Naval Architecture & Ocean Engineering

[2008] [09-1]

Engineering Mathematics 2

October, 2008

Prof. Kyu-Yeul Lee

Department of Naval Architecture and Ocean Engineering,
Seoul National University of College of Engineering

Matrices(3) : Eigenvalues, Eigenvectors and Diagonalization

Eigenvalues and Eigenvectors

Example of Eigenvalue Problems

- Stretching of an Elastic Membrane
- Markov Matrix
- Rotating String
- Buckling

Inverse of a matrix.

Gauss-Jordan Elimination

Diagonalization

Singular Value Decomposition

Quadratic Form

Positive Definite Matrix & Optimization

Eigenvalues, Eigenvectors

Eigenvalues, Eigenvectors

Let $\mathbf{A}=[a_{jk}]$ be a given matrix n x n matrix and consider the vector equation

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Our task is to determine $\mathbf{x}$'s and λ's that satisfy this equation. Clearly, the zero vector $\mathbf{x} = 0$ is a solution of this equation for any value λ, because $\mathbf{A0}=\mathbf{0}$. This is of no interest.

(a) Eigenvalue: A value of λ for which this equation has a solution $\mathbf{x}\neq\mathbf{0}$ (= characteristic value, latent root)

(b) Eigenvector: The corresponding solutions $\mathbf{x}\neq\mathbf{0}$ of this equation

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

Determine eigenvalues and eigenvectors

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

(a) Eigenvalues

$$\mathbf{Ax} = \lambda \mathbf{x}$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

Determine eigenvalues and eigenvectors

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

(a) Eigenvalues

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

Determine eigenvalues and eigenvectors

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

$$-5x_1 + 2x_2 = \lambda x_1$$

$$2x_1 - 2x_2 = \lambda x_2$$

(a) Eigenvalues

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

Determine eigenvalues and eigenvectors

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

(a) Eigenvalues

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$-5x_1 + 2x_2 = \lambda x_1$$

$$2x_1 - 2x_2 = \lambda x_2$$

$$(-5-\lambda)x_1 + 2x_2 = 0$$

$$2x_1 + (-2-\lambda)x_2 = 0$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

Determine eigenvalues and eigenvectors

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

(a) Eigenvalues

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$-5x_1 + 2x_2 = \lambda x_1$$

$$2x_1 - 2x_2 = \lambda x_2$$

$$(-5-\lambda)x_1 + 2x_2 = 0$$

$$2x_1 + (-2-\lambda)x_2 = 0$$

This can be written in matrix notation.

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

Determine eigenvalues and eigenvectors

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

(a) Eigenvalues

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$-5x_1 + 2x_2 = \lambda x_1$$

$$2x_1 - 2x_2 = \lambda x_2$$

$$(-5-\lambda)x_1 + 2x_2 = 0$$

$$2x_1 + (-2-\lambda)x_2 = 0$$

This can be written in matrix notation.

$$\mathbf{Ax} - \lambda \mathbf{x} = 0$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

Determine eigenvalues and eigenvectors

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

(a) Eigenvalues

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$-5x_1 + 2x_2 = \lambda x_1$$

$$2x_1 - 2x_2 = \lambda x_2$$

$$(-5-\lambda)x_1 + 2x_2 = 0$$

$$2x_1 + (-2-\lambda)x_2 = 0$$

This can be written in matrix notation.

$$\mathbf{Ax} - \lambda \mathbf{x} = 0$$

$$\mathbf{Ax} - \lambda \mathbf{Ix} = 0$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

Determine eigenvalues and eigenvectors

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

(a) Eigenvalues

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$-5x_1 + 2x_2 = \lambda x_1$$

$$2x_1 - 2x_2 = \lambda x_2$$

$$(-5-\lambda)x_1 + 2x_2 = 0$$

$$2x_1 + (-2-\lambda)x_2 = 0$$

This can be written in matrix notation.

$$\mathbf{Ax} - \lambda \mathbf{x} = 0$$

$$\mathbf{Ax} - \lambda \mathbf{Ix} = 0$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors)

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

We see that this is a homogeneous linear system. By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq 0$ if and only if its coefficient determinant is zero.

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors)

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

We see that this is a homogeneous linear system. By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq 0$ if and only if its coefficient determinant is zero.

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors)

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

We see that this is a homogeneous linear system. By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq 0$ if and only if its coefficient determinant is zero.

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}=0$$

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors)

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

We see that this is a homogeneous linear system. By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq 0$ if and only if its coefficient determinant is zero.

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}=0$$

$$(-5-\lambda)(-2-\lambda)-4=0$$

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors)

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

We see that this is a homogeneous linear system. By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq 0$ if and only if its coefficient determinant is zero.

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}=0$$

$$(-5-\lambda)(-2-\lambda)-4=0$$

$$\lambda^2+7\lambda+6=0$$

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors)

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

We see that this is a homogeneous linear system. By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq 0$ if and only if its coefficient determinant is zero.

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}=0$$

$$(-5-\lambda)(-2-\lambda)-4=0$$

$$\lambda^2+7\lambda+6=0$$

$$\lambda_1=-1$$

$$\lambda_2=-6$$

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors)

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

We see that this is a homogeneous linear system. By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq 0$ if and only if its coefficient determinant is zero.

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}=0$$

$$(-5-\lambda)(-2-\lambda)-4=0$$

$$\lambda^2+7\lambda+6=0$$

$$\lambda_1=-1$$

$$\lambda_2=-6$$

$$(-5-\lambda)x_1+2x_2=0\cdots(a)$$

$$2x_1+(-2-\lambda)x_2=0\cdots(b)$$

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors)

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

We see that this is a homogeneous linear system. By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq 0$ if and only if its coefficient determinant is zero.

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}=0$$

$$(-5-\lambda)(-2-\lambda)-4=0$$

$$\lambda^2+7\lambda+6=0$$

$$\lambda_1=-1$$

$$\lambda_2=-6$$

$$(-5-\lambda)x_1+2x_2=0\cdots(a)$$

$$2x_1+(-2-\lambda)x_2=0\cdots(b)$$

$$x_1=0, x_2=0$$

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors)

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

We see that this is a homogeneous linear system. By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq 0$ if and only if its coefficient determinant is zero.

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}=0$$

$$(-5-\lambda)(-2-\lambda)-4=0$$

$$\lambda^2+7\lambda+6=0$$

$$\lambda_1=-1$$

$$\lambda_2=-6$$

$$(-5-\lambda)x_1+2x_2=0\cdots(a)$$

$$2x_1+(-2-\lambda)x_2=0\cdots(b)$$

$x_1=0, x_2=0$: Trivial solution

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors)

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

We see that this is a homogeneous linear system. By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq 0$ if and only if its coefficient determinant is zero.

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}=0$$

$$(-5-\lambda)(-2-\lambda)-4=0$$

$$\lambda^2+7\lambda+6=0$$

$$\lambda_1=-1$$

$$\lambda_2=-6$$

$$(-5-\lambda)x_1+2x_2=0\cdots(a)$$

$$2x_1+(-2-\lambda)x_2=0\cdots(b)$$

$x_1=0, x_2=0$: Trivial solution

To have solution non-trivial solution

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors)

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

We see that this is a homogeneous linear system. By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq 0$ if and only if its coefficient determinant is zero.

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}=0$$

$$(-5-\lambda)(-2-\lambda)-4=0$$

$$\lambda^2+7\lambda+6=0$$

$$\lambda_1=-1$$

$$\lambda_2=-6$$

$$(-5-\lambda)x_1+2x_2=0\cdots(a)$$

$$2x_1+(-2-\lambda)x_2=0\cdots(b)$$

$x_1=0, x_2=0$: Trivial solution

To have solution non-trivial solution

No. of Linearly independent equation < Unknowns

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors)

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

We see that this is a homogeneous linear system. By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq 0$ if and only if its coefficient determinant is zero.

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}=0$$

$$(-5-\lambda)(-2-\lambda)-4=0$$

$$\lambda^2+7\lambda+6=0$$

$$\lambda_1=-1$$

$$\lambda_2=-6$$

$$(-5-\lambda)x_1+2x_2=0\cdots(a)$$

$$2x_1+(-2-\lambda)x_2=0\cdots(b)$$

$x_1=0, x_2=0$: Trivial solution

To have solution non-trivial solution

No. of Linearly independent equation < Unknowns

$$(a)\times n=(b)$$

$$(-5-\lambda)n=2\cdots(c)$$

$$2n=(-2-\lambda)\cdots(d)$$

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors)

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

We see that this is a homogeneous linear system. By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq 0$ if and only if its coefficient determinant is zero.

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}=0$$

$$(-5-\lambda)(-2-\lambda)-4=0$$

$$\lambda^2+7\lambda+6=0$$

$$\lambda_1=-1$$

$$\lambda_2=-6$$

$$(-5-\lambda)x_1+2x_2=0\cdots(a)$$

$$2x_1+(-2-\lambda)x_2=0\cdots(b)$$

$x_1=0, x_2=0$: Trivial solution

To have solution non-trivial solution

No. of Linearly independent equation < Unknowns

$(a)\times n=(b)$ Linearly dependent, $n\neq 0$

$(-5-\lambda)n=2\cdots(c)$

$2n=(-2-\lambda)\cdots(d)$

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors)

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

We see that this is a homogeneous linear system. By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq 0$ if and only if its coefficient determinant is zero.

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}=0$$

$$(-5-\lambda)(-2-\lambda)-4=0$$

$$\lambda^2+7\lambda+6=0$$

$$\lambda_1=-1$$

$$\lambda_2=-6$$

$$(-5-\lambda)x_1+2x_2=0\cdots(a)$$

$$2x_1+(-2-\lambda)x_2=0\cdots(b)$$

$x_1=0, x_2=0$: Trivial solution

To have solution non-trivial solution

No. of Linearly independent equation < Unknowns

$(a)\times n=(b)$ Linearly dependent, $n\neq 0$

$(-5-\lambda)n=2\cdots(c)$

$2n=(-2-\lambda)\cdots(d)$

$(c)/(d)$

$$\frac{(-5-\lambda)}{2}=\frac{2}{(-2-\lambda)}$$

$$\therefore(-5-\lambda)(-2-\lambda)-4=0$$

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors)

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

We see that this is a homogeneous linear system. By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq 0$ if and only if its coefficient determinant is zero.

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}=0$$

$$(-5-\lambda)(-2-\lambda)-4=0$$

$$\lambda^2+7\lambda+6=0$$

$$\lambda_1=-1$$

$$\lambda_2=-6$$

$$(-5-\lambda)x_1+2x_2=0\cdots(a)$$

$$2x_1+(-2-\lambda)x_2=0\cdots(b)$$

$x_1=0, x_2=0$: Trivial solution

To have solution non-trivial solution

No. of Linearly independent equation < Unknowns

$(a)\times n=(b)$ Linearly dependent, $n\neq 0$

$(-5-\lambda)n=2\cdots(c)$

$2n=(-2-\lambda)\cdots(d)$

$(c)/(d)$

$$\frac{(-5-\lambda)}{2}=\frac{2}{(-2-\lambda)}$$

$$\therefore(-5-\lambda)(-2-\lambda)-4=0$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1 + 2x_2 = 0 \qquad \lambda_1 = -1$$

$$2x_1 + (-2-\lambda)x_2 = 0 \qquad \lambda_2 = -6$$

-Eigenvector of A corresponding to λ_1

$$\lambda_1 = -1$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1 + 2x_2 = 0 \qquad \lambda_1 = -1$$

$$2x_1 + (-2-\lambda)x_2 = 0 \qquad \lambda_2 = -6$$

-Eigenvector of A corresponding to λ_1

$$\lambda_1 = -1$$

The equation is

$$-4x_1 + 2x_2 = 0$$

$$2x_1 - x_2 = 0$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1 + 2x_2 = 0 \qquad \lambda_1 = -1$$

$$2x_1 + (-2-\lambda)x_2 = 0 \qquad \lambda_2 = -6$$

-Eigenvector of A corresponding to λ_1

$$\lambda_1 = -1$$

The equation is

$$-4x_1 + 2x_2 = 0$$

$$2x_1 - x_2 = 0$$

$$\therefore x_2 = 2x_1$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1+2x_2=0 \qquad \lambda_1=-1$$

$$2x_1+(-2-\lambda)x_2=0 \qquad \lambda_2=-6$$

-Eigenvector of A corresponding to λ_1

$$\lambda_1=-1$$

The equation is

$$-4x_1+2x_2=0$$

$$2x_1-x_2=0$$

$$\therefore x_2=2x_1$$

From this, we determine eigenvector corresponding to λ_1 up to a scalar multiple. If we choose x_1=1, we obtain

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1+2x_2=0 \qquad \lambda_1=-1$$

$$2x_1+(-2-\lambda)x_2=0 \qquad \lambda_2=-6$$

-Eigenvector of A corresponding to λ_1

$$\lambda_1=-1$$

The equation is

$$-4x_1+2x_2=0$$

$$2x_1-x_2=0$$

$$\therefore x_2=2x_1$$

From this, we determine eigenvector corresponding to λ_1 up to a scalar multiple. If we choose $x_1=1$, we obtain

$$\mathbf{x}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\2x_1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1+2x_2=0 \qquad \lambda_1=-1$$

$$2x_1+(-2-\lambda)x_2=0 \qquad \lambda_2=-6$$

What happen if choose x_1=something else?

-Eigenvector of A corresponding to λ_1

$$\lambda_1=-1$$

The equation is

$$-4x_1+2x_2=0$$

$$2x_1-x_2=0$$

$$\therefore x_2=2x_1$$

From this, we determine eigenvector corresponding to λ_1 up to a scalar multiple. If we choose x_1=1, we obtain

$$\mathbf{x}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\2x_1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1+2x_2=0 \qquad \lambda_1=-1$$

$$2x_1+(-2-\lambda)x_2=0 \qquad \lambda_2=-6$$

-Eigenvector of A corresponding to λ_1

$$\lambda_1=-1$$

The equation is

$$-4x_1+2x_2=0$$

$$2x_1-x_2=0$$

$$\therefore x_2=2x_1$$

From this, we determine eigenvector corresponding to λ_1 up to a scalar multiple. If we choose x_1=1, we obtain

$$\mathbf{x}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\2x_1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

What happen if choose x_1=something else?

$$1:\mathbf{x}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\2x_1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

$$2:\mathbf{x}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\2x_1\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}=2\begin{bmatrix}1\\2\end{bmatrix}$$

$$3:\mathbf{x}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\2x_1\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix}=3\begin{bmatrix}1\\2\end{bmatrix}$$

$$\vdots$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1+2x_2=0 \qquad \lambda_1=-1$$

$$2x_1+(-2-\lambda)x_2=0 \qquad \lambda_2=-6$$

-Eigenvector of A corresponding to λ_1

$$\lambda_1=-1$$

The equation is

$$-4x_1+2x_2=0$$

$$2x_1-x_2=0$$

$$\therefore x_2=2x_1$$

From this, we determine eigenvector corresponding to λ_1 up to a scalar multiple. If we choose $x_1=1$, we obtain

$$\mathbf{x}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\2x_1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

What happen if choose x_1=something else?

$$1:\mathbf{x}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\2x_1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

$$2:\mathbf{x}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\2x_1\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}=2\begin{bmatrix}1\\2\end{bmatrix}$$

$$3:\mathbf{x}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\2x_1\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix}=3\begin{bmatrix}1\\2\end{bmatrix}$$

$$\vdots$$

'basis'

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1 + 2x_2 = 0 \qquad \lambda_1 = -1$$

$$2x_1 + (-2-\lambda)x_2 = 0 \qquad \lambda_2 = -6$$

-Eigenvector of A corresponding to λ_2

$$\lambda_2 = -6$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1 + 2x_2 = 0 \qquad \lambda_1 = -1$$

$$2x_1 + (-2-\lambda)x_2 = 0 \qquad \lambda_2 = -6$$

-Eigenvector of A corresponding to λ_2

$$\lambda_2 = -6$$

The equation is

$$x_1 + 2x_2 = 0$$

$$2x_1 + 4x_2 = 0$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1 + 2x_2 = 0 \qquad \lambda_1 = -1$$

$$2x_1 + (-2-\lambda)x_2 = 0 \qquad \lambda_2 = -6$$

-Eigenvector of A corresponding to λ_2

$$\lambda_2 = -6$$

The equation is

$$x_1 + 2x_2 = 0$$

$$2x_1 + 4x_2 = 0$$

$$\therefore x_2 = -x_1/2$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1+2x_2=0 \qquad \lambda_1=-1$$

$$2x_1+(-2-\lambda)x_2=0 \qquad \lambda_2=-6$$

-Eigenvector of A corresponding to λ_2

$$\lambda_2=-6$$

The equation is

$$x_1+2x_2=0$$

$$2x_1+4x_2=0$$

$$\therefore x_2=-x_1/2$$

From this, we determine eigenvector corresponding to λ_2 up to a scalar multiple. If we choose x_1=2, we obtain

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1+2x_2=0 \qquad \lambda_1=-1$$

$$2x_1+(-2-\lambda)x_2=0 \qquad \lambda_2=-6$$

-Eigenvector of A corresponding to λ_2

$$\lambda_2=-6$$

The equation is

$$x_1+2x_2=0$$

$$2x_1+4x_2=0$$

$$\therefore x_2=-x_1/2$$

From this, we determine eigenvector corresponding to λ_2 up to a scalar multiple. If we choose x_1=2, we obtain

$$\mathbf{x}_2=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\-x_1/2\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$$

Eigenvalues, Eigenvectors

$$D(\lambda)=\begin{vmatrix}-5-\lambda & 2\\ 2 & -2-\lambda\end{vmatrix}=0$$

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1+2x_2=0 \qquad \lambda_1=-1$$

$$2x_1+(-2-\lambda)x_2=0 \qquad \lambda_2=-6$$

-Eigenvector of A corresponding to λ_2

$$\lambda_2=-6$$

The equation is

$$x_1+2x_2=0$$

$$2x_1+4x_2=0$$

$$\therefore x_2=-x_1/2$$

From this, we determine eigenvector corresponding to λ_2 up to a scalar multiple. If we choose x_1=2, we obtain

$$\mathbf{x}_2=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ -x_1/2\end{bmatrix}=\begin{bmatrix}2\\ -1\end{bmatrix}$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1 + 2x_2 = 0 \qquad \lambda_1 = -1$$

$$2x_1 + (-2-\lambda)x_2 = 0 \qquad \lambda_2 = -6$$

-Eigenvector of A corresponding to λ_1

$$\lambda_2 = -6$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1 + 2x_2 = 0 \qquad \lambda_1 = -1$$

$$2x_1 + (-2-\lambda)x_2 = 0 \qquad \lambda_2 = -6$$

-Eigenvector of A corresponding to λ_1

$$\lambda_2 = -6$$

The equation is

$$x_1 + 2x_2 = 0$$

$$2x_1 + 4x_2 = 0$$

$$\therefore x_2 = -x_1/2$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1 + 2x_2 = 0 \qquad \lambda_1 = -1$$

$$2x_1 + (-2-\lambda)x_2 = 0 \qquad \lambda_2 = -6$$

-Eigenvector of A corresponding to λ_1

$$\lambda_2 = -6$$

The equation is

$$x_1 + 2x_2 = 0$$

$$2x_1 + 4x_2 = 0$$

$$\therefore x_2 = -x_1/2$$

From this, we determine eigenvector corresponding to λ_2 up to a scalar multiple. If we choose x_1=2, we obtain

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1+2x_2=0 \qquad \lambda_1=-1$$

$$2x_1+(-2-\lambda)x_2=0 \qquad \lambda_2=-6$$

-Eigenvector of A corresponding to λ_1

$$\lambda_2=-6$$

The equation is

$$x_1+2x_2=0$$

$$2x_1+4x_2=0$$

$$\therefore x_2=-x_1/2$$

From this, we determine eigenvector corresponding to λ_2 up to a scalar multiple. If we choose x_1=2, we obtain

$$\mathbf{x}_2=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\-x_1/2\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1+2x_2=0 \qquad \lambda_1=-1$$
$$2x_1+(-2-\lambda)x_2=0 \qquad \lambda_2=-6$$

-Eigenvector of A corresponding to λ_1

$$\lambda_2=-6$$

The equation is

$$x_1+2x_2=0$$
$$2x_1+4x_2=0$$
$$\therefore x_2=-x_1/2$$

From this, we determine eigenvector corresponding to λ_2 up to a scalar multiple. If we choose $x_1=2$, we obtain

$$\mathbf{x}_2=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\-x_1/2\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$$

What happen if choose x_2=something else?

$$1:\mathbf{x}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\-x_1/2\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$$

$$2:\mathbf{x}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\-x_1/2\end{bmatrix}=\begin{bmatrix}4\\-2\end{bmatrix}=2\begin{bmatrix}2\\-1\end{bmatrix}$$

$$3:\mathbf{x}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\-x_1/2\end{bmatrix}=\begin{bmatrix}6\\-3\end{bmatrix}=3\begin{bmatrix}2\\-1\end{bmatrix}$$

$$\vdots$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1+2x_2=0 \qquad \lambda_1=-1$$
$$2x_1+(-2-\lambda)x_2=0 \qquad \lambda_2=-6$$

-Eigenvector of A corresponding to λ_1

$$\lambda_2=-6$$

The equation is

$$x_1+2x_2=0$$
$$2x_1+4x_2=0$$
$$\therefore x_2=-x_1/2$$

From this, we determine eigenvector corresponding to λ_2 up to a scalar multiple. If we choose $x_1=2$, we obtain

$$\mathbf{x}_2=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\-x_1/2\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$$

What happen if choose x_2=something else?

$$1:\mathbf{x}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\-x_1/2\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$$
$$2:\mathbf{x}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\-x_1/2\end{bmatrix}=\begin{bmatrix}4\\-2\end{bmatrix}=2\begin{bmatrix}2\\-1\end{bmatrix}$$
$$3:\mathbf{x}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\-x_1/2\end{bmatrix}=\begin{bmatrix}6\\-3\end{bmatrix}=3\begin{bmatrix}2\\-1\end{bmatrix}$$
$$\vdots$$

'basis'

General case of Eigenvectors, Eigenvalues

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\mathbf{Ax} = \lambda \mathbf{x}$$

General case of Eigenvectors, Eigenvalues

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= \lambda x_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= \lambda x_2 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= \lambda x_n \end{aligned}$$

General case of Eigenvectors, Eigenvalues

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= \lambda x_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= \lambda x_2 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= \lambda x_n \end{aligned}$$

$$\begin{aligned} (a_{11} - \lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn} - \lambda)x_n &= 0 \end{aligned}$$

General case of Eigenvectors, Eigenvalues

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= \lambda x_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= \lambda x_2 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= \lambda x_n \end{aligned}$$

$$\begin{aligned} (a_{11} - \lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn} - \lambda)x_n &= 0 \end{aligned}$$

In matrix notation,

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

General case of Eigenvectors, Eigenvalues

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

General case of Eigenvectors, Eigenvalues

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}=0$$

General case of Eigenvectors, Eigenvalues

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}=0$$

(a) Characteristic matrix: $\mathbf{A}-\lambda\mathbf{I}$

(b) Characteristic determinant: $D(\lambda)$

(c) Characteristic polynomial: a polynomial of nth degree in λ by developing $D(\lambda)$

Eigenvalues, Eigenvectors

Theorem 8.1 Eigenvalues

The eigenvalues of a square matrix A are the roots of the characteristic equation of A.

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Hence an n x n matrix has at least one eigenvalue and at most n numerically different eigenvalues.

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}$$

$$=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}$$

$$=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}$$

$$=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}$$

$$=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

Choosing

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}$$

$$=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing

$x_3=-1$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}$$

$$=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing

$$x_3=-1$$

we have

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}$$

$$=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing

$$x_3=-1$$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}$$

$$=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing

$$x_3=-1$$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=2$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}$$

$$=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing

$$x_3=-1$$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=2$$

$$-7x_1+2x_2-3x_3=0$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}$$

$$=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing

$$x_3=-1$$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=1$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}$$

$$=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing

$$x_3=-1$$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=1$$

$$\mathbf{x}_1=\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathrm{T}}$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}$$

$$=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing

$$x_3=-1$$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=1$$

$$\mathbf{x}_1=\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathrm{T}}$$

if choose

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing

$x_3=-1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=1$$

$$\mathbf{x}_1=\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathrm{T}}$$

if choose

$x_3=1$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing

$x_3=-1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=1$$

$$\mathbf{x}_1=\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathrm{T}}$$

if choose

$x_3=1$

we have

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing

$x_3=-1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=1$$

$$\mathbf{x}_1=\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathrm{T}}$$

if choose

$x_3=1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing

$x_3=-1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=1$$

$$\mathbf{x}_1=\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathrm{T}}$$

if choose

$x_3=1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=-2$$

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

Ex.)Multiple Eigenvalues

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing $x_3=-1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=1$$

$$\mathbf{x}_1=\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathrm{T}}$$

if choose $x_3=1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=-2$$

$$-7x_1+2x_2-3x_3=0$$

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

Ex.)Multiple Eigenvalues

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing $x_3=-1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=1$$

$$\mathbf{x}_1=\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathrm{T}}$$

if choose $x_3=1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=-2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=-1$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing

$x_3=-1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=1$$

$$\mathbf{x}_1=\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathbf{T}}$$

if choose

$x_3=1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=-2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=-1$$

$$\mathbf{x}_1=\begin{bmatrix}-1 & -2 & 1\end{bmatrix}^{\mathbf{T}}$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing

$x_3=-1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=1$$

$$\mathbf{x}_1=\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathbf{T}}$$

if choose

$x_3=1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=-2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=-1$$

$$\mathbf{x}_1=\begin{bmatrix}-1 & -2 & 1\end{bmatrix}^{\mathbf{T}}$$

$$\mathbf{x}_1=(-1)\times\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathbf{T}}$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing $x_3=-1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=1$$

$$\mathbf{x}_1=\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathbf{T}}$$

if choose $x_3=1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=-2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=-1$$

$$\mathbf{x}_1=\begin{bmatrix}-1 & -2 & 1\end{bmatrix}^{\mathbf{T}}$$

$$\mathbf{x}_1=(-1)\times\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathbf{T}}$$

'basis'

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

Ex.)Multiple Eigenvalues

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(2) $\lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(2) $\lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

Ex.)Multiple Eigenvalues

$(2)\ \lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

Ex.)Multiple Eigenvalues

(2) $\lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

Ex.)Multiple Eigenvalues

(2) $\lambda=-3$

$$x_1+2x_2-3x_3=0$$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

Ex.)Multiple Eigenvalues

$(2)\ \lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1+2x_2-3x_3=0$$

$$x_1=-2x_2+3x_3$$

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

Ex.)Multiple Eigenvalues

$(2)\ \lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1+2x_2-3x_3=0$$

$$x_1=-2x_2+3x_3$$

Choosing

$$x_2=1,\ x_3=0$$

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

Ex.)Multiple Eigenvalues

$(2)\ \lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1+2x_2-3x_3=0$$

$$x_1=-2x_2+3x_3$$

Choosing

$$x_2=1,\ x_3=0$$

we have

$$x_1=-2$$

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

Ex.)Multiple Eigenvalues

$(2)\ \lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1+2x_2-3x_3=0$$

$$x_1=-2x_2+3x_3$$

Choosing

$$x_2=1,\ x_3=0$$

we have

$$x_1=-2$$

$$\mathbf{x}_2=\begin{bmatrix}-2 & 1 & 0\end{bmatrix}^T$$

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

Ex.)Multiple Eigenvalues

$(2)\ \lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1+2x_2-3x_3=0$$

$$x_1=-2x_2+3x_3$$

Choosing

$$x_2=1,\ x_3=0$$

we have

$$x_1=-2$$

$$\mathbf{x}_2=\begin{bmatrix}-2 & 1 & 0\end{bmatrix}^T$$

Choosing

$$x_2=0,\ x_3=1$$

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

Ex.)Multiple Eigenvalues

$(2)\ \lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1+2x_2-3x_3=0$$

$$x_1=-2x_2+3x_3$$

Choosing

$$x_2=1,\ x_3=0$$

we have

$$x_1=-2$$

$$\mathbf{x}_2=\begin{bmatrix}-2 & 1 & 0\end{bmatrix}^T$$

Choosing

$$x_2=0,\ x_3=1$$

we have

$$x_1=3$$

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

Ex.)Multiple Eigenvalues

$(2)\ \lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1+2x_2-3x_3=0$$

$$x_1=-2x_2+3x_3$$

Choosing

$$x_2=1,\ x_3=0$$

we have

$$x_1=-2$$

$$\mathbf{x}_2=\begin{bmatrix}-2 & 1 & 0\end{bmatrix}^T$$

Choosing

$$x_2=0,\ x_3=1$$

we have

$$x_1=3$$

$$\mathbf{x}_3=\begin{bmatrix}3 & 0 & 1\end{bmatrix}^T$$

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

Ex.)Multiple Eigenvalues

$(2)\ \lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1+2x_2-3x_3=0$$

$$x_1=-2x_2+3x_3$$

Choosing

$$x_2=1,\ x_3=0$$

we have

$$x_1=-2$$

$$\mathbf{x}_2=\begin{bmatrix}-2 & 1 & 0\end{bmatrix}^T$$

Choosing

$$x_2=0,\ x_3=1$$

we have

$$x_1=3$$

$$\mathbf{x}_3=\begin{bmatrix}3 & 0 & 1\end{bmatrix}^T$$

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

Ex.)Multiple Eigenvalues

(2) $\lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1+2x_2-3x_3=0$$

$$x_1=-2x_2+3x_3$$

Choosing

$x_2=1,\ x_3=0$

we have

$x_1=-2$

$$\mathbf{x}_2=\begin{bmatrix}-2 & 1 & 0\end{bmatrix}^T$$

Choosing

$x_2=0,\ x_3=1$

we have

$x_1=3$

$$\mathbf{x}_3=\begin{bmatrix}3 & 0 & 1\end{bmatrix}^T$$

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(2) $\lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1+2x_2-3x_3=0$$

$$x_1=-2x_2+3x_3$$

Choosing

$x_2=1,\ x_3=0$

we have

$x_1=-2$

$$\mathbf{x}_2=\begin{bmatrix}-2 & 1 & 0\end{bmatrix}^T$$

Choosing

$x_2=0,\ x_3=1$

we have

$x_1=3$

$$\mathbf{x}_3=\begin{bmatrix}3 & 0 & 1\end{bmatrix}^T$$

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\Rightarrow\begin{cases}x_1+2x_2-3x_3=0\\ 2x_1+4x_2-6x_3=0\\ -_1-2x_2+3x_3=0\end{cases}$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

$(2)\ \lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1+2x_2-3x_3=0$$

$$x_1=-2x_2+3x_3$$

Choosing

$x_2=1,\ x_3=0$

we have

$x_1=-2$

$$\mathbf{x}_2=\begin{bmatrix}-2 & 1 & 0\end{bmatrix}^T$$

Choosing

$x_2=0,\ x_3=1$

we have

$x_1=3$

$$\mathbf{x}_3=\begin{bmatrix}3 & 0 & 1\end{bmatrix}^T$$

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\Rightarrow\begin{cases}x_1+2x_2-3x_3=0\\ 2x_1+4x_2-6x_3=0\\ -_1-2x_2+3x_3=0\end{cases}$$

$$\Downarrow$$

$$x_1+2x_2-3x_3=0$$

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(2) $\lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1+2x_2-3x_3=0$$

$$x_1=-2x_2+3x_3$$

Choosing

$$x_2=1,\ x_3=0$$

we have

$$x_1=-2$$

$$\mathbf{x}_2=\begin{bmatrix}-2 & 1 & 0\end{bmatrix}^T$$

Choosing

$$x_2=0,\ x_3=1$$

we have

$$x_1=3$$

$$\mathbf{x}_3=\begin{bmatrix}3 & 0 & 1\end{bmatrix}^T$$

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\Rightarrow\begin{cases}x_1+2x_2-3x_3=0\\ 2x_1+4x_2-6x_3=0\\ -_1-2x_2+3x_3=0\end{cases}$$

$$\Downarrow$$

$$x_1+2x_2-3x_3=0$$

we have only one equation

and three variables

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

$(2)\ \lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1+2x_2-3x_3=0$$

$$x_1=-2x_2+3x_3$$

Choosing

$x_2=1,\ x_3=0$

we have

$x_1=-2$

$$\mathbf{x}_2=\begin{bmatrix}-2 & 1 & 0\end{bmatrix}^T$$

Choosing

$x_2=0,\ x_3=1$

we have

$x_1=3$

$$\mathbf{x}_3=\begin{bmatrix}3 & 0 & 1\end{bmatrix}^T$$

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\Rightarrow\begin{cases}x_1+2x_2-3x_3=0\\ 2x_1+4x_2-6x_3=0\\ -_1-2x_2+3x_3=0\end{cases}$$

$$\Downarrow$$

$$x_1+2x_2-3x_3=0$$

we have only one equation

and three variables

two free variables (x_2, x_3)

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\therefore \lambda = \pm i$$

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(1) $\lambda = i$

Characteristic Equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\therefore \lambda = \pm i$$

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\therefore \lambda = \pm i$$

(1) $\lambda = i$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda \mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\therefore \lambda = \pm i$$

(1) $\lambda = i$

$$\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

$$-ix_1 + x_2 = 0$$

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda \mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\therefore \lambda = \pm i$$

(1) $\lambda = i$

$$\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

$$-ix_1 + x_2 = 0$$

Choosing $x_1 = 1$

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda \mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\therefore \lambda = \pm i$$

(1) $\lambda = i$

$$\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

$$-ix_1 + x_2 = 0$$

Choosing $x_1 = 1$

we have $x_2 = i$

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda \mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\therefore \lambda = \pm i$$

(1) $\lambda = i$

$$\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

$$-ix_1 + x_2 = 0$$

Choosing $x_1 = 1$

we have $x_2 = i$

$$\mathbf{x}_1 = \begin{bmatrix} 1 & i \end{bmatrix}^{\mathrm{T}}$$

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

$(2)\ \lambda = -i$

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

(2) $\lambda = -i$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$$

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

(2) $\lambda = -i$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$$

$$ix_1 + x_2 = 0$$

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

$(2)\ \lambda = -i$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$$

$$ix_1 + x_2 = 0$$

Choosing $x_1 = 1$

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

$(2)\ \lambda = -i$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$$

$$ix_1 + x_2 = 0$$

Choosing $x_1 = 1$

we have $x_2 = -i$

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

$(2)\ \lambda = -i$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$$

$$ix_1 + x_2 = 0$$

Choosing $x_1 = 1$

we have $x_2 = -i$

$$\mathbf{x}_2 = \begin{bmatrix} 1 & -i \end{bmatrix}^{\mathrm{T}}$$

Eigenvalues of the Transpose

Theorem 8.3 Eigenvalues of the Transpose

The transpose A^T of a square matrix A has the same eigenvalues as A

Proof

Transposition does not change the value of the characteristic determinant

Theorem 2. Further Properties of *n*th-Order Determinants

(d) Transposition leaves the value of a determinant unaltered.

