not all zero, we call these vectors linearly dependent, because then we can express (at least) one of them as a linear combination of the others. For instance, if (1) holds with, say, c_1=0, we can solve (1) for $\mathbf{a}_{(1)}$:

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(Some k_j's may be zero. Or even all of them, namely, if $\mathbf{a}_{(1)}$=0.)

Linear Independence and Dependence of Vectors

Ex 1) Linear Independence and Dependence

Vector

$$\mathbf{a}_{(1)} = [\ 3,\ \ 0,\ \ 2,\ \ 2]$$
$$\mathbf{a}_{(2)} = [-6,\ \ 42,\ \ 24,\ \ 54]$$
$$\mathbf{a}_{(3)} = [21,\ -21,\ \ 0,\ -15]$$

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$$-\mathbf{a}_{(3)} = [-21,\ 21,\ \ 0,\ \ 15]$$

$$6\mathbf{a}_{(1)} - \frac{1}{2}\mathbf{a}_{(2)} - \mathbf{a}_{(3)} = [0, 0, 0, 0]$$

Linear Systems

$$\begin{cases} ① & 3x_1 + 0\cdot x_2 + 2x_3 = 2 \\ ② & -6x_1 + 42x_2 + 24x_3 = 54 \\ ③ & 21x_1 - 21x_2 + 0\cdot x_3 = -15 \end{cases}$$

⇩ ①x2+②

$$\begin{cases} 3x_1 + 0\cdot x_2 + 2x_3 = 2 \\ 0\cdot x_1 + 42x_2 + 28x_3 = 58 \\ 21x_1 - 21x_2 + 0\cdot x_3 = -15 \end{cases}$$

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$$\begin{cases} 3x_1 + 0\cdot x_2 + 2x_3 = 2 \\ 0\cdot x_1 + 42x_2 + 28x_3 = 58 \\ 0\cdot x_1 - 21x_2 - 14x_3 = -29 \end{cases}$$

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Matrix

$$\left[\begin{array}{ccc:c} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{array}\right] \qquad \begin{bmatrix} 3 & 0 & 2 \\ -6 & 42 & 24 \\ 21 & -21 & 0 \end{bmatrix}$$

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Linear Independence and Dependence of Vectors

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The three vectors are **linearly dependent**

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The three equations are linearly dependent

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Linear Independence and Dependence of Vectors

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The three vectors are **linearly dependent**

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$$\begin{cases} ①\ \ 3x_1+0\cdot x_2+2x_3=2 \\ ②\ -6x_1+42x_2+24x_3=54 \\ ③\ \ 21x_1-21x_2+0\cdot x_3=-15 \end{cases}$$

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The three rows are **linearly dependent**

Rank of a Matrix

Rank of a Matrix

The rank of a matrix A

: “<u>the maximum number of linearly independent row vectors” of</u> A. rank A.

Rank of a Matrix

Rank of a Matrix

The rank of a matrix $\mathbf{A}$

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Ex 2) Rank

The matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \cdots (2)$$

Rank of a Matrix

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The rank of a matrix $\mathbf{A}$

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The matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \cdots (2)$$

has rank 2, because Example 1 shows that the first two row vectors are linearly independent, whereas all three row vectors are linearly dependent.

Rank of a Matrix

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The rank of a matrix $\mathbf{A}$

: "<u>the maximum number of linearly independent row vectors" of</u> $\mathbf{A}$. rank $\mathbf{A}$.

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The matrix

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has rank 2, because Example 1 shows that the first two row vectors are linearly independent, whereas all three row vectors are linearly dependent.

Note further that rank $\mathbf{A}=0$ if and only if $\mathbf{A}=0$ (zero matrix).

Rank of a Matrix

☑ Example 1
Rank of 3 x 4 Matrix

Consider the 3 x 4 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & -2 & 6 & 8 \\ 3 & 5 & -7 & 8 \end{pmatrix}.$$

With $\mathbf{u}_1$=(-1 1 -1 3), $\mathbf{u}_2$=(2 -2 6 8), and $\mathbf{u}_3$=(3 5 -7 8), we see that
$4\mathbf{u}_1 - 1/2\mathbf{u}_2 + \mathbf{u}_3 = 0$.
the set $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ is linearly dependent.

On the other hand, since neither $\mathbf{u}_1$ nor $\mathbf{u}_2$ is a constant multiple of the other set of row vectors $\mathbf{u}_1$, $\mathbf{u}_2$ is linearly independent.
Hence by Definition, $\text{rank}(\mathbf{A}) = 2$.

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & -2 & 6 & 8 \\ 3 & 5 & -7 & 8 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 - x_3 = 3 \\ 2x_1 - 2x_2 + 6x_3 = 8 \\ 3x_1 + 5x_2 - 7x_3 = 8 \end{cases}$$

⇩ ①x(-2)+②

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 8 & 2 \\ 3 & 5 & -7 & 8 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 - x_3 = 3 \\ 0 \cdot x_1 - 4x_2 + 8x_3 = 2 \\ 3x_1 + 5x_2 - 7x_3 = 8 \end{cases}$$

⇩ ①x(-3)+③

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 8 & 2 \\ 0 & 2 & -4 & -1 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 - x_3 = 3 \\ 0 \cdot x_1 - 4x_2 + 8x_3 = 2 \\ 0 \cdot x_1 + 2x_2 - 4x_3 = -1 \end{cases}$$

⇩ ②x(0.5)+③

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 8 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 - x_3 = 3 \\ 0 \cdot x_1 - 4x_2 + 8x_3 = 2 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \end{cases}$$

$\text{rank}(\mathbf{A}) = 2$

Rank and Row-Equivalent Matrices

Theorem : Row-Equivalent Matrices

Row-equivalent matrices have the same rank.

A_1 row-equivalent to a matrix A_2
→ rank is invariant under elementary row operations.

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix} \Downarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 10 & 25 \\ 0 & 0 & -95 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 \\ 0.6 & 1.5 & 1.5 & -5.4 \\ 1.2 & -0.3 & -0.3 & 2.4 \end{bmatrix} \Downarrow \begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 \\ 0 & 1.1 & 1.1 & -4.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{bmatrix} \Downarrow \begin{bmatrix} 3 & 2 & 1 \\ 0 & -1/3 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank and Linear System Solutions

Rank and Row-Equivalent Matrices

Theorem 8.4 **Rank of a Matrix by Row Reduction**

If a matrix A is row equivalent to a row-echelon form B, then
i) the row space of A = the row space of B
ii) the nonzero rows of B from a basis for the row space of A, and
iii) <u>rank(A) = the number of nonzero rows in B</u>

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + 11x_2 + 24x_3 = 90 \\ 3x_1 - 3x_2 - 92x_3 = -190 \\ 2x_1 - 2x_2 + 2x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 0\cdot x_1 + 10x_2 + 25x_3 = 90 \\ 0\cdot x_1 + 0\cdot x_2 - 95x_3 = -190 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

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i) the row space of A = the row space of B
ii) the nonzero rows of B from a basis for the row space of A, and
iii) <u>rank(A) = the number of nonzero rows in B</u>

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

$$\begin{array}{c} ① \\ ② \\ \\ ③ \end{array}\left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rank : 3

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + 11x_2 + 24x_3 = 90 \\ 3x_1 - 3x_2 - 92x_3 = -190 \\ 2x_1 - 2x_2 + 2x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 0 \cdot x_1 + 10x_2 + 25x_3 = 90 \\ 0 \cdot x_1 + 0 \cdot x_2 - 95x_3 = -190 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \end{cases}$$

$$\left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$$\left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & \boxed{12} \end{array}\right]$$

Rank and Row-Equivalent Matrices

Theorem 8.4 **Rank of a Matrix by Row Reduction**

If a matrix A is row equivalent to a row-echelon form B, then
i) the row space of A = the row space of B
ii) the nonzero rows of B from a basis for the row space of A, and
iii) <u>rank(A) = the number of nonzero rows in B</u>

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

$$\Downarrow$$

$$\begin{array}{c} ① \\ ② \\ \\ ③ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rank : 3

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + 11x_2 + 24x_3 = 90 \\ 3x_1 - 3x_2 - 92x_3 = -190 \\ 2x_1 - 2x_2 + 2x_3 = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 0 \cdot x_1 + 10x_2 + 25x_3 = 90 \\ 0 \cdot x_1 + 0 \cdot x_2 - 95x_3 = -190 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$$\Downarrow$$

$$\begin{array}{c} ① \\ ② \\ \\ \end{array} \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rank : 2

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

Rank and Row-Equivalent Matrices

Theorem 8.4 **Rank of a Matrix by Row Reduction**

If a matrix A is row equivalent to a row-echelon form B, then
i) the row space of A = the row space of B
ii) the nonzero rows of B from a basis for the row space of A, and
iii) <u>rank(A) = the number of nonzero rows in B</u>

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

$$\begin{array}{c} ① \\ ② \\ \\ ③ \end{array} \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rank : 3

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + 11x_2 + 24x_3 = 90 \\ 3x_1 - 3x_2 - 92x_3 = -190 \\ 2x_1 - 2x_2 + 2x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 0 \cdot x_1 + 10x_2 + 25x_3 = 90 \\ 0 \cdot x_1 + 0 \cdot x_2 - 95x_3 = -190 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \end{cases}$$

$$\left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$$\begin{array}{c} ① \\ ② \\ \\ \end{array} \left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rank : 2

$$\begin{cases} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0 \\ -x_1 - 0.3x_2 - 0.3x_3 + 0.2x_4 = -2.3 \\ 1.5x_1 + 1.0x_2 + 1.0x_3 - 2.5x_4 = 4.0 \end{cases}$$

$$\begin{cases} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0 \\ 0 \cdot x_1 + 1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 0.0 \end{cases}$$

$$\left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & \boxed{12} \end{array}\right]$$

rank : 3

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ 9x_1 + 7x_2 + 2x_3 = 15 \\ 3x_1 + 2x_2 + x_3 = -9 \end{cases}$$

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ -1/3x_2 + 1/3x_3 = -2 \\ 0 = 12 \end{cases}$$

Rank and Row-Equivalent Matrices

☑ Example 3
Linear Independence /Dependence

Determine whether the set of vectors

$$\mathbf{u}_1 = <2,1,1>$$

$$\mathbf{u}_2 = <0,3,0>$$

$$\mathbf{u}_3 = <3,1,2>$$

in R^3 in linearly dependent or linearly independent.

Solution)

If we form a matrix $\mathbf{A}$ with the given vectors as rows, and if we row reduce $\mathbf{A}$ to a row-echelon form $\mathbf{B}$ with rank 3, then the set of vectors is linearly independent.
If $\mathrm{rank}(\mathbf{A})<3$, then the set of vectors is linearly dependent.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 3 & 1 & 2 \end{pmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus $\mathrm{rank}(\mathbf{A})=3$ and the set of vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ is linearly independent.

Rank and Linear Systems

different

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \boxed{1} \end{array}\right]$$

No solution case

rank(A)≠rank(A|B)

False statement

Rank and Linear Systems

$$\begin{aligned} x_1 + x_2 &= 1 \\ 4x_1 - x_2 &= -6 \\ 2x_1 - 3x_2 &= 8 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 8 \end{bmatrix} \quad \Rightarrow \quad \mathbf{Ax} = \mathbf{B}$$

different

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \boxed{1} \end{array}\right]$$

No solution case

rank(A)≠rank(A|B)

False statement

Rank and Linear Systems

$$\begin{aligned} x_1 + x_2 &= 1 \\ 4x_1 - x_2 &= -6 \\ 2x_1 - 3x_2 &= 8 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 8 \end{bmatrix} \Rightarrow \mathbf{Ax} = \mathbf{B}$$

$$[\mathbf{A} \mid \mathbf{B}] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & -1 & -6 \\ 2 & -3 & 8 \end{array}\right]$$

different

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \boxed{1} \end{array}\right]$$

No solution case

rank(A)≠rank(A|B) False statement

Rank and Linear Systems

$$\begin{aligned} x_1 + x_2 &= 1 \\ 4x_1 - x_2 &= -6 \\ 2x_1 - 3x_2 &= 8 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 8 \end{bmatrix} \quad \Rightarrow \quad \mathbf{Ax} = \mathbf{B}$$

$$[\mathbf{A} \,|\, \mathbf{B}] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & -1 & -6 \\ 2 & -3 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 16 \end{array}\right]$$

row operation

different

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \boxed{0 \quad 0} & & \boxed{1} \end{array}\right]$$

No solution case

rank(A)≠rank(A|B) — False statement

Rank and Linear Systems

$$
\begin{aligned}
x_1 + x_2 &= 1 \\
4x_1 - x_2 &= -6 \\
2x_1 - 3x_2 &= 8
\end{aligned}
\quad \Rightarrow \quad
\begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
=
\begin{bmatrix} 1 \\ -6 \\ 8 \end{bmatrix}
\quad \Rightarrow \quad
\mathbf{Ax} = \mathbf{B}
$$

$$
[\mathbf{A} \mid \mathbf{B}] =
\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & -1 & -6 \\ 2 & -3 & 8 \end{array}\right]
\xrightarrow[\text{row operation}]{}
\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 16 \end{array}\right]
\xrightarrow[\text{row operation}]{}
\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]
$$

different

$$
\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \boxed{1} \end{array}\right]
$$

No solution case

rank(A)≠rank(A|B) False statement

Rank and Linear Systems

$$\begin{aligned} x_1 + x_2 &= 1 \\ 4x_1 - x_2 &= -6 \\ 2x_1 - 3x_2 &= 8 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 8 \end{bmatrix} \quad \Rightarrow \quad \mathbf{A}\mathbf{x} = \mathbf{B}$$

$$[\mathbf{A} \mid \mathbf{B}] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & -1 & -6 \\ 2 & -3 & 8 \end{array}\right] \xrightarrow[\text{row operation}]{} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 16 \end{array}\right] \xrightarrow[\text{row operation}]{} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

rank(A|B)=3

different

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \boxed{1} \end{array}\right]$$

No solution case

rank(A)≠rank(A|B) — **False statement**

Rank and Linear Systems

$$\begin{aligned} x_1 + x_2 &= 1 \\ 4x_1 - x_2 &= -6 \\ 2x_1 - 3x_2 &= 8 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 8 \end{bmatrix} \quad \Rightarrow \quad \mathbf{Ax} = \mathbf{B}$$

$$[\mathbf{A} \,|\, \mathbf{B}] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & -1 & -6 \\ 2 & -3 & 8 \end{array}\right] \xrightarrow[\text{row operation}]{} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 16 \end{array}\right] \xrightarrow[\text{row operation}]{} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

rank(A|B)=3

different

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \boxed{1} \end{array}\right]$$

No solution case

rank(A)≠rank(A|B) — **False statement**

Rank and Linear Systems

$$\begin{aligned} x_1 + x_2 &= 1 \\ 4x_1 - x_2 &= -6 \\ 2x_1 - 3x_2 &= 8 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 8 \end{bmatrix} \quad \Rightarrow \quad \mathbf{Ax} = \mathbf{B}$$

$$[\mathbf{A} \,|\, \mathbf{B}] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & -1 & -6 \\ 2 & -3 & 8 \end{array}\right] \xrightarrow{\text{row operation}} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 16 \end{array}\right] \xrightarrow{\text{row operation}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

rank(A|B)=3

different

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix} \xrightarrow{\text{row operation}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \boxed{1} \end{array}\right]$$

No solution case

rank(A)≠rank(A|B) — **False statement**

Rank and Linear Systems

$$\begin{aligned} x_1 + x_2 &= 1 \\ 4x_1 - x_2 &= -6 \\ 2x_1 - 3x_2 &= 8 \end{aligned} \;\Rightarrow\; \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 8 \end{bmatrix} \;\Rightarrow\; \mathbf{Ax} = \mathbf{B}$$

$$[\mathbf{A} \mid \mathbf{B}] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & -1 & -6 \\ 2 & -3 & 8 \end{array}\right] \xrightarrow[\text{row operation}]{} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 16 \end{array}\right] \xrightarrow[\text{row operation}]{} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

rank(A|B)=3

different

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix} \xrightarrow[\text{row operation}]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

rank(A)=2

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \boxed{1} \end{array}\right]$$

No solution case

rank(A)≠rank(A|B) — **False statement**

Rank and Linear Systems

$$\begin{aligned} x_1 + x_2 &= 1 \\ 4x_1 - x_2 &= -6 \\ 2x_1 - 3x_2 &= 8 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 8 \end{bmatrix} \Rightarrow \mathbf{Ax} = \mathbf{B}$$

$$[\mathbf{A} \,|\, \mathbf{B}] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & -1 & -6 \\ 2 & -3 & 8 \end{array}\right] \xrightarrow[\text{row operation}]{} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 16 \end{array}\right] \xrightarrow[\text{row operation}]{} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

rank(A|B)=3

different

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix} \xrightarrow[\text{row operation}]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

rank(A)=2

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

No solution case

$$\begin{cases} x_1 + 0 \cdot x_2 = 1 \\ 0 \cdot x_1 + x_2 = 0 \\ \boxed{0 \cdot x_1 + 0 \cdot x_2 = 1} \end{cases}$$

rank(A)≠rank(A|B) **False statement**

Rank and Row-Equivalent Matrices

Theorem 8.5 **Consistency of AX=B**

A linear system of equations AX=B is consistent if and only if the rank of the coefficient matrix A is the same as the rank of the augmented matrix of the system