(e) A zero row or column renders the value of a determinant zero.

(f) Proportional rows or columns render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.

Sum and Multiplication of Eigenvalues

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}$$

Sum and Multiplication of Eigenvalues

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}$$

$$=(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}\cdot a_{21}$$

Sum and Multiplication of Eigenvalues

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}$$

$$=(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}\cdot a_{21}$$

$$=\lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}\cdot a_{21}$$

Sum and Multiplication of Eigenvalues

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}$$

$$=(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}\cdot a_{21}$$

$$=\lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}\cdot a_{21}$$

$$=\lambda^2-(a_{11}+a_{22})\lambda+\det\mathbf{A}$$

Sum and Multiplication of Eigenvalues

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}$$

$$=(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}\cdot a_{21}$$

$$=\lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}\cdot a_{21}$$

$$=\lambda^2-(a_{11}+a_{22})\lambda+\det\mathbf{A}$$

$$=(\lambda-\lambda_1)(\lambda-\lambda_2)$$

$$=\lambda^2-(\lambda_1+\lambda_2)\lambda+\lambda_1\lambda_2=0$$

Sum and Multiplication of Eigenvalues

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}$$

$$=(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}\cdot a_{21}$$

$$=\lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}\cdot a_{21}$$

$$=\lambda^2-(a_{11}+a_{22})\lambda+\det\mathbf{A}$$

$$=(\lambda-\lambda_1)(\lambda-\lambda_2)$$

$$=\lambda^2-(\lambda_1+\lambda_2)\lambda+\lambda_1\lambda_2=0$$

$$\therefore a_{11}+a_{12}=\lambda_1+\lambda_2$$

Sum and Multiplication of Eigenvalues

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}$$

$$=(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}\cdot a_{21}$$

$$=\lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}\cdot a_{21}$$

$$=\lambda^2-(a_{11}+a_{22})\lambda+\det\mathbf{A}$$

$$=(\lambda-\lambda_1)(\lambda-\lambda_2)$$

$$=\lambda^2-(\lambda_1+\lambda_2)\lambda+\lambda_1\lambda_2=0$$

$$\therefore a_{11}+a_{12}=\lambda_1+\lambda_2$$

$$\therefore \det\mathbf{A}=\lambda_1\cdot\lambda_2$$

Sum and Multiplication of Eigenvalues

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}$$

$$=(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}\cdot a_{21}$$

$$=\lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}\cdot a_{21}$$

$$=\lambda^2-(a_{11}+a_{22})\lambda+\det\mathbf{A}$$

$$=(\lambda-\lambda_1)(\lambda-\lambda_2)$$

$$=\lambda^2-(\lambda_1+\lambda_2)\lambda+\lambda_1\lambda_2=0$$

$$\therefore a_{11}+a_{12}=\lambda_1+\lambda_2$$

$$\therefore \det\mathbf{A}=\lambda_1\cdot\lambda_2$$

The sum of these *n* eigenvalues equals the sum of the entries on the main diagonal of A, called trace of A;

$$\therefore \text{trace}\,\mathbf{A}=\sum_{j=1}^{n}a_{jj}=\sum_{k=1}^{n}\lambda_k$$

Sum and Multiplication of Eigenvalues

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}$$

$$=(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}\cdot a_{21}$$

$$=\lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}\cdot a_{21}$$

$$=\lambda^2-(a_{11}+a_{22})\lambda+\det\mathbf{A}$$

$$=(\lambda-\lambda_1)(\lambda-\lambda_2)$$

$$=\lambda^2-(\lambda_1+\lambda_2)\lambda+\lambda_1\lambda_2=0$$

$$\therefore a_{11}+a_{12}=\lambda_1+\lambda_2$$

$$\therefore \det\mathbf{A}=\lambda_1\cdot\lambda_2$$

The sum of these *n* eigenvalues equals the sum of the entries on the main diagonal of A, called trace of A;

$$\therefore \text{trace}\,\mathbf{A}=\sum_{j=1}^{n}a_{jj}=\sum_{k=1}^{n}\lambda_k$$

The product of the eigenvalues equals the determinant of A,

$$\det\mathbf{A}=\lambda_1\lambda_2\cdots\lambda_n$$

Example of Eigenvalue Problems

-Stretching of an Elastic Membrane

-Markov Matrix

-Boundary-value problem

- **-Rotating string**
- **-Buckling of a Thin Vertical Column**

Matrix as a transformation

Solving linear systems

$$x_1 + 2x_2 + x_3 = 1$$
$$0x_1 + x_2 + 3x_3 = 5$$
$$0x_1 + 2x_2 + 6x_3 = 10$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix} \quad \mathbf{Ax} = \mathbf{B}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 2 & 6 & 10 \end{array}\right] \quad [\mathbf{A}\,|\,\mathbf{B}]$$

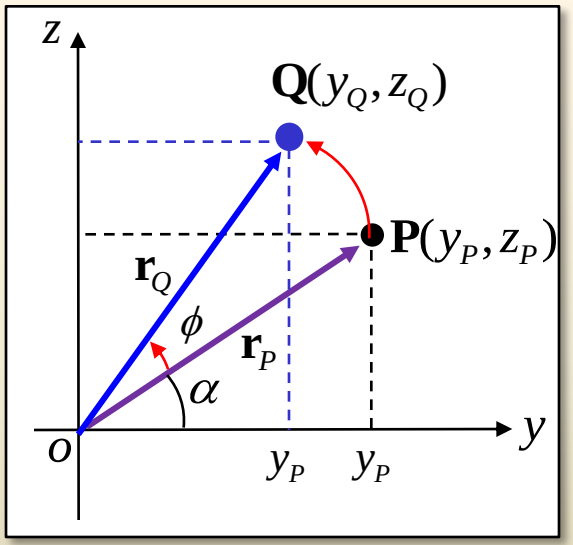
$$\mathbf{y} = \mathbf{Az}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

z : input

y : output

A : Transformation



$$\begin{bmatrix} y_Q \\ z_Q \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y_P \\ z_P \end{bmatrix}$$

점의 회전 변환

$$\mathbf{Ax} = \lambda\mathbf{x}$$

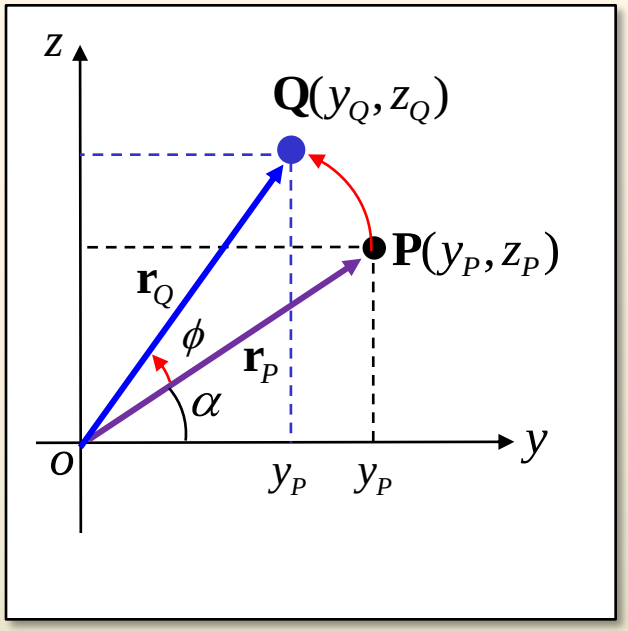
To see transformation properties

2차원 평면에서의 점의 회전과 좌표계 회전

✓ 고정된 좌표계에서 물체의 회전

Given: oyz 에서 정의된 점 P의 좌표값

Find : 점 P 을oyz 에 대해 ϕ 만큼 회전시킨 점 Q구하기



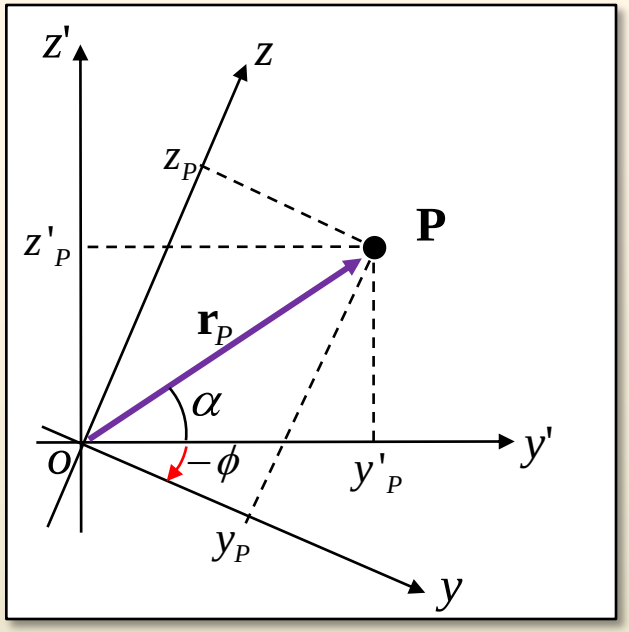
$$\begin{bmatrix} y_Q \\ z_Q \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y_P \\ z_P \end{bmatrix}$$

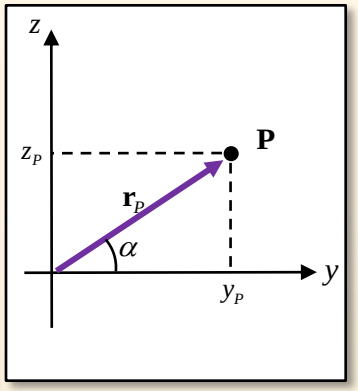
점의 회전 변환

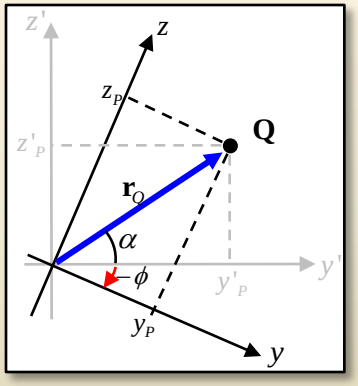
✓ 좌표계 회전

Given: $oy'z'$ 에서 정의된 점 P의 좌표값

Find: $oy'z'$ 에 대해$-\phi$ 만큼 회전한 새로운 좌표계 oyz 에서의 P의 좌표값







$$\begin{bmatrix} y_P \\ z_P \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y'_P \\ z'_P \end{bmatrix}$$

좌표계 회전 변환

$oy'z'$: Body fixed coordinate

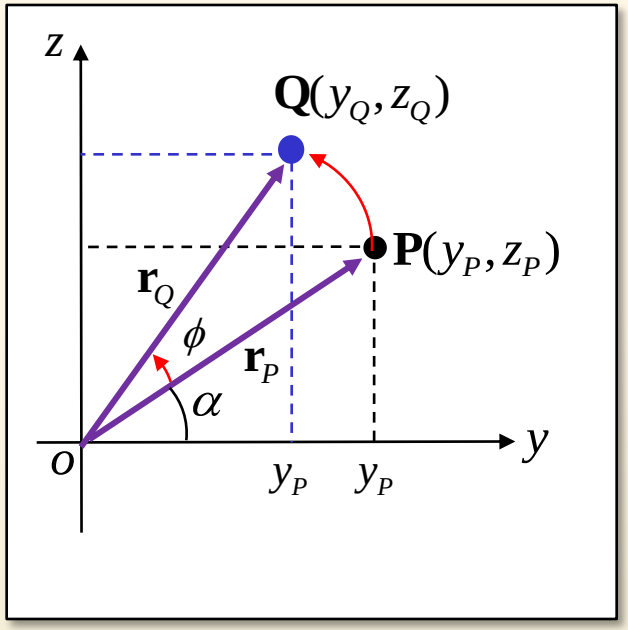
oyz : Global coordinate

점을 ϕ 만큼 회전시키는 변환 행렬과 좌표계를 $-\phi$ 만큼 회전시키는 변환 행렬이 동일함

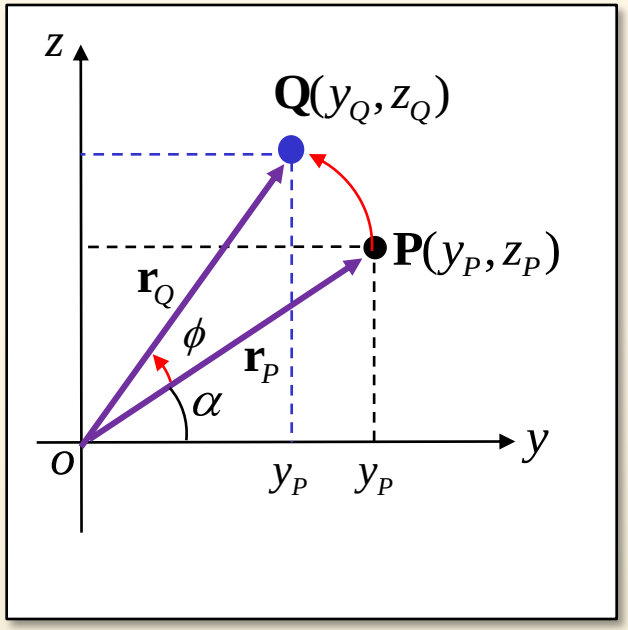
$$\mathbf{r}_p = \mathbf{r}_Q$$

Given: oyz 에서 정의된 점 P의 좌표값

Find: 점 P 을 oyz 에 대해 ϕ 만큼 회전시킨 점 Q구하기



Given: oyz 에서 정의된 점 P의 좌표값
Find: 점 P 을 oyz 에 대해 ϕ 만큼 회전시킨 점 Q구하기



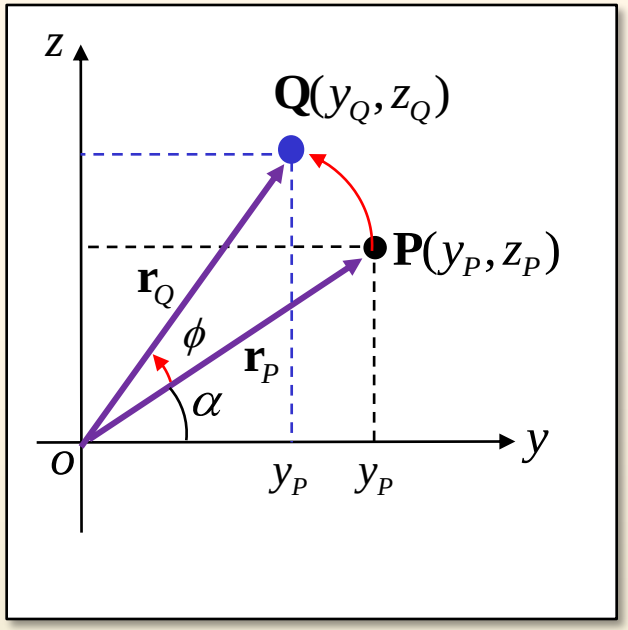
① 점 P,Q의 좌표를 각으로 표현하면,

$$y_P = |\mathbf{r}_P|\cos\alpha \qquad y_Q = |\mathbf{r}_Q|\cos(\alpha+\phi)$$

$$z_P = |\mathbf{r}_P|\sin\alpha \qquad z_Q = |\mathbf{r}_Q|\sin(\alpha+\phi)$$

Given: oyz 에서 정의된 점 P의 좌표값

Find: 점 P 을 oyz 에 대해 ϕ 만큼 회전시킨 점 Q구하기



① 점 P,Q의 좌표를 각으로 표현하면,

$$y_P = |\mathbf{r}_P|\cos\alpha \qquad y_Q = |\mathbf{r}_Q|\cos(\alpha+\phi)$$

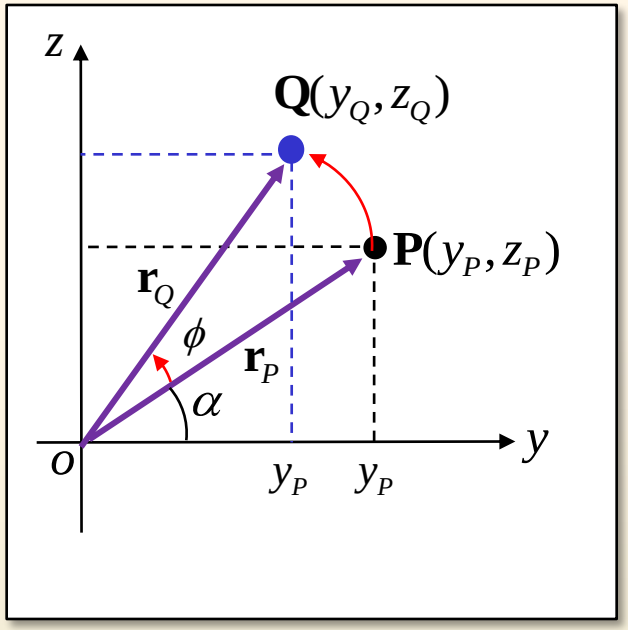
$$z_P = |\mathbf{r}_P|\sin\alpha \qquad z_Q = |\mathbf{r}_Q|\sin(\alpha+\phi)$$

② 삼각함수 합공식

$$\sin(\alpha+\phi) = \sin\alpha\cos\phi + \cos\alpha\sin\phi$$

$$\cos(\alpha+\phi) = \cos\alpha\cos\phi - \sin\alpha\sin\phi$$

Given: oyz 에서 정의된 점 P의 좌표값
Find: 점 P 을 oyz 에 대해 ϕ 만큼 회전시킨 점 Q구하기



① 점 P,Q의 좌표를 각으로 표현하면,

$$y_P = |\mathbf{r}_P|\cos\alpha \qquad y_Q = |\mathbf{r}_Q|\cos(\alpha+\phi)$$
$$z_P = |\mathbf{r}_P|\sin\alpha \qquad z_Q = |\mathbf{r}_Q|\sin(\alpha+\phi)$$

② 삼각함수 합공식

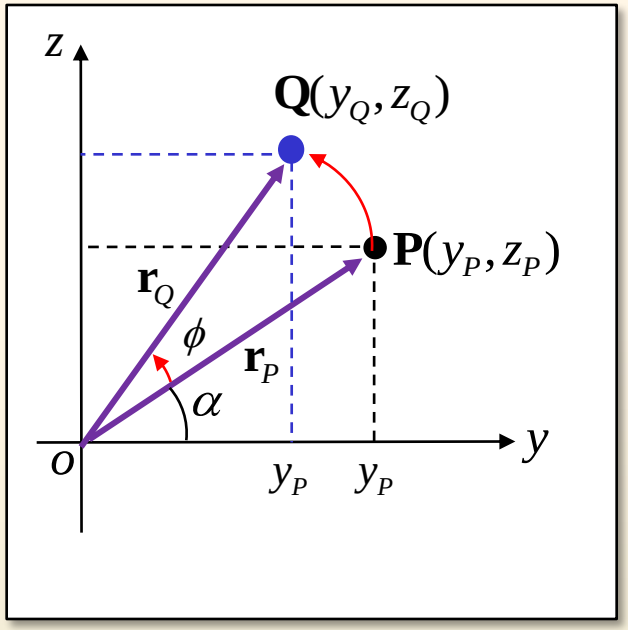
$$\sin(\alpha+\phi) = \sin\alpha\cos\phi + \cos\alpha\sin\phi$$
$$\cos(\alpha+\phi) = \cos\alpha\cos\phi - \sin\alpha\sin\phi$$

③ 점 Q의 좌표를 삼각함수의 차공식으로 전개하면,

$$\begin{aligned} y_Q &= |\mathbf{r}_Q|\cos(\alpha+\phi) \\ &= |\mathbf{r}_Q|\cos\alpha\cos\phi - |\mathbf{r}_Q|\sin\alpha\sin\phi \\ &= (|\mathbf{r}_P|\cos\alpha)\cos\phi - (|\mathbf{r}_P|\sin\alpha)\sin\phi \quad ,(|\mathbf{r}_P| = |\mathbf{r}_Q|) \\ &= y_P\cos\phi - z_P\sin\phi \end{aligned}$$

$$\begin{aligned} z_Q &= |\mathbf{r}_Q|\sin(\alpha+\phi) \\ &= |\mathbf{r}_Q|\sin\alpha\cos\phi + |\mathbf{r}_Q|\cos\alpha\sin\phi \\ &= (|\mathbf{r}_P|\sin\alpha)\cos\phi + (|\mathbf{r}_P|\cos\alpha)\sin\phi \quad ,(|\mathbf{r}_P| = |\mathbf{r}_Q|) \\ &= z_P\cos\phi + y_P\sin\phi \end{aligned}$$

Given: oyz 에서 정의된 점 P의 좌표값
Find: 점 P 을 oyz 에 대해 ϕ 만큼 회전시킨 점 Q구하기



① 점 P,Q의 좌표를 각으로 표현하면,

$$y_P = |\mathbf{r}_P|\cos\alpha \qquad y_Q = |\mathbf{r}_Q|\cos(\alpha+\phi)$$
$$z_P = |\mathbf{r}_P|\sin\alpha \qquad z_Q = |\mathbf{r}_Q|\sin(\alpha+\phi)$$

② 삼각함수 합공식

$$\sin(\alpha+\phi) = \sin\alpha\cos\phi + \cos\alpha\sin\phi$$
$$\cos(\alpha+\phi) = \cos\alpha\cos\phi - \sin\alpha\sin\phi$$

③ 점 Q의 좌표를 삼각함수의 차공식으로 전개하면,

$$\begin{aligned} y_Q &= |\mathbf{r}_Q|\cos(\alpha+\phi) \\ &= |\mathbf{r}_Q|\cos\alpha\cos\phi - |\mathbf{r}_Q|\sin\alpha\sin\phi \\ &= (|\mathbf{r}_P|\cos\alpha)\cos\phi - (|\mathbf{r}_P|\sin\alpha)\sin\phi \quad ,(|\mathbf{r}_P| = |\mathbf{r}_Q|) \\ &= y_P\cos\phi - z_P\sin\phi \end{aligned}$$

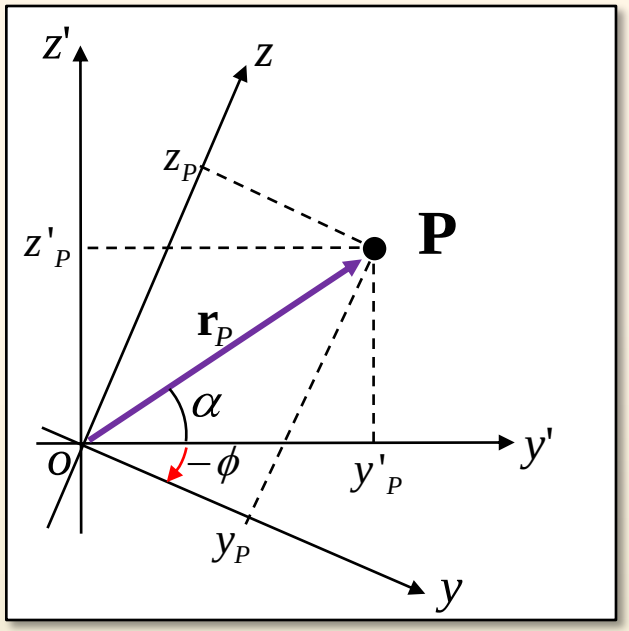
$$\begin{aligned} z_Q &= |\mathbf{r}_Q|\sin(\alpha+\phi) \\ &= |\mathbf{r}_Q|\sin\alpha\cos\phi + |\mathbf{r}_Q|\cos\alpha\sin\phi \\ &= (|\mathbf{r}_P|\sin\alpha)\cos\phi + (|\mathbf{r}_P|\cos\alpha)\sin\phi \quad ,(|\mathbf{r}_P| = |\mathbf{r}_Q|) \\ &= z_P\cos\phi + y_P\sin\phi \end{aligned}$$

④ 행렬로 표현하면,

$$\begin{bmatrix} y_Q \\ z_Q \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y_P \\ z_P \end{bmatrix}$$

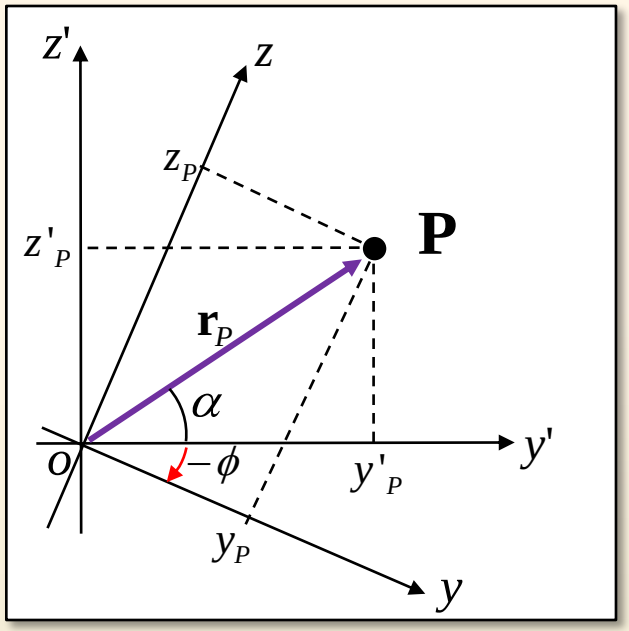
Given: $oy'z'$ 에서 정의된 점 P의 좌표값

Find: $oy'z'$ 에 대해 $-\phi$ 만큼 회전한 새로운 좌표계 oyz 에서의 P의 좌표값



Given: $oy'z'$ 에서 정의된 점 P의 좌표값

Find: $oy'z'$ 에 대해 $-\phi$ 만큼 회전한 새로운 좌표계 oyz 에서의 P의 좌표값



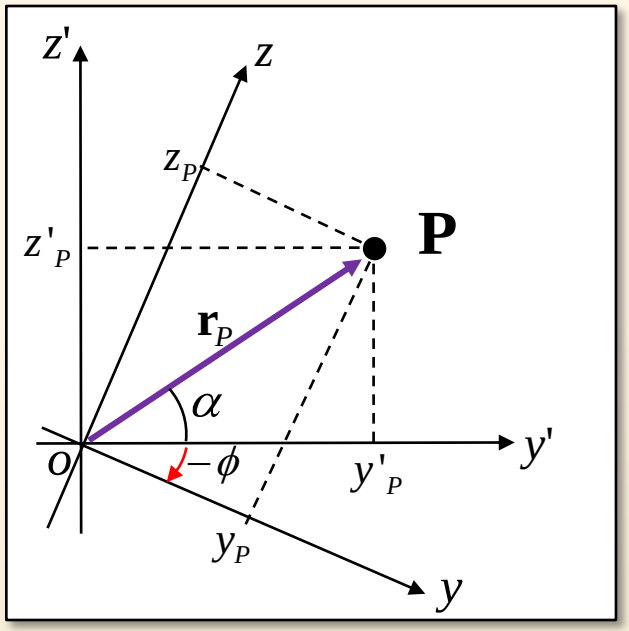
① 점 P의 좌표를 각으로 표현하면,

$$y'_P = |\mathbf{r}_P|\cos\alpha \qquad y_P = |\mathbf{r}_P|\cos(\alpha+\phi)$$

$$z'_P = |\mathbf{r}_P|\sin\alpha \qquad z_P = |\mathbf{r}_P|\sin(\alpha+\phi)$$

Given: $oy'z'$ 에서 정의된 점 P의 좌표값

Find: $oy'z'$ 에 대해 $-\phi$ 만큼 회전한 새로운 좌표계 oyz 에서의 P의 좌표값



① 점 P의 좌표를 각으로 표현하면,

$$y'_P = |\mathbf{r}_P|\cos\alpha \qquad y_P = |\mathbf{r}_P|\cos(\alpha+\phi)$$

$$z'_P = |\mathbf{r}_P|\sin\alpha \qquad z_P = |\mathbf{r}_P|\sin(\alpha+\phi)$$

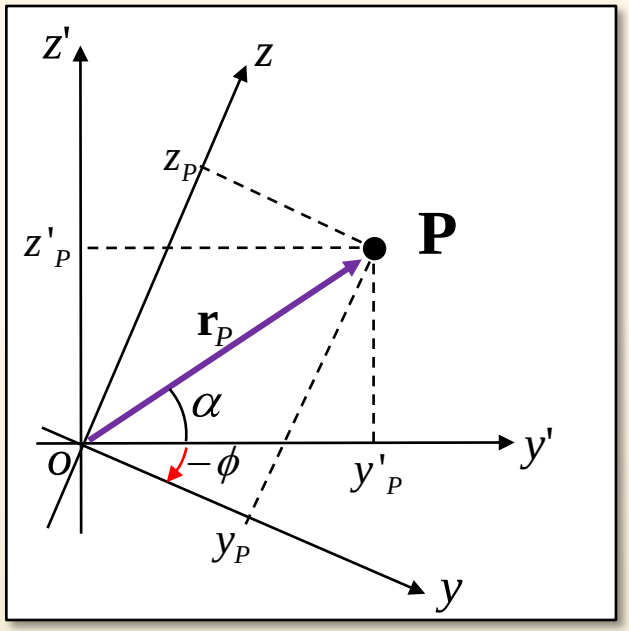
② 삼각함수 합공식

$$\sin(\alpha+\phi) = \sin\alpha\cos\phi + \cos\alpha\sin\phi$$

$$\cos(\alpha+\phi) = \cos\alpha\cos\phi - \sin\alpha\sin\phi$$

Given: $oy'z'$ 에서 정의된 점 P의 좌표값

Find: $oy'z'$ 에 대해 $-\phi$ 만큼 회전한 새로운 좌표계 oyz 에서의 P의 좌표값



① 점 P의 좌표를 각으로 표현하면,

$$y'_P = |\mathbf{r}_P|\cos\alpha \qquad y_P = |\mathbf{r}_P|\cos(\alpha+\phi)$$
$$z'_P = |\mathbf{r}_P|\sin\alpha \qquad z_P = |\mathbf{r}_P|\sin(\alpha+\phi)$$

② 삼각함수 합공식

$$\sin(\alpha+\phi) = \sin\alpha\cos\phi + \cos\alpha\sin\phi$$
$$\cos(\alpha+\phi) = \cos\alpha\cos\phi - \sin\alpha\sin\phi$$

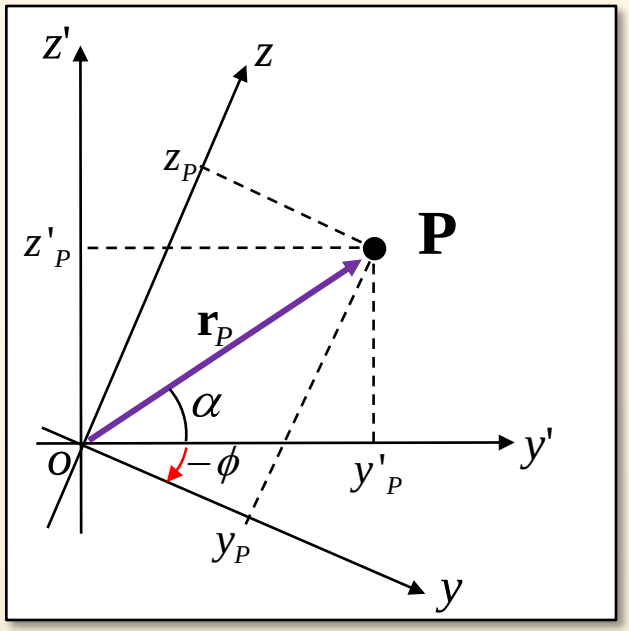
③ 점 P의 좌표를 삼각함수의 차공식으로 전개하면,

$$\begin{aligned} y_P &= |\mathbf{r}_P|\cos(\alpha+\phi) \\ &= |\mathbf{r}_P|\cos\alpha\cos\phi - |\mathbf{r}_P|\sin\alpha\sin\phi \\ &= (|\mathbf{r}_P|\cos\alpha)\cos\phi - (|\mathbf{r}_P|\sin\alpha)\sin\phi \\ &= y'_P\cos\phi - z'_P\sin\phi \end{aligned}$$

$$\begin{aligned} z_P &= |\mathbf{r}_P|\sin(\alpha+\varphi) \\ &= |\mathbf{r}_P|\sin\alpha\cos\varphi + |\mathbf{r}_P|\cos\alpha\sin\varphi \\ &= (|\mathbf{r}_P|\sin\alpha)\cos\varphi + (|\mathbf{r}_P|\cos\alpha)\sin\varphi \\ &= z'_P\cos\varphi + y'_P\sin\varphi \end{aligned}$$