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & \boxed{12} \end{array}\right]$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + 11x_2 + 24x_3 = 90 \\ 3x_1 - 3x_2 - 92x_3 = -190 \\ 2x_1 - 2x_2 + 2x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 0 \cdot x_1 + 10x_2 + 25x_3 = 90 \\ 0 \cdot x_1 + 0 \cdot x_2 - 95x_3 = -190 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \end{cases}$$

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$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

$$\Downarrow$$

$$\begin{array}{c} ① \\ ② \\ \\ ③ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$$\Downarrow$$

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$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + 11x_2 + 24x_3 = 90 \\ 3x_1 - 3x_2 - 92x_3 = -190 \\ 2x_1 - 2x_2 + 2x_3 = 0 \end{cases}$$

$$\Downarrow$$

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Rank and Row-Equivalent Matrices

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A linear system of equations AX=B is consistent if and only if the rank of the coefficient matrix A is the same as the rank of the augmented matrix of the system

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

$$\begin{array}{c} ① \\ ② \\ \\ ③ \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$$\begin{array}{c} ① \\ ② \\ \\ \end{array}\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & \boxed{12} \end{array}\right]$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + 11x_2 + 24x_3 = 90 \\ 3x_1 - 3x_2 - 92x_3 = -190 \\ 2x_1 - 2x_2 + 2x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 0 \cdot x_1 + 10x_2 + 25x_3 = 90 \\ 0 \cdot x_1 + 0 \cdot x_2 - 95x_3 = -190 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \end{cases}$$

Rank and Row-Equivalent Matrices

Theorem 8.5 **Consistency of AX=B**

A linear system of equations AX=B is consistent if and only if the rank of the coefficient matrix A is the same as the rank of the augmented matrix of the system

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

$$\begin{array}{c} ① \\ ② \\ \\ ③ \end{array}\left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rank (A|B): 3
rank (A): 3

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + 11x_2 + 24x_3 = 90 \\ 3x_1 - 3x_2 - 92x_3 = -190 \\ 2x_1 - 2x_2 + 2x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 0 \cdot x_1 + 10x_2 + 25x_3 = 90 \\ 0 \cdot x_1 + 0 \cdot x_2 - 95x_3 = -190 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \end{cases}$$

$$\left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

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rank (A|B) : 2
rank (A): 2

$$\begin{cases} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0 \\ -x_1 - 0.3x_2 - 0.3x_3 + 0.2x_4 = -2.3 \\ 1.5x_1 + 1.0x_2 + 1.0x_3 - 2.5x_4 = 4.0 \end{cases}$$

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$$\begin{array}{c} ① \\ ② \\ ③ \end{array}\left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

rank (A|B) : 3
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$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ 9x_1 + 7x_2 + 2x_3 = 15 \\ 3x_1 + 2x_2 + x_3 = -9 \end{cases}$$

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Rank and Row-Equivalent Matrices

Theorem 8.5 **Consistency of AX=B**

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Solution
- One solution

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Solution
- Many solutions

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No Solution

Rank in Terms of Column Vectors

Theorem : Rank in Terms of Column Vectors

The rank r of a matrix $\mathbf{A}$ equals the maximum number of linearly independent column vectors of $\mathbf{A}$. Hence $\mathbf{A}$ and its transpose $\mathbf{A}^T$ have the same rank.

Proof)

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Then by definition of rank, $\mathbf{A}$ has r linearly independent rows which we denote by $\mathbf{v}_{(1)}, \cdots, \mathbf{v}_{(r)}$ and all the rows $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(m)}$ of $\mathbf{A}$ are linear combinations of those.

Rank in Terms of Column Vectors - 3 by 3 matrix

let $\text{rank}\,\mathbf{A} = 3$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix},$$

Note

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A의 rank가 3이므로 행벡터는 linearly independent 하다.

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그러므로 A의 행벡터가 이루는 공간은 3개의 Basis를 갖는다. (ex : $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$)

즉 그 공간의 임의의 벡터(b)는 A의 행벡터로 표현될 수 있다. $\mathbf{b} = l\mathbf{a}_1 + m\mathbf{a}_2 + n\mathbf{a}_3 \quad (l, m, n : const)$

Ex) 3차원 공간상의 벡터 b를 basis $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ 의 linear combination으로 표현

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⇦ A의 rank가 3이므로 행벡터는 linearly independent 하다.

⇦ 그러므로 A의 행벡터가 이루는 공간은 3개의 Basis를 갖는다. (ex : $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$)

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Rank in Terms of Column Vectors
- 3 by 3 matrix

$\text{let rank } \mathbf{A} = 3$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix},$$

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Ex) 만으로는 3차원 공간상의 벡터 b를 표현할 수 없음

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행벡터를 v1, v2, v3의 일차결합으로 표현함

Note

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Rank in Terms of Column Vectors - 3 by 3 matrix

$$\mathbf{A}=\begin{bmatrix}\mathbf{a}_1\\\mathbf{a}_2\\\mathbf{a}_3\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ \hline a_{21} & a_{22} & a_{23}\\ \hline a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}\mathbf{v}_2+c_{13}\mathbf{v}_3\\ \hline c_{21}\mathbf{v}_1+c_{22}\mathbf{v}_2+c_{23}\mathbf{v}_3\\ \hline c_{31}\mathbf{v}_1+c_{32}\mathbf{v}_2+c_{33}\mathbf{v}_3\end{bmatrix}$$

$$\begin{aligned}a_{11}&=c_{11}v_{11}+c_{12}v_{21}+c_{13}v_{31}\\a_{21}&=c_{21}v_{11}+c_{22}v_{21}+c_{23}v_{31}\\a_{31}&=c_{31}v_{11}+c_{32}v_{21}+c_{33}v_{31}\end{aligned}$$

$$\begin{aligned}a_{12}&=c_{11}v_{12}+c_{12}v_{22}+c_{13}v_{32}\\a_{22}&=c_{21}v_{12}+c_{22}v_{22}+c_{23}v_{32}\\a_{32}&=c_{31}v_{12}+c_{32}v_{22}+c_{33}v_{32}\end{aligned}$$

$$\begin{aligned}a_{13}&=c_{11}v_{13}+c_{12}v_{23}+c_{13}v_{33}\\a_{23}&=c_{21}v_{13}+c_{22}v_{23}+c_{23}v_{33}\\a_{33}&=c_{31}v_{13}+c_{32}v_{23}+c_{33}v_{33}\end{aligned}$$

A의 **열벡터**

이 열벡터들은 linearly independent 한 basis

열벡터도 3개의 basis (c의 성분의 수는 v의 수와 동일함) 존재. **따라서 rank A^T = rank A**

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행벡터 a의 성분을 새로운 basis v의 성분으로 표현하면

$$\begin{aligned}[\mathbf{a}_1]&=[a_{11},a_{12},a_{13}]=c_{11}\mathbf{v}_1+c_{12}\mathbf{v}_2+c_{13}\mathbf{v}_3\\ &=c_{11}[v_{11},v_{12},v_{13}]+c_{12}[v_{21},v_{22},v_{23}]+c_{13}[v_{31},v_{32},v_{33}]\\ &=[(c_{11}v_{11}+c_{12}v_{21}+c_{13}v_{31}),(c_{11}v_{12}+c_{12}v_{22}+c_{13}v_{32}),(c_{11}v_{13}+c_{12}v_{22}+c_{13}v_{32})]\end{aligned}$$

$$\begin{aligned}a_{11}&=c_{11}v_{11}+c_{12}v_{21}+c_{13}v_{31}\\ a_{21}&=c_{21}v_{11}+c_{22}v_{21}+c_{23}v_{31}\\ a_{31}&=c_{31}v_{11}+c_{32}v_{21}+c_{33}v_{31}\end{aligned}$$

$$\begin{aligned}a_{12}&=c_{11}v_{12}+c_{12}v_{22}+c_{13}v_{32}\\ a_{22}&=c_{21}v_{12}+c_{22}v_{22}+c_{23}v_{32}\\ a_{32}&=c_{31}v_{12}+c_{32}v_{22}+c_{33}v_{32}\end{aligned}$$

$$\begin{aligned}a_{13}&=c_{11}v_{13}+c_{12}v_{23}+c_{13}v_{33}\\ a_{23}&=c_{21}v_{13}+c_{22}v_{23}+c_{23}v_{33}\\ a_{33}&=c_{31}v_{13}+c_{32}v_{23}+c_{33}v_{33}\end{aligned}$$

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$$\boxed{a_{11} = c_{11}v_{11} + c_{12}v_{21} + c_{13}v_{31}}$$
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$$[\mathbf{a}_1] = [a_{11}, a_{12}, a_{13}] = c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + c_{13}\mathbf{v}_3$$
$$= c_{11}[v_{11}, v_{12}, v_{13}] + c_{12}[v_{21}, v_{22}, v_{23}] + c_{13}[v_{31}, v_{32}, v_{33}]$$
$$= [(c_{11}v_{11} + c_{12}v_{21} + c_{13}v_{31}), (c_{11}v_{12} + c_{12}v_{22} + c_{13}v_{32}), (c_{11}v_{13} + c_{12}v_{22} + c_{13}v_{32})]$$

$$a_{12} = c_{11}v_{12} + c_{12}v_{22} + c_{13}v_{32}$$
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$$a_{13} = c_{11}v_{13} + c_{12}v_{23} + c_{13}v_{33}$$
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$$= [(c_{11}v_{11} + c_{12}v_{21} + c_{13}v_{31}), (c_{11}v_{12} + c_{12}v_{22} + c_{13}v_{32}), (c_{11}v_{13} + c_{12}v_{22} + c_{13}v_{32})]$$

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$$=[(c_{11}v_{11}+c_{12}v_{21}+c_{13}v_{31}),(c_{11}v_{12}+c_{12}v_{22}+c_{13}v_{32}),(c_{11}v_{13}+c_{12}v_{22}+c_{13}v_{32})]$$

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A의 열벡터

이 열벡터들은 linearly independent 한 basis

$$\begin{bmatrix}a_{1k}\\ a_{2k}\\ a_{3k}\end{bmatrix}=v_{1k}\begin{bmatrix}c_{11}\\ c_{21}\\ c_{31}\end{bmatrix}+v_{2k}\begin{bmatrix}c_{12}\\ c_{22}\\ c_{32}\end{bmatrix}+v_{3k}\begin{bmatrix}c_{13}\\ c_{23}\\ c_{33}\end{bmatrix}$$

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A의 열벡터

이 열벡터들이 linearly dependent 하다면?

$$\begin{bmatrix}a_{1k}\\ a_{2k}\\ a_{3k}\end{bmatrix}=v_{1k}\begin{bmatrix}c_{11}\\ c_{21}\\ c_{31}\end{bmatrix}+v_{2k}\begin{bmatrix}c_{12}\\ c_{22}\\ c_{32}\end{bmatrix}+v_{3k}\begin{bmatrix}c_{13}\\ c_{23}\\ c_{33}\end{bmatrix}$$

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$$\begin{bmatrix}a_{1k}\\ a_{2k}\\ a_{3k}\end{bmatrix}=v_{1k}\begin{bmatrix}c_{11}\\ c_{21}\\ c_{31}\end{bmatrix}+v_{2k}\begin{bmatrix}c_{12}\\ c_{22}\\ c_{32}\end{bmatrix}+v_{3k}u\begin{bmatrix}c_{12}\\ c_{22}\\ c_{23}\end{bmatrix}$$

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$$\begin{bmatrix}a_{1k}\\a_{2k}\\a_{3k}\end{bmatrix}=v_{1k}\begin{bmatrix}c_{11}\\c_{21}\\c_{31}\end{bmatrix}+(v_{2k}+v_{3k}u)\begin{bmatrix}c_{12}\\c_{22}\\c_{32}\end{bmatrix}$$

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$$\begin{bmatrix}a_{1k}\\a_{2k}\\a_{3k}\end{bmatrix}=v_{1k}\begin{bmatrix}c_{11}\\c_{21}\\c_{31}\end{bmatrix}+(v_{2k}+v_{3k}u)\begin{bmatrix}c_{12}\\c_{22}\\c_{32}\end{bmatrix}$$

A의 행벡터 관점에서 basis의 개수가 줄어들게 되어 모순이 됨 $\mathbf{v}_1, \mathbf{v}_{new}$

$$\mathbf{A}=\begin{bmatrix}\mathbf{a}_1\\\mathbf{a}_2\\\mathbf{a}_3\end{bmatrix}=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}\mathbf{v}_2+uc_{12}\mathbf{v}_3\\c_{21}\mathbf{v}_1+c_{22}\mathbf{v}_2+uc_{22}\mathbf{v}_3\\c_{31}\mathbf{v}_1+c_{32}\mathbf{v}_2+uc_{33}\mathbf{v}_3\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}(1+u)(\mathbf{v}_2+\mathbf{v}_3)\\c_{21}\mathbf{v}_1+c_{22}(1+u)(\mathbf{v}_2+\mathbf{v}_3)\\c_{31}\mathbf{v}_1+c_{32}(1+u)(\mathbf{v}_2+\mathbf{v}_3)\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}(1+u)\mathbf{v}_{new}\\c_{21}\mathbf{v}_1+c_{22}(1+u)\mathbf{v}_{new}\\c_{31}\mathbf{v}_1+c_{32}(1+u)\mathbf{v}_{new}\end{bmatrix}$$

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$$\mathbf{A}=\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}\mathbf{v}_2+c_{13}\mathbf{v}_3\\ c_{21}\mathbf{v}_1+c_{22}\mathbf{v}_2+c_{23}\mathbf{v}_3\\ c_{31}\mathbf{v}_1+c_{32}\mathbf{v}_2+c_{33}\mathbf{v}_3\end{bmatrix}$$

A의 열벡터

$$\begin{bmatrix}a_{1k}\\ a_{2k}\\ a_{3k}\end{bmatrix}=v_{1k}\begin{bmatrix}c_{11}\\ c_{21}\\ c_{31}\end{bmatrix}+v_{2k}\begin{bmatrix}c_{12}\\ c_{22}\\ c_{32}\end{bmatrix}+v_{3k}\begin{bmatrix}c_{13}\\ c_{23}\\ c_{33}\end{bmatrix}$$

이 열벡터들이 linearly dependent 하다면?

$$\begin{bmatrix}a_{1k}\\ a_{2k}\\ a_{3k}\end{bmatrix}=v_{1k}\begin{bmatrix}c_{11}\\ c_{21}\\ c_{31}\end{bmatrix}+v_{2k}\begin{bmatrix}c_{12}\\ c_{22}\\ c_{32}\end{bmatrix}+v_{3k}u\begin{bmatrix}c_{12}\\ c_{22}\\ c_{23}\end{bmatrix}$$

$$\begin{bmatrix}a_{1k}\\ a_{2k}\\ a_{3k}\end{bmatrix}=v_{1k}\begin{bmatrix}c_{11}\\ c_{21}\\ c_{31}\end{bmatrix}+(v_{2k}+v_{3k}u)\begin{bmatrix}c_{12}\\ c_{22}\\ c_{32}\end{bmatrix}$$

A의 행백터 관점에서 basis의 개수가 줄어들게 되어 모순이 됨 $\mathbf{v}_1, \mathbf{v}_{new}$

$$\mathbf{A}=\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}\mathbf{v}_2+uc_{12}\mathbf{v}_3\\ c_{21}\mathbf{v}_1+c_{22}\mathbf{v}_2+uc_{22}\mathbf{v}_3\\ c_{31}\mathbf{v}_1+c_{32}\mathbf{v}_2+uc_{33}\mathbf{v}_3\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}(1+u)(\mathbf{v}_2+\mathbf{v}_3)\\ c_{21}\mathbf{v}_1+c_{22}(1+u)(\mathbf{v}_2+\mathbf{v}_3)\\ c_{31}\mathbf{v}_1+c_{32}(1+u)(\mathbf{v}_2+\mathbf{v}_3)\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}(1+u)\mathbf{v}_{new}\\ c_{21}\mathbf{v}_1+c_{22}(1+u)\mathbf{v}_{new}\\ c_{31}\mathbf{v}_1+c_{32}(1+u)\mathbf{v}_{new}\end{bmatrix}$$

Rank in Terms of Column Vectors
- 3 by 3 matrix

$$\mathbf{A}=\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}\mathbf{v}_2+c_{13}\mathbf{v}_3\\ c_{21}\mathbf{v}_1+c_{22}\mathbf{v}_2+c_{23}\mathbf{v}_3\\ c_{31}\mathbf{v}_1+c_{32}\mathbf{v}_2+c_{33}\mathbf{v}_3\end{bmatrix}$$

A의 열벡터

$$\begin{bmatrix}a_{1k}\\ a_{2k}\\ a_{3k}\end{bmatrix}=v_{1k}\begin{bmatrix}c_{11}\\ c_{21}\\ c_{31}\end{bmatrix}+v_{2k}\begin{bmatrix}c_{12}\\ c_{22}\\ c_{32}\end{bmatrix}+v_{3k}\begin{bmatrix}c_{13}\\ c_{23}\\ c_{33}\end{bmatrix}$$

∴ 이 열벡터들은 linearly independent 한 basis

이 열벡터들이 linearly dependent 하다면?