Given: $oy'z'$ 에서 정의된 점 P의 좌표값

Find: $oy'z'$ 에 대해 $-\phi$ 만큼 회전한 새로운 좌표계 oyz 에서의 P의 좌표값



① 점 P의 좌표를 각으로 표현하면,

$$y'_P = |\mathbf{r}_P|\cos\alpha \qquad y_P = |\mathbf{r}_P|\cos(\alpha+\phi)$$

$$z'_P = |\mathbf{r}_P|\sin\alpha \qquad z_P = |\mathbf{r}_P|\sin(\alpha+\phi)$$

② 삼각함수 합공식

$$\sin(\alpha+\phi) = \sin\alpha\cos\phi + \cos\alpha\sin\phi$$

$$\cos(\alpha+\phi) = \cos\alpha\cos\phi - \sin\alpha\sin\phi$$

③ 점 P의 좌표를 삼각함수의 차공식으로 전개하면,

$$\begin{aligned} y_P &= |\mathbf{r}_P|\cos(\alpha+\phi) \\ &= |\mathbf{r}_P|\cos\alpha\cos\phi - |\mathbf{r}_P|\sin\alpha\sin\phi \\ &= (|\mathbf{r}_P|\cos\alpha)\cos\phi - (|\mathbf{r}_P|\sin\alpha)\sin\phi \\ &= y'_P\cos\phi - z'_P\sin\phi \end{aligned}$$

$$\begin{aligned} z_P &= |\mathbf{r}_P|\sin(\alpha+\varphi) \\ &= |\mathbf{r}_P|\sin\alpha\cos\varphi + |\mathbf{r}_P|\cos\alpha\sin\varphi \\ &= (|\mathbf{r}_P|\sin\alpha)\cos\varphi + (|\mathbf{r}_P|\cos\alpha)\sin\varphi \\ &= z'_P\cos\varphi + y'_P\sin\varphi \end{aligned}$$

④ 행렬로 표현하면,

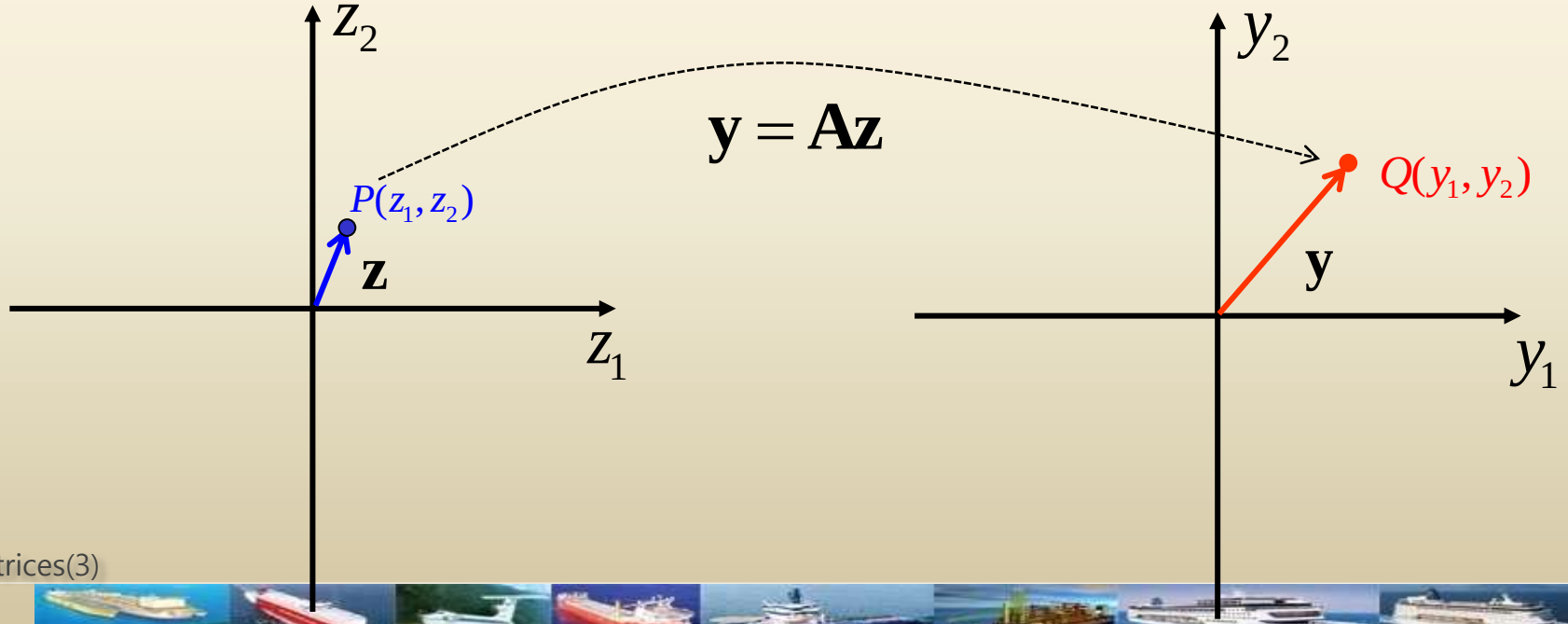
$$\begin{bmatrix} y_P \\ z_P \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y'_P \\ z'_P \end{bmatrix}$$

Stretching of an Elastic Membrane

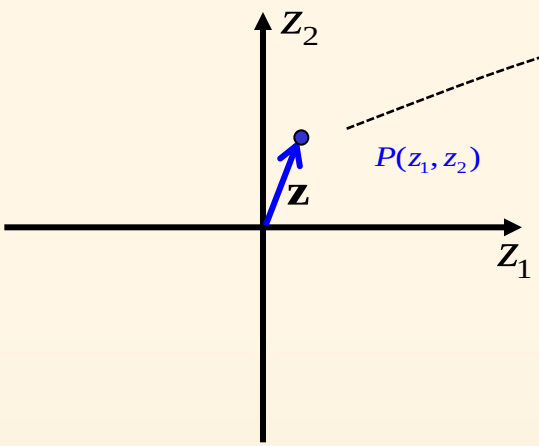
An elastic membrane on the z_1z_2-plane with boundary circle $z_1^2+z_2^2=1$ is stretched so that a point $P(z_1, z_2)$ goes over into the point $Q(y_1, y_2)$ given by

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

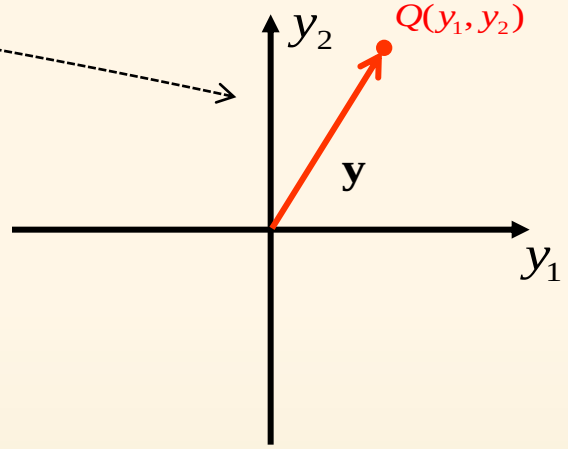
Find the principle directions, that is, the directions of the position vector y of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?



Stretching of an Elastic Membrane



$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$



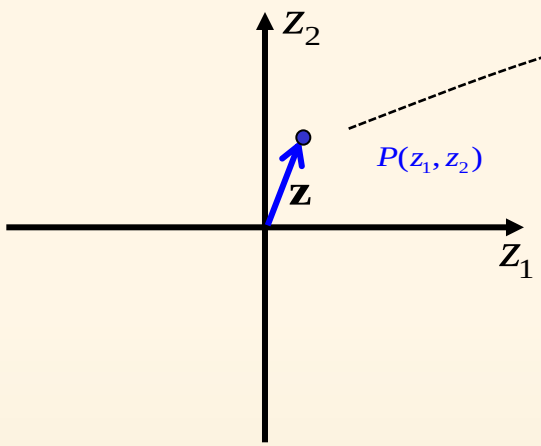
Given:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

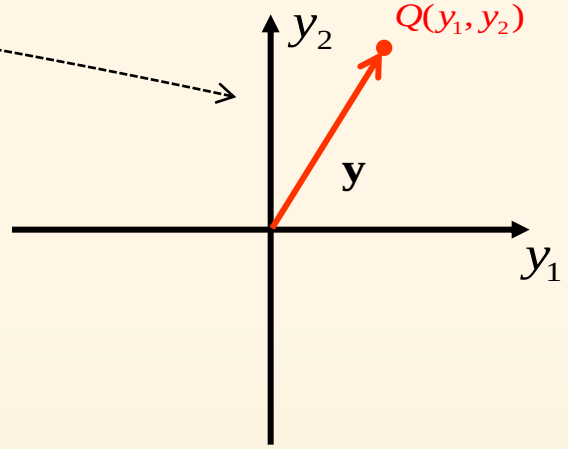
Transformed :

$$z_1^2 + z_2^2 = 1$$

Stretching of an Elastic Membrane



$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$



Given:

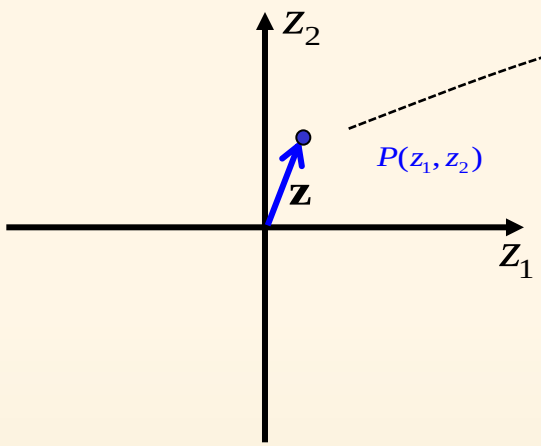
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Transformed :

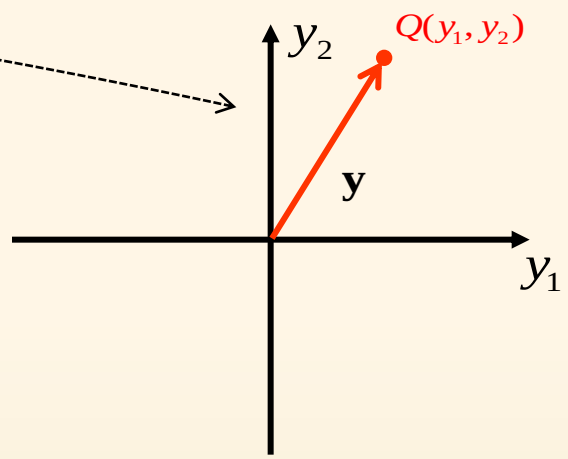
$$\begin{cases} y_1 = 5z_1 + 3z_2 \\ y_2 = 3z_1 + 5z_2 \end{cases}$$

$$z_1^2 + z_2^2 = 1 \quad \Rightarrow \quad \begin{cases} z_1 = \dfrac{1}{16}(5y_1 - 3y_2) \\ z_2 = \dfrac{1}{16}(-3y_1 + 5y_2) \end{cases}$$

Stretching of an Elastic Membrane



$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$



Given:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

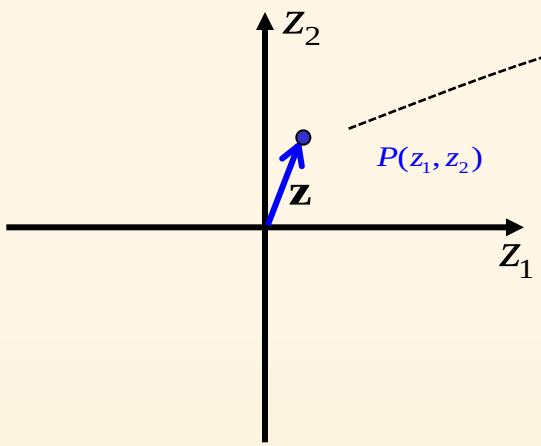
$$\begin{cases} y_1 = 5z_1 + 3z_2 \\ y_2 = 3z_1 + 5z_2 \end{cases}$$

Transformed :

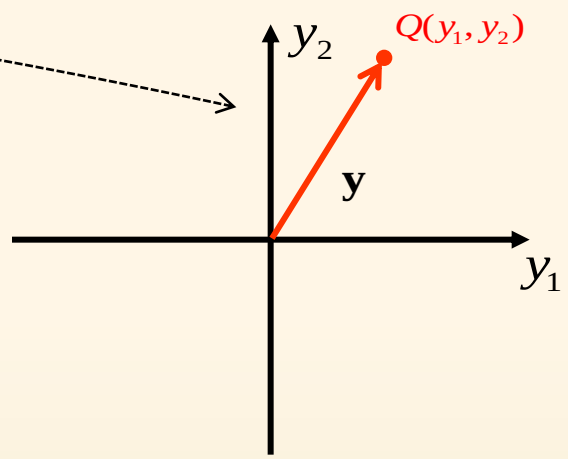
$$\frac{1}{16^2}(5y_1 - 3y_2)^2 + \frac{1}{16^2}(-3y_1 + 5y_2)^2 = 1$$

$$z_1^2 + z_2^2 = 1 \quad \Rightarrow \quad \begin{cases} z_1 = \frac{1}{16}(5y_1 - 3y_2) \\ z_2 = \frac{1}{16}(-3y_1 + 5y_2) \end{cases} \quad \Rightarrow$$

Stretching of an Elastic Membrane



$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$



Given:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

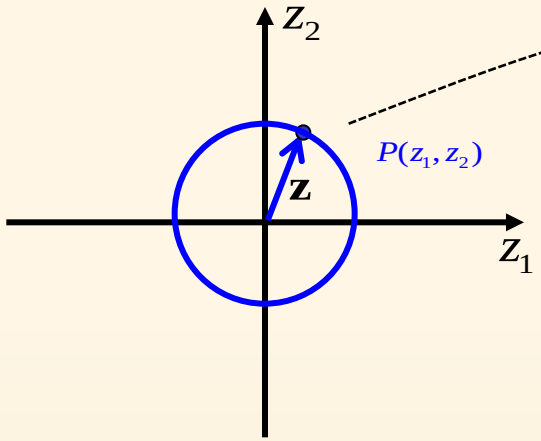
$$\begin{cases} y_1 = 5z_1 + 3z_2 \\ y_2 = 3z_1 + 5z_2 \end{cases}$$

Transformed :

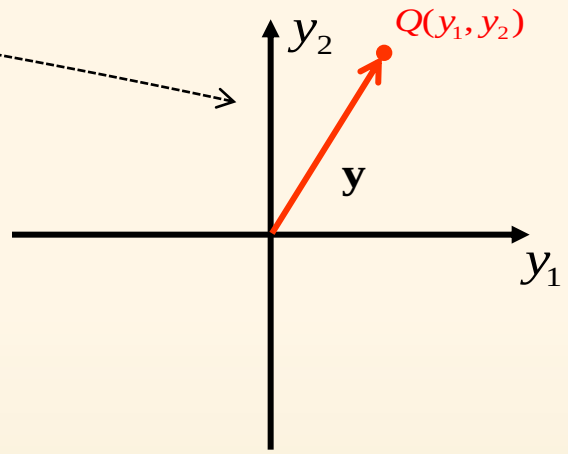
$$\frac{1}{16^2}(5y_1 - 3y_2)^2 + \frac{1}{16^2}(-3y_1 + 5y_2)^2 = 1$$

$$z_1^2 + z_2^2 = 1 \quad \Rightarrow \quad \begin{cases} z_1 = \frac{1}{16}(5y_1 - 3y_2) \\ z_2 = \frac{1}{16}(-3y_1 + 5y_2) \end{cases} \quad \Rightarrow \quad 34y_1^2 - 60y_1y_2 + 34y_2^2 = 256$$

Stretching of an Elastic Membrane



$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$



Given:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

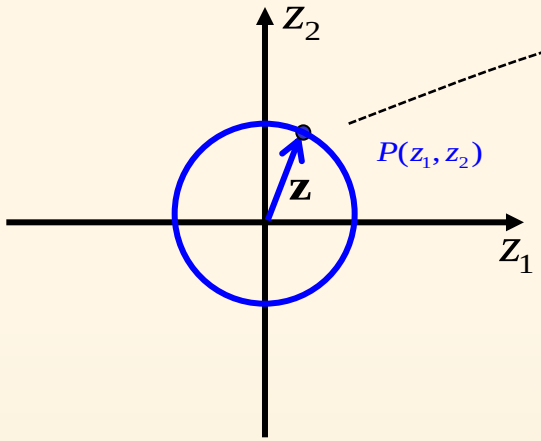
$$\begin{cases} y_1 = 5z_1 + 3z_2 \\ y_2 = 3z_1 + 5z_2 \end{cases}$$

Transformed :

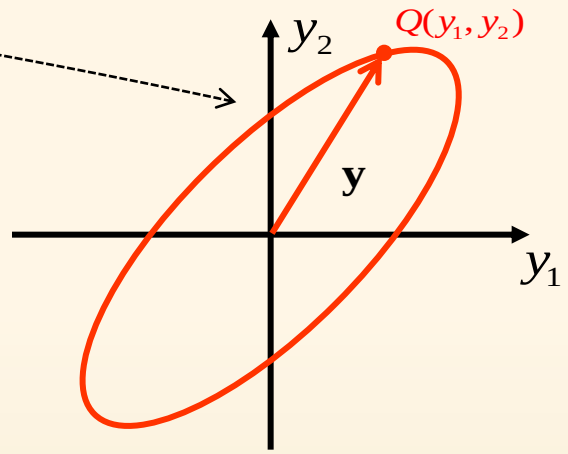
$$\frac{1}{16^2}(5y_1 - 3y_2)^2 + \frac{1}{16^2}(-3y_1 + 5y_2)^2 = 1$$

$$z_1^2 + z_2^2 = 1 \quad \Rightarrow \quad \begin{cases} z_1 = \frac{1}{16}(5y_1 - 3y_2) \\ z_2 = \frac{1}{16}(-3y_1 + 5y_2) \end{cases} \quad \Rightarrow \quad 34y_1^2 - 60y_1y_2 + 34y_2^2 = 256$$

Stretching of an Elastic Membrane



$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$



Given:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\begin{cases} y_1 = 5z_1 + 3z_2 \\ y_2 = 3z_1 + 5z_2 \end{cases}$$

Transformed :

$$\frac{1}{16^2}(5y_1 - 3y_2)^2 + \frac{1}{16^2}(-3y_1 + 5y_2)^2 = 1$$

$$z_1^2 + z_2^2 = 1 \quad \Rightarrow \quad \begin{cases} z_1 = \frac{1}{16}(5y_1 - 3y_2) \\ z_2 = \frac{1}{16}(-3y_1 + 5y_2) \end{cases} \quad \Rightarrow \quad 34y_1^2 - 60y_1y_2 + 34y_2^2 = 256$$

Stretching of an Elastic Membrane

$$\frac{y_1'^2}{8^2}+\frac{y_2'^2}{2^2}=1 \quad \longleftarrow \quad y_1'=\frac{1}{\sqrt{2}}(y_1+y_2), \quad y_2'=\frac{1}{\sqrt{2}}(-y_1+y_2)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix}=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix}=\begin{bmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

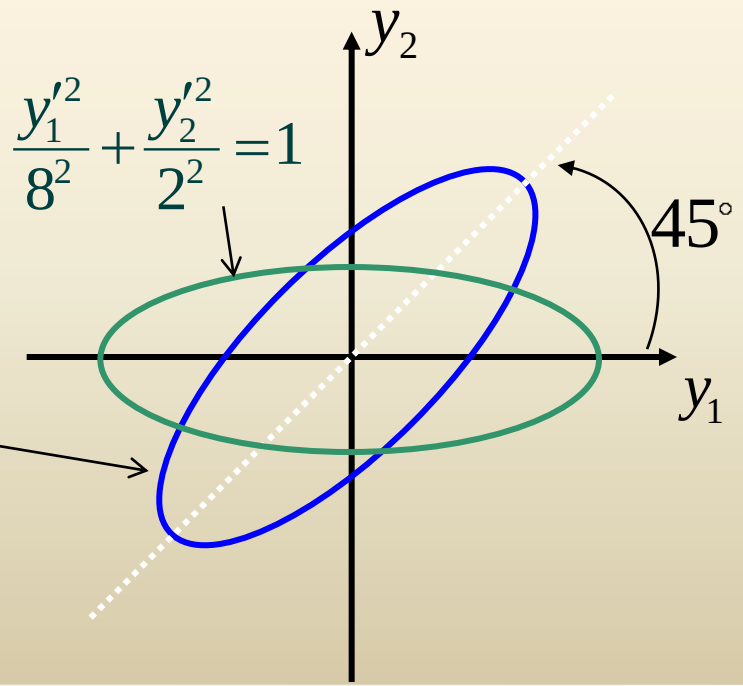
$$\frac{1}{2}\left(\frac{(y_1+y_2)^2}{8^2}+\frac{(-y_1+y_2)^2}{2^2}\right)=1$$

$$\frac{(y_1+y_2)^2}{8^2}+\frac{(-y_1+y_2)^2}{2^2}=2$$

$$4(y_1^2+2y_1y_2+y_2^2)+64(y_1^2-2y_1y_2+y_2^2)=2$$

$$68y_1^2-120y_1y_2+68y_2^2=512$$

$$34y_1^2-60y_1y_2+34y_2^2=256$$

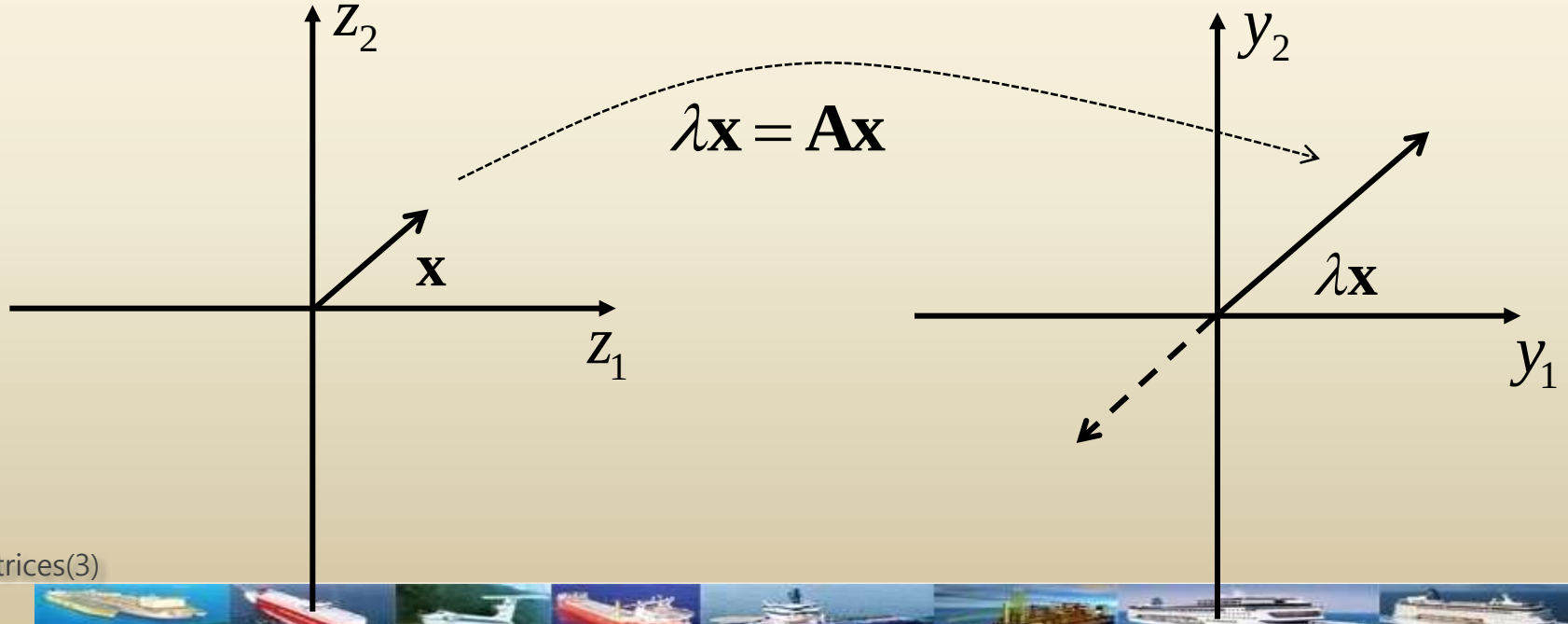


Stretching of an Elastic Membrane

An elastic membrane on the z_1z_2-plane with boundary circle $z_1^2+z_2^2=1$ is stretched so that a point $P(z_1, z_2)$ goes over into the point $Q(y_1, y_2)$ given by

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Find the principle directions, that is, the directions of the position vector y of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?



Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

(1) $\lambda=\lambda_1=8$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

(1) $\lambda=\lambda_1=8$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

(1) $\lambda=\lambda_1=8$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}=0$$

(1) $\lambda=\lambda_1=8$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

(1) $\lambda=\lambda_1=8$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

(1) $\lambda=\lambda_1=8$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of **x** *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

(1) $\lambda=\lambda_1=8$

$$-3x_1+3x_2=0$$

$$3x_1-3x_2=0$$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of **x** *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

(1) $\lambda=\lambda_1=8$

$$-3x_1+3x_2=0$$

$$3x_1-3x_2=0$$

$$\therefore x_1=x_2$$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

(1) $\lambda=\lambda_1=8$

$$-3x_1+3x_2=0$$

$$3x_1-3x_2=0$$

$$\therefore x_1=x_2$$

For instance, $x_1=x_2=1$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

(1) $\lambda=\lambda_1=8$

$$-3x_1+3x_2=0$$

$$3x_1-3x_2=0$$

$$\therefore x_1=x_2$$

For instance, $x_1=x_2=1$

$$\mathbf{x}_1=\begin{bmatrix}1 & 1\end{bmatrix}^T$$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

(1) $\lambda=\lambda_1=8$

$$-3x_1+3x_2=0$$

$$3x_1-3x_2=0$$

$$\therefore x_1=x_2$$

For instance, $x_1=x_2=1$

$$\mathbf{x}_1=\begin{bmatrix}1 & 1\end{bmatrix}^T$$

$$\mathbf{Ax}_1=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}8\\ 8\end{bmatrix}$$

$$=8\begin{bmatrix}1\\ 1\end{bmatrix}=\lambda_1\mathbf{x}_1$$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x after the stretch is the same direction before the stretch.

(2) $\lambda=\lambda_2=2$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

(2) $\lambda=\lambda_2=2$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

(2) $\lambda=\lambda_2=2$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}=0$$

(2) $\lambda=\lambda_2=2$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

(2) $\lambda=\lambda_2=2$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x after the stretch is the same direction before the stretch.

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

(2) $\lambda=\lambda_2=2$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of **x** *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

(2) $\lambda=\lambda_2=2$

$$3x_1+3x_2=0$$

$$3x_1+3x_2=0$$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

(2) $\lambda=\lambda_2=2$

$$3x_1+3x_2=0$$

$$3x_1+3x_2=0$$

$$\therefore x_1=-x_2$$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

(2) $\lambda=\lambda_2=2$

$$3x_1+3x_2=0$$

$$3x_1+3x_2=0$$

$$\therefore x_1=-x_2$$

For instance, $x_1=1,\ x_2=-1$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

(2) $\lambda=\lambda_2=2$

$$3x_1+3x_2=0$$

$$3x_1+3x_2=0$$

$$\therefore x_1=-x_2$$

For instance, $x_1=1,\ x_2=-1$

$$\mathbf{x}_2=\begin{bmatrix}1 & -1\end{bmatrix}^T$$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x *after the stretch is the same direction before the stretch.*

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

(2) $\lambda=\lambda_2=2$

$$3x_1+3x_2=0$$

$$3x_1+3x_2=0$$

$$\therefore x_1=-x_2$$

For instance, $x_1=1,\ x_2=-1$

$$\mathbf{x}_2=\begin{bmatrix}1 & -1\end{bmatrix}^T$$

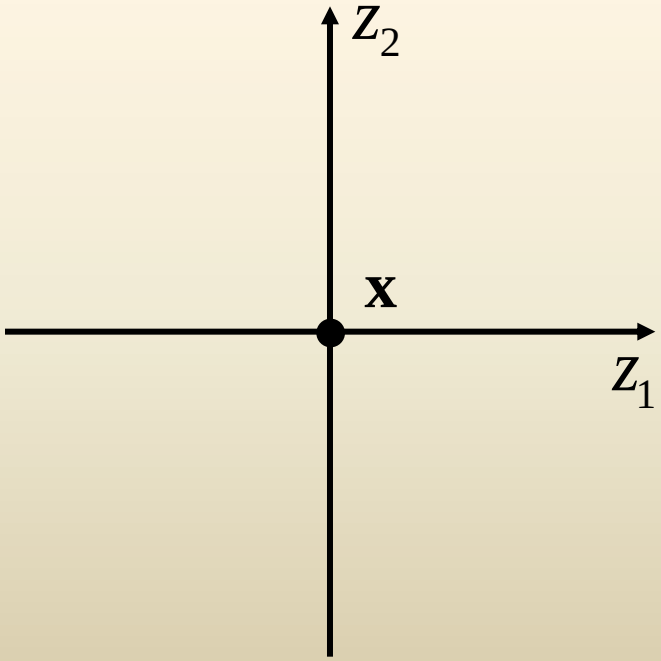
$$\mathbf{Ax}_2=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}2\\ -2\end{bmatrix}$$

$$=2\begin{bmatrix}1\\ -1\end{bmatrix}=\lambda_2\mathbf{x}_2$$

Transformation and trivial solution

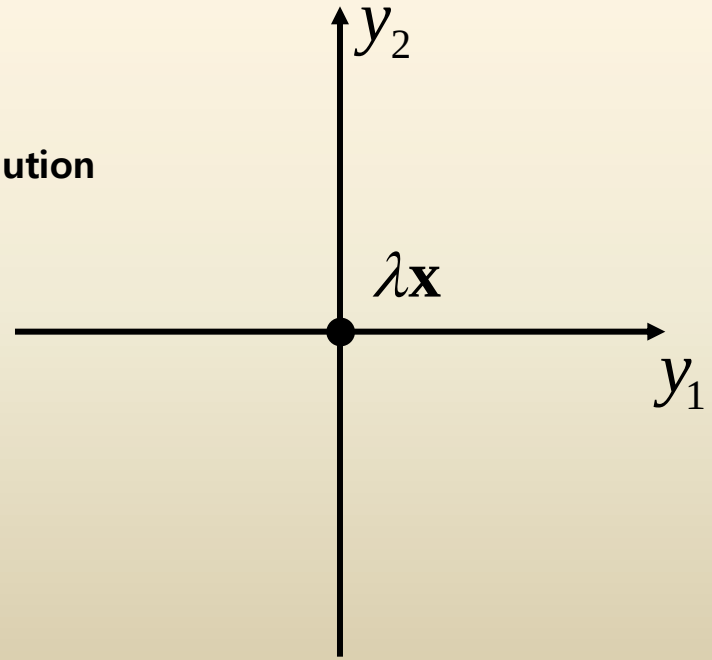
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Find the principle directions, that is, the directions of the position vector y of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?



$$\lambda\mathbf{x} = \mathbf{Ax}$$

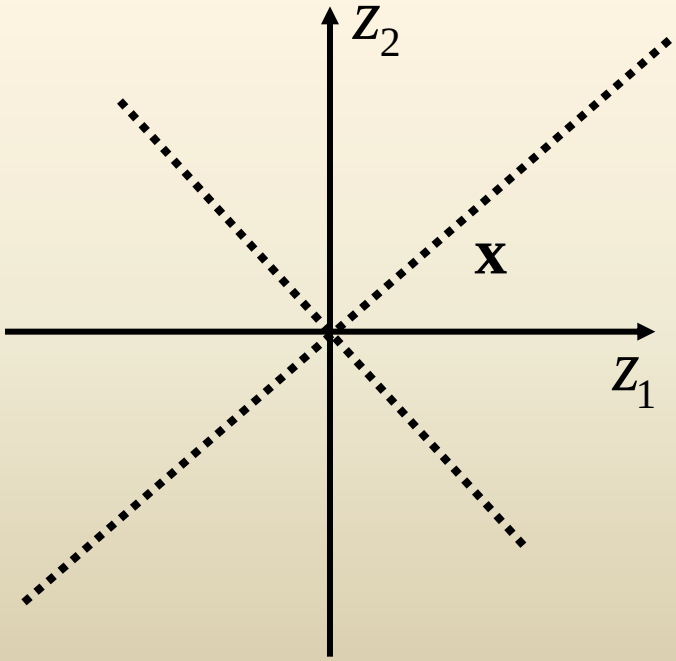
$\mathbf{x} = \mathbf{0}$:Trivial solution



Transformation and trivial solution

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A}\mathbf{z} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Find the principle directions, that is, the directions of the position vector y of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?