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$$\begin{bmatrix}a_{1k}\\ a_{2k}\\ a_{3k}\end{bmatrix}=v_{1k}\begin{bmatrix}c_{11}\\ c_{21}\\ c_{31}\end{bmatrix}+(v_{2k}+v_{3k}u)\begin{bmatrix}c_{12}\\ c_{22}\\ c_{32}\end{bmatrix}$$

A의 행백터 관점에서 basis의 개수가 줄어들게 되어 모순이 됨 $\mathbf{v}_1, \mathbf{v}_{new}$

$$\mathbf{A}=\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}\mathbf{v}_2+uc_{12}\mathbf{v}_3\\ c_{21}\mathbf{v}_1+c_{22}\mathbf{v}_2+uc_{22}\mathbf{v}_3\\ c_{31}\mathbf{v}_1+c_{32}\mathbf{v}_2+uc_{33}\mathbf{v}_3\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}(1+u)(\mathbf{v}_2+\mathbf{v}_3)\\ c_{21}\mathbf{v}_1+c_{22}(1+u)(\mathbf{v}_2+\mathbf{v}_3)\\ c_{31}\mathbf{v}_1+c_{32}(1+u)(\mathbf{v}_2+\mathbf{v}_3)\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}(1+u)\mathbf{v}_{new}\\ c_{21}\mathbf{v}_1+c_{22}(1+u)\mathbf{v}_{new}\\ c_{31}\mathbf{v}_1+c_{32}(1+u)\mathbf{v}_{new}\end{bmatrix}$$

Rank in Terms of Column Vectors
- 3 by 3 matrix

$$\mathbf{A}=\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}\mathbf{v}_2+c_{13}\mathbf{v}_3\\ c_{21}\mathbf{v}_1+c_{22}\mathbf{v}_2+c_{23}\mathbf{v}_3\\ c_{31}\mathbf{v}_1+c_{32}\mathbf{v}_2+c_{33}\mathbf{v}_3\end{bmatrix}$$

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A의 열벡터

$$\begin{bmatrix}a_{1k}\\ a_{2k}\\ a_{3k}\end{bmatrix}=v_{1k}\begin{bmatrix}c_{11}\\ c_{21}\\ c_{31}\end{bmatrix}+v_{2k}\begin{bmatrix}c_{12}\\ c_{22}\\ c_{32}\end{bmatrix}+v_{3k}\begin{bmatrix}c_{13}\\ c_{23}\\ c_{33}\end{bmatrix}$$

∴ 이 열벡터들은 linearly independent 한 basis

열벡터도 3개의 basis (c의 성분의 수는 v의 수와 동일함) 존재.
따라서 rank $\mathbf{A}^T$ = rank A

A의 행백터 관점에서 basis의 개수가 줄어들게 되어 모순이 됨 $\mathbf{v}_1, \mathbf{v}_{new}$

$$\mathbf{A}=\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}\mathbf{v}_2+uc_{12}\mathbf{v}_3\\ c_{21}\mathbf{v}_1+c_{22}\mathbf{v}_2+uc_{22}\mathbf{v}_3\\ c_{31}\mathbf{v}_1+c_{32}\mathbf{v}_2+uc_{33}\mathbf{v}_3\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}(1+u)(\mathbf{v}_2+\mathbf{v}_3)\\ c_{21}\mathbf{v}_1+c_{22}(1+u)(\mathbf{v}_2+\mathbf{v}_3)\\ c_{31}\mathbf{v}_1+c_{32}(1+u)(\mathbf{v}_2+\mathbf{v}_3)\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}(1+u)\mathbf{v}_{new}\\ c_{21}\mathbf{v}_1+c_{22}(1+u)\mathbf{v}_{new}\\ c_{31}\mathbf{v}_1+c_{32}(1+u)\mathbf{v}_{new}\end{bmatrix}$$

Solutions of Homogeneous Linear Systems

Homogeneous Linear System with Fewer Equations Than Unknowns (2)

$$\boxed{\begin{aligned} \mathbf{A}\mathbf{x} &= \lambda\mathbf{x} \\ \mathbf{A}\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\ (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} &= \mathbf{0} \end{aligned}}$$

$$\begin{cases} \mathbf{A} : \text{matrix} \\ \lambda : \text{scalar} \\ \mathbf{x} : \text{vector} \end{cases}$$

Homogeneous Linear System with Fewer Equations Than Unknowns (2)

$$\boxed{\begin{aligned} \mathbf{A}\mathbf{x} &= \lambda\mathbf{x} \\ \mathbf{A}\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\ (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} &= \mathbf{0} \end{aligned}} \qquad \begin{cases} \mathbf{A} : \text{matrix} \\ \lambda : \text{scalar} \\ \mathbf{x} : \text{vector} \end{cases}$$

- **If the rank $(\mathbf{A} - \lambda\mathbf{I})$ is equal to n, the number of component of $\mathbf{x}$, (the determinant of $(\mathbf{A} - \lambda\mathbf{I})$ is nonzero), we have a trivial solution ($\mathbf{x} = \mathbf{0}$).**

Homogeneous Linear System with Fewer Equations Than Unknowns (2)

$$\boxed{\begin{aligned} \mathbf{Ax} &= \lambda\mathbf{x} \\ \mathbf{Ax} - \lambda\mathbf{x} &= \mathbf{0} \\ (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} &= \mathbf{0} \end{aligned}}
\qquad
\begin{cases} \mathbf{A} : \text{matrix} \\ \lambda : \text{scalar} \\ \mathbf{x} : \text{vector} \end{cases}$$

- **If the rank $(\mathbf{A} - \lambda\mathbf{I})$ is equal to n, the number of component of $\mathbf{x}$, (the determinant of $(\mathbf{A} - \lambda\mathbf{I})$ is nonzero), we have a trivial solution ($\mathbf{x} = \mathbf{0}$).**

- **If the rank $(\mathbf{A} - \lambda\mathbf{I})$ is less than n, the number of component of $\mathbf{x}$, (the determinant of $(\mathbf{A} - \lambda\mathbf{I})$ is zero), we have Infinitely many solutions ($\mathbf{x} \neq \mathbf{0}$).**

Homogeneous Linear System with Fewer Equations Than Unknowns (2)

$$\boxed{\begin{aligned} \mathbf{Ax} &= \lambda\mathbf{x} \\ \mathbf{Ax} - \lambda\mathbf{x} &= \mathbf{0} \\ (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} &= \mathbf{0} \end{aligned}}
\qquad
\begin{cases} \mathbf{A} : \text{matrix} \\ \lambda : \text{scalar} \\ \mathbf{x} : \text{vector} \end{cases}$$

- **If the rank $(\mathbf{A} - \lambda\mathbf{I})$ is equal to n, the number of component of $\mathbf{x}$, (the determinant of $(\mathbf{A} - \lambda\mathbf{I})$ is nonzero), we have a trivial solution ($\mathbf{x} = \mathbf{0}$).**

- **If the rank $(\mathbf{A} - \lambda\mathbf{I})$ is less than n, the number of component of $\mathbf{x}$, (the determinant of $(\mathbf{A} - \lambda\mathbf{I})$ is zero), we have Infinitely many solutions ($\mathbf{x} \neq \mathbf{0}$).**

- **A scalar λ such that the equation holds for some vector $\mathbf{x} \neq \mathbf{0}$ is called an eigenvalue of $\mathbf{A}$.**

Homogeneous Linear System with Fewer Equations Than Unknowns (2)

$$\boxed{\begin{aligned} \mathbf{Ax} &= \lambda\mathbf{x} \\ \mathbf{Ax} - \lambda\mathbf{x} &= \mathbf{0} \\ (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} &= \mathbf{0} \end{aligned}}$$

$$\begin{cases} \mathbf{A} : \text{matrix} \\ \lambda : \text{scalar} \\ \mathbf{x} : \text{vector} \end{cases}$$

- **If the rank $(\mathbf{A} - \lambda\mathbf{I})$ is equal to n, the number of component of $\mathbf{x}$, (the determinant of $(\mathbf{A} - \lambda\mathbf{I})$ is nonzero), we have a trivial solution ($\mathbf{x} = \mathbf{0}$).**

- **If the rank $(\mathbf{A} - \lambda\mathbf{I})$ is less than n, the number of component of $\mathbf{x}$, (the determinant of $(\mathbf{A} - \lambda\mathbf{I})$ is zero), we have Infinitely many solutions ($\mathbf{x} \neq \mathbf{0}$).**

- **A scalar λ such that the equation holds for some vector $\mathbf{x} \neq \mathbf{0}$ is called an eigenvalue of $\mathbf{A}$.**

- **At that time, vector $\mathbf{x}$ is called eigenvector of $\mathbf{A}$.**

Second- and Third-Order Determinants

Determinant of second- and third order

Determinant of second order

Determinant of third order

Determinant of second- and third order

Determinant of second order

$$D = \det A = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of third order

Determinant of second- and third order

Determinant of second order

$$D = \det A = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of third order

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Determinant of second- and third order

Determinant of second order

$$D = \det A = \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of third order

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Determinant of second- and third order

Determinant of second order

$$D = \det A = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of third order

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \quad a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Determinant of Order n

Terms

In D we have n^2 entries a_{jk}, also n rows and n columns, and a main diagonal on which $a_{11}, a_{12}, \ldots, a_{nn}$ stand.

M_{jk} is called the minor of a_{jk} in D, and C_{jk} the cofactor of a_{jk} in D

For later use we note that D may also be written in terms of minors

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \qquad (j = 1, 2, \cdots, n)$$

$$D = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \qquad (k = 1, 2, \cdots, n)$$

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Determinant of Order n

A determinant of order n is a scalar associated with an n x n matrix $\mathrm{A}=[a_{jk}]$, which is written

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

and is defined for n=1 by

$$D = a_{11}$$

Determinant of Order n

For $n \geq 2$ by

$$D = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn} \quad (j = 1,\ 2,\ \cdots,\ n)$$

or

$$D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \quad (k = 1,\ 2,\ \cdots,\ n)$$

Here,

$$C_{jk} = (-1)^{j+k} M_{jk}$$

M_{jk} is a determinant of order n-1, namely, the determinant of the submatrix of A obtained A by omitting the row and column of the entry a_{jk}, that is, the jth row and the kth column.

Determinant of Order n

1) n=1

$$\mathbf{A} = \left[a_{11} \right]$$

2) n=2

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Determinant of Order n

1) n**=1**

$$\mathbf{A} = \begin{bmatrix} a_{11} \end{bmatrix} \qquad \therefore \det \mathbf{A} = a_{11}$$

2) n**=2**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Determinant of Order n

1) n=1

$$\mathbf{A} = \left[a_{11}\right] \qquad \therefore \det \mathbf{A} = a_{11}$$

2) n=2

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det \mathbf{A} =$$

Determinant of Order n

1) n**=1**

$$\mathbf{A} = \left[a_{11}\right] \qquad \therefore \det \mathbf{A} = a_{11}$$

2) n**=2**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det \mathbf{A} = \; a_{11}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Determinant of Order *n*

1) *n*=1

$$\mathbf{A} = \begin{bmatrix} a_{11} \end{bmatrix} \qquad \therefore \det \mathbf{A} = a_{11}$$

2) *n*=2

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det \mathbf{A} = a_{11} \begin{vmatrix} & \\ & a_{22} \end{vmatrix}$$

Determinant of Order n

1) n=1

$$\mathbf{A} = \left[a_{11} \right] \qquad \therefore \det \mathbf{A} = a_{11}$$

2) n=2

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det \mathbf{A} = a_{11} \begin{vmatrix} & \\ & a_{22} \end{vmatrix} - a_{12} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Determinant of Order n

1) n=1

$$\mathbf{A} = \left[a_{11} \right] \qquad \therefore \det \mathbf{A} = a_{11}$$

2) n=2

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

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Determinant of Order *n*

1) *n*=1

$$\mathbf{A} = \begin{bmatrix} a_{11} \end{bmatrix} \qquad \therefore \det \mathbf{A} = a_{11}$$

2) *n*=2

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det \mathbf{A} = a_{11} \begin{vmatrix} & \\ & a_{22} \end{vmatrix} - a_{12} \begin{vmatrix} & \\ a_{21} & \end{vmatrix}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

Determinant of Order n

3) n=3

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det \mathbf{A} =$$

Determinant of Order n

3) n=3

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det \mathbf{A} = a_{11} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Determinant of Order n

3) n=3

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det \mathbf{A} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Determinant of Order n

3) n**=3**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det \mathbf{A} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Determinant of Order n

3) n=3

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det \mathbf{A} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Determinant of Order n

3) n=3

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det \mathbf{A} = a_{11} \begin{vmatrix} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} & & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Determinant of Order n

3) n=3

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det \mathbf{A} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinant of Order n

3) n=3

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det \mathbf{A} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right)$$
$$+ a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

1) 1^{st} row

Determinant :

$$C_{jk} = (-1)^{j+k} M_{jk}$$

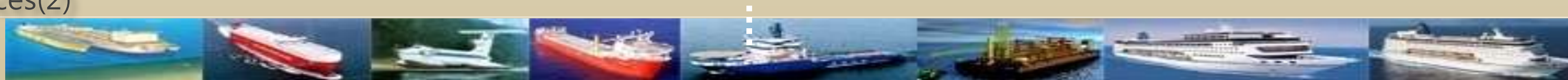
(Minors and Cofactors of a Third-Order Determinant)

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

1) 1st row

$$M_{11} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$



Determinant :
(Minors and Cofactors of a Third-Order Determinant)

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Find minors and cofactors.

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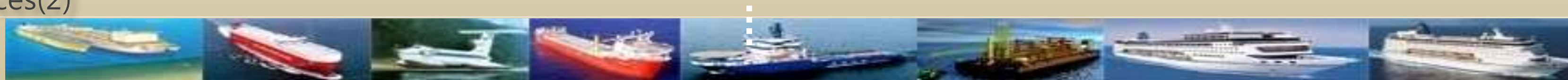
1) 1st row

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$



$$C_{jk} = (-1)^{j+k} M_{jk}$$

Determinant :
(Minors and Cofactors of a Third-Order Determinant)

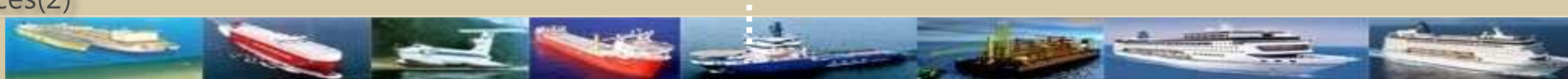
Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

1) 1st row

$$M_{11} = \begin{vmatrix} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$



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1) 1st row

$$M_{11} = \begin{vmatrix} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = \begin{vmatrix} & & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix}$$

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1) 1st row

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Determinant :

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1) 1st row

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

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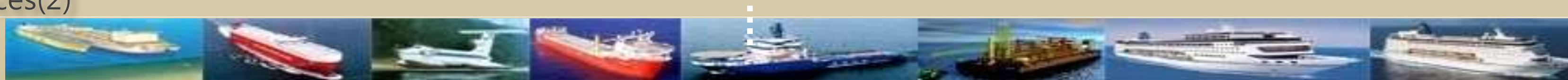
1) 1st row

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{11} = (-1)^{1+1} M_{11} = M_{11}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$



Determinant :

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Find minors and cofactors.