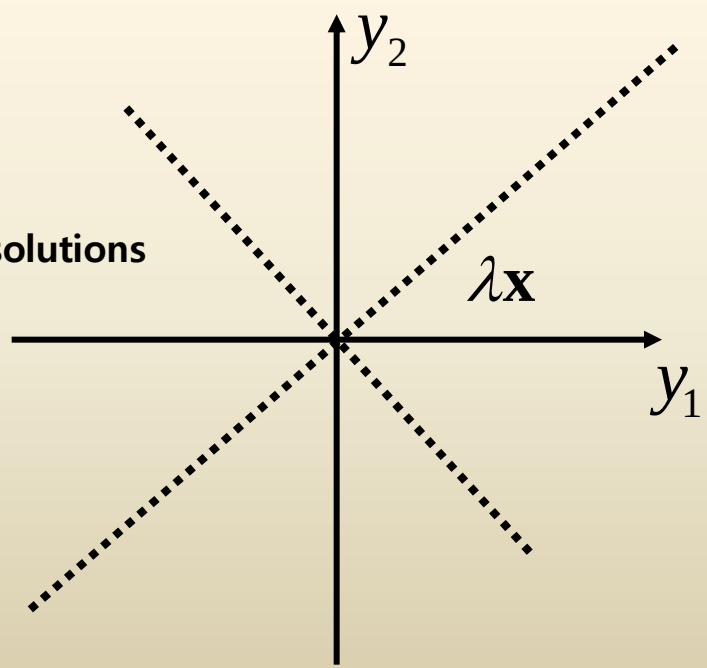


$$\lambda\mathbf{x} = \mathbf{A}\mathbf{x}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

x :Nontrivial many solutions

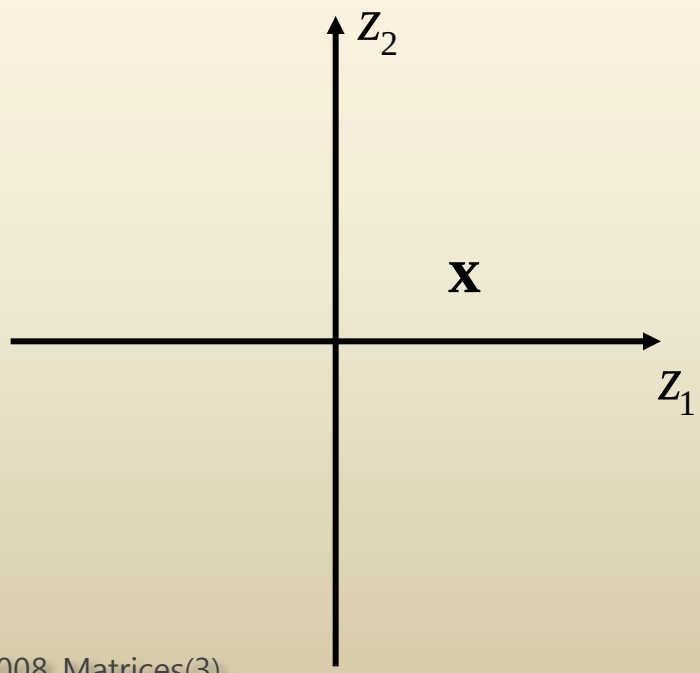


Stretching of an Elastic Membrane

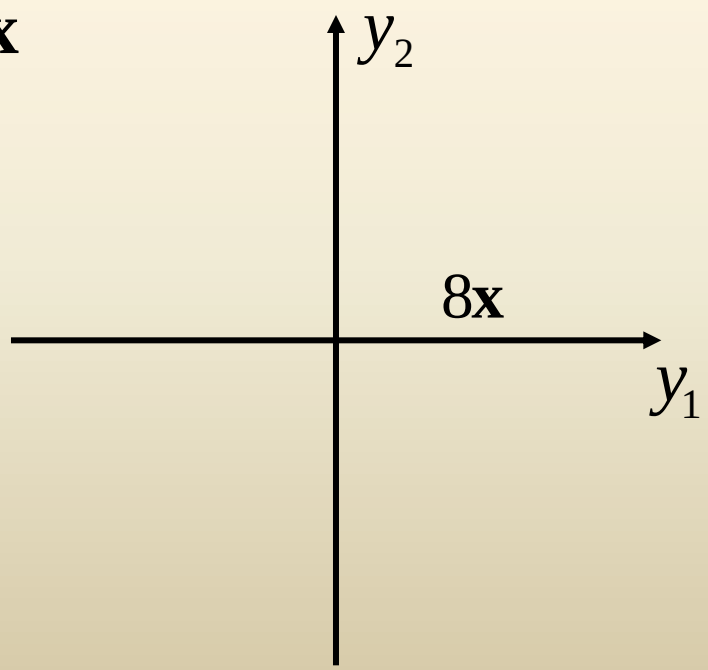
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x after the stretch is the same direction before the stretch.



$$\mathbf{Ax} = \lambda\mathbf{x}$$



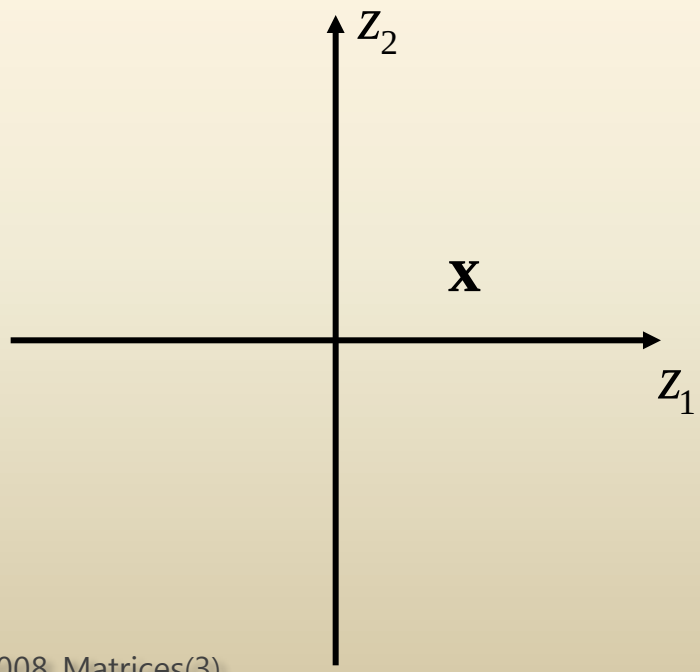
Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

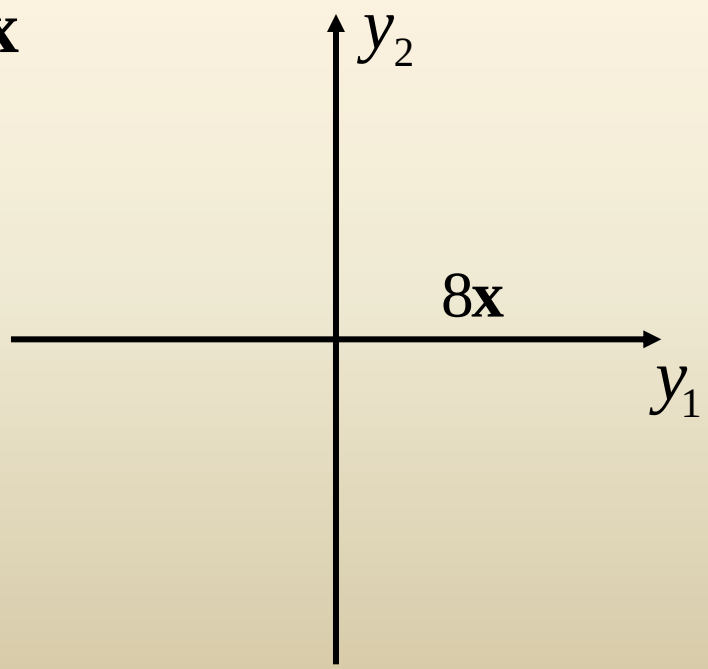
1) $\lambda_1 = 8$

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x after the stretch is the same direction before the stretch.



$$\mathbf{Ax} = \lambda\mathbf{x}$$



Stretching of an Elastic Membrane

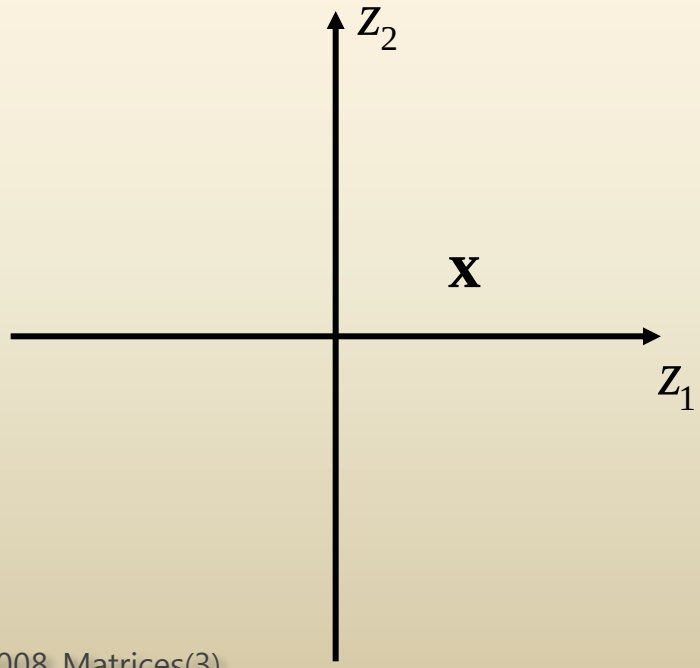
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

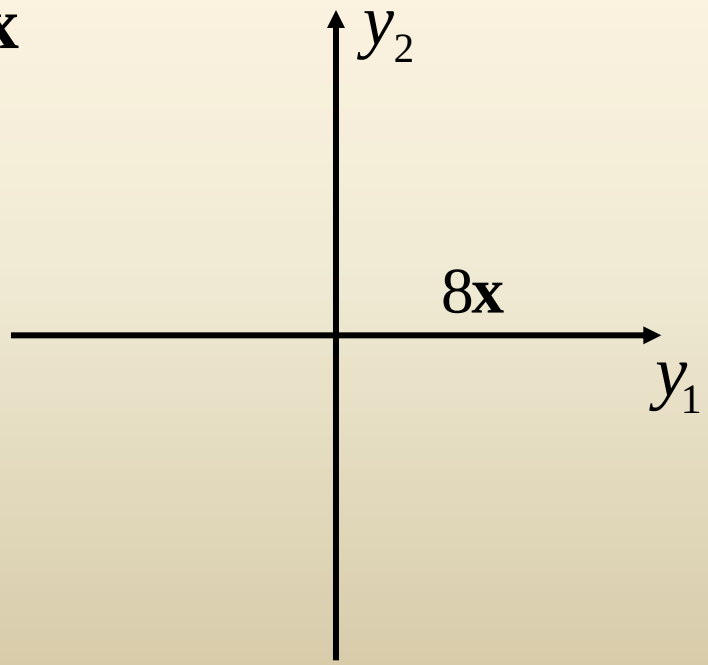
It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$1)\ \lambda_1 = 8$$

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$



$$\mathbf{Ax} = \lambda\mathbf{x}$$



Stretching of an Elastic Membrane

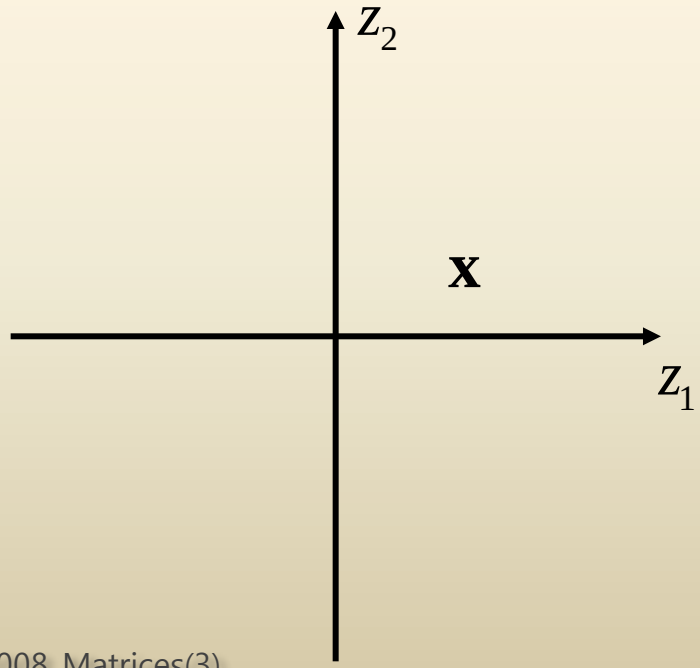
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

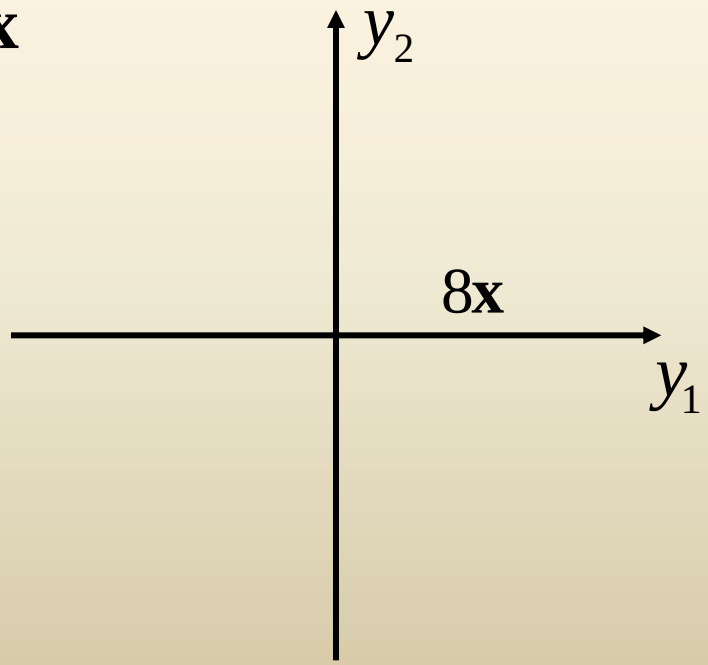
$$1)\ \lambda_1 = 8$$

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$



$$\mathbf{Ax} = \lambda\mathbf{x}$$

when, $x_1 = 1$



Stretching of an Elastic Membrane

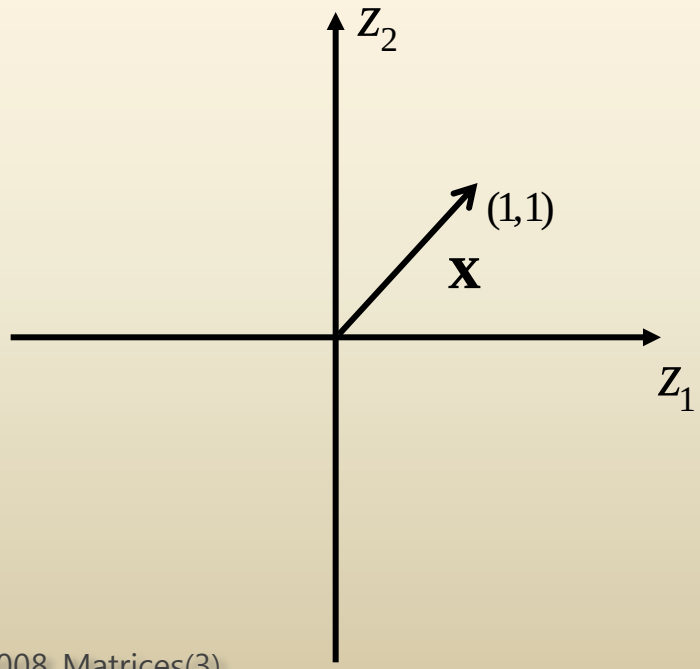
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x *after the stretch is the same direction before the stretch.*

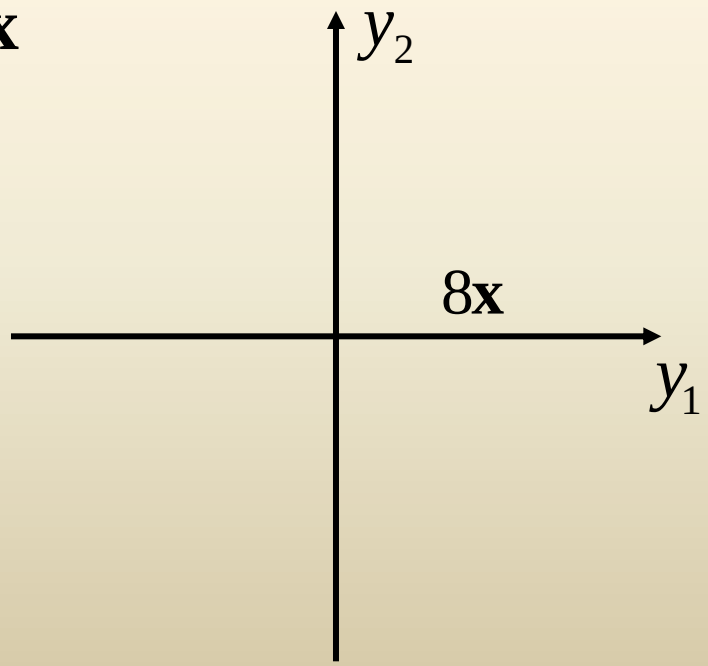
1) $\lambda_1 = 8$

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$



$$\mathbf{Ax} = \lambda\mathbf{x}$$

when, $x_1 = 1$



Stretching of an Elastic Membrane

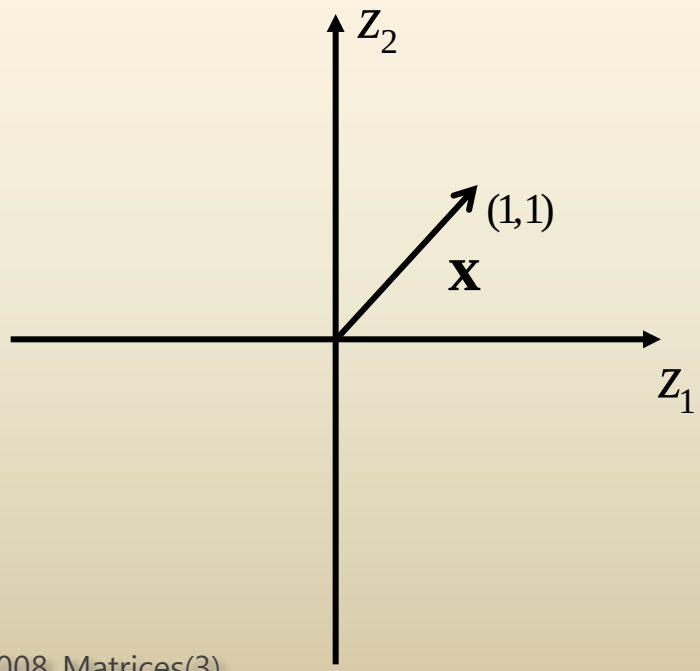
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x *after the stretch is the same direction before the stretch.*

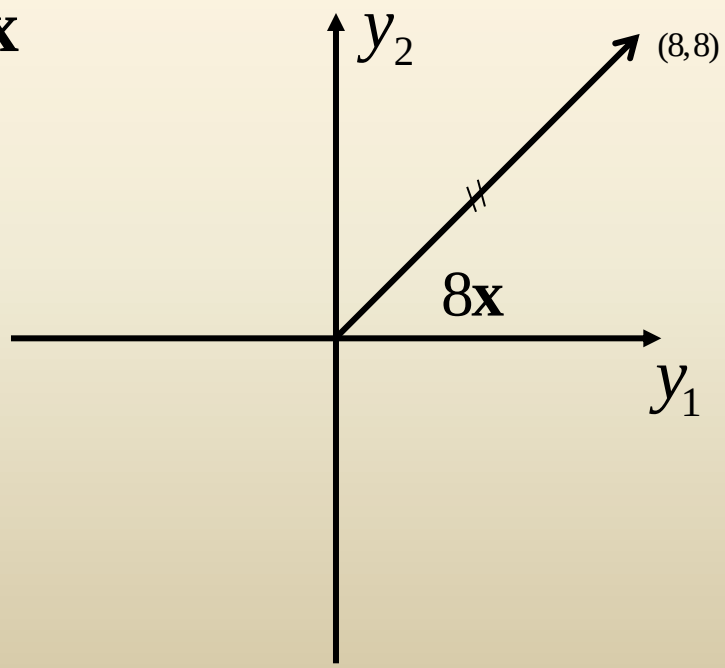
1) $\lambda_1 = 8$

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$



$\mathbf{Ax} = \lambda\mathbf{x}$

when, $x_1 = 1$



Stretching of an Elastic Membrane

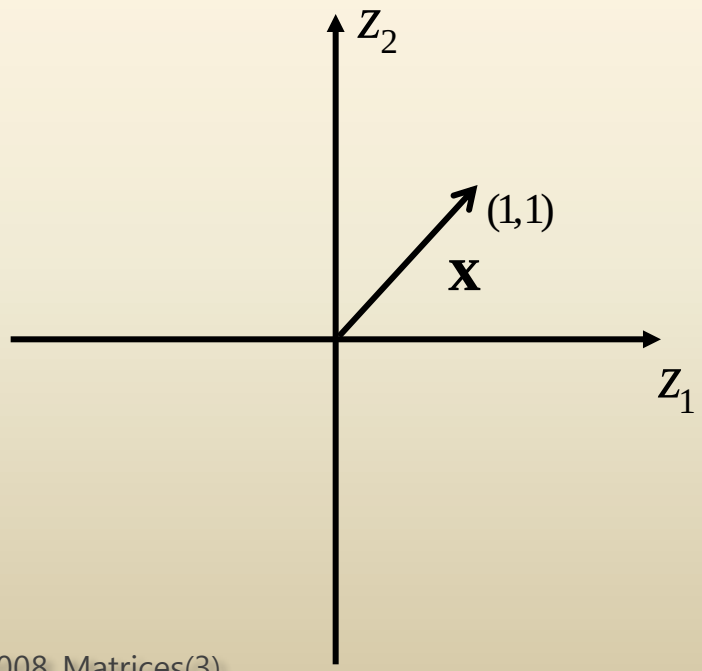
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x *after the stretch is the same direction before the stretch.*

1) $\lambda_1 = 8$

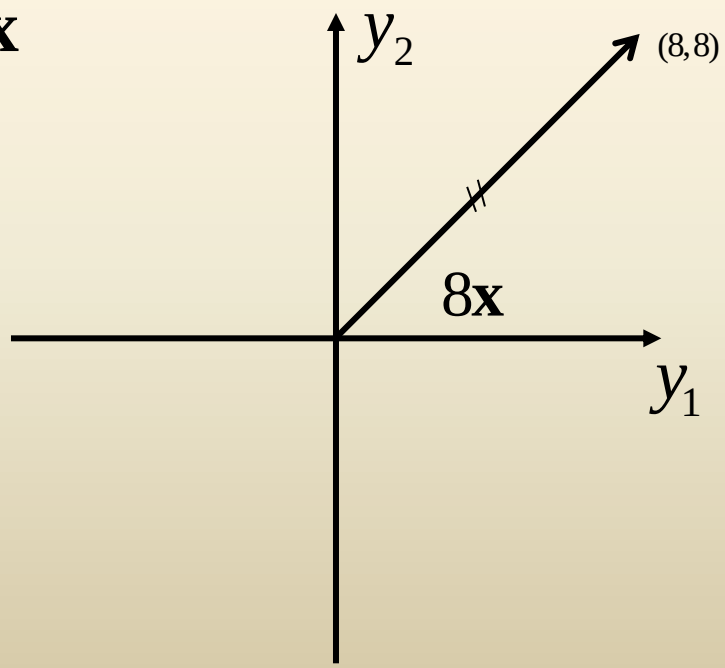
$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$



$\mathbf{Ax} = \lambda\mathbf{x}$

when, $x_1 = 1$

when, $x_1 = 2$



Stretching of an Elastic Membrane

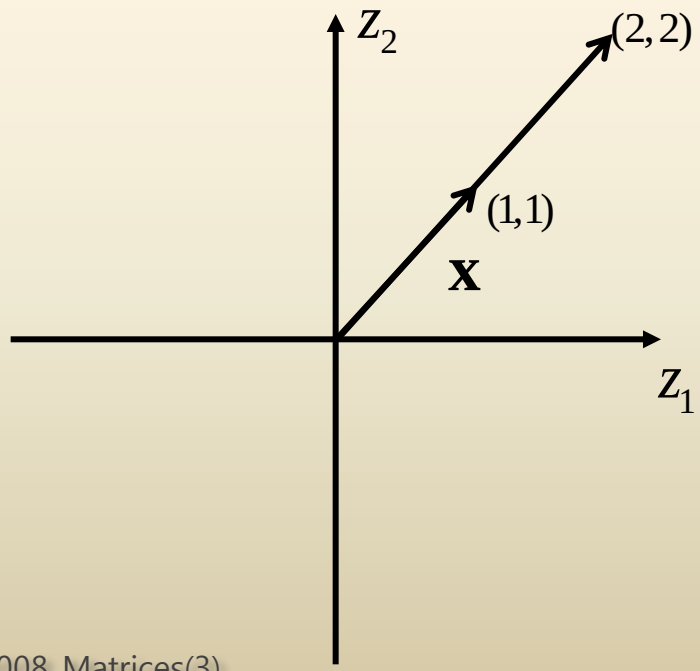
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x *after the stretch is the same direction before the stretch.*

1) $\lambda_1 = 8$

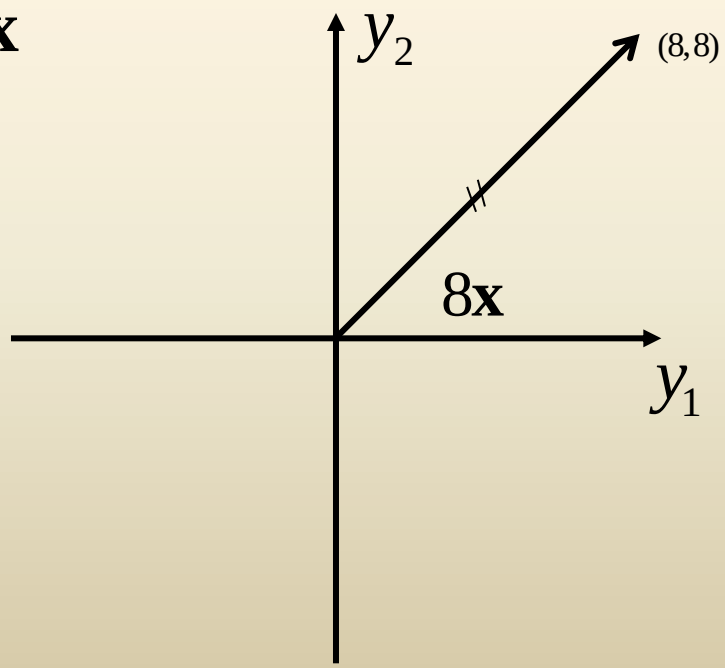
$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$



$\mathbf{Ax} = \lambda\mathbf{x}$

when, $x_1 = 1$

when, $x_1 = 2$



Stretching of an Elastic Membrane

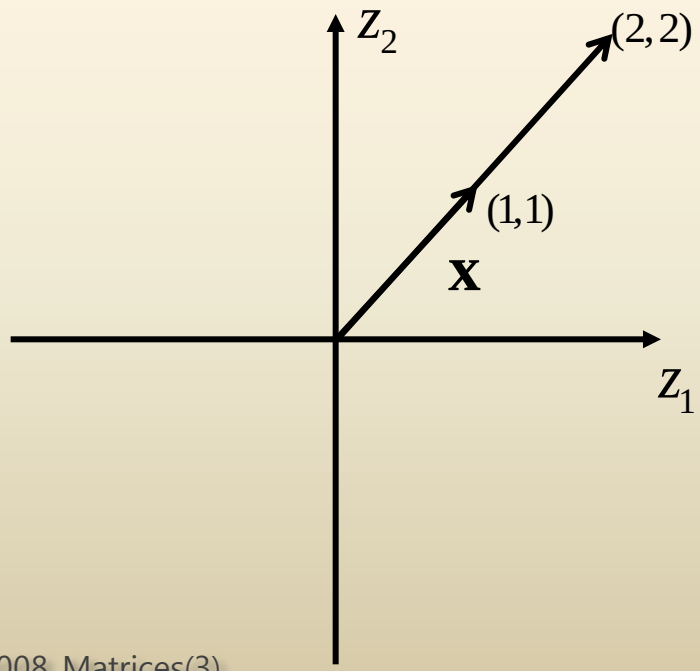
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x *after the stretch is the same direction before the stretch.*

1) $\lambda_1 = 8$

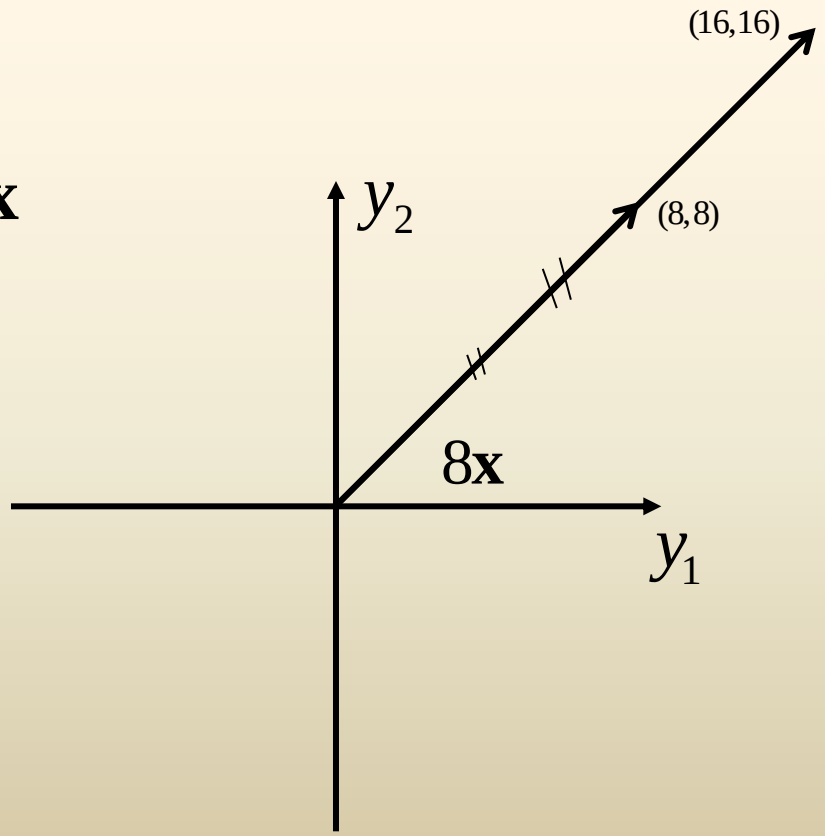
$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$



$\mathbf{Ax} = \lambda\mathbf{x}$

when, $x_1 = 1$

when, $x_1 = 2$



Stretching of an Elastic Membrane

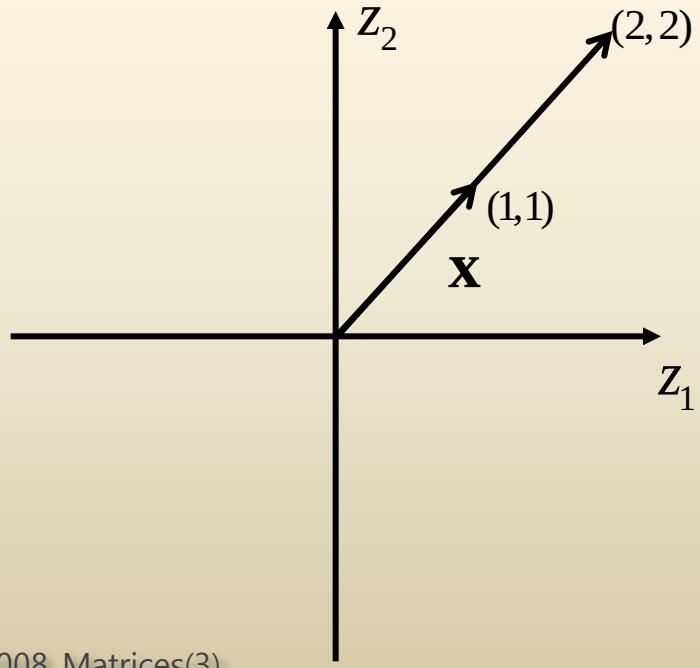
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

1) $\lambda_1 = 8$

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

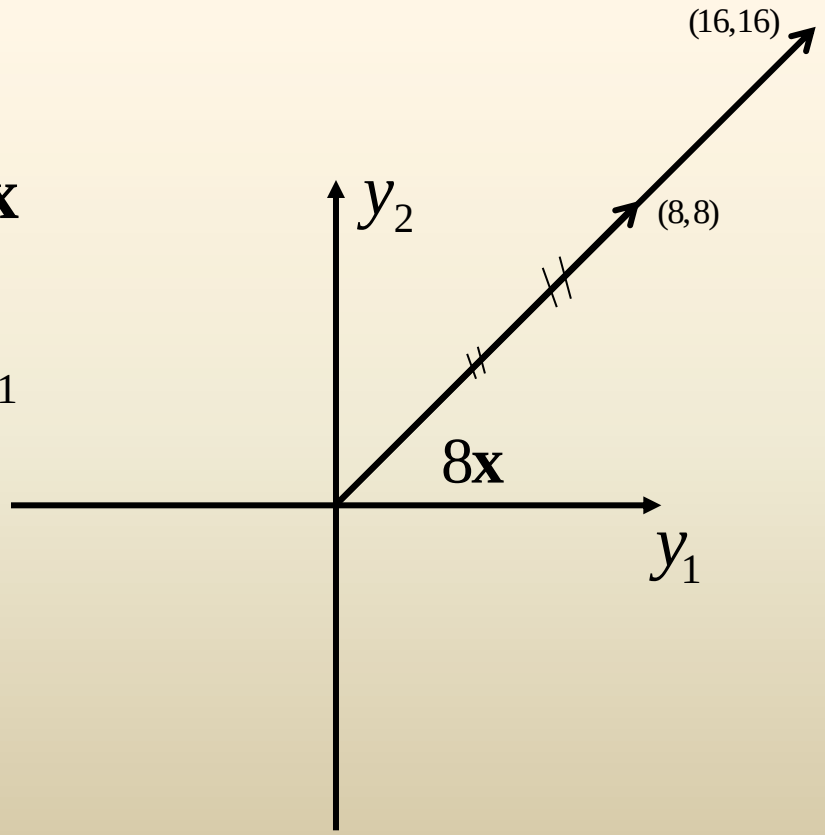


$\mathbf{Ax} = \lambda\mathbf{x}$

when, $x_1 = 1$

when, $x_1 = 2$

when, $x_1 = -1$



Stretching of an Elastic Membrane

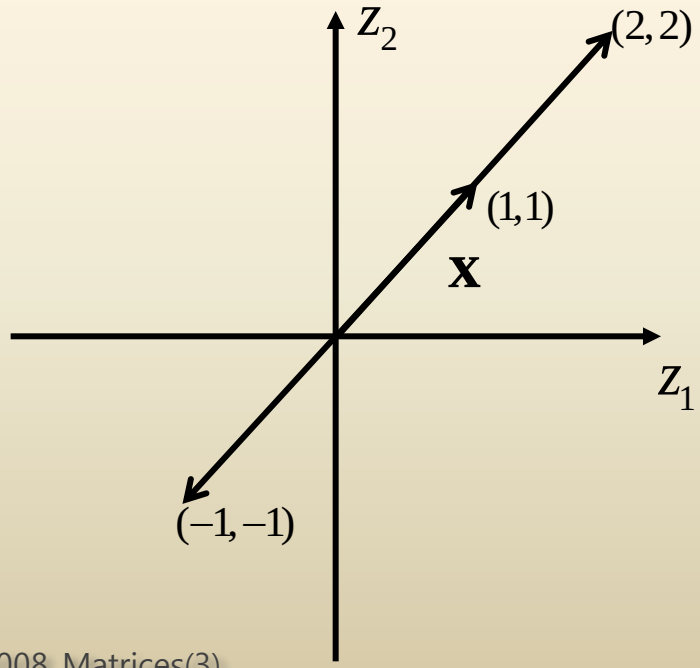
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