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1) 1st row

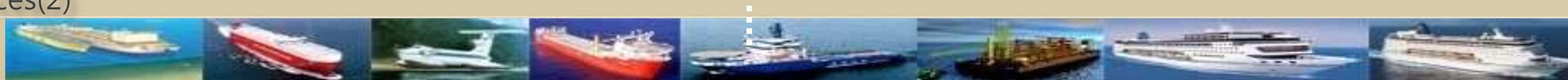
$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{11} = (-1)^{1+1} M_{11} = M_{11}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{12} = (-1)^{1+2} M_{12} = -M_{12}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$



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(Minors and Cofactors of a Third-Order Determinant)

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

1) 1st row

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

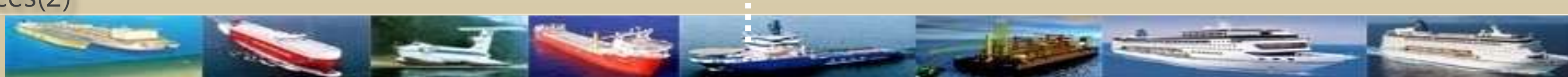
$$C_{11} = (-1)^{1+1} M_{11} = M_{11}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{12} = (-1)^{1+2} M_{12} = -M_{12}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{13} = (-1)^{1+3} M_{13} = M_{13}$$



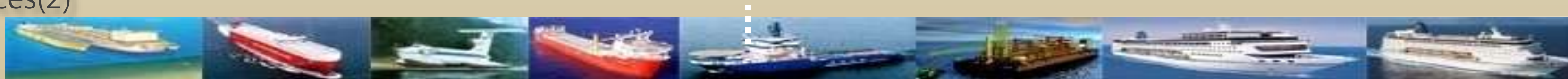
Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

2) 2^{nd} row



Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

2) 2nd row

$$M_{21} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Determinant :
(Minors and Cofactors of a Third-Order Determinant)

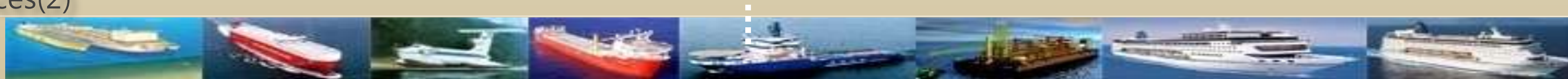
$$C_{jk} = (-1)^{j+k} M_{jk}$$

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

2) 2nd row

$$M_{21} = \begin{vmatrix} & a_{12} & a_{13} \\ & & \\ & a_{32} & a_{33} \end{vmatrix}$$



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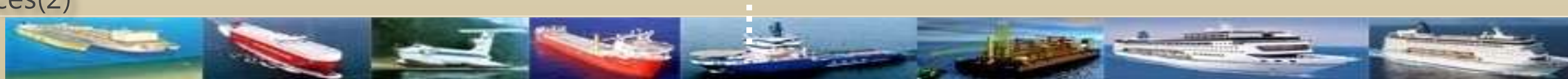
2) 2nd row

$$M_{21} = \begin{vmatrix} & a_{12} & a_{13} \\ & & \\ & a_{32} & a_{33} \end{vmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$



Determinant :
(Minors and Cofactors of a Third-Order Determinant)

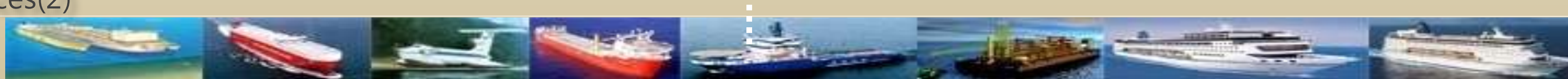
Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

2) 2nd row

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$



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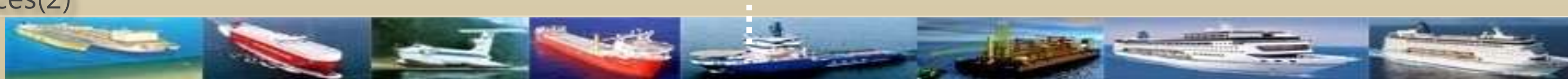
2) 2nd row

$$M_{21} = \begin{vmatrix} & a_{12} & a_{13} \\ & & \\ & a_{32} & a_{33} \end{vmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & & a_{13} \\ & & \\ a_{31} & & a_{33} \end{vmatrix}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$



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2) 2nd row

$$M_{21} = \begin{vmatrix} & a_{12} & a_{13} \\ & & \\ & a_{32} & a_{33} \end{vmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & & a_{13} \\ & & \\ a_{31} & & a_{33} \end{vmatrix}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} & \\ & & \\ a_{31} & a_{32} & \end{vmatrix}$$

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$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

2) 2nd row

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

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Find minors and cofactors.

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2) 2nd row

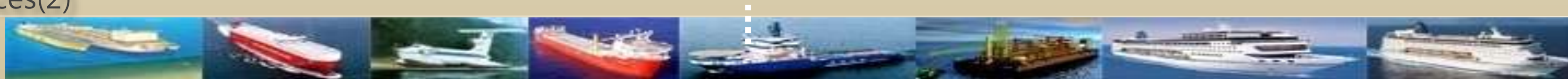
$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{22} = (-1)^{2+2} M_{22} = M_{22}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$



Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

2) 2nd row

$$M_{21} = \begin{vmatrix} & a_{12} & a_{13} \\ & & \\ & a_{32} & a_{33} \end{vmatrix}$$

$$C_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

$$M_{22} = \begin{vmatrix} a_{11} & & a_{13} \\ & & \\ a_{31} & & a_{33} \end{vmatrix}$$

$$C_{22} = (-1)^{2+2} M_{22} = M_{22}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} & \\ & & \\ a_{31} & a_{32} & \end{vmatrix}$$

$$C_{23} = (-1)^{2+3} M_{23} = -M_{23}$$

Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3) 3rd row

Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3) 3rd row

$$M_{31} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3) 3rd row

$$M_{31} = \begin{vmatrix} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & \end{vmatrix}$$

Determinant :
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3) 3rd row

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

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$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

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3) 3rd row

$$M_{31} = \begin{vmatrix} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & \end{vmatrix}$$

$$M_{32} = \begin{vmatrix} a_{11} & & a_{13} \\ a_{21} & & a_{23} \\ & & \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

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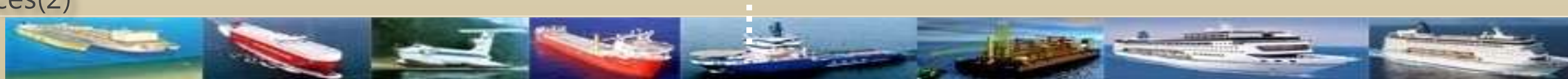
3) 3rd row

$$M_{31} = \begin{vmatrix} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & \end{vmatrix}$$

$$M_{32} = \begin{vmatrix} a_{11} & & a_{13} \\ a_{21} & & a_{23} \\ & & \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ & & \end{vmatrix}$$



Determinant :
(Minors and Cofactors of a Third-Order Determinant)

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Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3) 3rd row

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$C_{31} = (-1)^{3+1} M_{31} = M_{31}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3) 3rd row

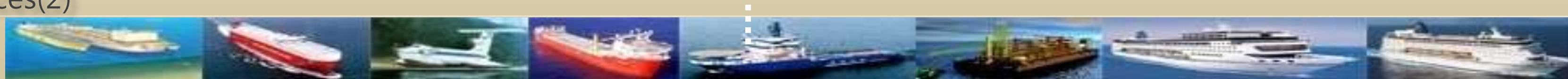
$$M_{31} = \begin{vmatrix} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & \end{vmatrix}$$

$$C_{31} = (-1)^{3+1} M_{31} = M_{31}$$

$$M_{32} = \begin{vmatrix} a_{11} & & a_{13} \\ a_{21} & & a_{23} \\ & & \end{vmatrix}$$

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32}$$

$$M_{13} = \begin{vmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ & & \end{vmatrix}$$



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3) 3rd row

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

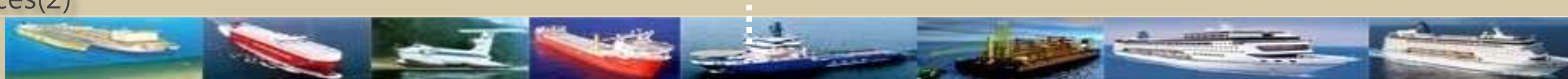
$$C_{31} = (-1)^{3+1} M_{31} = M_{31}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32}$$

$$M_{13} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$C_{33} = (-1)^{3+3} M_{33} = M_{33}$$



Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$D=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}M_{jk}\quad (j=1,\ 2,\ \cdots,\ n)$$

(Expansions of a Third-Order Determinant)

Find determinant

$$\det \mathbf{A}=\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

1) 1st rows

Determinant :

$$D=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}M_{jk}\quad (j=1,\,2,\,\cdots,\,n)$$

(Minors and Cofactors of a Third-Order Determinant)

(Expansions of a Third-Order Determinant)

Find determinant

$$\det \mathbf{A}=\begin{vmatrix}1 & 3 & 0\\ 2 & 6 & 4\\ -1 & 0 & 2\end{vmatrix}$$

1) 1st rows

$$=1\begin{vmatrix}1 & 3 & 0\\ 2 & 6 & 4\\ -1 & 0 & 2\end{vmatrix}$$

Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$D=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}M_{jk}\quad (j=1,\,2,\,\cdots,\,n)$$

(Expansions of a Third-Order Determinant)

Find determinant

$$\det \mathbf{A}=\begin{vmatrix}1 & 3 & 0\\ 2 & 6 & 4\\ -1 & 0 & 2\end{vmatrix}$$

1) 1st rows

$$=1\begin{vmatrix}6 & 4\\ 0 & 2\end{vmatrix}$$

Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

(Expansions of a Third-Order Determinant)

Find determinant

$$\det \mathbf{A} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

1) 1st rows

$$= 1 \begin{vmatrix} & & \\ & 6 & 4 \\ & 0 & 2 \end{vmatrix} - 3 \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1,\ 2,\ \cdots,\ n)$$

(Expansions of a Third-Order Determinant)

Find determinant

$$\det \mathbf{A} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

1) 1st rows

$$= 1 \begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3 \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$D=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}M_{jk}\quad (j=1,\,2,\,\cdots,\,n)$$

(Expansions of a Third-Order Determinant)

Find determinant

$$\det\mathbf{A}=\begin{vmatrix}1 & 3 & 0\\ 2 & 6 & 4\\ -1 & 0 & 2\end{vmatrix}$$

1) 1st rows

$$=1\begin{vmatrix}6 & 4\\ 0 & 2\end{vmatrix}-3\begin{vmatrix}2 & 4\\ -1 & 2\end{vmatrix}+0\begin{vmatrix}1 & 3 & 0\\ 2 & 6 & 4\\ -1 & 0 & 2\end{vmatrix}$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$D=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}M_{jk}\quad (j=1,\ 2,\ \cdots,\ n)$$

(Expansions of a Third-Order Determinant)

Find determinant

$$\det \mathbf{A}=\begin{vmatrix}1 & 3 & 0\\ 2 & 6 & 4\\ -1 & 0 & 2\end{vmatrix}$$

1) 1st rows

$$=1\begin{vmatrix}6 & 4\\ 0 & 2\end{vmatrix}-3\begin{vmatrix}2 & 4\\ -1 & 2\end{vmatrix}+0\begin{vmatrix}2 & 6\\ -1 & 0\end{vmatrix}$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$D=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}M_{jk} \quad (j=1,\,2,\,\cdots,\,n)$$

(Expansions of a Third-Order Determinant)

Find determinant

$$\det\mathbf{A}=\begin{vmatrix}1 & 3 & 0\\ 2 & 6 & 4\\ -1 & 0 & 2\end{vmatrix}$$

1) 1st rows

$$=1\begin{vmatrix}6 & 4\\ 0 & 2\end{vmatrix}-3\begin{vmatrix}2 & 4\\ -1 & 2\end{vmatrix}+0\begin{vmatrix}2 & 6\\ -1 & 0\end{vmatrix}$$

$$=1(12-0)-3(4+4)+0(0+6)=-12$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

2) 2nd rows

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

2) 2nd rows

$$= -2 \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

Determinant :

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1,\ 2,\ \cdots,\ n)$$

(Minors and Cofactors of a Third-Order Determinant)

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

2) 2nd rows

$$= -2 \begin{vmatrix} & 3 & 0 \\ & & \\ & 0 & 2 \end{vmatrix}$$

Determinant :

$$D=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}M_{jk}\quad (j=1,\,2,\,\cdots,\,n)$$

(Minors and Cofactors of a Third-Order Determinant)

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det\mathbf{A}=\begin{vmatrix}1&3&0\\2&6&4\\-1&0&2\end{vmatrix}$$

2) 2nd rows

$$=-2\begin{vmatrix} &3&0\\ & & \\ &0&2\end{vmatrix}+6\begin{vmatrix}1&3&0\\2&6&4\\-1&0&2\end{vmatrix}$$

Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

2) 2nd rows

$$= -2 \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} + 6 \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix}$$

Determinant :

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

(Minors and Cofactors of a Third-Order Determinant)

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

2) 2nd rows

$$= -2\begin{vmatrix} & 3 & 0 \\ & & \\ & 0 & 2 \end{vmatrix} + 6\begin{vmatrix} 1 & & 0 \\ & & \\ -1 & & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1,\ 2,\ \cdots,\ n)$$

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

2) 2nd rows

$$= -2 \begin{vmatrix} & 3 & 0 \\ & & \\ & 0 & 2 \end{vmatrix} + 6 \begin{vmatrix} 1 & & 0 \\ & & \\ -1 & & 2 \end{vmatrix} - 4 \begin{vmatrix} 1 & 3 & \\ & & \\ -1 & 0 & \end{vmatrix}$$

Determinant :

$$D=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}M_{jk} \quad (j=1,\ 2,\ \cdots,\ n)$$

(Minors and Cofactors of a Third-Order Determinant)

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A}=\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

2) 2nd rows

$$=-2\begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix}+6\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix}-4\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix}$$

$$=-2(6-0)+6(2+0)-4(0+3)=-12$$

$$D=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}M_{jk} \quad (j=1,\ 2,\ \cdots,\ n)$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A}=\begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix}$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$D=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}M_{jk} \quad (j=1,\,2,\,\cdots,\,n)$$

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A}=\begin{vmatrix}-3 & 0 & 0\\ 6 & 4 & 0\\ -1 & 2 & 5\end{vmatrix}$$

$$=-3\begin{vmatrix}-3 & 0 & 0\\ 6 & 4 & 0\\ -1 & 2 & 5\end{vmatrix}$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1,\ 2,\ \cdots,\ n)$$

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A} = \begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix}$$

Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A} = \begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix}$$

$$= -3 \begin{vmatrix} & & \\ & 4 & 0 \\ & 2 & 5 \end{vmatrix} - 0 \begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix}$$

Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A} = \begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix} - 0 \begin{vmatrix} 6 & 0 \\ -1 & 5 \end{vmatrix}$$

Determinant :

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

(Minors and Cofactors of a Third-Order Determinant)

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A} = \begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix}$$

$$= -3 \begin{vmatrix} & & \\ & 4 & 0 \\ & 2 & 5 \end{vmatrix} - 0 \begin{vmatrix} & & \\ 6 & & 0 \\ -1 & & 5 \end{vmatrix} + 0 \begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix}$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$D=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}M_{jk}\quad (j=1,\,2,\,\cdots,\,n)$$

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det\mathbf{A}=\begin{vmatrix}-3 & 0 & 0\\ 6 & 4 & 0\\ -1 & 2 & 5\end{vmatrix}$$

$$=-3\begin{vmatrix}4 & 0\\ 2 & 5\end{vmatrix}-0\begin{vmatrix}6 & 0\\ -1 & 5\end{vmatrix}+0\begin{vmatrix}6 & 4\\ -1 & 2\end{vmatrix}$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A} = \begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix} - 0 \begin{vmatrix} 6 & 0 \\ -1 & 5 \end{vmatrix} + 0 \begin{vmatrix} 6 & 4 \\ -1 & 2 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix}$$

Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A} = \begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix} - 0 \begin{vmatrix} 6 & 0 \\ -1 & 5 \end{vmatrix} + 0 \begin{vmatrix} 6 & 4 \\ -1 & 2 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix} = -3 \cdot 4 \cdot 5$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$D=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}M_{jk} \quad (j=1,\,2,\,\cdots,\,n)$$

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A}=\begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix}$$

$$=-3\begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix}-0\begin{vmatrix} 6 & 0 \\ -1 & 5 \end{vmatrix}+0\begin{vmatrix} 6 & 4 \\ -1 & 2 \end{vmatrix}$$

$$=-3\begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix}=-3\cdot 4\cdot 5=-60$$

Solving linear systems of two equations

Solve the linear systems of two equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \cdots①$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \quad \cdots②$$

1. General Solution

Solving linear systems of two equations

Solve the linear systems of two equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \cdots①$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \quad \cdots②$$

1. General Solution

$$①\times a_{22} - ②\times a_{12}:$$

Solving linear systems of two equations

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$$a_{21}x_1 + a_{22}x_2 = b_2 \quad \cdots ②$$

1. General Solution

$$① \times a_{22} - ② \times a_{12} :$$

$$(a_{11}a_{22} - a_{12}a_{21})x_1$$

$$= b_1 a_{22} - a_{12} b_2$$

Solving linear systems of two equations

Solve the linear systems of two equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \qquad \cdots ①$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \qquad \cdots ②$$

1. General Solution

$$① \times a_{22} - ② \times a_{12} :$$

$$\left(a_{11}a_{22} - a_{12}a_{21}\right)x_1$$

$$= b_1a_{22} - a_{12}b_2$$

$$\therefore\ x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$\left(a_{11}a_{22} - a_{12}a_{21} \neq 0\right)$$

Solving linear systems of two equations

Solve the linear systems of two equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \cdots ①$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \quad \cdots ②$$