1) $\lambda_1 = 8$

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

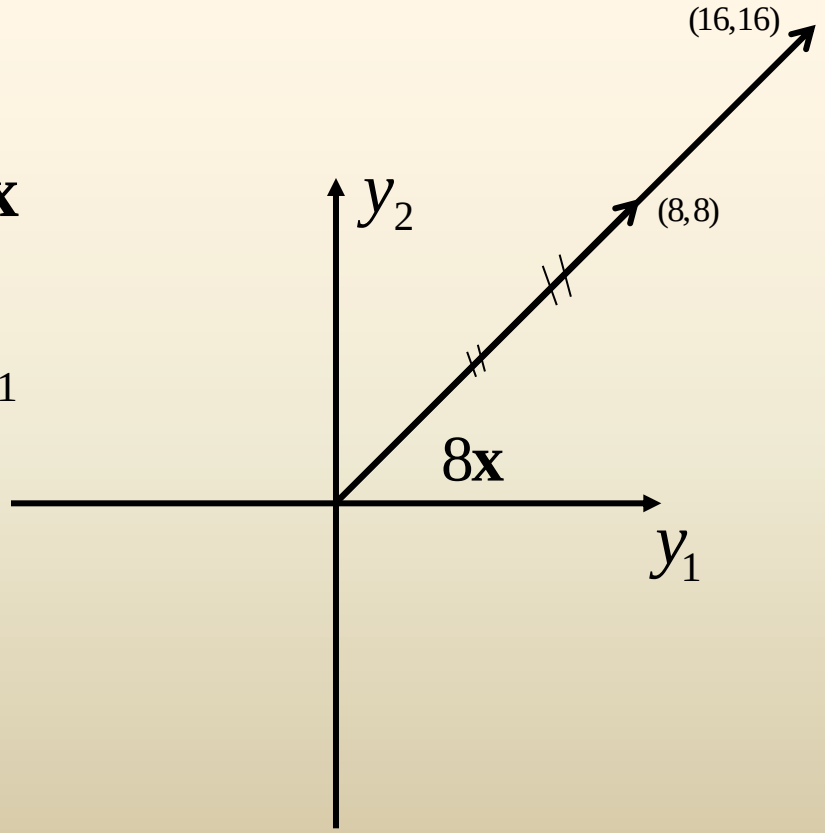


$\mathbf{Ax} = \lambda\mathbf{x}$

when, $x_1 = 1$

when, $x_1 = 2$

when, $x_1 = -1$



Stretching of an Elastic Membrane

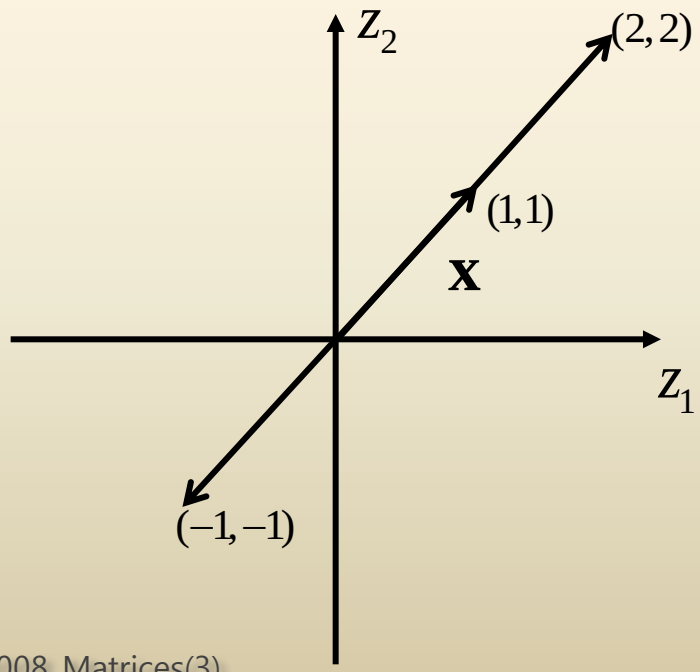
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x *after the stretch is the same direction before the stretch.*

1) $\lambda_1 = 8$

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

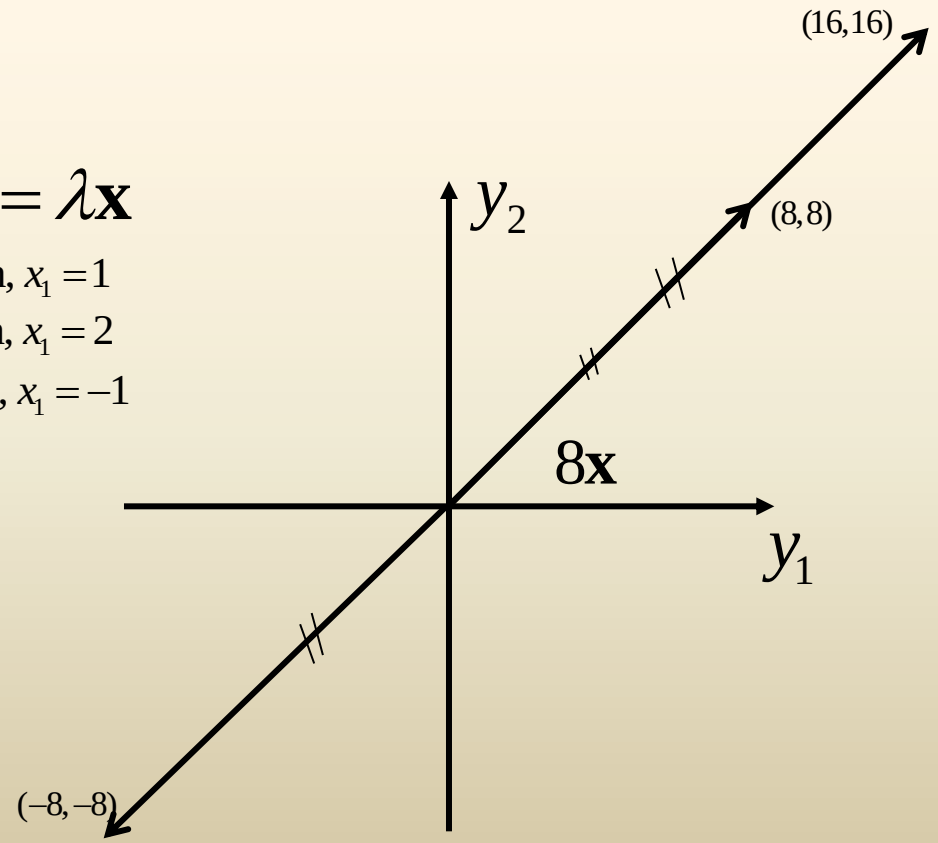


$\mathbf{Ax} = \lambda\mathbf{x}$

when, $x_1 = 1$

when, $x_1 = 2$

when, $x_1 = -1$

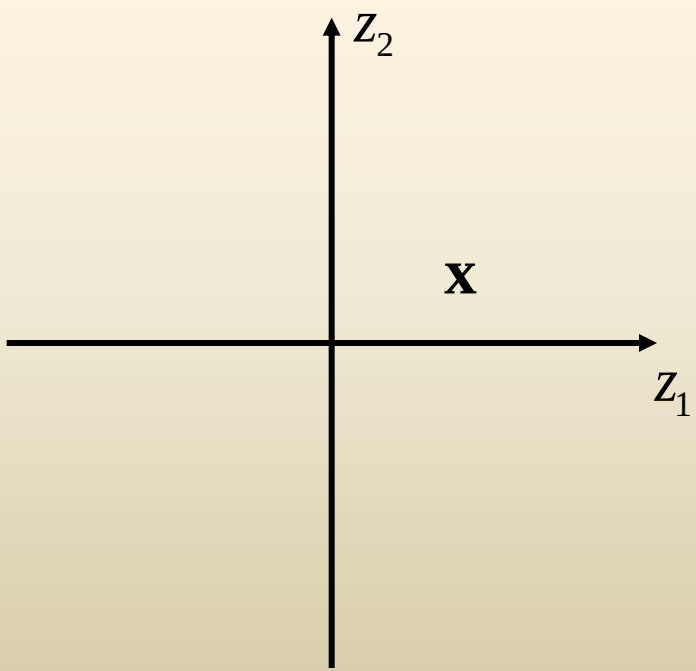


Stretching of an Elastic Membrane

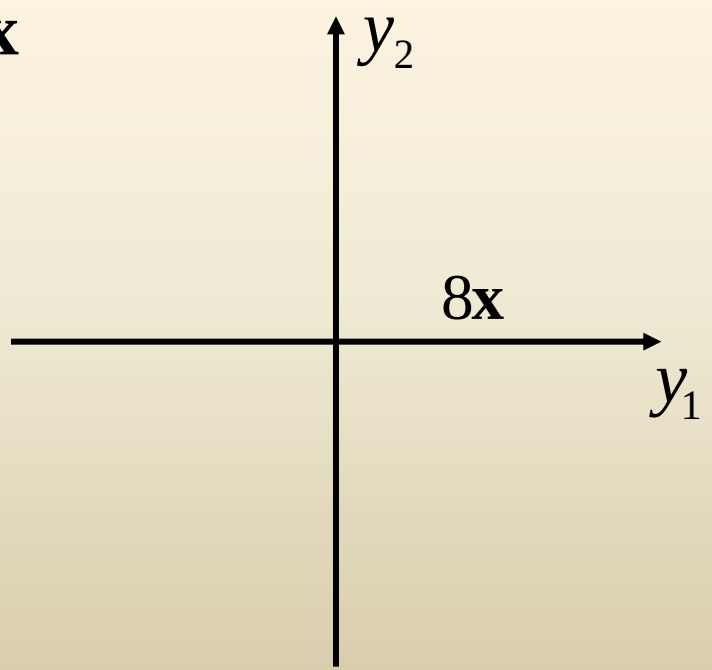
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x *after the stretch is the same direction before the stretch.*



$$\mathbf{Ax} = \lambda\mathbf{x}$$



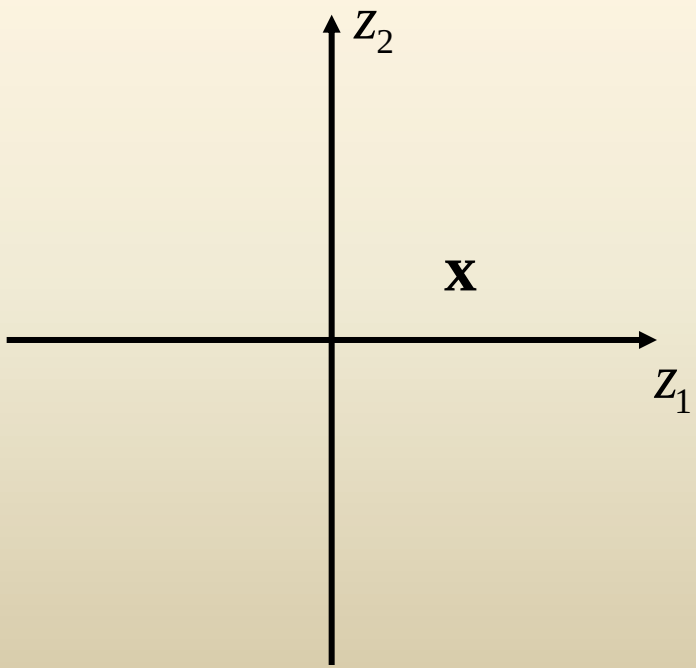
Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

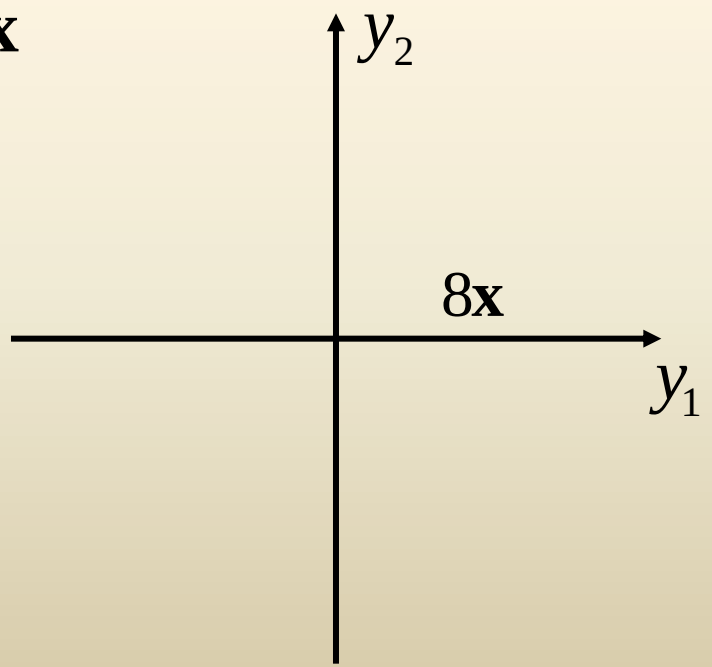
$1)\ \lambda_1 = 2$

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x after the stretch is the same direction before the stretch.



$$\mathbf{Ax} = \lambda\mathbf{x}$$



Stretching of an Elastic Membrane

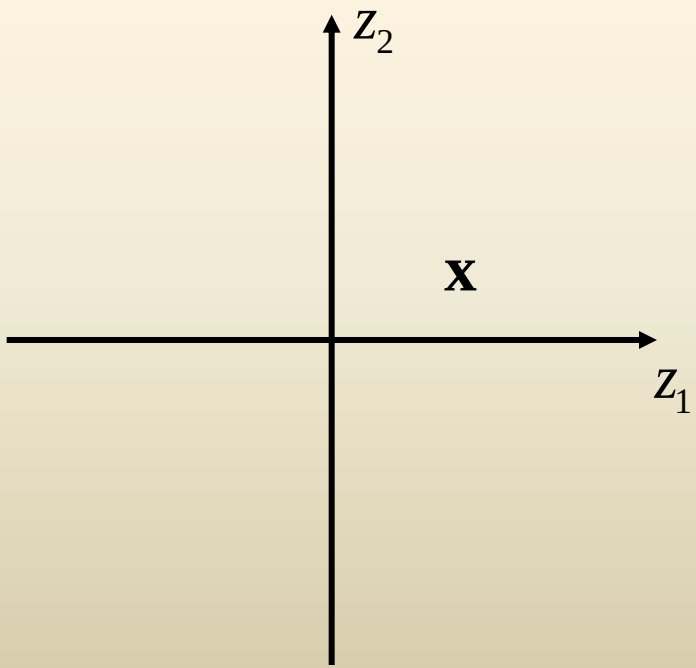
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

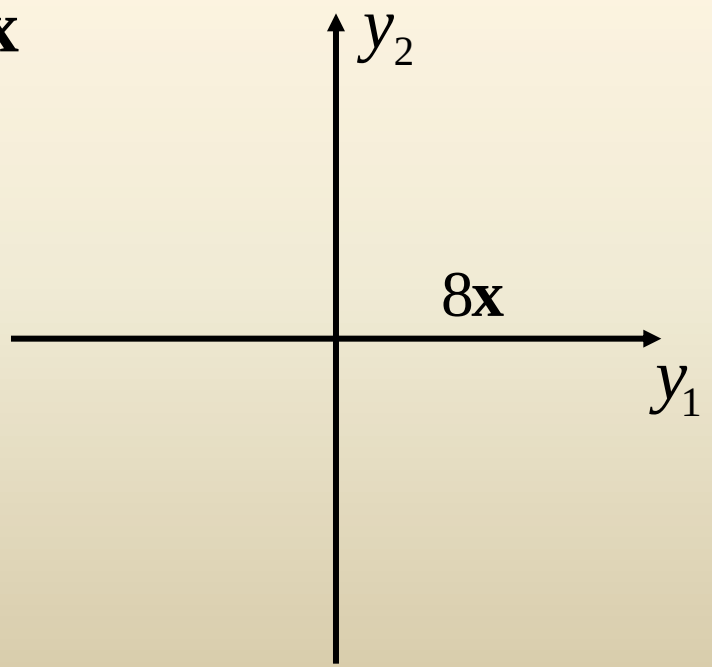
It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

1) $\lambda_1 = 2$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$



$$\mathbf{Ax} = \lambda\mathbf{x}$$



Stretching of an Elastic Membrane

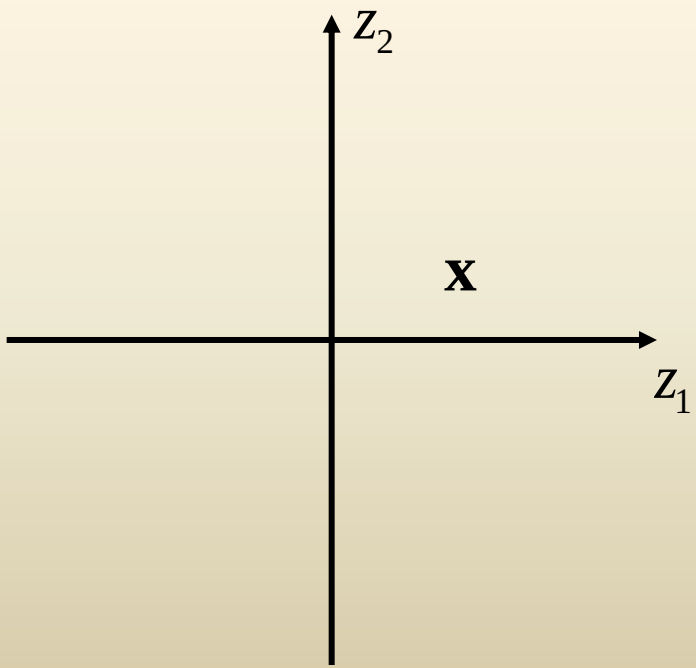
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

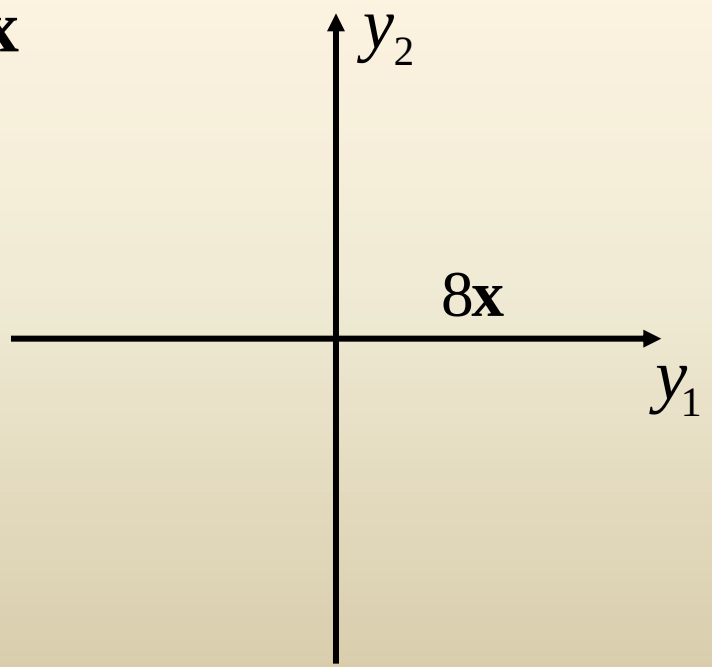
$$1)\ \lambda_1 = 2$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$



$$\mathbf{Ax} = \lambda\mathbf{x}$$

when, $x_1 = 1$



Stretching of an Elastic Membrane

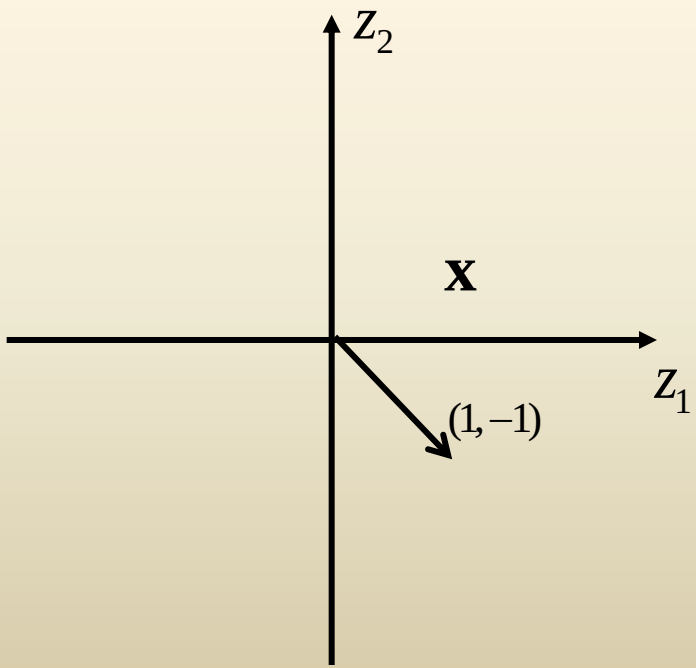
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

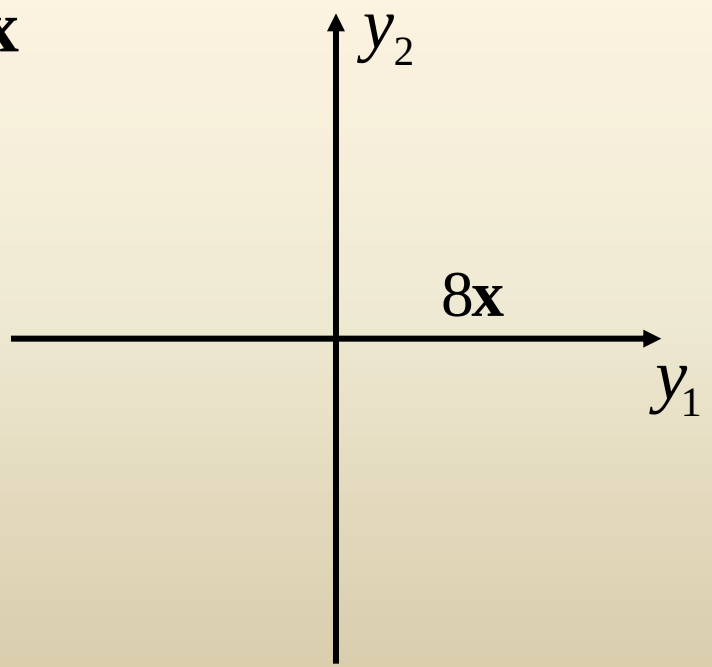
1) $\lambda_1 = 2$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$



$$\mathbf{Ax} = \lambda\mathbf{x}$$

when, $x_1 = 1$



Stretching of an Elastic Membrane

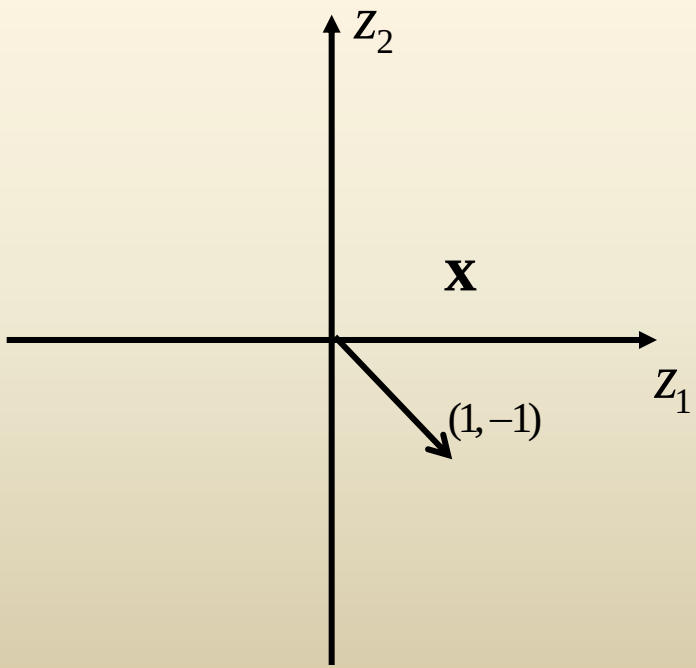
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

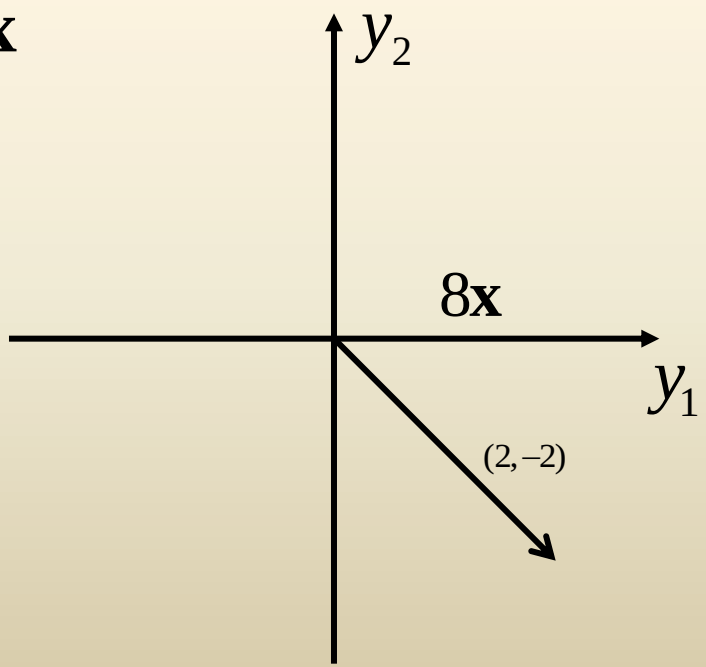
1) $\lambda_1 = 2$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$



$$\mathbf{Ax} = \lambda\mathbf{x}$$

when, $x_1 = 1$



Stretching of an Elastic Membrane

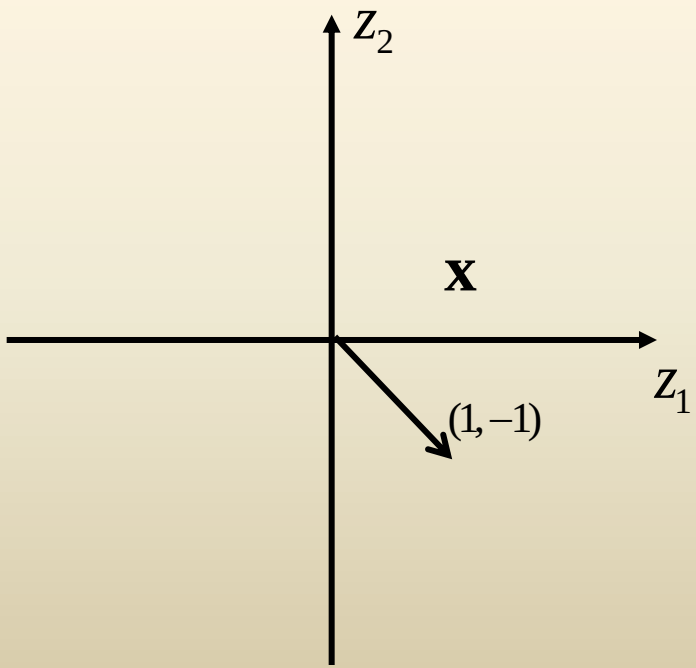
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

1) $\lambda_1 = 2$

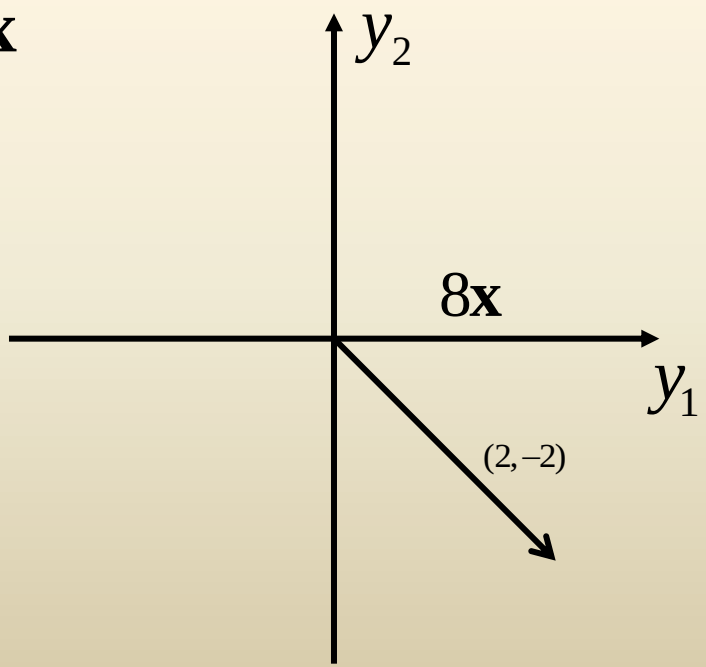
$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$



$$\mathbf{Ax} = \lambda\mathbf{x}$$

when, $x_1 = 1$

when, $x_1 = 2$



Stretching of an Elastic Membrane

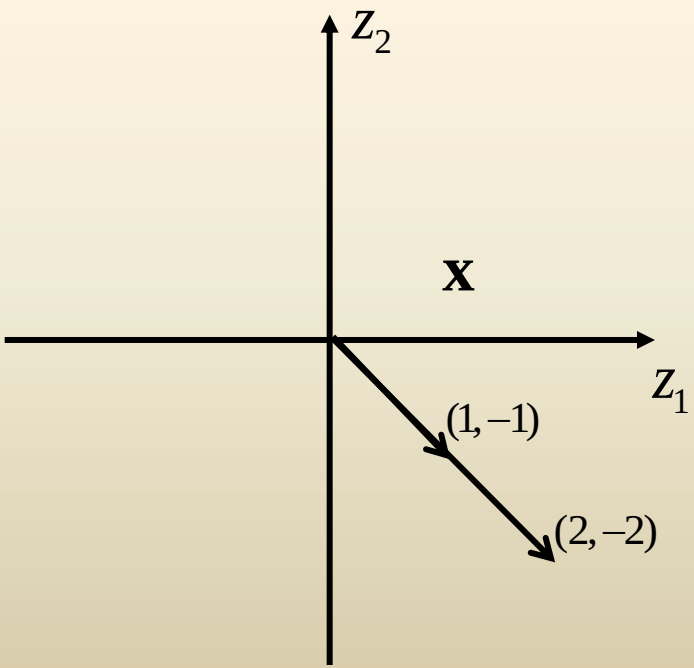
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

1) $\lambda_1 = 2$

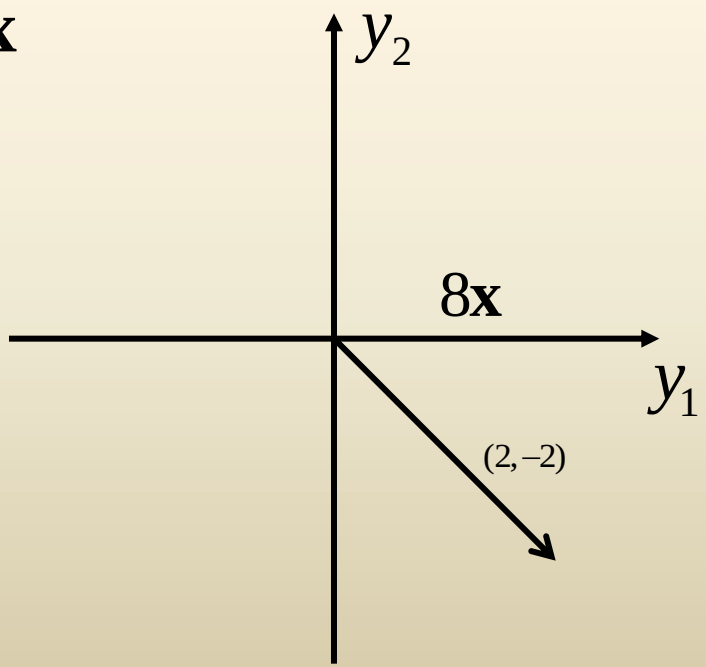
$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$



$$\mathbf{Ax} = \lambda\mathbf{x}$$

when, $x_1 = 1$

when, $x_1 = 2$



Stretching of an Elastic Membrane

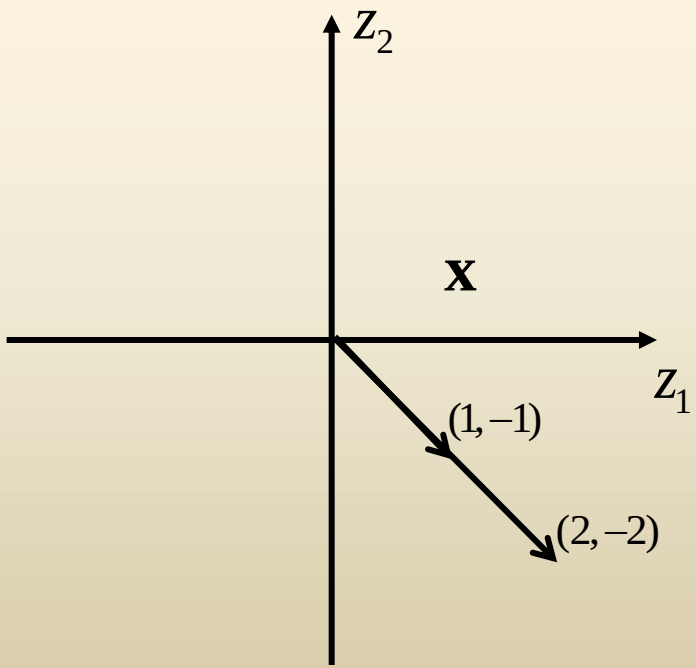
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

1) $\lambda_1 = 2$

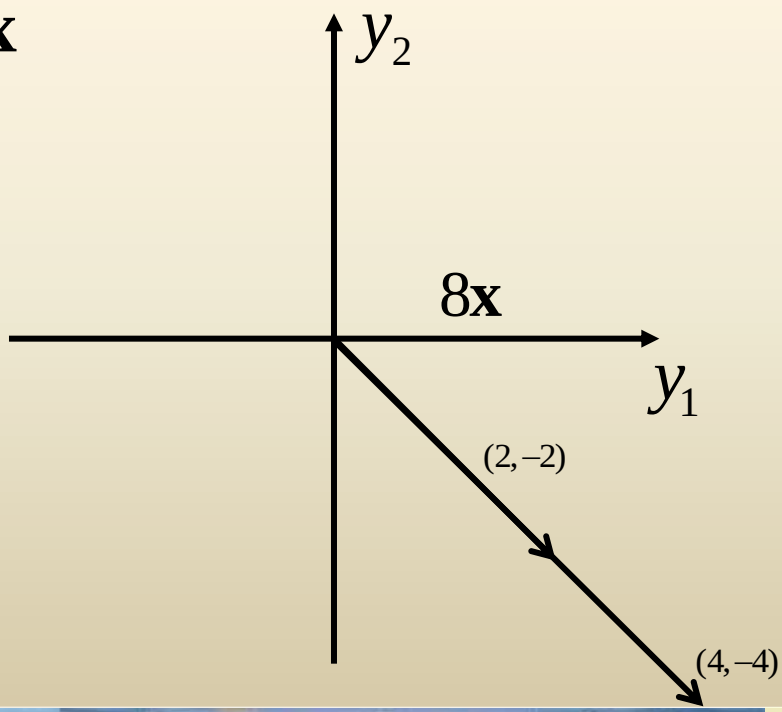
$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$



$$\mathbf{Ax} = \lambda\mathbf{x}$$

when, $x_1 = 1$

when, $x_1 = 2$



Stretching of an Elastic Membrane

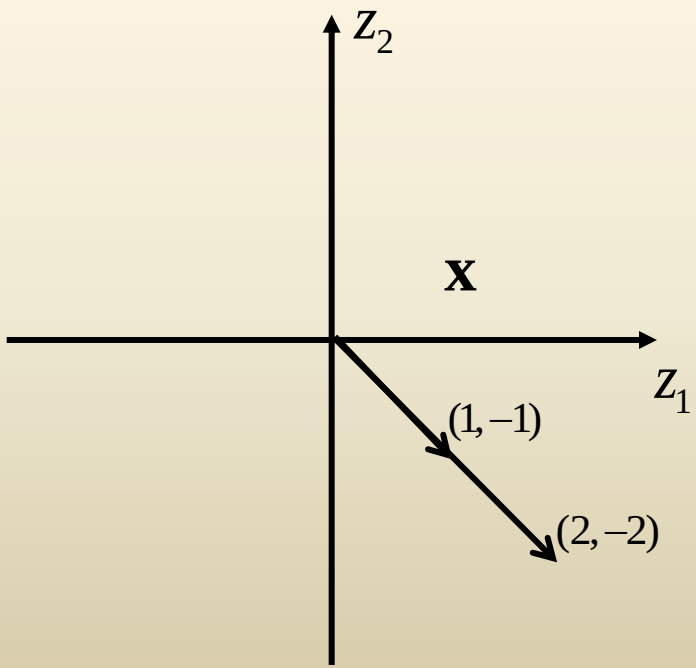
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

1) $\lambda_1 = 2$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

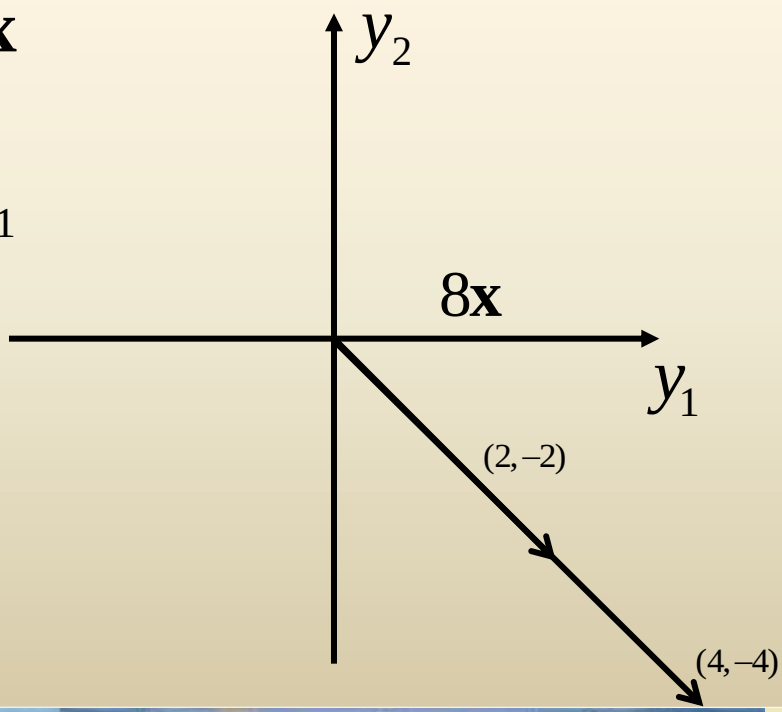


$$\mathbf{Ax} = \lambda\mathbf{x}$$

when, $x_1 = 1$

when, $x_1 = 2$

when, $x_1 = -1$



Stretching of an Elastic Membrane

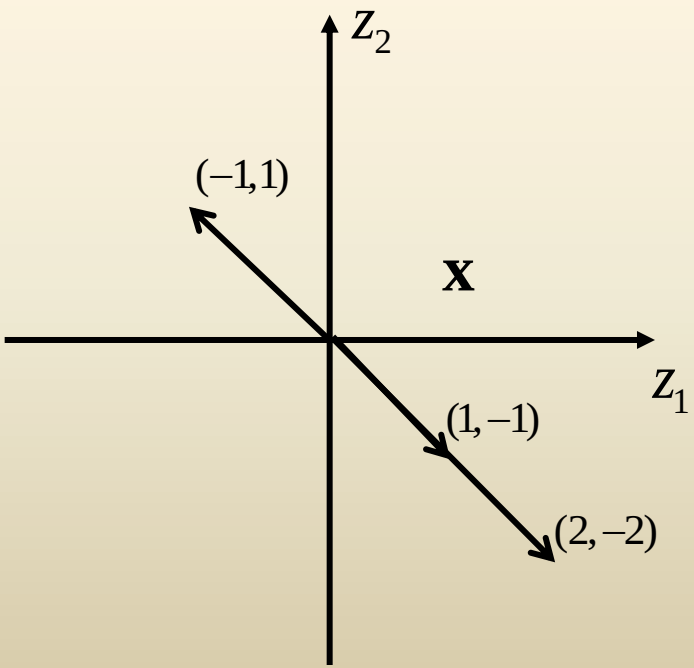
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

1) $\lambda_1 = 2$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

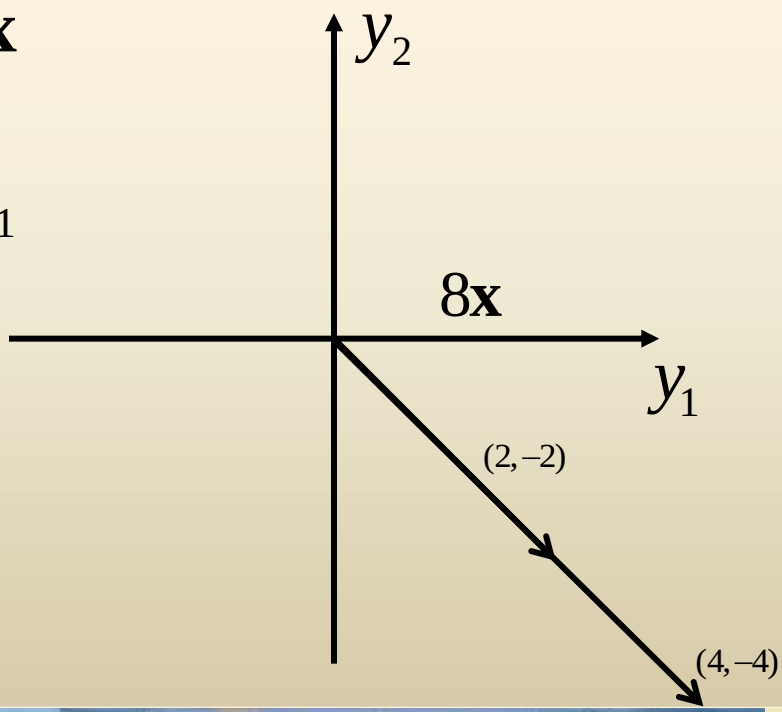


$$\mathbf{Ax} = \lambda\mathbf{x}$$

when, $x_1 = 1$

when, $x_1 = 2$

when, $x_1 = -1$



Stretching of an Elastic Membrane

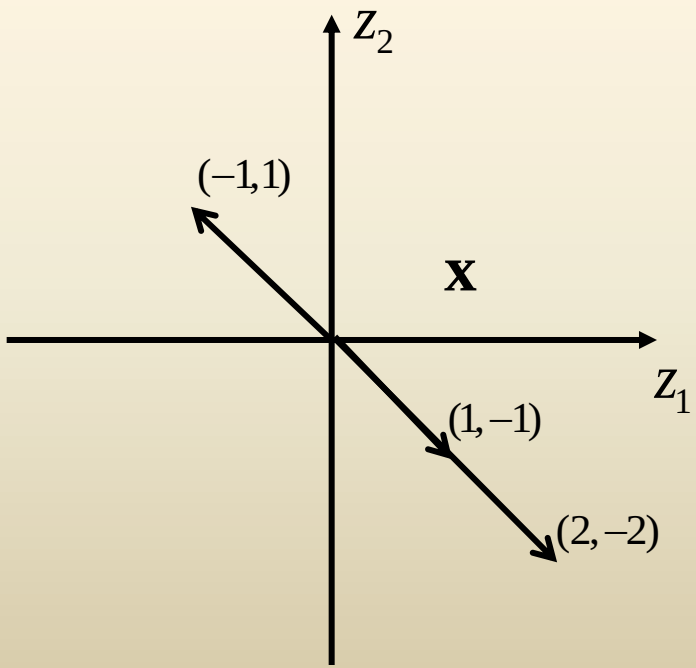
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$1)\ \lambda_1 = 2$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

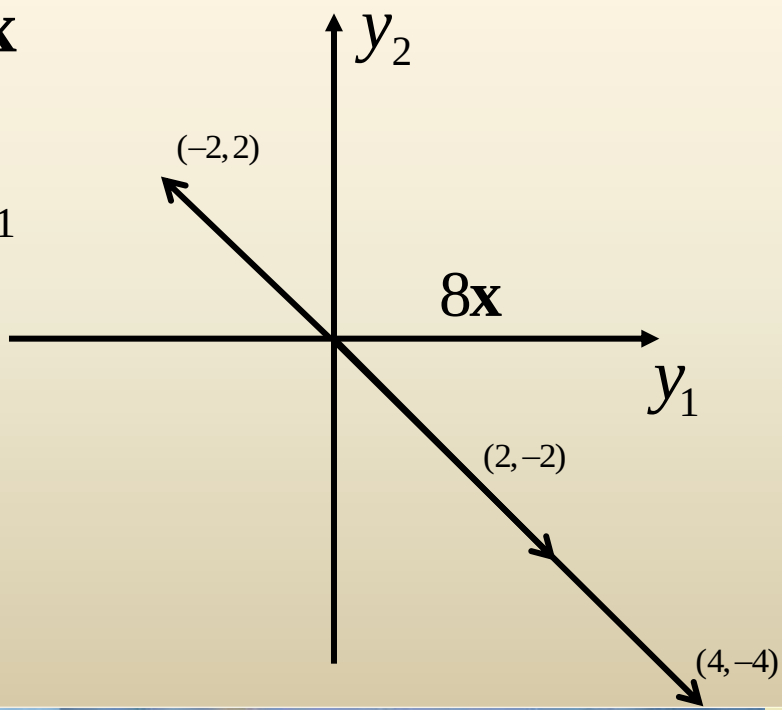


$$\mathbf{Ax} = \lambda\mathbf{x}$$

when, $x_1 = 1$

when, $x_1 = 2$

when, $x_1 = -1$



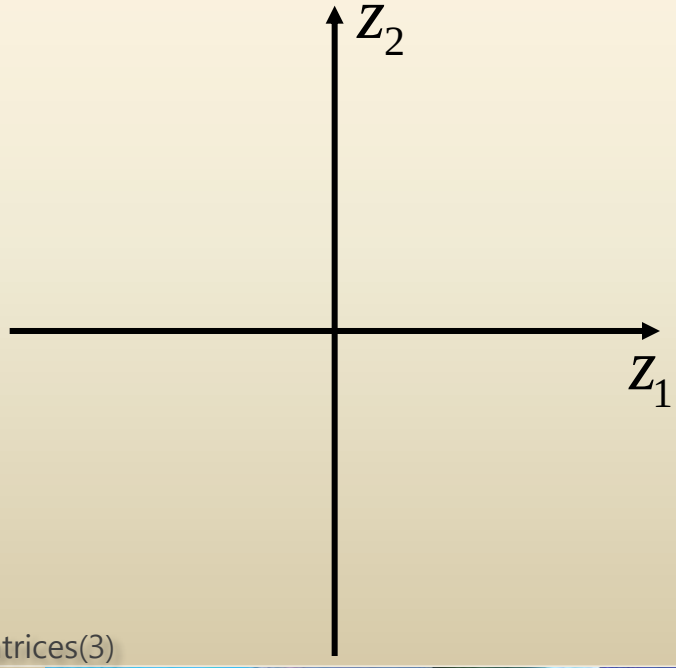
Stretching of an Elastic Membrane

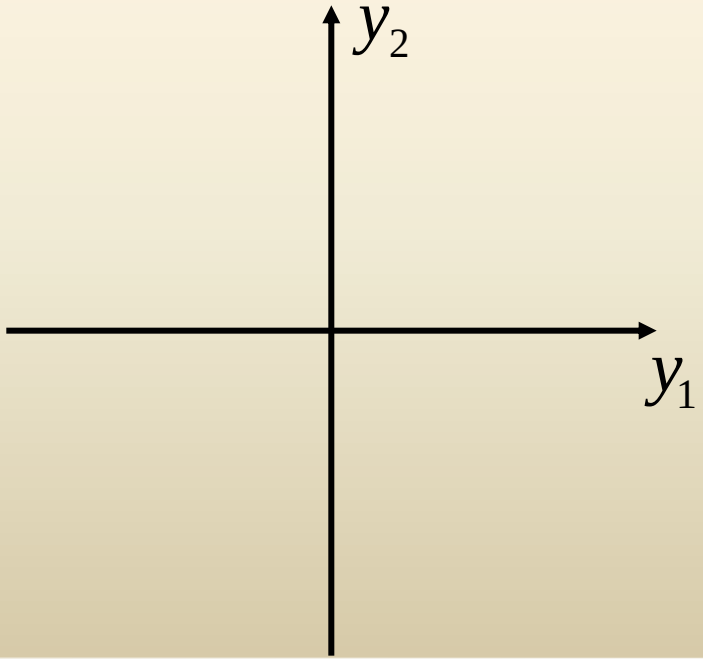
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x after the stretch is the same direction before the stretch.

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

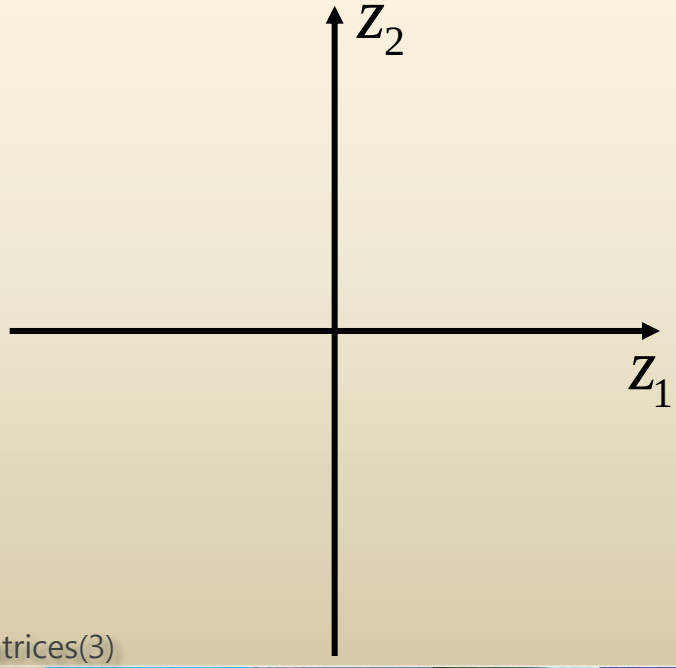
Use Eigenvalue, Eigenvector

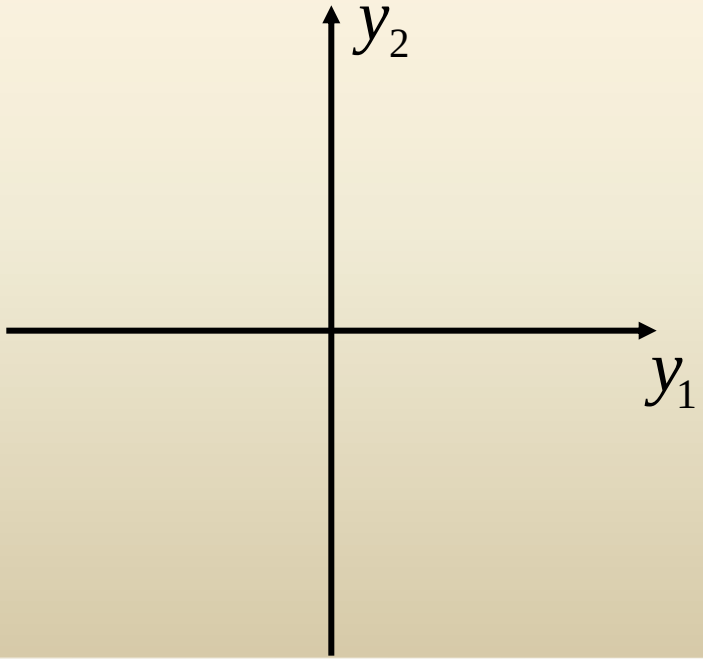
$$\mathbf{A}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x after the stretch is the same direction before the stretch.

$$\lambda\mathbf{x} = \mathbf{A}\mathbf{x}$$





Stretching of an Elastic Membrane

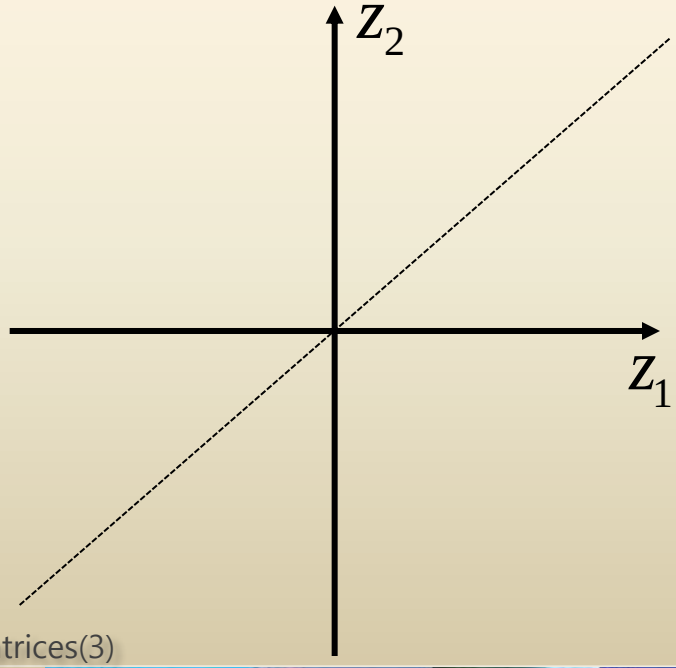
Use Eigenvalue, Eigenvector

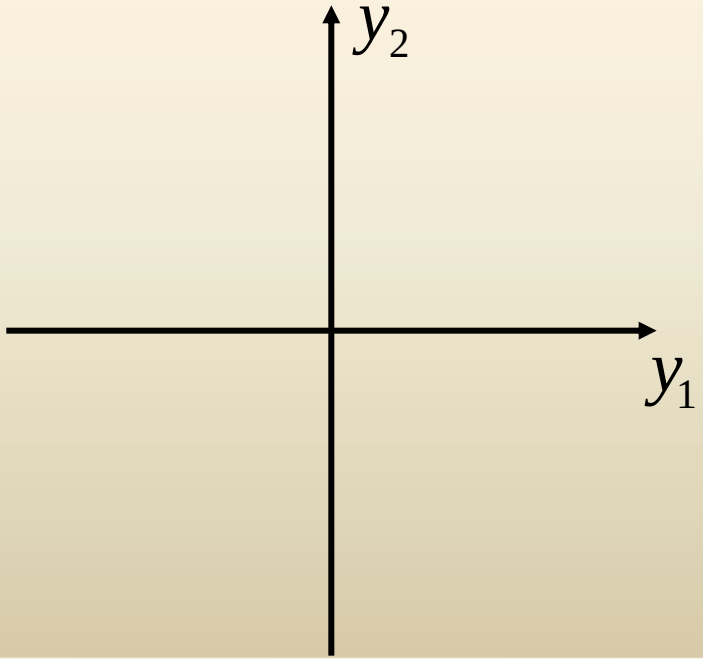
$$\mathbf{A}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x after the stretch is the same direction before the stretch.

$$\lambda\mathbf{x} = \mathbf{A}\mathbf{x}$$





Stretching of an Elastic Membrane

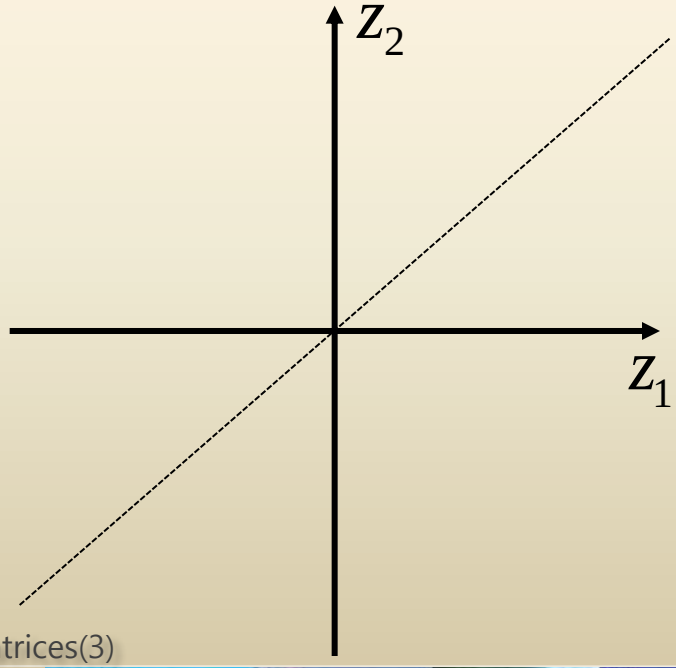
Use Eigenvalue, Eigenvector

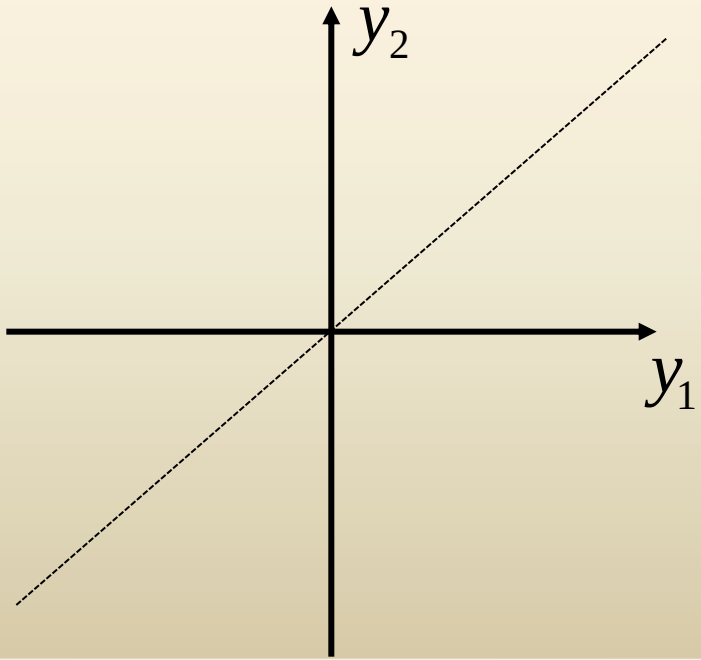
$$\mathbf{A}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x after the stretch is the same direction before the stretch.

$$\lambda\mathbf{x} = \mathbf{A}\mathbf{x}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

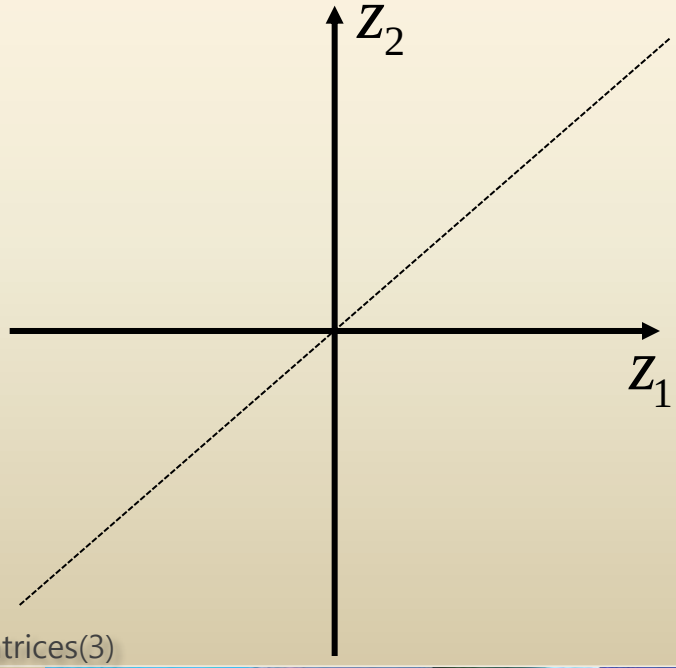
When $\mathbf{Ax}=\lambda\mathbf{x}$.

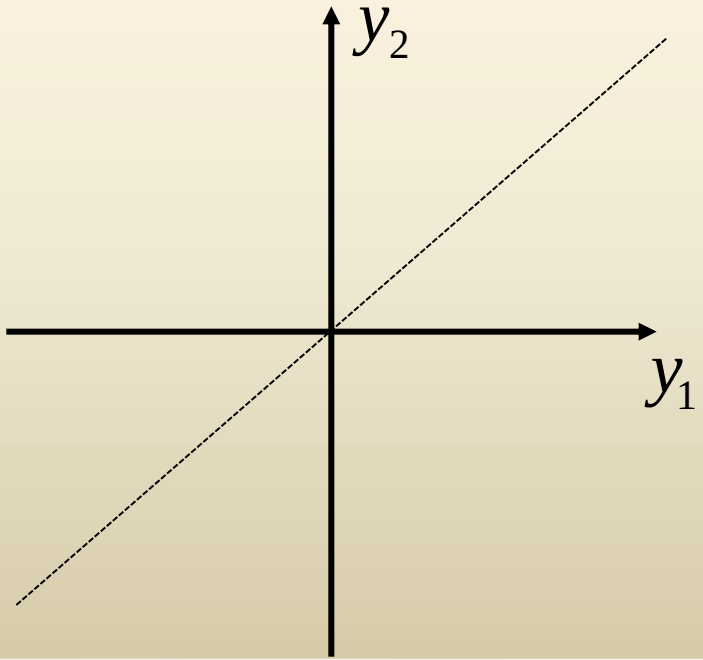
It means that the direction of x *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

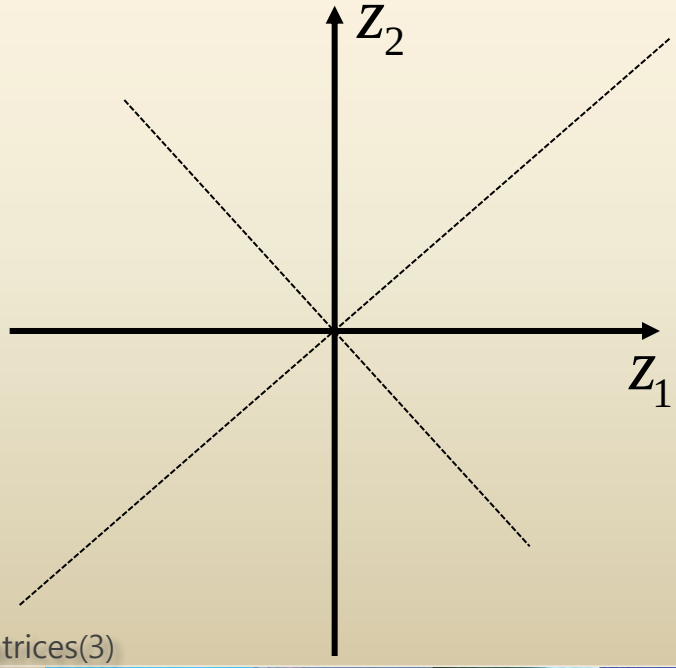
When $\mathbf{Ax}=\lambda\mathbf{x}$.

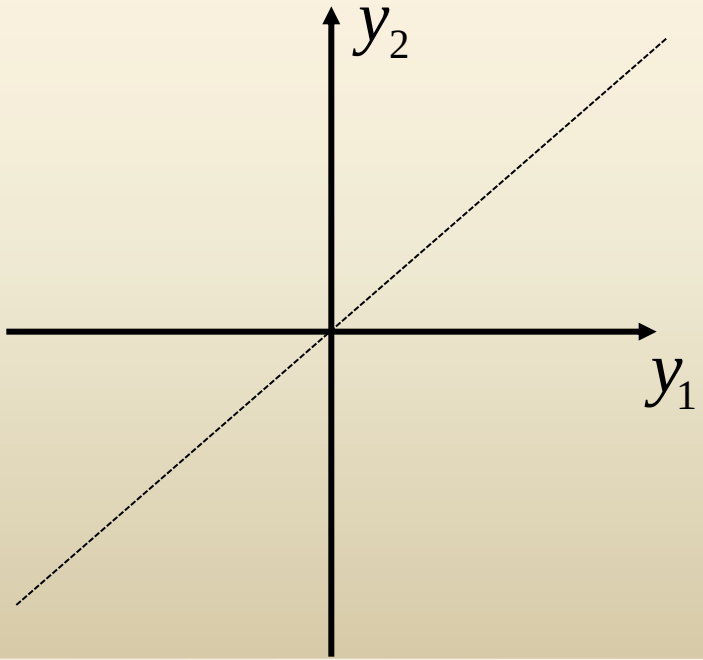
It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

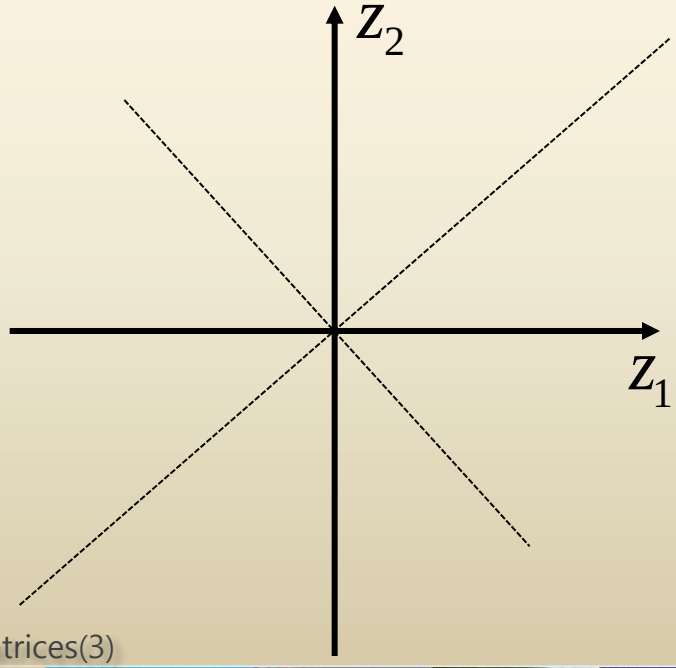
When $\mathbf{Ax}=\lambda\mathbf{x}$.