1. General Solution

$$① \times a_{22} - ② \times a_{12} :$$

$$(a_{11}a_{22} - a_{12}a_{21})x_1$$

$$= b_1 a_{22} - a_{12} b_2$$

$$\therefore x_1 = \frac{b_1 a_{22} - a_{12} b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$(a_{11}a_{22} - a_{12}a_{21} \neq 0)$$

$$① \times (-a_{21}) + ② \times a_{11} :$$

Solving linear systems of two equations

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1. General Solution

$$①\times a_{22} - ②\times a_{12}:$$

$$(a_{11}a_{22} - a_{12}a_{21})x_1$$

$$= b_1 a_{22} - a_{12}b_2$$

$$\therefore x_1 = \frac{b_1 a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$(a_{11}a_{22} - a_{12}a_{21} \neq 0)$$

$$①\times(-a_{21}) + ②\times a_{11}:$$

$$(a_{11}a_{22} - a_{12}a_{21})x_1$$

$$= a_{11}b_2 - b_1 a_{21}$$

Solving linear systems of two equations

Solve the linear systems of two equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \cdots ①$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \quad \cdots ②$$

1. General Solution

$$① \times a_{22} - ② \times a_{12} :$$

$$(a_{11}a_{22} - a_{12}a_{21})x_1 = b_1a_{22} - a_{12}b_2$$

$$\therefore x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$(a_{11}a_{22} - a_{12}a_{21} \neq 0)$$

$$① \times (-a_{21}) + ② \times a_{11} :$$

$$(a_{11}a_{22} - a_{12}a_{21})x_1 = a_{11}b_2 - b_1a_{21}$$

$$\therefore x_2 = \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$(a_{11}a_{22} - a_{12}a_{21} \neq 0)$$

Solving linear systems of two equations

Solve the linear systems of two equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \cdots ①$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \quad \cdots ②$$

2. Use Cramer's rule

$$x_1 = \frac{b_1 a_{22} - a_{12} b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

Solving linear systems of two equations

Solve the linear systems of two equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \qquad \cdots ①$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \qquad \cdots ②$$

2. Use Cramer's rule

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D}$$

$$x_1 = \frac{b_1 a_{22} - a_{12} b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

Solving linear systems of two equations

Solve the linear systems of two equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \qquad \cdots ①$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \qquad \cdots ②$$

2. Use Cramer's rule

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{b_1 a_{22} - a_{12} b_2}{D}$$

$$(D \neq 0)$$

$$x_1 = \frac{b_1 a_{22} - a_{12} b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

Solving linear systems of two equations

Solve the linear systems of two equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \qquad \cdots①$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \qquad \cdots②$$

2. Use Cramer's rule

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{b_1a_{22} - a_{12}b_2}{D} \qquad (D \neq 0)$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D}$$

$$x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Solving linear systems of two equations

Solve the linear systems of two equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \qquad \cdots①$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \qquad \cdots②$$

2. Use Cramer's rule

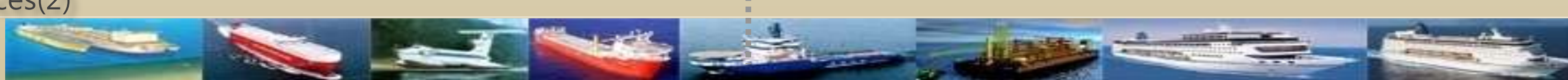
$$x_1 = \frac{b_1 a_{22} - a_{12} b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{b_1 a_{22} - a_{12} b_2}{D} \quad (D \neq 0)$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D} = \frac{a_{11} b_2 - b_1 a_{21}}{D} \quad (D \neq 0)$$



Solving linear systems of two equations

Solve the linear systems of two equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \qquad \cdots ①$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \qquad \cdots ②$$

2. Use Cramer's rule

$$x_1 = \frac{b_1 a_{22} - a_{12} b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{b_1 a_{22} - a_{12} b_2}{D} \qquad (D \neq 0)$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D} = \frac{a_{11} b_2 - b_1 a_{21}}{D} \qquad (D \neq 0)$$

$(D = 0)$

Solving linear systems of three equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$x_1 = \frac{D_1}{D}, \qquad x_2 = \frac{D_2}{D}; \qquad x_3 = \frac{D_3}{D}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Solving linear systems of three equations

Note that D_1, D_2, D_3 are obtained by replacing Columns 1, 2, 3.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

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\quad \Rightarrow \quad
\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\mathbf{A}}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}}
=
\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}}
$$

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
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$$

$$
\mathbf{A} = \begin{bmatrix} \boxed{\phantom{a_{11}}} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
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=
\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}}
$$

$$
\mathbf{A} = \begin{bmatrix} \boxed{b_1} & a_{12} & a_{13} \\ \boxed{b_2} & a_{22} & a_{23} \\ \boxed{b_3} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
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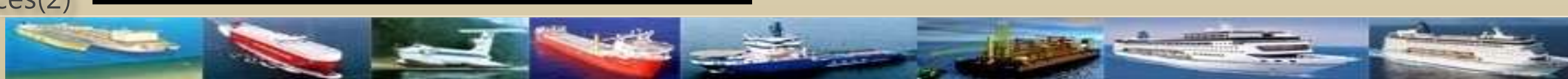
$$

$$
\mathbf{A} = \begin{bmatrix} \boxed{\begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}} & \begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix} & \begin{matrix} a_{13} \\ a_{23} \\ a_{33} \end{matrix} \end{bmatrix}, \quad
\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
\Rightarrow
D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}
$$



Solving linear systems of three equations

Note that D_1, D_2, D_3 are obtained by replacing Columns 1, 2, 3.

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\mathbf{A} = \begin{bmatrix} a_{11} & \boxed{\phantom{a_{12}}} & a_{13} \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
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$$\mathbf{A} = \begin{bmatrix} a_{11} & \boxed{b_1} & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

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Note that D_1, D_2, D_3 are obtained by replacing Columns 1, 2, 3.

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$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Solving linear systems of three equations

Note that *D_1, D_2, D_3* are obtained by replacing Columns 1, 2, 3.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}}$$

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\quad \Rightarrow \quad
\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\mathbf{A}}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}}
=
\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}}
$$

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \boxed{b_1} \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
\quad \Rightarrow \quad
D_2 = \begin{vmatrix} a_{11} & a_{21} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{23} & b_3 \end{vmatrix}
$$

Cramer's Rule

Cramer's Theorem (Solution of Linear Systems by Determinants)

(a) If a linear system of n equations in the same number of unknowns $x_1, \dots, x_n$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

has a nonzero coefficient determinant D=det($\mathbf{A}$), the system has precisely one solution. This solution is given by the formulas

$$
x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad , \quad \cdots \quad , \quad x_n = \frac{D_n}{D}
$$

Where D_k is the determinant obtained from D by replacing in D the kth column by the column with the entries $b_1, \dots, b_n$.

Cramer's Rule

Cramer's Theorem (Solution of Linear Systems by Determinants)

$$D_k = \begin{vmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2k} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nk} & \cdots & a_{nn} \end{vmatrix} \qquad \begin{matrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{matrix}$$

Cramer's Rule

Cramer's Theorem (Solution of Linear Systems by Determinants)

$$D_k = \begin{vmatrix} a_{11} & \cdots & & \cdots & a_{1n} \\ a_{21} & \cdots & & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_{n1} & \cdots & & \cdots & a_{nn} \end{vmatrix} \qquad \begin{matrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{matrix}$$

Cramer's Rule

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$$D_k = \begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix} \qquad \begin{matrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{matrix}$$

replace

Cramer's Rule

Cramer's Theorem (Solution of Linear Systems by Determinants)

$$
D_k = \begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix} \qquad \begin{matrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{matrix}
$$

replace

$$
D_k = b_1 C_{1k} + b_2 C_{2k} + \cdots + b_n C_{nk}
$$

Cramer's Rule

Cramer's Theorem (Solution of Linear Systems by Determinants)

$$D_k = \begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix} \qquad \begin{matrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{matrix}$$

replace

$$D_k = b_1 C_{1k} + b_2 C_{2k} + \cdots + b_n C_{nk}$$

(b) Hence if the system is homogeneous and D≠0, it has only the trivial solution $x_1=0, \ldots, x_n=0$. If D=0, the homogeneous system also has nontrivial solutions.

(참고) 3차 연립방정식의 해

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = p \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = q \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = r \end{cases}$$

$$\begin{array}{rl} & a_{21}a_{11}x_1 + a_{21}a_{12}x_2 + a_{21}a_{13}x_3 = a_{21}p \\ - & \underline{\left| a_{11}a_{21}x_1 + a_{11}a_{22}x_2 + a_{11}a_{23}x_3 = a_{11}q \right.} \end{array}$$

$$(a_{21}a_{12} - a_{11}a_{22})x_2 + (a_{21}a_{13} - a_{11}a_{23})x_3 = a_{21}p - a_{11}q$$

(참고) 3차 연립방정식의 해

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = p \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = q \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = r \end{cases}$$

$$\begin{array}{rl} & a_{21}a_{11}x_1 + a_{21}a_{12}x_2 + a_{21}a_{13}x_3 = a_{21}p \\ - & a_{11}a_{21}x_1 + a_{11}a_{22}x_2 + a_{11}a_{23}x_3 = a_{11}q \\ \hline & (a_{21}a_{12} - a_{11}a_{22})x_2 + (a_{21}a_{13} - a_{11}a_{23})x_3 = a_{21}p - a_{11}q \end{array}$$

$$\begin{array}{rl} & a_{31}a_{11}x_1 + a_{31}a_{12}x_2 + a_{31}a_{13}x_3 = a_{31}p \\ - & a_{11}a_{31}x_1 + a_{11}a_{32}x_2 + a_{11}a_{33}x_3 = a_{11}r \\ \hline & (a_{31}a_{12} - a_{11}a_{32})x_2 + (a_{31}a_{13} - a_{11}a_{33})x_3 = a_{31}p - a_{11}r \end{array}$$

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Solve the linear systems of two equations

$$\left(D=0\right) \quad D=\det A=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}=0$$

$$a_{11}=\frac{a_{12}a_{21}}{a_{22}}$$

$$\begin{cases} ① \; a_{11}x_1+a_{12}x_2=b_1 \\ ② \; a_{21}x_1+a_{22}x_2=b_2 \end{cases}$$

$$\frac{a_{12}a_{21}}{a_{22}}x_1+a_{12}x_2=b_1$$

$$\begin{cases} a_{21}x_1+a_{22}x_2=\dfrac{a_{22}}{a_{12}}b_1 \\ a_{21}x_1+a_{22}x_2=b_2 \end{cases}$$

$$\begin{cases} a_{21}x_1+a_{22}x_2=\dfrac{a_{22}}{a_{12}}b_1 \\ 0\cdot x_1+0\cdot x_2=\dfrac{a_{22}}{a_{12}}b_1-b_2 \end{cases}$$

when $\dfrac{a_{22}}{a_{12}}b_1-b_2=0$

$$\begin{cases} a_{21}x_1+a_{22}x_2=\dfrac{a_{22}}{a_{12}}b_1 \\ 0\cdot x_1+0\cdot x_2=0 \end{cases}$$

Linearly independent equation : 1
Variables : 2

$$\left[\begin{array}{cc:c} a_{21} & a_{22} & \dfrac{a_{22}}{a_{12}}b_1 \\ 0 & 0 & 0 \end{array}\right]$$

rank(A)=1=rank(A|B)
rank(A)=1<2 unknowns

$\dfrac{a_{22}}{a_{12}}b_1-b_2\neq 0=b'$

$$\begin{cases} a_{21}x_1+a_{22}x_2=\dfrac{a_{22}}{a_{12}}b_1 \\ 0\cdot x_1+0\cdot x_2=b' \end{cases}$$

False statement

$$\left[\begin{array}{cc:c} a_{21} & a_{22} & \dfrac{a_{22}}{a_{12}}b_1 \\ 0 & 0 & b' \end{array}\right]$$

rank(A)=1≠2=rank(A|B)

Homogeneous linear systems $\begin{cases} a_{11}x_1+a_{12}x_2=0 \\ a_{21}x_1+a_{22}x_2=0 \end{cases}$

$$A\mathbf{x}=\mathbf{0} \qquad A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Trivial Solution $\mathbf{x}=\mathbf{0}$

$$\det A=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=0$$

$$A\mathbf{x}=\mathbf{0} \qquad A=\begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix}$$

Nontrivial many solutions

Behavior of an nth-Order Determinant under Elementary Row Operations

Theorem 1. Behavior of an nth-Order Determinant under Elementary Row Operations

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(a) Interchange of two rows multiplies the value of the determinant by -1.

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(b) Addition of a multiple of a row to another row does not alter the value of the determinant.

Behavior of an nth-Order Determinant under Elementary Row Operations

Theorem 1. Behavior of an nth-Order Determinant under Elementary Row Operations

(a) **Interchange of two rows multiplies the value of the determinant by -1.**

(b) **Addition of a multiple of a row to another row does not alter the value of the determinant.**

(c) **Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c.**

Behavior of an nth-Order Determinant under Elementary Row Operations

Proof. ***(a) Interchange of two rows multiplies the value of the determinant by -1*** **by induction.**

Behavior of an nth-Order Determinant under Elementary Row Operations

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Behavior of an nth-Order Determinant under Elementary Row Operations

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$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

Behavior of an nth-Order Determinant under Elementary Row Operations

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Behavior of an nth-Order Determinant under Elementary Row Operations

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(a) holds for determinants of order n-1≥2 and show that it then holds determinants of order n.

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(a) holds for determinants of order n-1≥2 and show that it then holds determinants of order n.
Let D be of order n. Let E be one of those interchanged. Expand D and E by a row that is not one of those interchanged

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk}, \qquad E = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} N_{jk}$$

Behavior of an nth-Order Determinant under Elementary Row Operations

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Behavior of an nth-Order Determinant under Elementary Row Operations

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Behavior of an nth-Order Determinant under Elementary Row Operations

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Behavior of an nth-Order Determinant under Elementary Row Operations

Proof. ***(b) Addition of a multiple of a row to another row does not alter the value of the determinant.***

Add c times Row i to Row j.

Let $\tilde{D}$ be the new determinant. Its entries in Row j are $a_{jk}+ca_{ik}$.

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$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

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Behavior of an *n*th-Order Determinant under Elementary Row Operations

We can write $\tilde{D}$ by the *j*th row.

$$\tilde{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ . & . & \cdots & . \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ . & . & \cdots & . \\ a_{j1} + ca_{i1} & a_{j2} + ca_{i2} & \cdots & a_{jn} + ca_{in} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

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Behavior of an nth-Order Determinant under Elementary Row Operations

We can write $\tilde{D}$ by the jth row.

$$\tilde{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} + ca_{i1} & a_{j2} + ca_{i2} & \cdots & a_{jn} + ca_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} (-1)^{j+k} \left(a_{jk} + ca_{ik} \right) M_{jk}$$

$$= \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} + c \sum_{k=1}^{n} (-1)^{j+k} a_{ik} M_{jk}$$

Behavior of an *n*th-Order Determinant under Elementary Row Operations

We can write $\tilde{D}$ by the *j*th row.

$$\tilde{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} + ca_{i1} & a_{j2} + ca_{i2} & \cdots & a_{jn} + ca_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} (-1)^{j+k} \left(a_{jk} + ca_{ik} \right) M_{jk}$$

$$= \underbrace{\sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk}}_{D_1} + c \underbrace{\sum_{k=1}^{n} (-1)^{j+k} a_{ik} M_{jk}}_{D_2}$$

Behavior of an nth-Order Determinant under Elementary Row Operations

We can write $\tilde{D}$ by the jth row.

$$\tilde{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} + ca_{i1} & a_{j2} + ca_{i2} & \cdots & a_{jn} + ca_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} (-1)^{j+k} \left(a_{jk} + ca_{ik}\right) M_{jk}$$

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Behavior of an nth-Order Determinant under Elementary Row Operations

$$D_1 = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk}$$

Behavior of an nth-Order Determinant under Elementary Row Operations

$$D_1 = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Behavior of an nth-Order Determinant under Elementary Row Operations

$$D_1 = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = D$$

Behavior of an nth-Order Determinant under Elementary Row Operations

Ⓘ

$$D_2 = \sum_{k=1}^{n} (-1)^{j+k} a_{ik} M_{jk}$$

ⓙ

Ⓘx(-1)+ⓙ

Behavior of an *n*th-Order Determinant under Elementary Row Operations

$$D_2 = \sum_{k=1}^{n} (-1)^{j+k} a_{ik} M_{jk} = \begin{array}{c} \\ \\ ⓘ \\ \\ ⓙ \\ \\ \\ \end{array} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ . & . & \cdots & . \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ . & . & \cdots & . \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

ⓘx(-1)+ⓙ

Behavior of an *n*th-Order Determinant under Elementary Row Operations

$$
D_2 = \sum_{k=1}^{n} (-1)^{j+k} a_{ik} M_{jk} =
\begin{array}{c}
\\ \\ \text{Ⓘ} \\ \\ \text{ⓙ} \\ \\ \\
\end{array}
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\cdot & \cdot & \cdots & \cdot \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\cdot & \cdot & \cdots & \cdot \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\cdot & \cdot & \cdots & \cdot \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
\begin{array}{l}
\\ \\ \text{Ⓘx(-1)+ⓙ} \\ \\ \\ \\ \\
\end{array}
$$

It has a_{ik} in both Row *i* and Row *j*.

Behavior of an nth-Order Determinant under Elementary Row Operations

$$D_2=\sum_{k=1}^{n}(-1)^{j+k}a_{ik}M_{jk}=\begin{array}{c} \\ \\ \textcircled{i} \\ \\ \textcircled{j} \\ \\ \\ \end{array}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}\begin{array}{l} \\ \\ \textcircled{i}\mathrm{x}(-1)+\textcircled{j} \\ \\ \\ \\ \\ \end{array}$$

It has a_{ik} in both Row i and Row j.
Interchanging these two rows gives D_2 back, but on the other hand it gives $-D_2$ by (a). ($D_2=-D_2=0$)

Behavior of an nth-Order Determinant under Elementary Row Operations

$$D_2=\sum_{k=1}^{n}(-1)^{j+k}a_{ik}M_{jk}=\begin{array}{c}\\ \\ ⓘ\\ \\ ⓙ\\ \\ \\ \end{array}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ . & . & \cdots & .\\ a_{i1} & a_{i2} & \cdots & a_{in}\\ . & . & \cdots & .\\ a_{i1} & a_{i2} & \cdots & a_{in}\\ . & . & \cdots & .\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix}\xrightarrow{ⓘ\text{x}(-1)+ⓙ}=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ . & . & \cdots & .\\ a_{i1} & a_{i2} & \cdots & a_{in}\\ . & . & \cdots & .\\ 0 & 0 & \cdots & 0\\ . & . & \cdots & .\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix}$$

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Behavior of an *n*th-Order Determinant under Elementary Row Operations

$$
D_2=\sum_{k=1}^{n}(-1)^{j+k}a_{ik}M_{jk}=
\begin{array}{c}
\\ \\ i \\ \\ j \\ \\ \\
\end{array}
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n}\\
\cdot & \cdot & \cdots & \cdot\\
\boxed{a_{i1}} & \boxed{a_{i2}} & \boxed{\cdots} & \boxed{a_{in}}\\
\cdot & \cdot & \cdots & \cdot\\
\boxed{a_{i1}} & \boxed{a_{i2}} & \boxed{\cdots} & \boxed{a_{in}}\\
\cdot & \cdot & \cdots & \cdot\\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
\xrightarrow{i\times(-1)+j}
=\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n}\\
\cdot & \cdot & \cdots & \cdot\\
a_{i1} & a_{i2} & \cdots & a_{in}\\
\cdot & \cdot & \cdots & \cdot\\
\boxed{0} & \boxed{0} & \boxed{\cdots} & \boxed{0}\\
\cdot & \cdot & \cdots & \cdot\\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
$$

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Behavior of an nth-Order Determinant under Elementary Row Operations

$$D_1 = D$$
$$D_2 = 0$$

$$\tilde{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1}+ca_{i1} & a_{j2}+ca_{i2} & \cdots & a_{jn}+ca_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Theorem (2.e) A zero row or column renders the value of a determinant zero

ix(-1)+j

Behavior of an nth-Order Determinant under Elementary Row Operations

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$$= D_1 + cD_2$$

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$$= D_1 + cD_2$$

ⓘx(-1)+ⓙ

$$= D + c \cdot 0$$

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$D_1 = D$

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Theorem (2.e) A zero row or column renders the value of a determinant zero

$$= D_1 + cD_2$$

ⓘx(-1)+ⓙ

$$= D + c \cdot 0$$

$$= D$$

Behavior of an nth-Order Determinant under Elementary Row Operations

$$D_1 = D$$
$$D_2 = 0$$

$$\tilde{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} + ca_{i1} & a_{j2} + ca_{i2} & \cdots & a_{jn} + ca_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Theorem (2.e) A zero row or column renders the value of a determinant zero

$$= D_1 + cD_2$$

ⓘx(-1)+ⓙ

$$= D + c \cdot 0$$

$$= D$$

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Behavior of an nth-Order Determinant under Elementary Row Operations

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$$D_2 = 0$$

$$\tilde{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} + ca_{i1} & a_{j2} + ca_{i2} & \cdots & a_{jn} + ca_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + c \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

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Behavior of an nth-Order Determinant under Elementary Row Operations

Proof. (c) Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

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Behavior of an *n*th-Order Determinant under Elementary Row Operations

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Expand the determinant by the *j*th row.

Behavior of an nth-Order Determinant under Elementary Row Operations

Proof. (c) Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \qquad \tilde{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ ca_{j1} & ca_{j2} & \cdots & ca_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Expand the determinant by the *j*th row.

$$\tilde{D} = \sum_{k=1}^{n} (-1)^{j+k} ca_{jk} M_{jk}$$

Behavior of an nth-Order Determinant under Elementary Row Operations

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Expand the determinant by the *j*th row.

$$\tilde{D} = \sum_{k=1}^{n} (-1)^{j+k} ca_{jk} M_{jk} = c \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk}$$

Behavior of an nth-Order Determinant under Elementary Row Operations

Proof. (c) Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \Rightarrow \quad \tilde{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ ca_{j1} & ca_{j2} & \cdots & ca_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

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$$\tilde{D} = \sum_{k=1}^{n} (-1)^{j+k} ca_{jk} M_{jk} = c\sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} = cD$$

Further Properties of *n*th-Order Determinants

Theorem 2. Further Properties of *n*th-Order Determinants

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(d) Transposition leaves the value of a determinant unaltered.

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Further Properties of nth-Order Determinants

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$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ . & . & \cdots & . \\ 0 & 0 & \cdots & 0 \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Further Properties of nth-Order Determinants

Proof.
(d) Transposition leaves the value of a determinant unaltered.

Transposition is defined as for matrices, that is, the jth row becomes the jth column of the transpose.

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Further Properties of nth-Order Determinants

Proof.

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$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ . & . & \cdots & . \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ . & . & \cdots & . \\ ca_{i1} & ca_{i2} & \cdots & ca_{in} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \begin{matrix} \\ \\ ⓘ \\ \\ ⓙ \\ \\ \\ \end{matrix}$$

ⓘx(-1)+ⓙ

Theorem (1.c) Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c

Theorem. ***(1.b) Addition of a multiple of a row to another row does not alter the value of the determinant.***

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Theorem (1.c) Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c

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Theorem (1.c) Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c

Theorem. ***(1.b) Addition of a multiple of a row to another row does not alter the value of the determinant.***

Further Properties of *n*th-Order Determinants

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Theorem (1.c) Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c

Theorem. ***(1.b) Addition of a multiple of a row to another row does not alter the value of the determinant.***

Further Properties of *n*th-Order Determinants

Proof.

(f) Proportional rows or columns render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ ca_{i1} & ca_{i2} & \cdots & ca_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = c\times \begin{array}{c} \\ \\ \textcircled{i} \\ \\ \textcircled{j} \\ \\ \\ \end{array} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ \boxed{a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}} \\ \cdot & \cdot & \cdots & \cdot \\ \boxed{a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \xrightarrow{\textcircled{i}\times(-1)+\textcircled{j}} = c\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = c\times 0 = 0$$

Theorem (1.c) Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c

Theorem. ***(1.b) Addition of a multiple of a row to another row does not alter the value of the determinant.***

Further Properties of *n*th-Order Determinants

Proof.

(f) Proportional rows or columns render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ ca_{i1} & ca_{i2} & \cdots & ca_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = c\times \begin{array}{c} \\ \\ ⓘ \\ \\ ⓙ \\ \\ \\ \end{array} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \xrightarrow{ⓘ\times(-1)+ⓙ} = c \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = c\times 0 = 0$$

Theorem (1.c) Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c

Theorem. ***(1.b) Addition of a multiple of a row to another row does not alter the value of the determinant.***

Determinant of a Triangular Matrix

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Determinant of a Triangular Matrix

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$$= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Determinant of a Triangular Matrix

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

$$= a_{11}C_{11} \qquad \left(\because a_{12} = a_{13} = \cdots = a_{1n} = 0\right)$$

Determinant of a Triangular Matrix

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

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$$= a_{11}M_{11}$$

Determinant of a Triangular Matrix

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \Rightarrow M_{11}$$

$$= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

$$= a_{11}C_{11} \qquad (\because a_{12} = a_{13} = \cdots = a_{1n} = 0)$$

$$= a_{11}M_{11}$$

Determinant of a Triangular Matrix

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

M_{11}

➔ It is also a determinant of a triangular matrix.

$$= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

$$= a_{11}C_{11} \qquad \left(\because a_{12} = a_{13} = \cdots = a_{1n} = 0\right)$$

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Determinant of a Triangular Matrix

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

M_{11}

➔ **It is also a determinant of a triangular matrix.**

$$= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

$$= a_{11}C_{11} \qquad (\because a_{12} = a_{13} = \cdots = a_{1n} = 0)$$

$$= a_{11}M_{11}$$

$$= a_{11} \times a_{22} \times \cdots \times a_{nn}$$

(참고) 3차 연립방정식의 해

$$(a_{21}a_{12} - a_{11}a_{22})x_2 + (a_{21}a_{13} - a_{11}a_{23})x_3 = a_{21}p - a_{11}q$$
$$(a_{31}a_{12} - a_{11}a_{32})x_2 + (a_{31}a_{13} - a_{11}a_{33})x_3 = a_{31}p - a_{11}r$$

(참고) 3차 연립방정식의 해

$$(a_{21}a_{12} - a_{11}a_{22})x_2 + (a_{21}a_{13} - a_{11}a_{23})x_3 = a_{21}p - a_{11}q$$

$$(a_{31}a_{12} - a_{11}a_{32})x_2 + (a_{31}a_{13} - a_{11}a_{33})x_3 = a_{31}p - a_{11}r$$

$$x_3 = \frac{(a_{22}a_{31} - a_{21}a_{32})p - (a_{31}a_{12} - a_{11}a_{32})q + (a_{21}a_{12} - a_{11}a_{22})r}{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}}$$

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$$x_3=\frac{(a_{22}a_{31}-a_{21}a_{32})p-(a_{31}a_{12}-a_{11}a_{32})q+(a_{21}a_{12}-a_{11}a_{22})r}{\boxed{a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}}}$$

$$\boxed{\begin{aligned}\det(\mathbf{A})=&\,a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}\\&-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}\end{aligned}}$$

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$$\Downarrow$$

$$\boxed{\begin{aligned} \det(\mathbf{A}) = {} & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}}$$

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$$\begin{vmatrix} a_{11} & a_{12} & p \\ a_{21} & a_{22} & q \\ a_{31} & a_{32} & r \end{vmatrix} = \begin{aligned}&(a_{22}a_{31} - a_{21}a_{32})p \\ &- (a_{31}a_{12} - a_{11}a_{32})q \\ &+ (a_{21}a_{12} - a_{11}a_{22})r\end{aligned}$$

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$$\begin{vmatrix}a_{11} & a_{12} & p\\ a_{21} & a_{22} & q\\ a_{31} & a_{32} & r\end{vmatrix}=\begin{aligned}&(a_{22}a_{31}-a_{21}a_{32})p\\&-(a_{31}a_{12}-a_{11}a_{32})q\\&+(a_{21}a_{12}-a_{11}a_{22})r\end{aligned}$$

$$\therefore x_3=\frac{D_3}{D}$$

Rank in Terms of Determinants

Theorem 3. Rank in Terms of Determinants

An m x n matrix $A=[a_{jk}]$ has rank $r \geq 1$ if and only if A has an r x r submatrix with nonzero determinant, whereas every square submatrix with more than r rows than A has (or does not have!) has determinant equal to zero.

In particular, if A is square, n x n, it has rank n if and only if

$$\det D \neq 0$$

Column Picture and Linear Equations*

*Strang G., Introduction to Linear Algebra, Third edition, Wellesley-Cambridge Press, 2003, Ch.2.1, p21

Vectors and Linear Equations

The central problem of linear algebra is to solve a system of equations. Those equations are linear, which means that the unknowns are only multiplied by numbers - we never see x times y.

First example)

$$x - 2y = 1$$
$$3x + 2y = 11$$

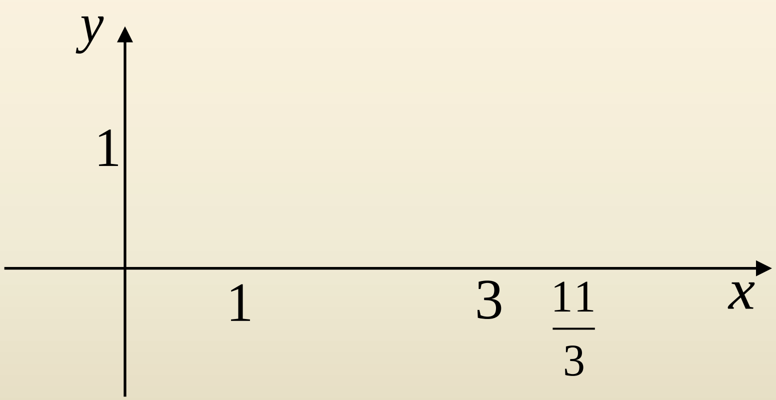

Figure 2.1 Row picture : The point (3, 1) where the lines meet is the solution

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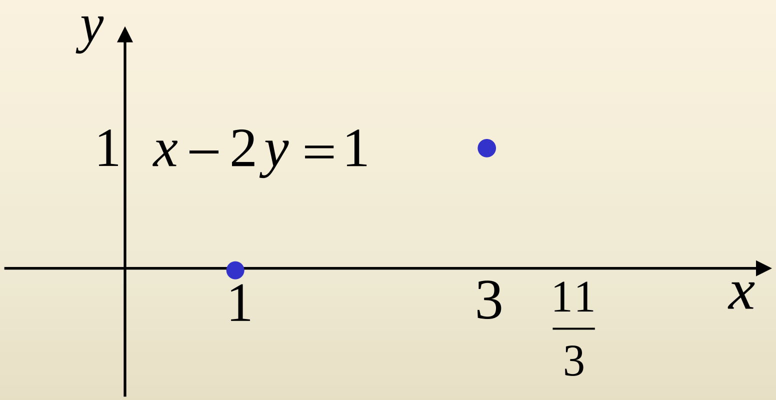

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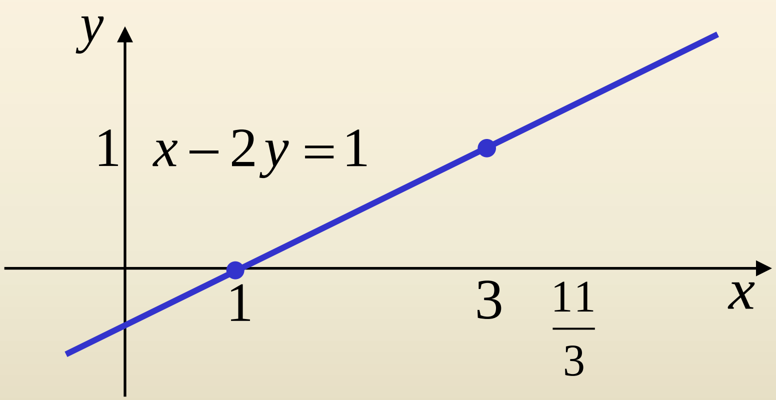

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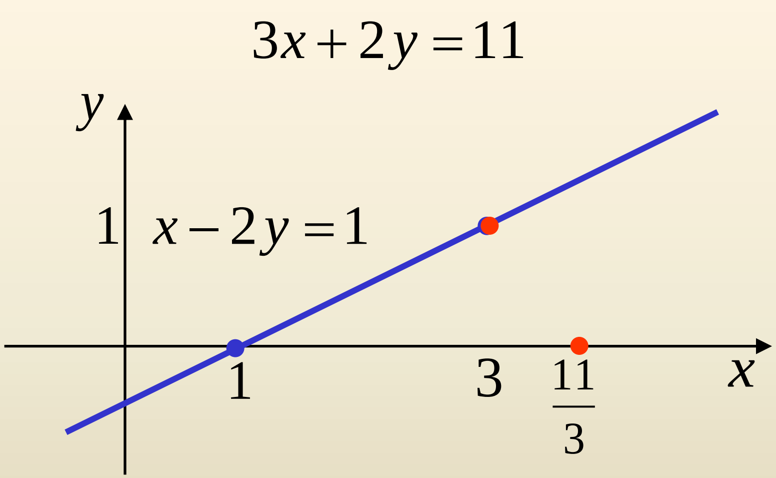

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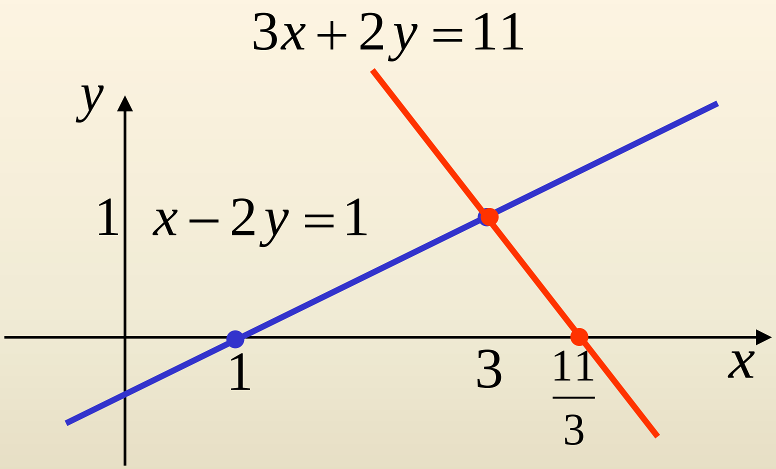