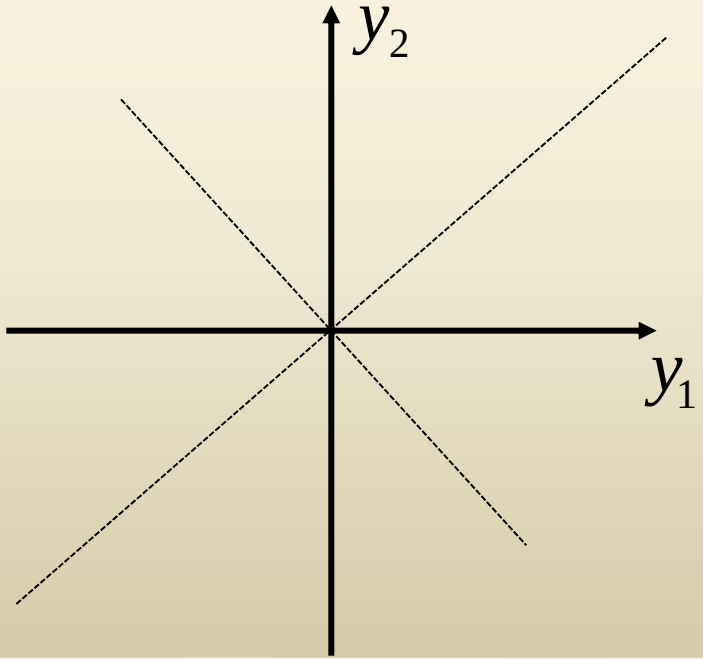
It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

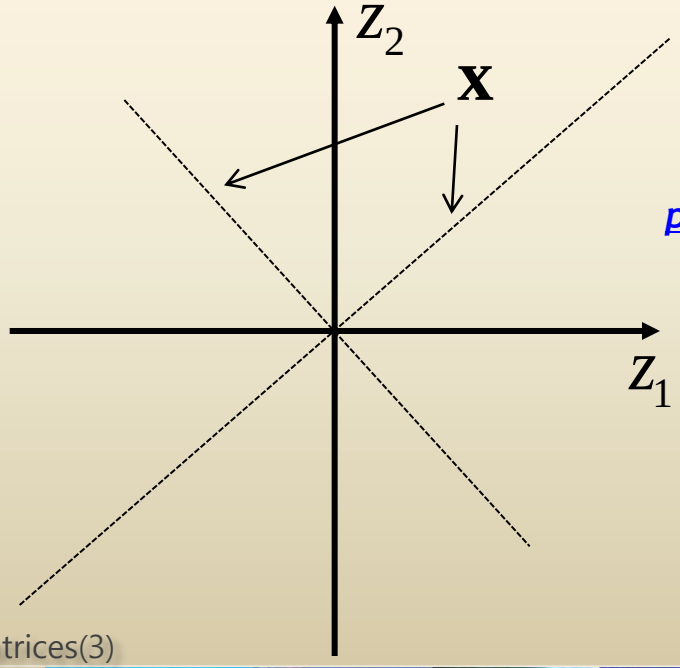
When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

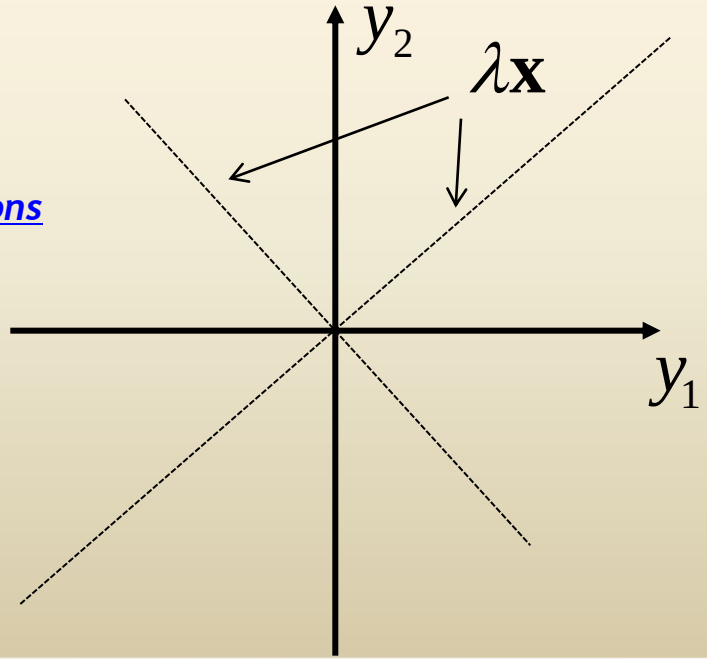
$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$



principle directions



Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

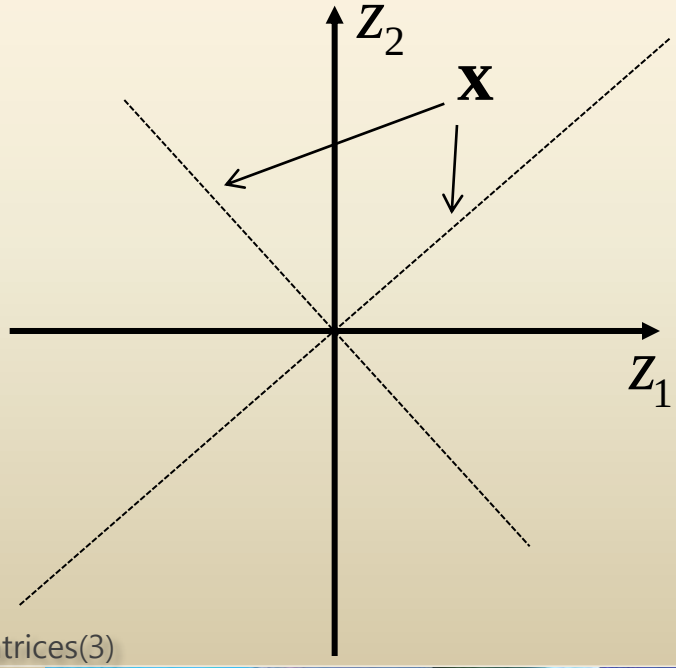
When $\mathbf{Ax}=\lambda\mathbf{x}$.

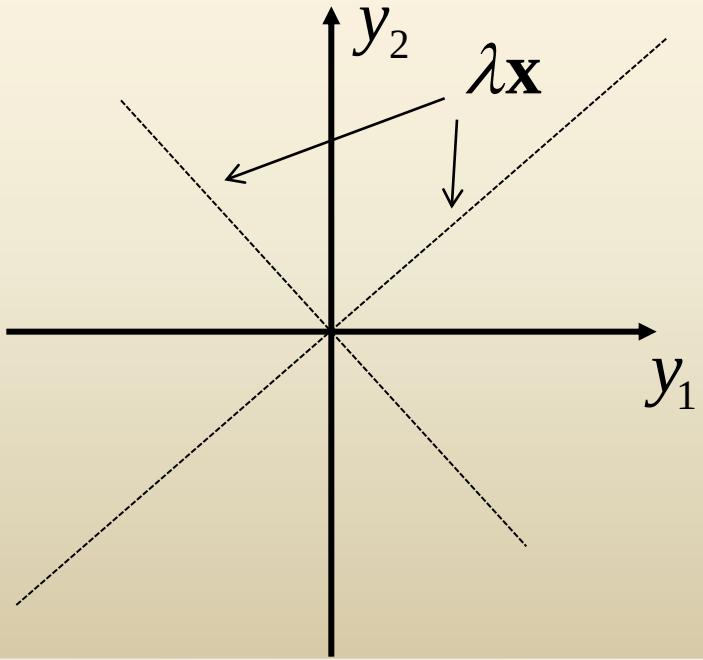
It means that the direction of x *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

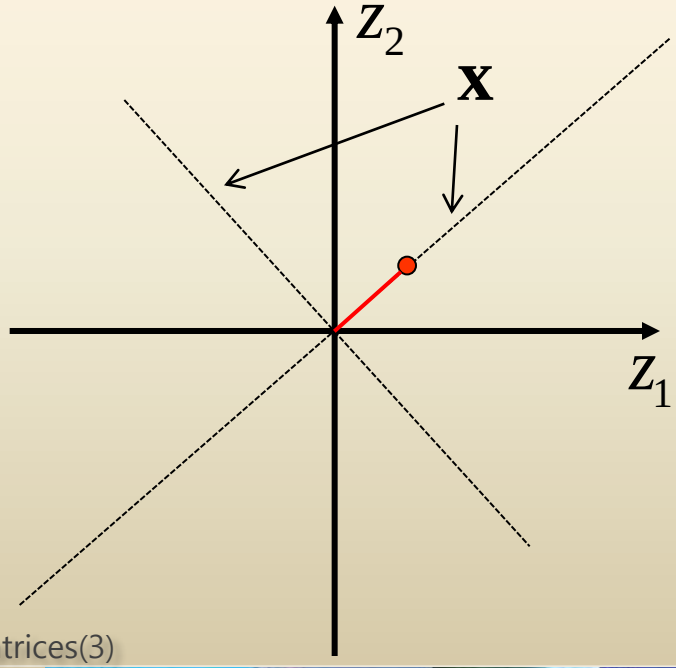
When $\mathbf{Ax}=\lambda\mathbf{x}$.

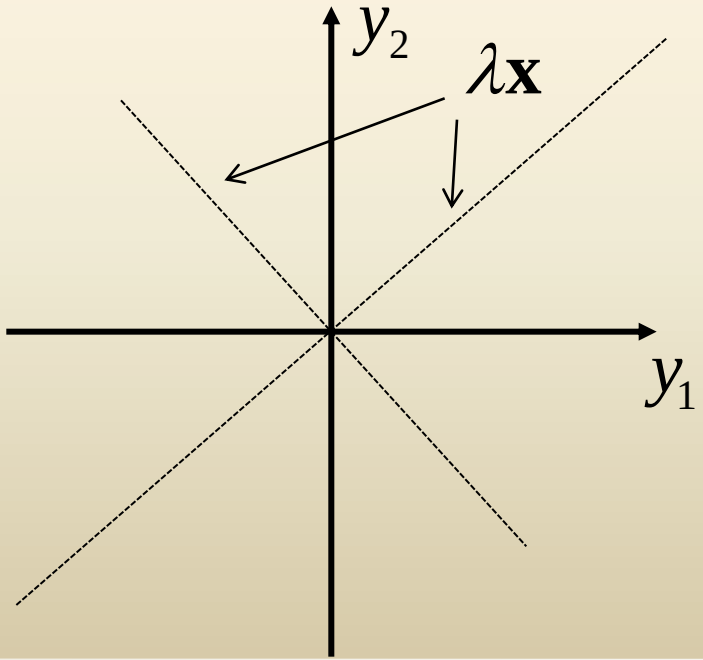
It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

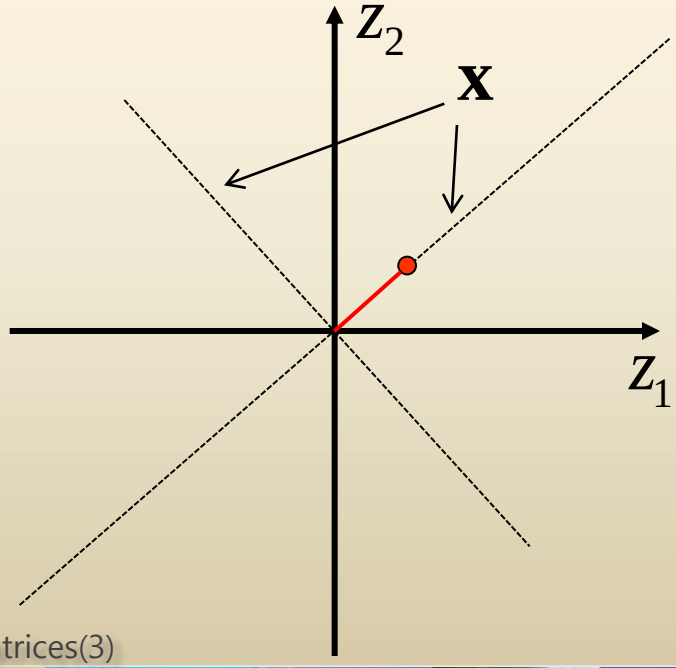
When $\mathbf{Ax}=\lambda\mathbf{x}$.

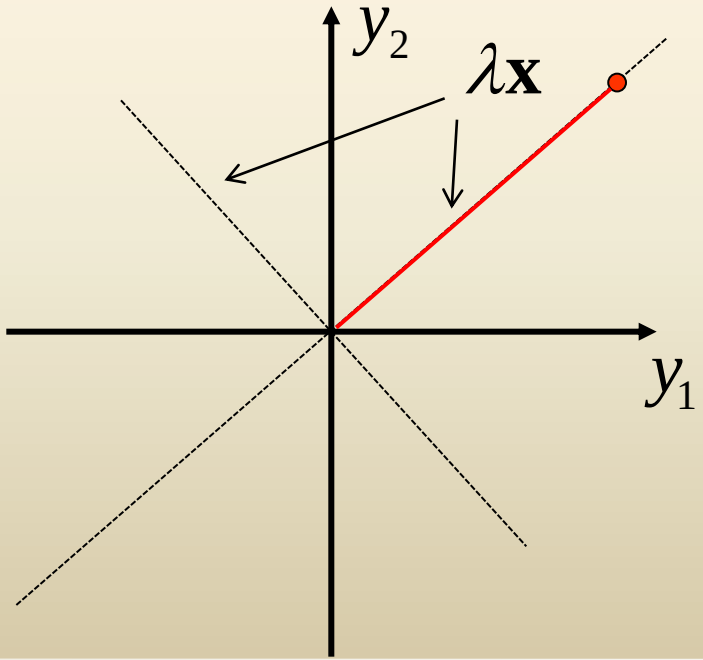
It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

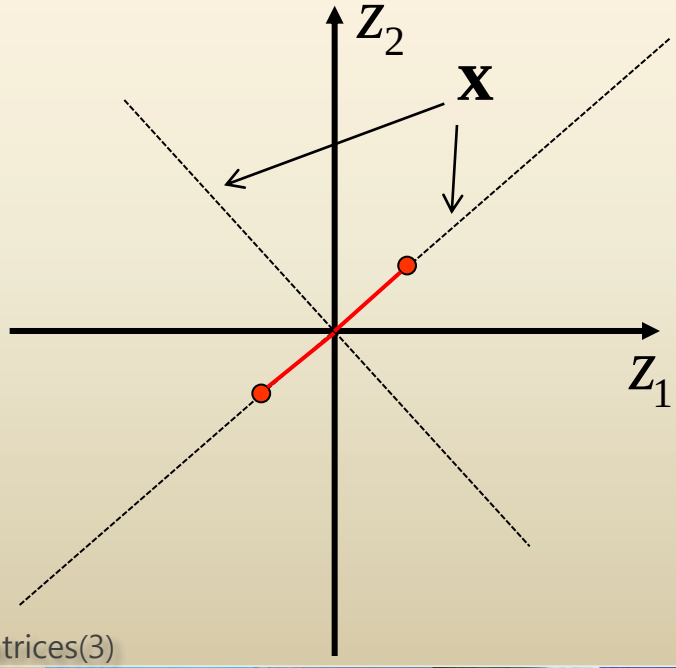
When $\mathbf{Ax}=\lambda\mathbf{x}$.

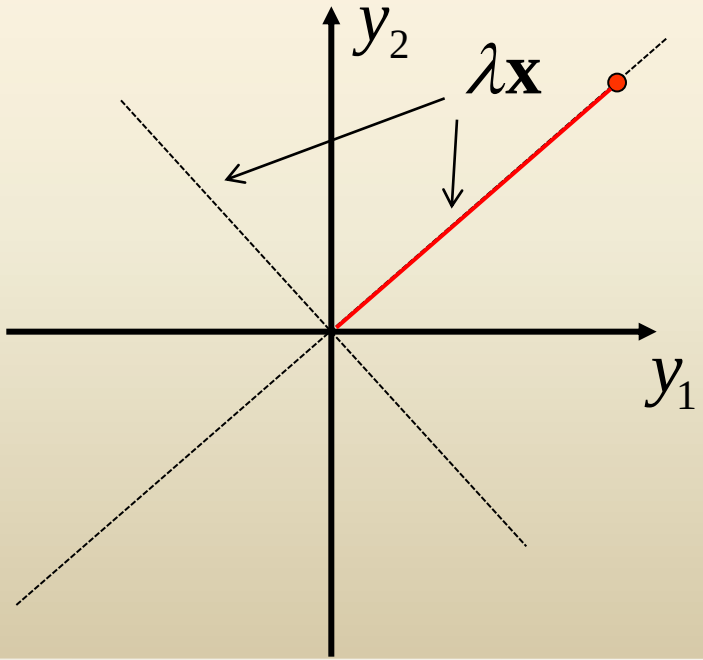
It means that the direction of x *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

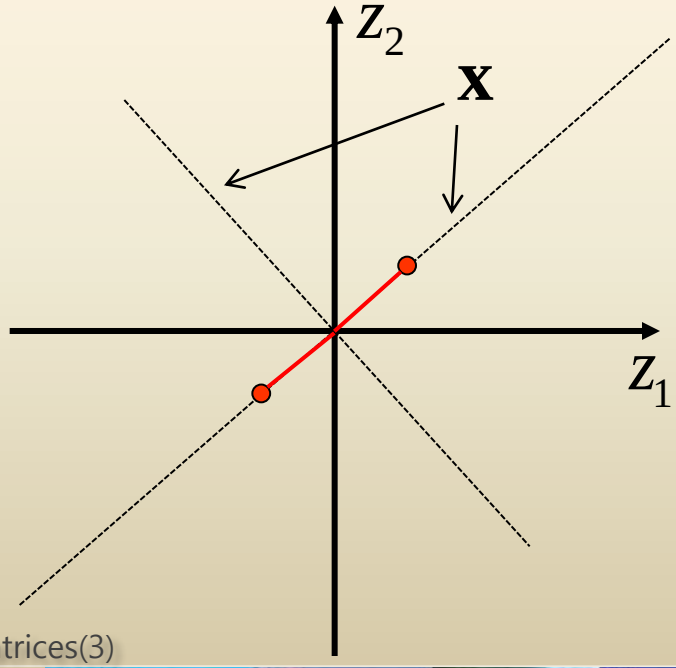
When $\mathbf{Ax}=\lambda\mathbf{x}$.

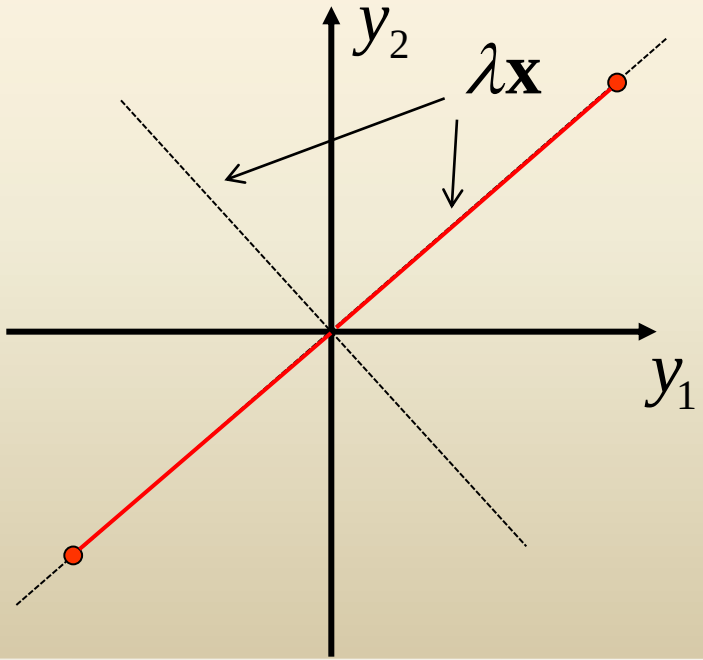
It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

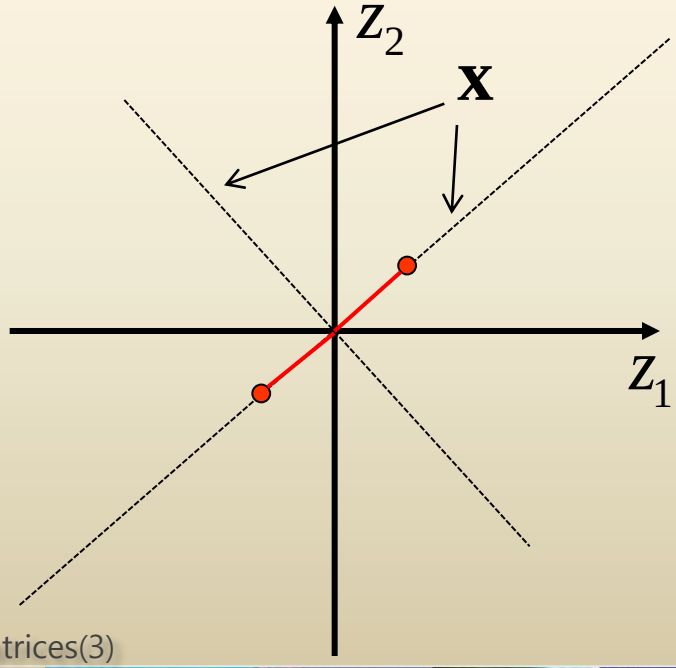
When $\mathbf{Ax}=\lambda\mathbf{x}$.

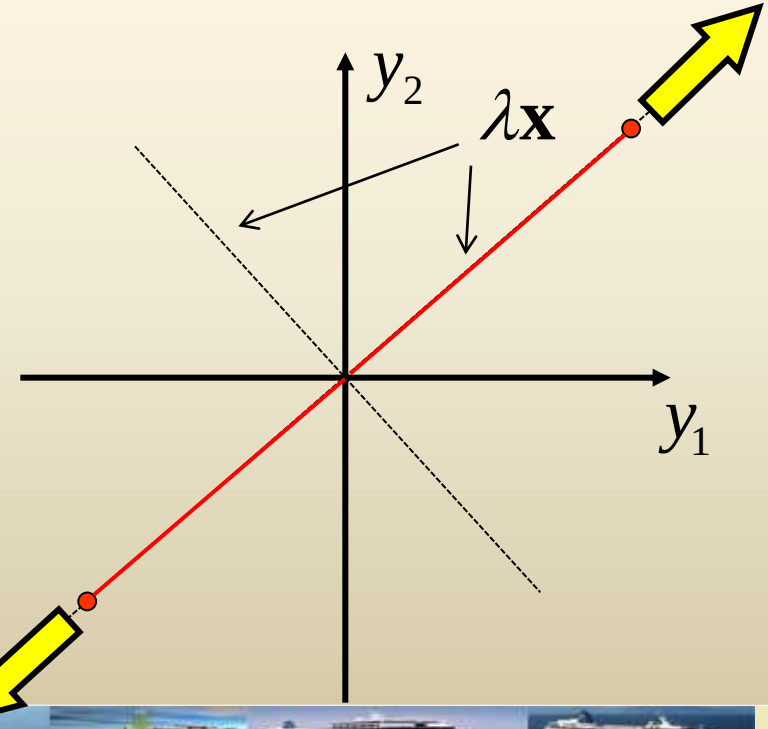
It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

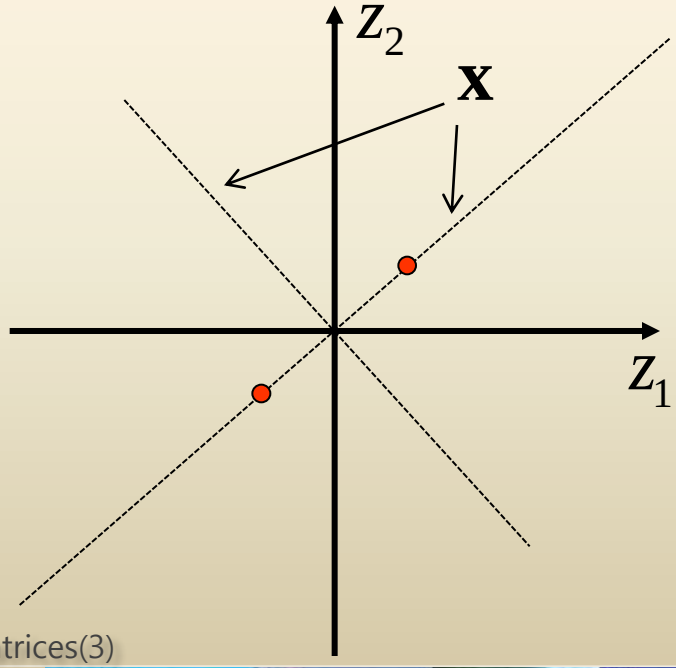
When $\mathbf{Ax}=\lambda\mathbf{x}$.

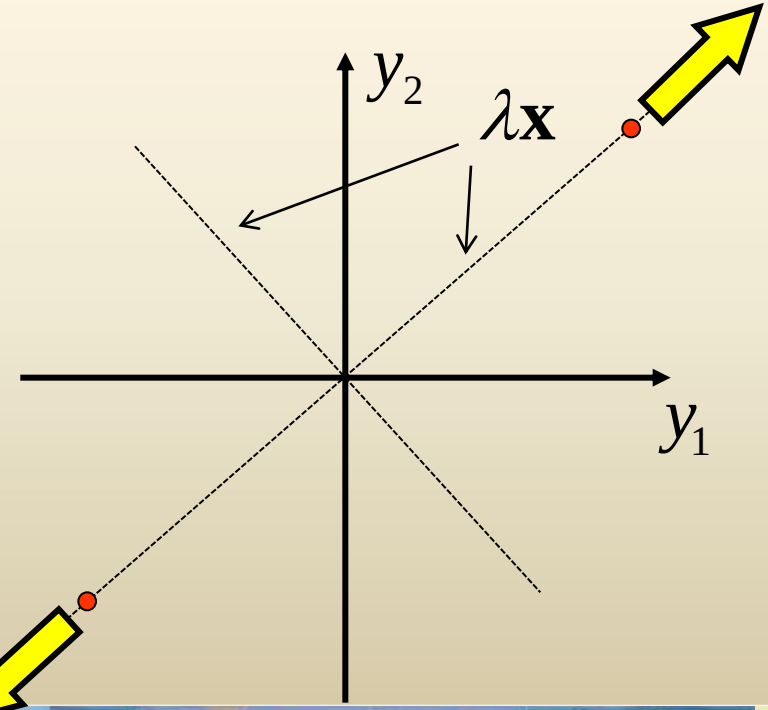
It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

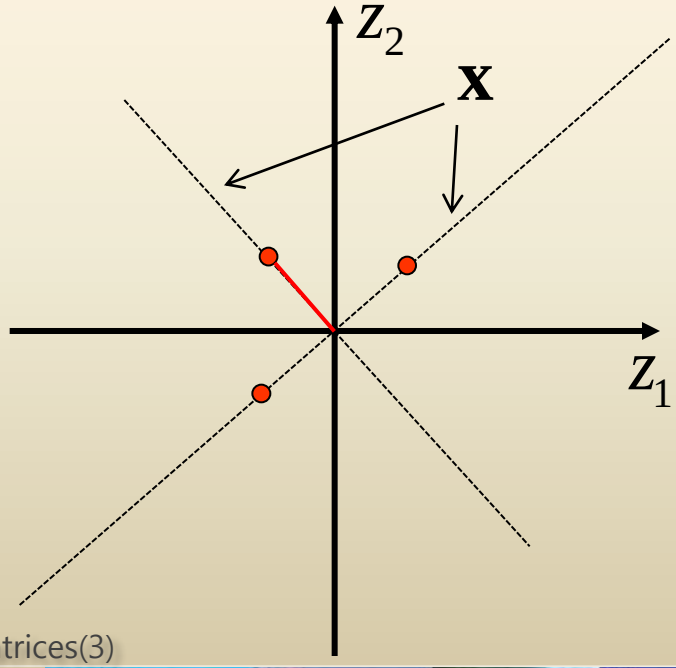
When $\mathbf{Ax}=\lambda\mathbf{x}$.

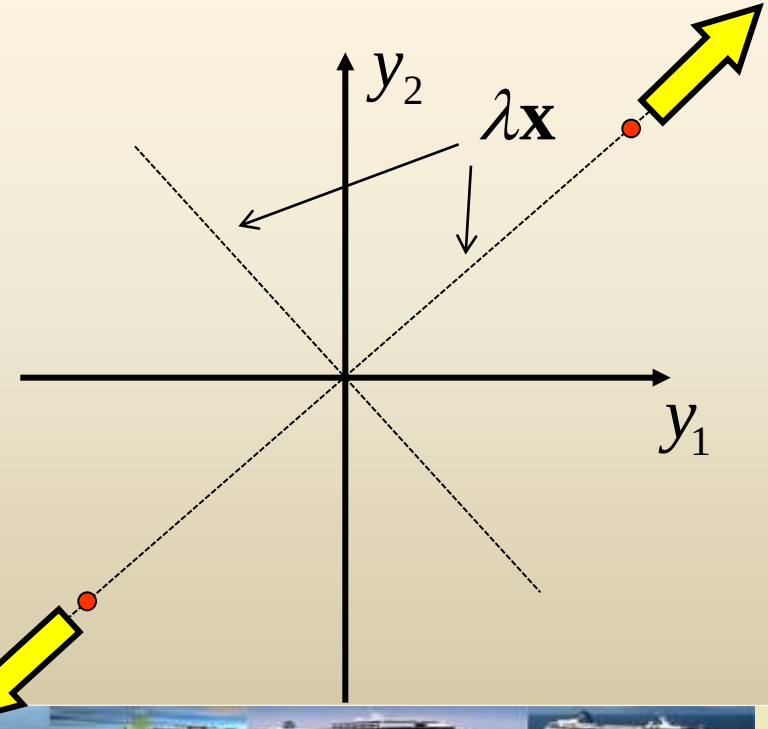
It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

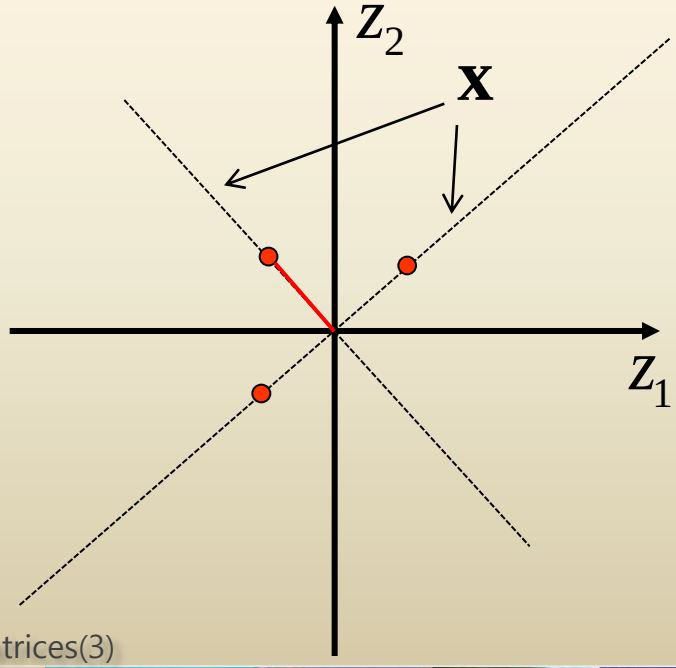
When $\mathbf{Ax}=\lambda\mathbf{x}$.

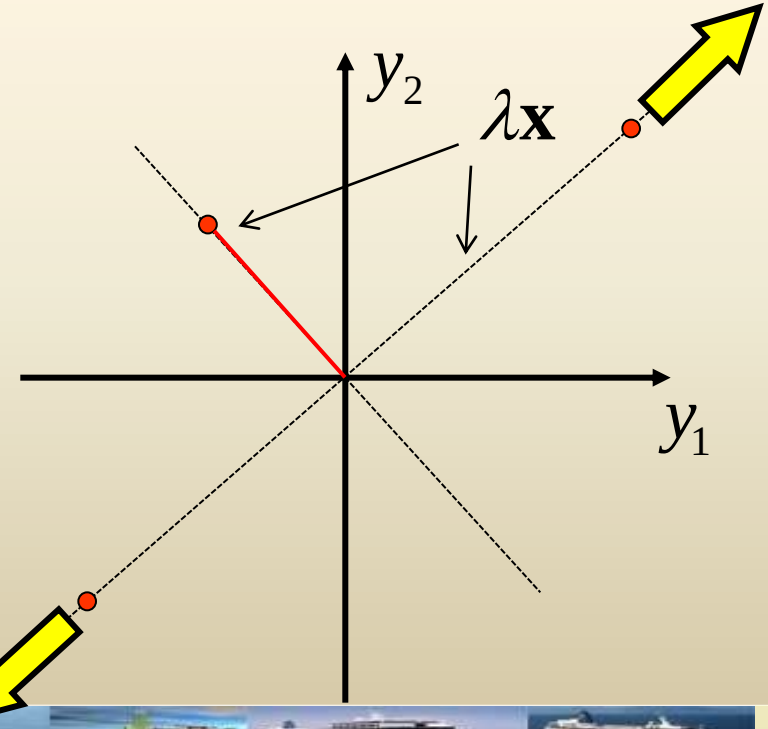
It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

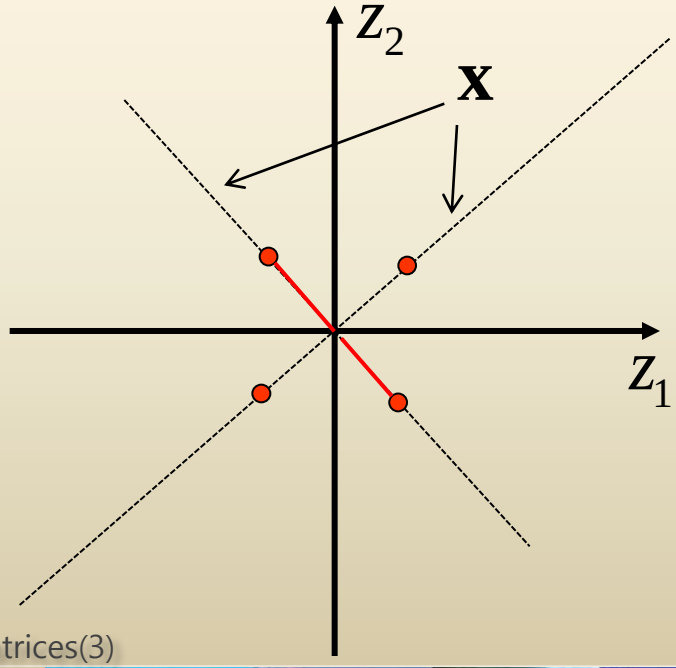
When $\mathbf{Ax}=\lambda\mathbf{x}$.