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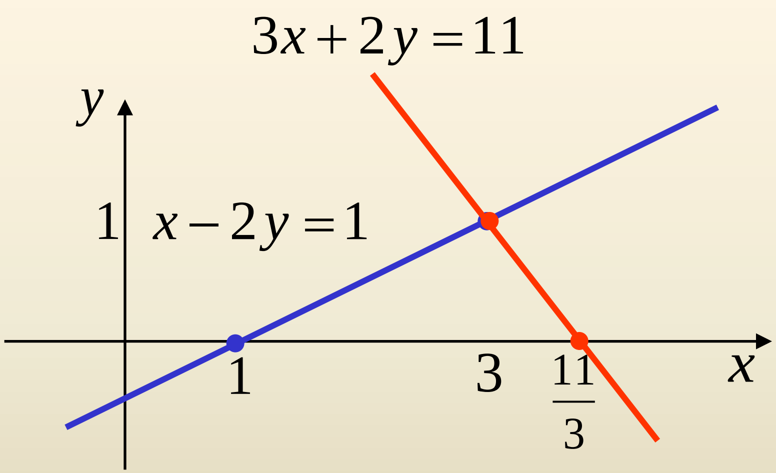

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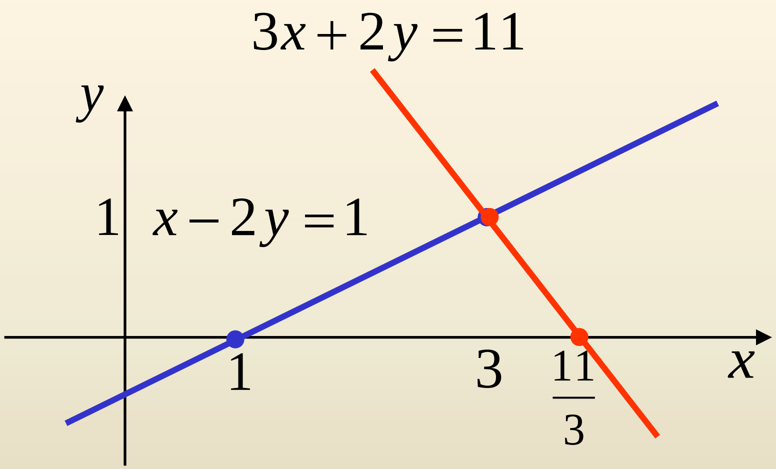

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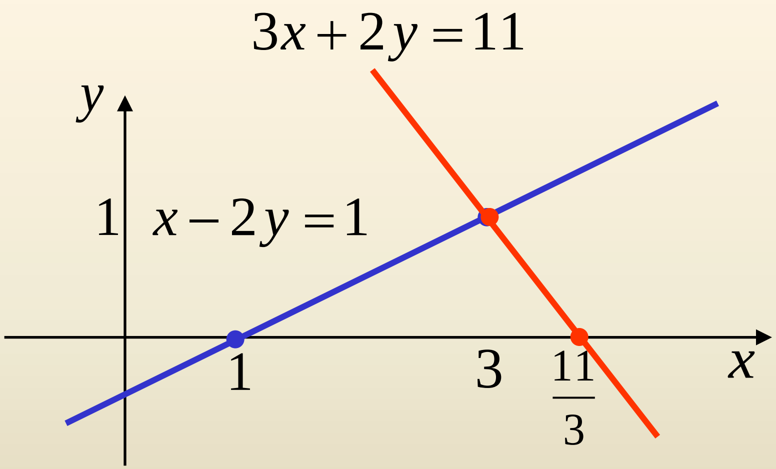

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The row picture show two lines meeting at a single point.

Vectors and Linear Equations

I want to recognize the linear system as a "vector equation". Instead of numbers we need to see vectors. If you separate the original system into its columns instead of its rows, you get

$$x\begin{bmatrix}1\\3\end{bmatrix}+y\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}=\mathbf{b}$$

This has two column vectors on the left side. The problem is to find the combination of those vectors that equals the vector on the right.

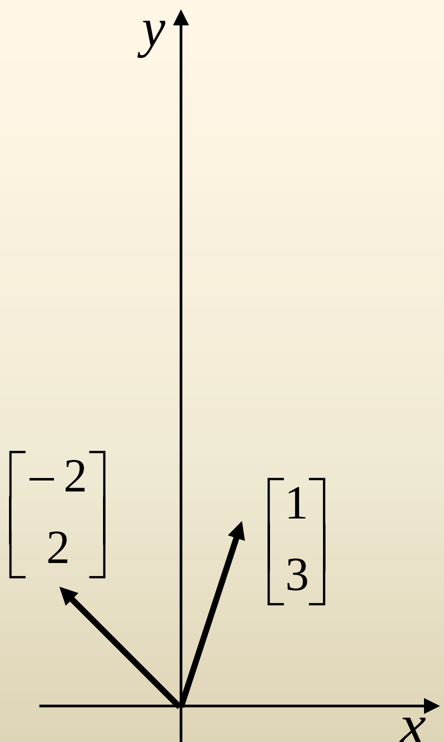
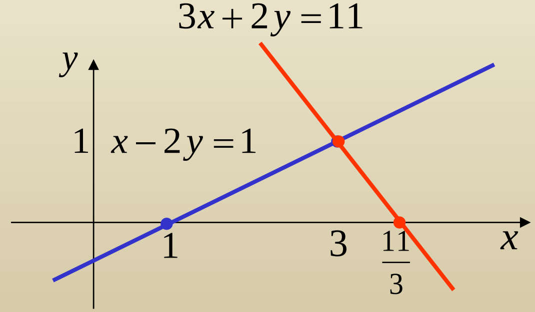


Figure 2.1 Column picture : A combination of columns produces the right side (1, 11).

비교



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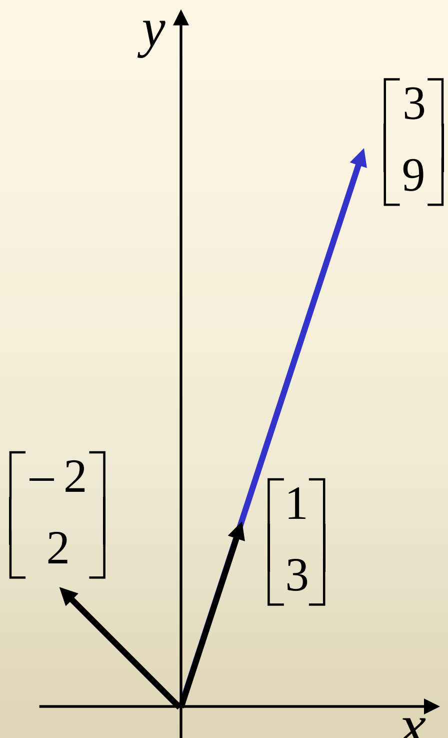

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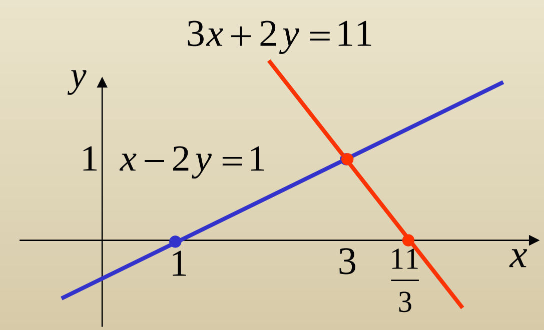

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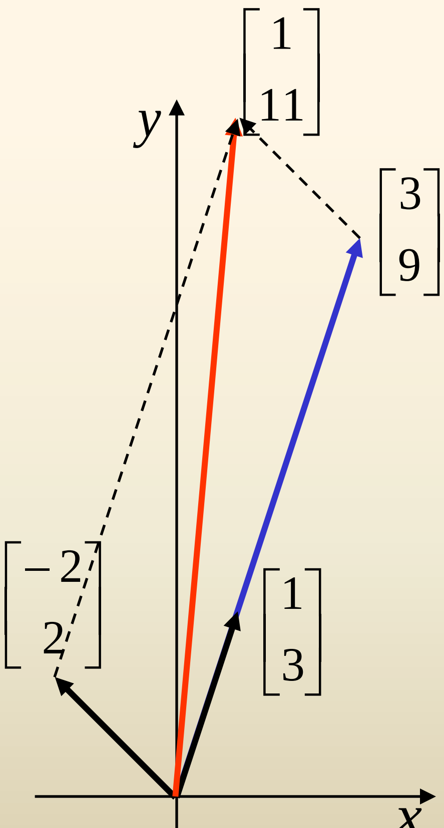

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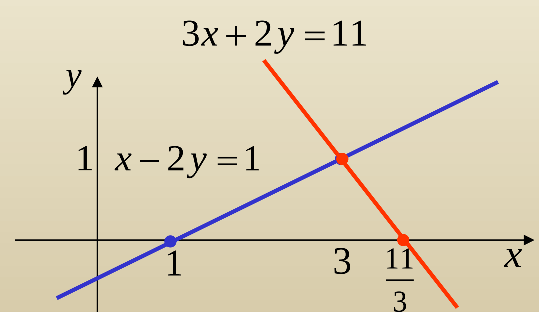

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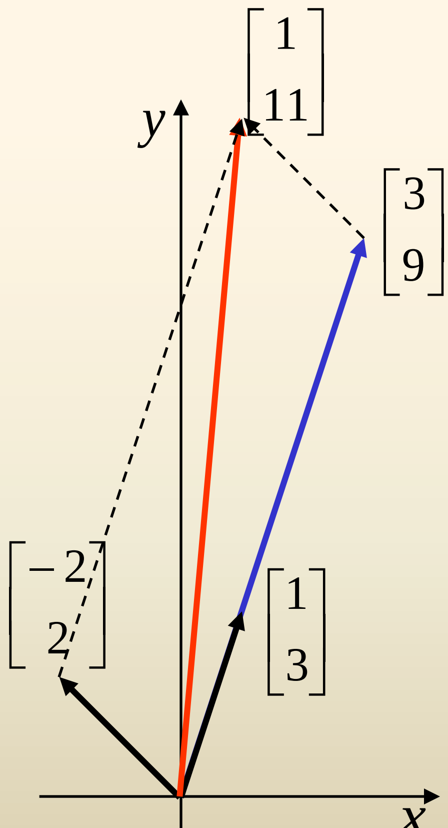
We are multiplying the first column by x and the second column by y, and adding. With the right choices $x = 3, y = 1$, this produces 3 (column 1) + 1 (column 2) = b.



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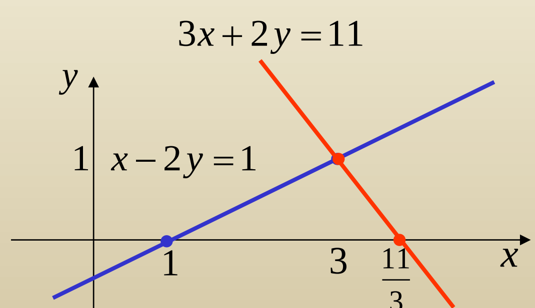

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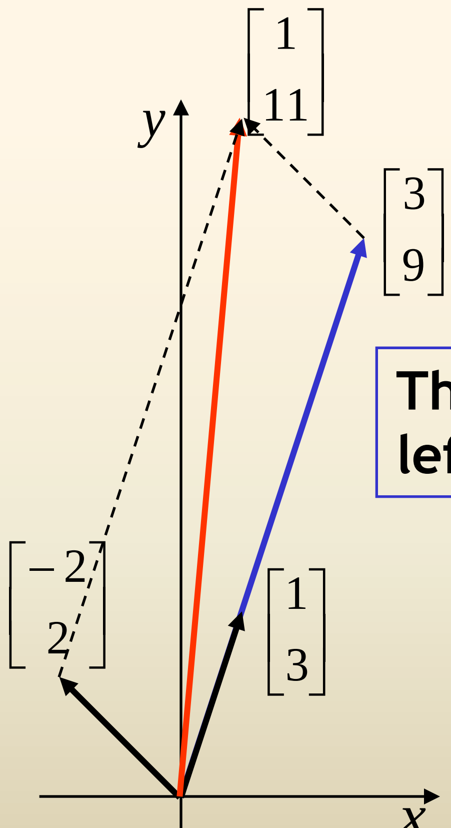
The column picture combines the column vectors on the left side to produce the vector b on the right side.



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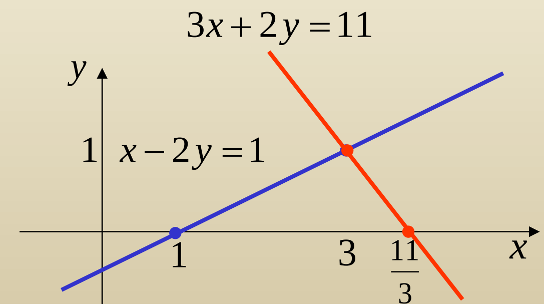

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Vectors and Linear Equations

$$x - 2y = 1$$
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$$x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix} = \mathbf{b} \qquad \therefore x = 3,\ y = 1$$

The left side of the vector equation is a linear combination of the columns. The problem is to find the right coefficients $x = 3$ and $y = 1$. We are combining scalar multiplication and vector addition into one step. That step is crucially important, because it contains both of basic operations :

$$\textbf{Linear combination}\quad 3\begin{bmatrix}1\\3\end{bmatrix} + 1\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$

Vectors and Linear Equations

The coefficient matrix on the left side of the equation is the 2 by 2 matrix $\mathbf{A}$:

$$\textbf{Coefficient matrix} \quad \mathbf{A} = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$$

This is very typical of linear algebra, to look at a matrix by rows and by columns. Its rows give the row picture and its columns give the column picture. Same numbers, different pictures, same equations. We write those equations as a matrix problem $\mathbf{Ax} = \mathbf{b}$.

$$\textbf{Matrix equation} \quad \mathbf{Ax} = \mathbf{b} : \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

The row picture deals with the two rows of $\mathbf{A}$. The column picture combines the columns. The numbers $x = 3$ and $y = 1$ go into the solution vector $\mathbf{x}$.

Vectors and Linear Equations
- Three Equations in Three Unknowns

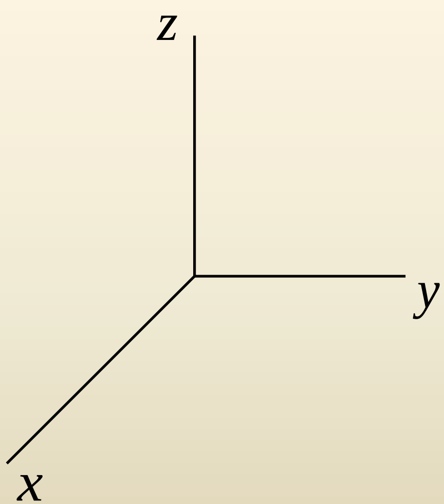
The three unknowns x, y, z. The linear equations $\mathbf{Ax} = \mathbf{b}$ are

$$x + 2y + 3z = 6$$

$$2x + 5y + 2z = 4$$

$$6x - 3y + z = 2$$

The row picture show three planes meeting at a single point.



The usual result of two equations in three unknowns is a intersect line L of solutions.

The third equation gives a third plane. It cuts the line L at a single point. That point lies on all three planes and it solves all three equations.

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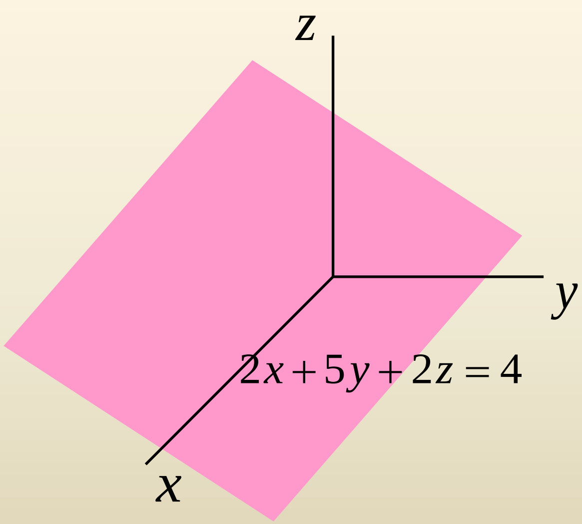

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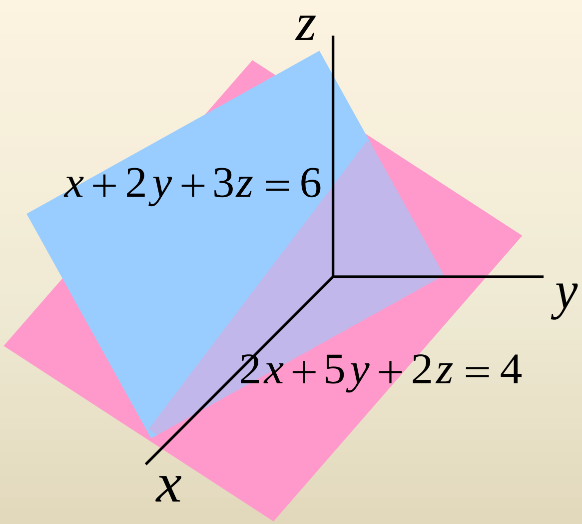

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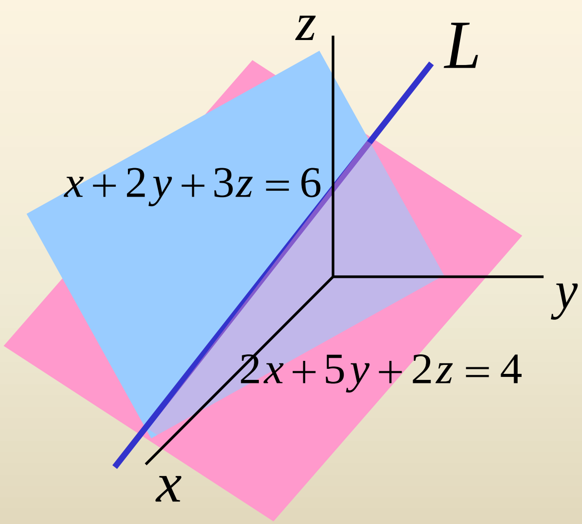

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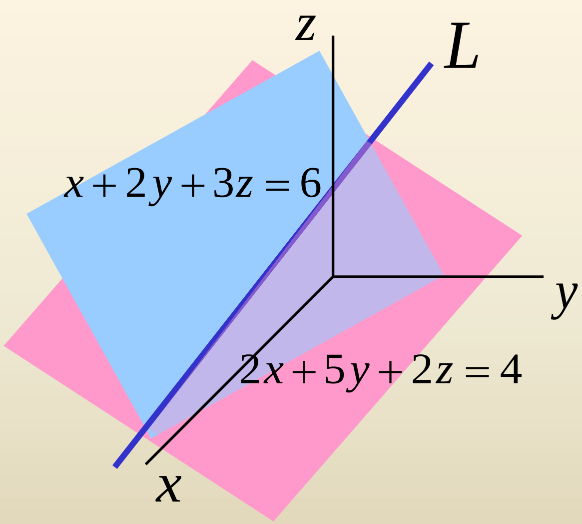

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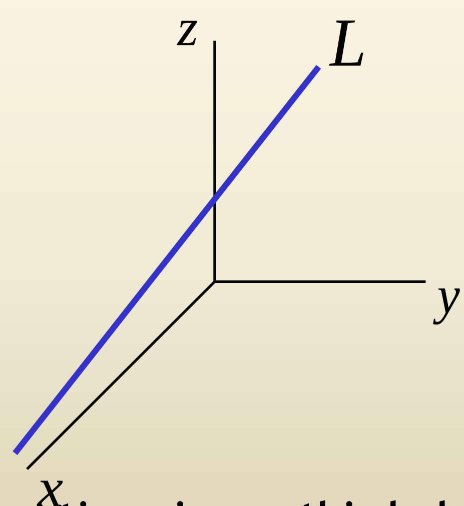

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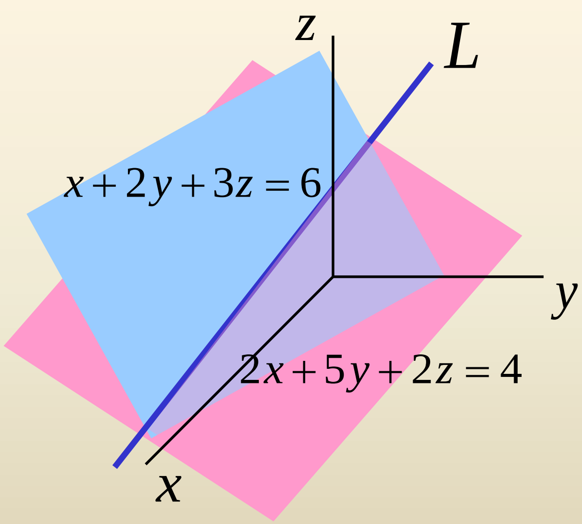
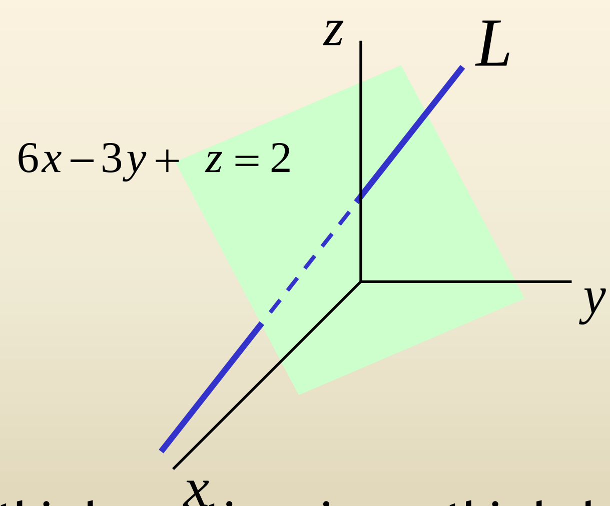


The usual result of two equations in three unknowns is a intersect line L of solutions.



The third equation gives a third plane. It cuts the line L at a single point. That point lies on all three planes and it solves all three equations.

Vectors and Linear Equations
- Three Equations in Three Unknowns

The three unknowns x, y, z. The linear equations $\mathbf{Ax} = \mathbf{b}$ are

$$x + 2y + 3z = 6$$

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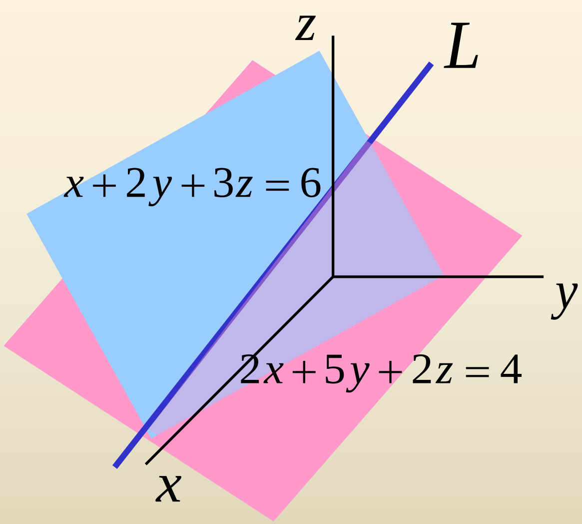

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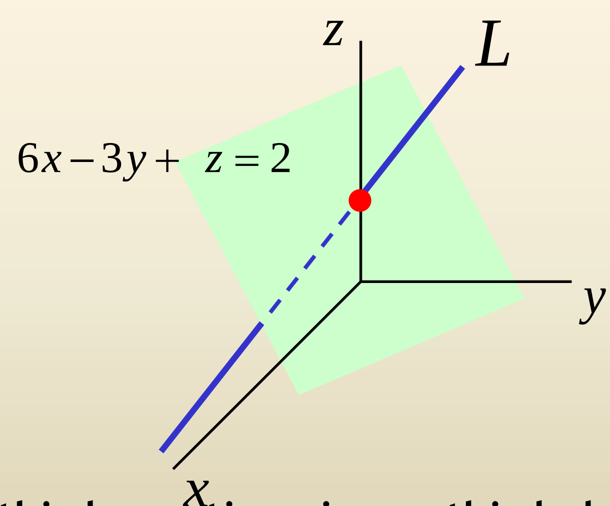

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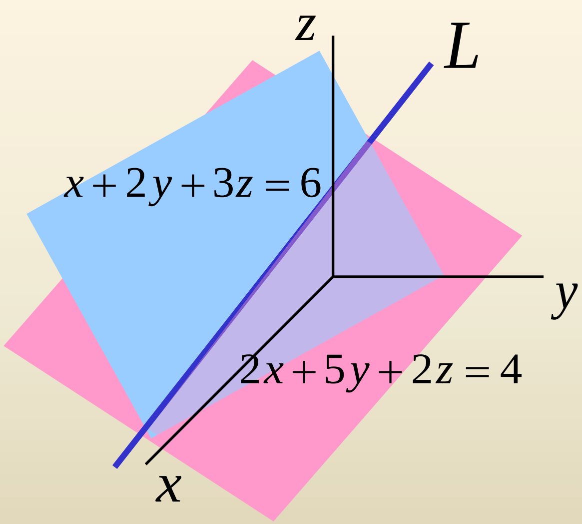

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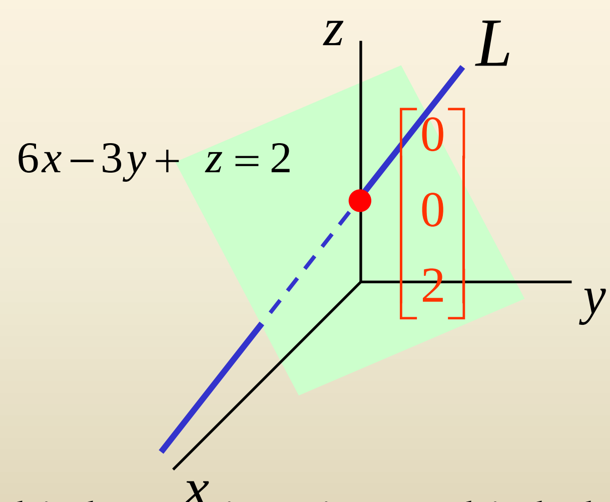

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Vectors and Linear Equations - Three Equations in Three Unknowns

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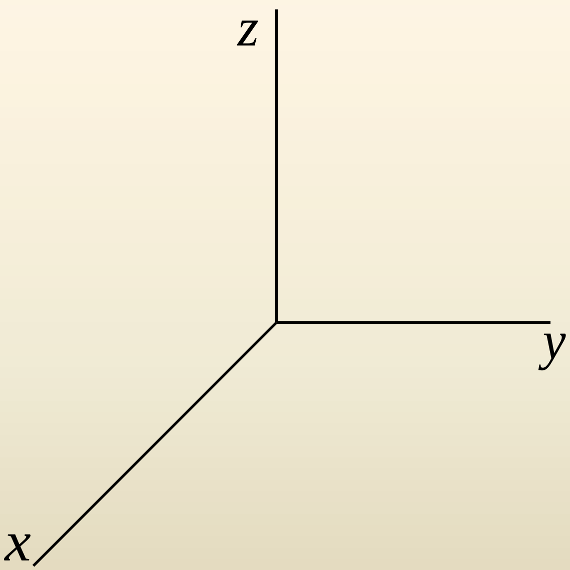
The column picture starts with the vector form of the equations :

$$x\begin{bmatrix}1\\2\\6\end{bmatrix}+y\begin{bmatrix}2\\5\\-3\end{bmatrix}+z\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}$$



C The column picture combines three columns to produce the vector (6,4,2)

Figure 2.4 Column picture : $(x, y, z) = (0, 0, 2)$ because $2(3, 2, 1) = (6, 4, 2) = \mathbf{b}$.

The coefficient we need are $x = 0$, $y = 0$ and $z = 2$. This is also the intersection point of the three planes in the row picture.

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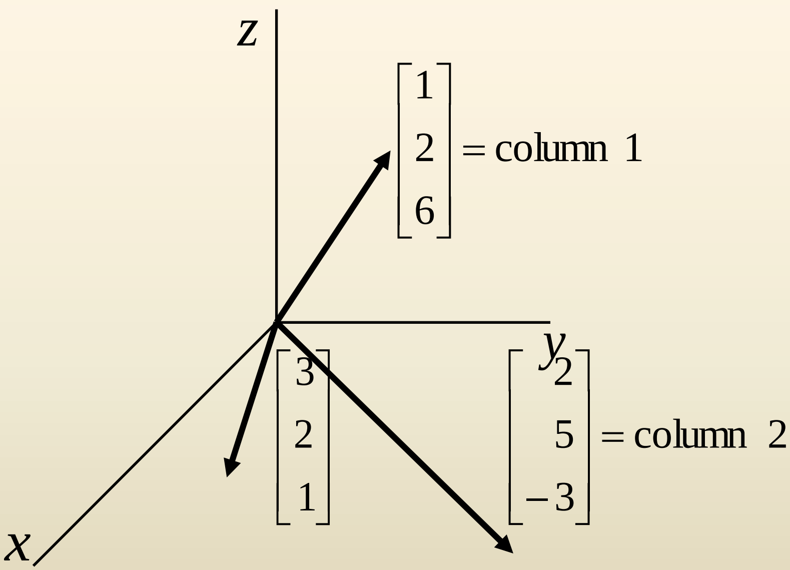

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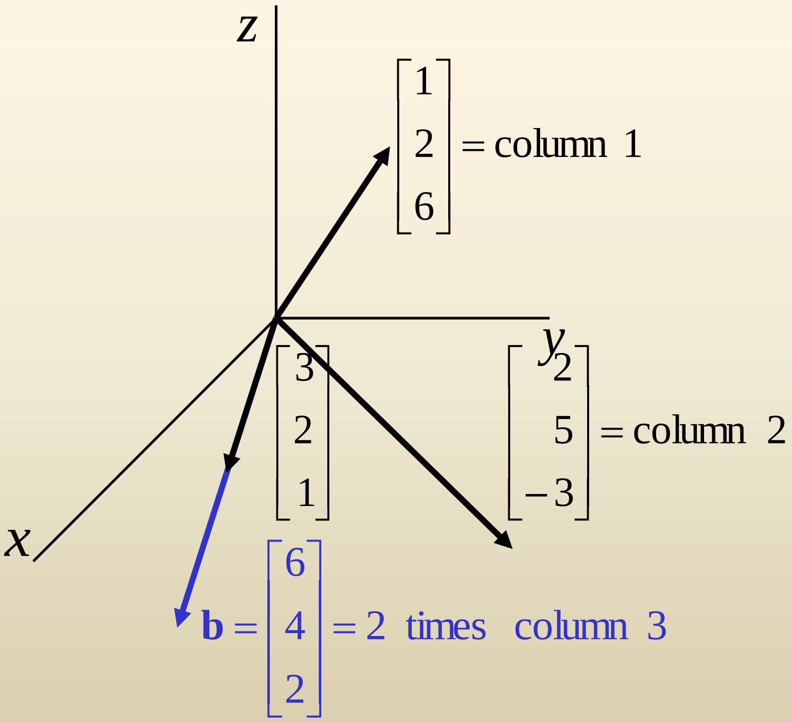

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Vectors and Linear Equations
- The Matrix Form of the Equations

$$\begin{aligned} x+2y+3z&=6\\ 2x+5y+2z&=4\\ 6x-3y+\ z&=2 \end{aligned} \qquad (3)$$

Matrix equation :

$$\mathbf{Ax}=\underbrace{\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 2\\ 6 & -3 & 1\end{bmatrix}}_{\text{Coefficient matrix}}\underbrace{\begin{bmatrix}x\\ y\\ z\end{bmatrix}}_{\text{unknown vector}}=\begin{bmatrix}6\\ 4\\ 2\end{bmatrix}=\mathbf{b}$$

We multiply the matrix $\mathbf{A}$ times the unknown vector $\mathbf{x}$ to get the right side $\mathbf{b}$.

Multiplication by rows : $\mathbf{Ax}$ comes from dot products, each row times the column $\mathbf{x}$:

$$\mathbf{Ax}=\begin{bmatrix}(\mathbf{row\ 1})\bullet\mathbf{x}\\ (\mathbf{row\ 2})\bullet\mathbf{x}\\ (\mathbf{row\ 3})\bullet\mathbf{x}\end{bmatrix}.$$

Multiplication by columns : $\mathbf{Ax}$ is a combination of column vectors :

$$\mathbf{Ax}=x(\mathbf{column\ \ 1}) + y(\mathbf{column\ \ 2})+z(\mathbf{column\ \ 3})$$

Vectors and Linear Equations - The Matrix Form of the Equations

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \mathbf{b}$$

$$\mathbf{Ax} = x\,(\mathbf{column\ \ 1}) + y\,(\mathbf{column\ \ 2}) + z\,(\mathbf{column\ \ 3})$$

When we substitute the solution $\mathbf{x} = (0, 0, 2)$, the multiplication $\mathbf{Ax}$ produces $\mathbf{b}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = 2 \text{ times column } 3 = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}.$$

The first dot product in row multiplication is $(1, 2, 3) \bullet (0, 0, 2) = 6$. The other dot products are 4 and 2. Multiplication by columns is simply 2 times column 3.

$\mathbf{Ax}$ as a combination of the columns of $\mathbf{A}$.