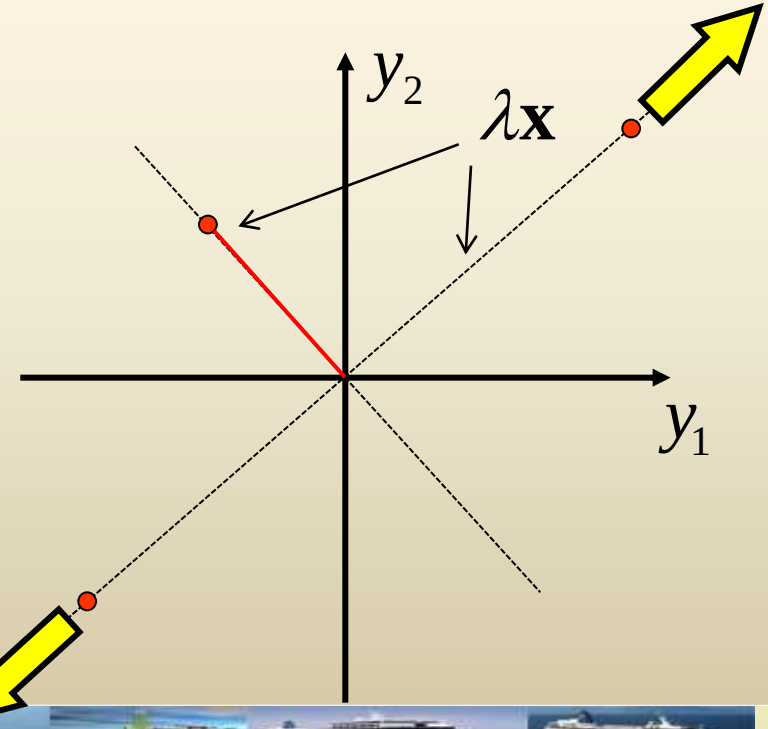
It means that the direction of x *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

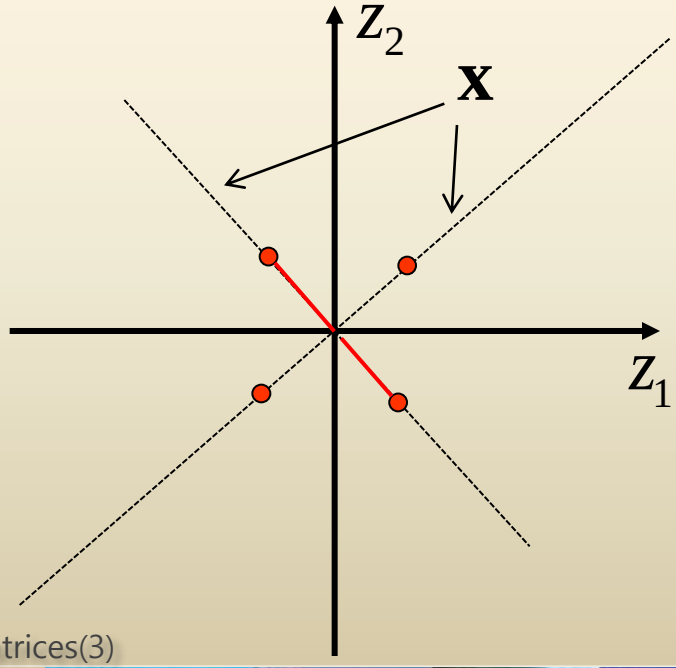
When $\mathbf{Ax}=\lambda\mathbf{x}$.

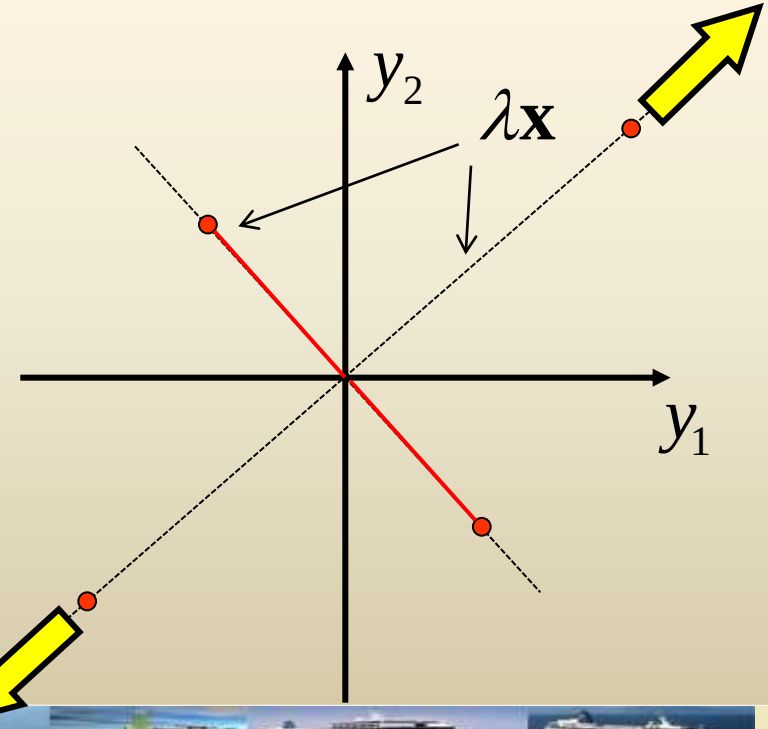
It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

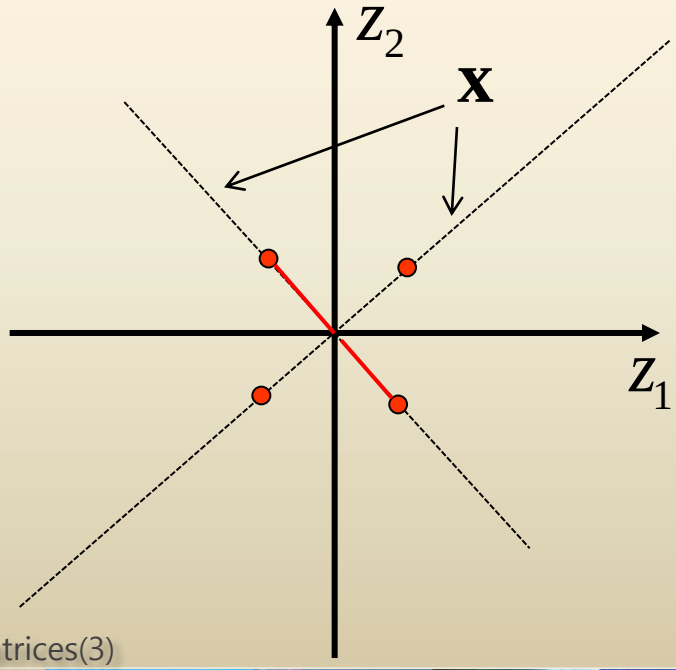
When $\mathbf{Ax}=\lambda\mathbf{x}$.

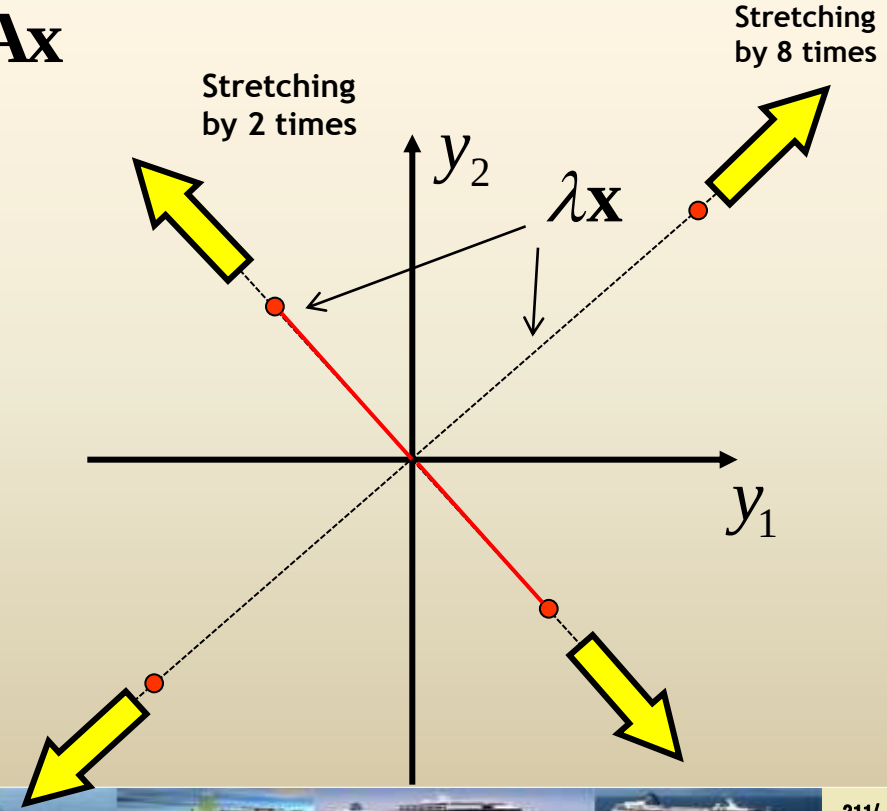
It means that the direction of x *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

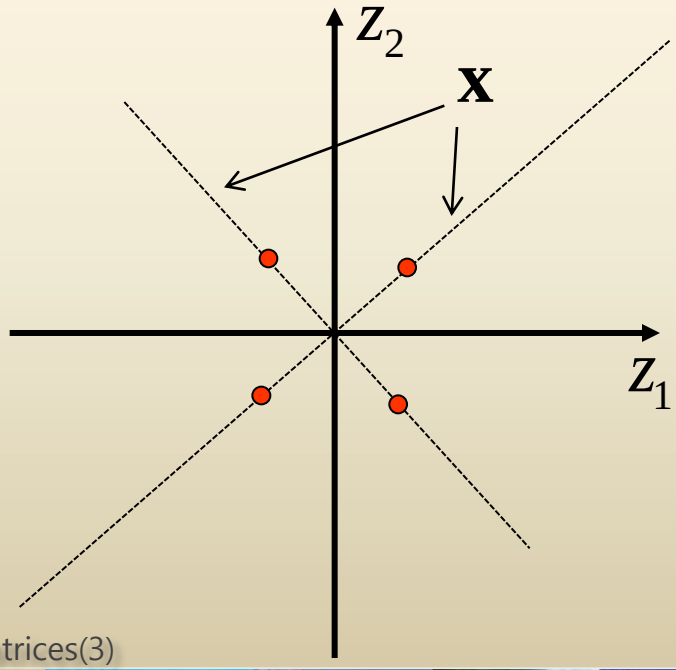
When $\mathbf{Ax}=\lambda\mathbf{x}$.

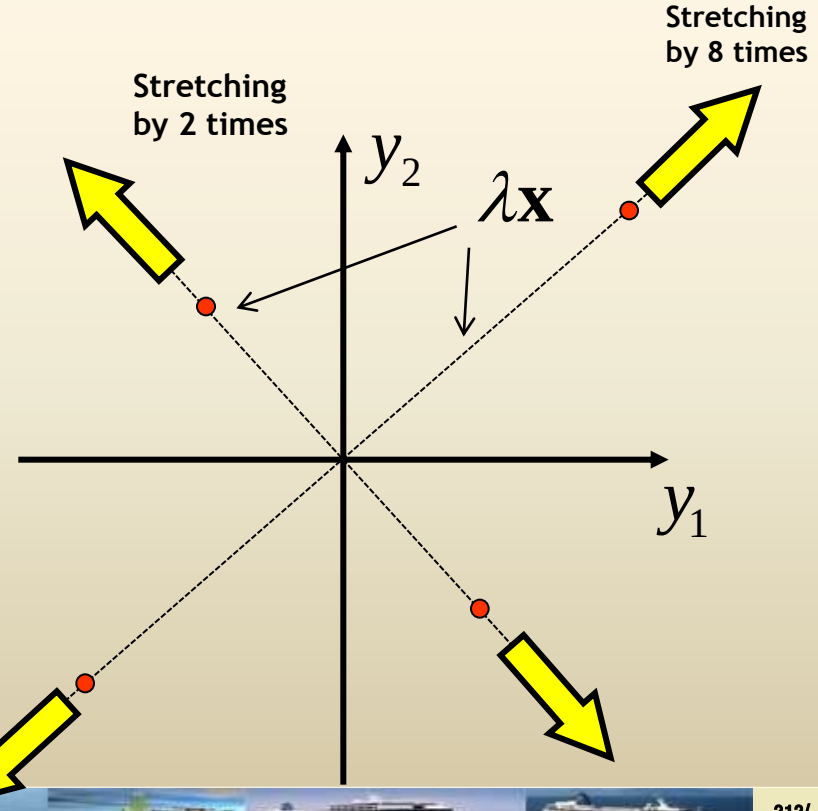
It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

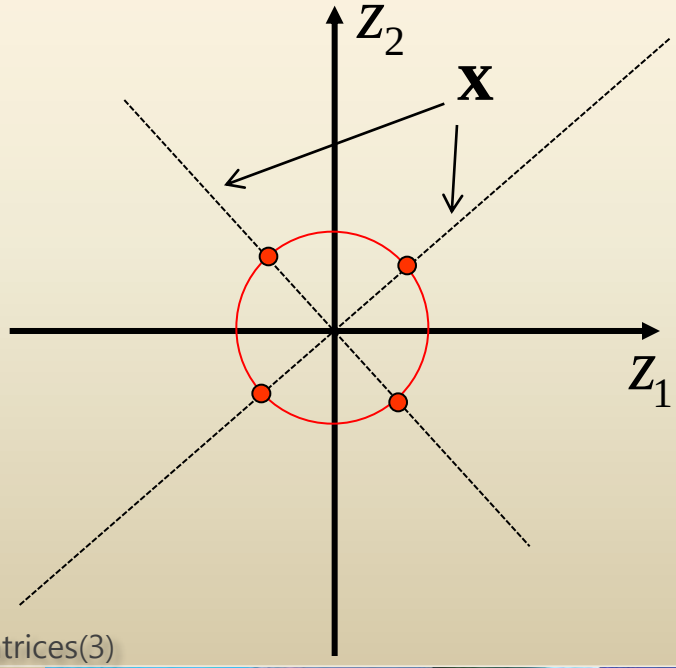
When $\mathbf{Ax}=\lambda\mathbf{x}$.

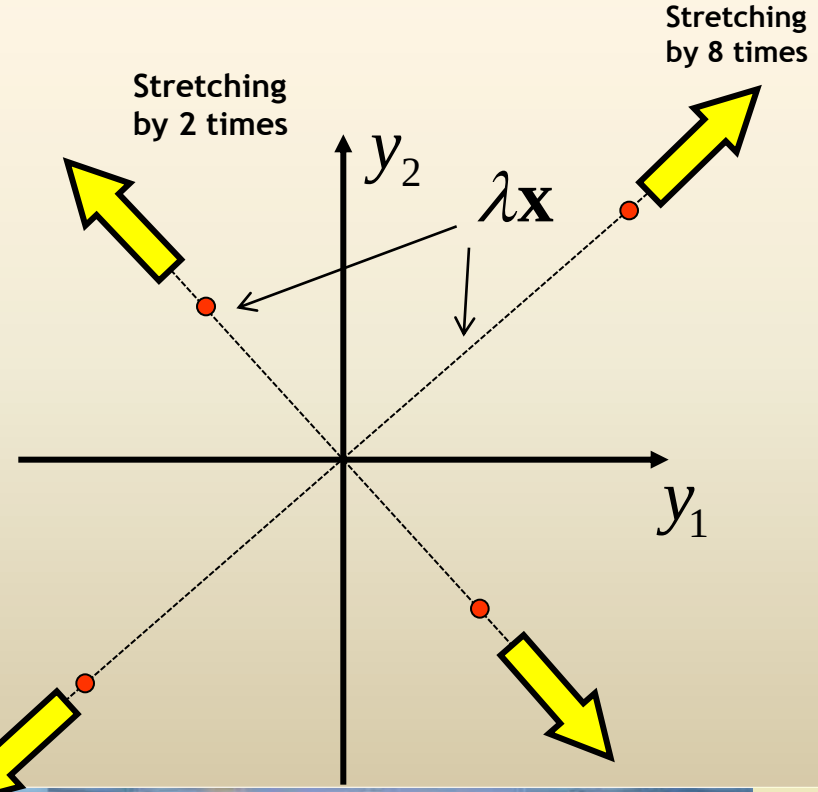
It means that the direction of x *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

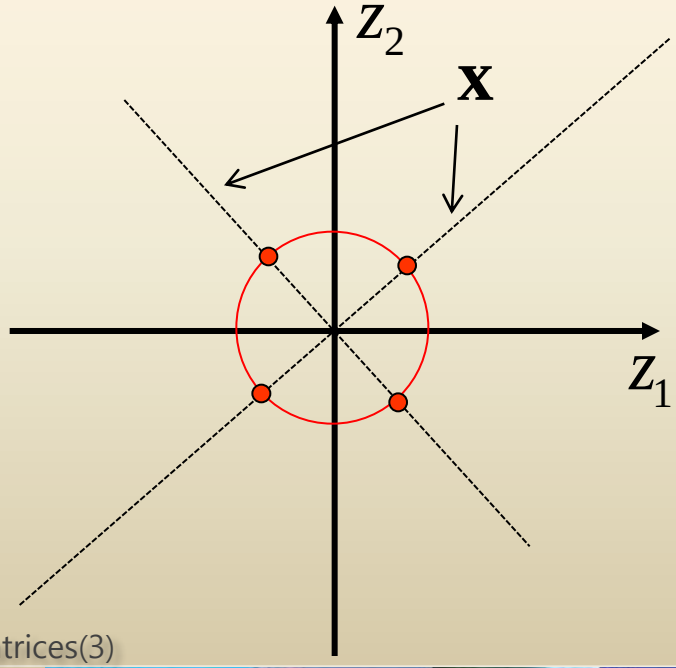
When $\mathbf{Ax}=\lambda\mathbf{x}$.

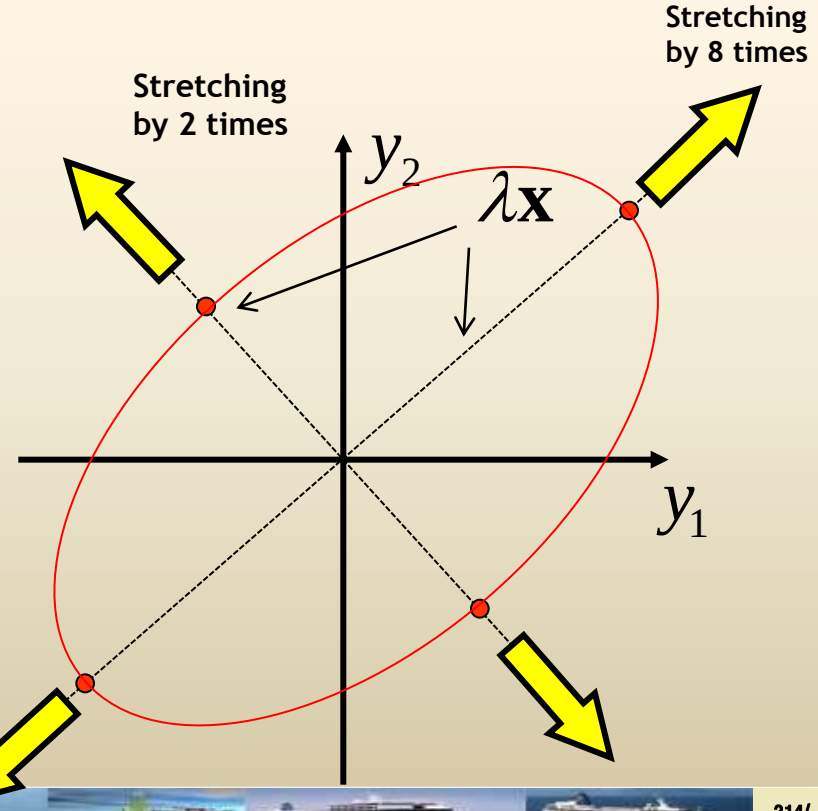
It means that the direction of x *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

So $\lambda_1 = 8,\ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$

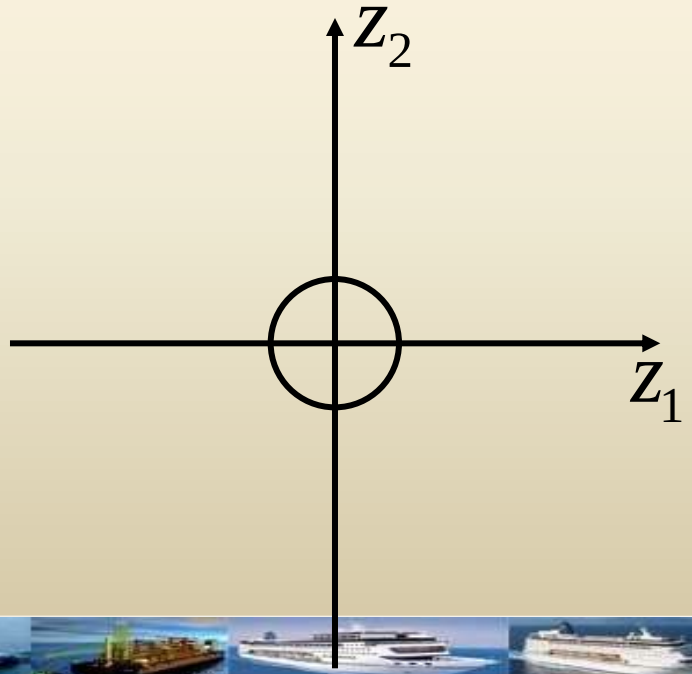
that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal

A real square matrix is orthogonal

if and only if column vectors $a_1, \ldots, a_n$ form an orthonormal system,

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j{}^{\mathrm{T}} \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

$$\mathbf{x}_1{}^{\mathrm{T}} \mathbf{x}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$



Stretching of an Elastic Membrane

So $\lambda_1 = 8,\ \mathbf{x}_1 = [1 \quad 1]^T$

➔ stretching by 8(= λ_1) *times* to the direction of $\mathbf{x}_1=[1\ \ 1]^T$.

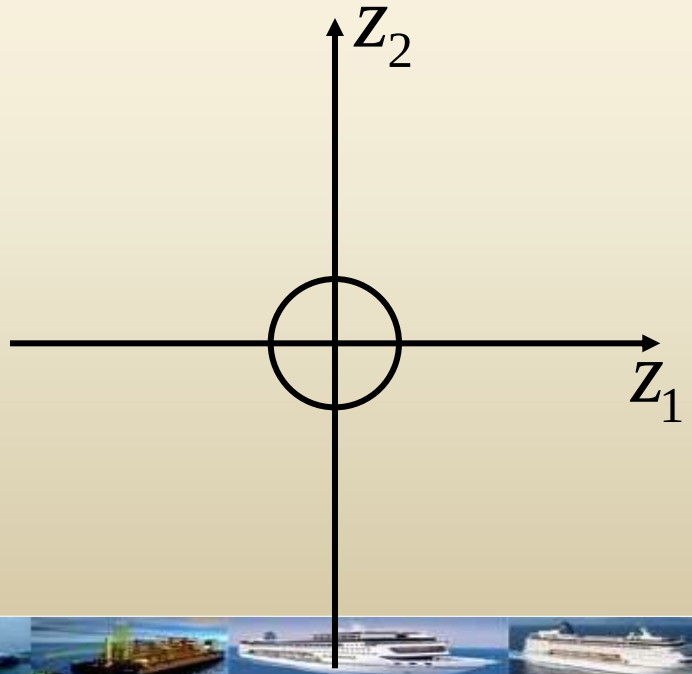
that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal

A real square matrix is orthogonal

if and only if column vectors $a_1, \ldots, a_n$ form an orthonormal system,

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j{}^T \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

$$\mathbf{x}_1{}^T \mathbf{x}_2 = [1 \quad 1] \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$



Stretching of an Elastic Membrane

So $\lambda_1 = 8,\ \mathbf{x}_1 = [1 \quad 1]^T$

➔ stretching by 8(= λ_1) *times* to the direction of $\mathbf{x}_1=[1\ \ 1]^T$.

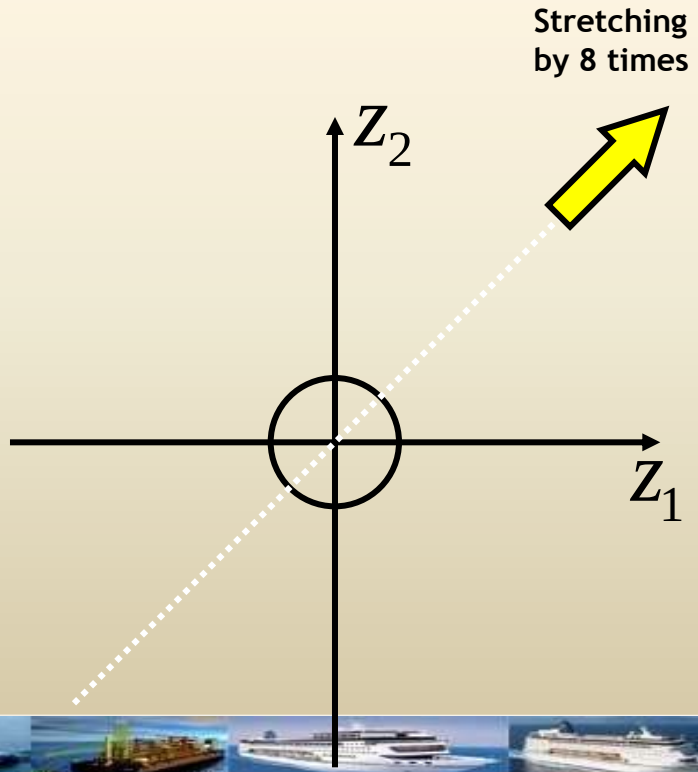
that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal

A real square matrix is orthogonal

if and only if column vectors $a_1, \ldots, a_n$ form an orthonormal system,

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j{}^T \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

$$\mathbf{x}_1{}^T \mathbf{x}_2 = [1 \quad 1] \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$



Stretching of an Elastic Membrane

So $\lambda_1 = 8,\ \mathbf{x}_1 = [1 \quad 1]^T$

➔ stretching by 8(= λ_1) *times* to the direction of $\mathbf{x}_1=[1\ \ 1]^T$.

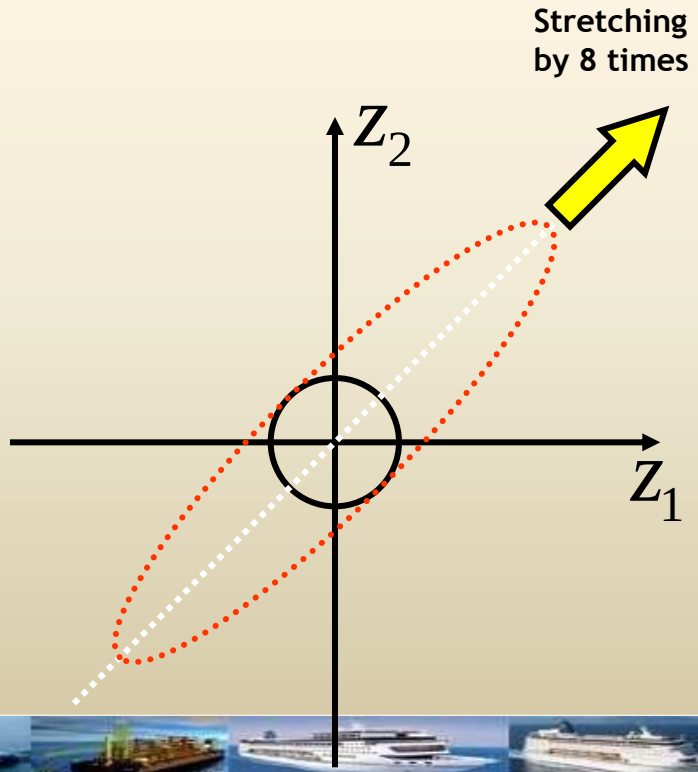
that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal

A real square matrix is orthogonal

if and only if column vectors $a_1, \ldots, a_n$ form an orthonormal system,

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j{}^T \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

$$\mathbf{x}_1{}^T \mathbf{x}_2 = [1 \quad 1] \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$



Stretching of an Elastic Membrane

So $\lambda_1 = 8,\ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$

➔ stretching by 8(= λ_1) *times* to the direction of $\mathbf{x}_1=[1\ \ 1]^T$.

$$\lambda_2 = 2,\ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$$

➔ stretching by 2(= λ_2) *times* to the direction of $\mathbf{x}_2=[1\ \ -1]^T$.

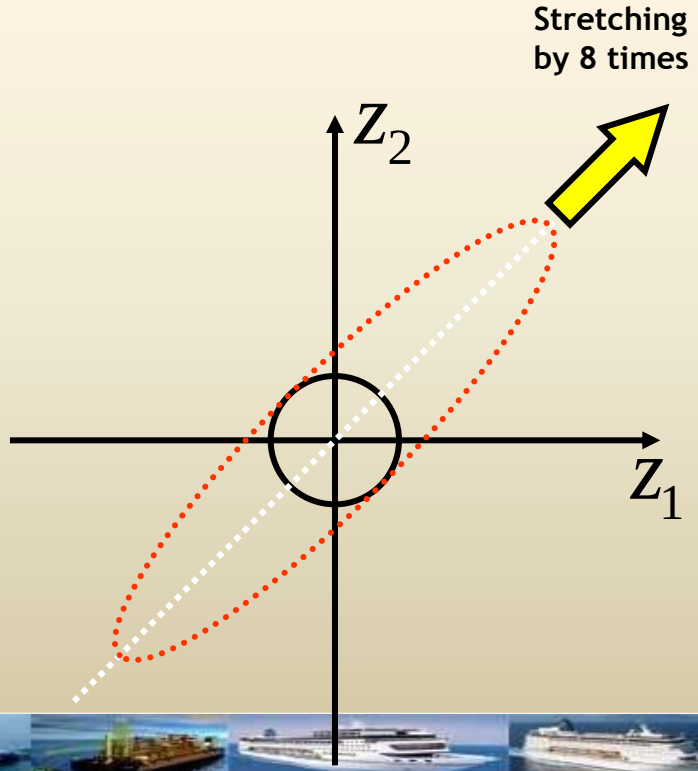
$$\mathbf{x}_1^T\mathbf{x}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal

A real square matrix is orthogonal

if and only if column vectors $a_1, \ldots, a_n$ form an orthonormal system,

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^T\mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$



Stretching of an Elastic Membrane

So $\lambda_1 = 8,\ \mathbf{x}_1 = [1 \quad 1]^T$

➔ stretching by 8(= λ_1) *times* to the direction of $\mathbf{x}_1=[1\ \ 1]^T$.

$$\lambda_2 = 2,\ \mathbf{x}_2 = [1 \quad -1]^T$$

➔ stretching by 2(= λ_2) *times* to the direction of $\mathbf{x}_2=[1\ \ -1]^T$.

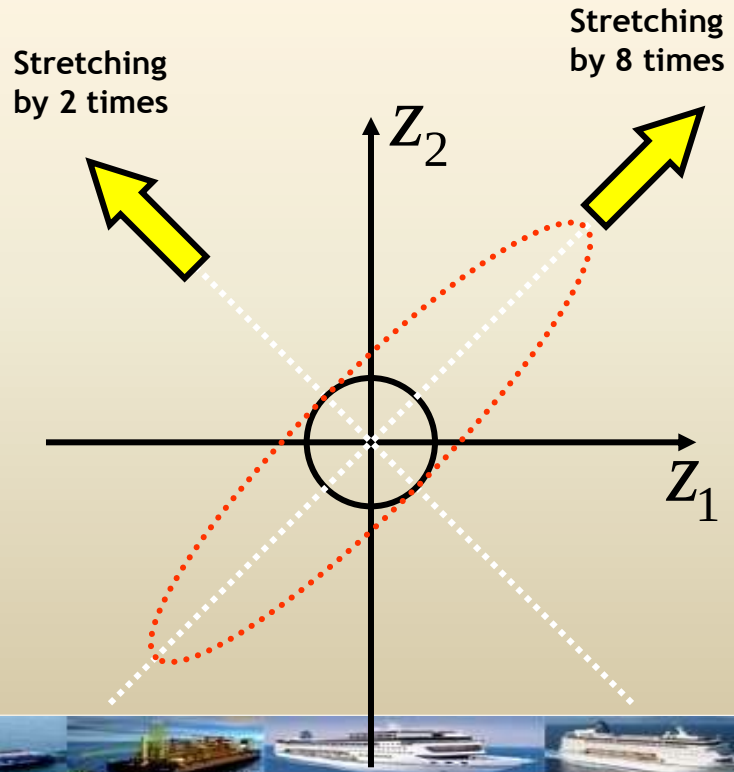
$$\mathbf{x}_1^T\mathbf{x}_2 = [1 \quad 1]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal

A real square matrix is orthogonal

if and only if column vectors $a_1, \ldots, a_n$ form an orthonormal system,

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^T\mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$



Stretching of an Elastic Membrane

So $\lambda_1 = 8,\ \mathbf{x}_1 = [1 \quad 1]^T$

➔ stretching by 8(= λ_1) *times* to the direction of $\mathbf{x}_1=[1\ \ 1]^T$.

$$\lambda_2 = 2,\ \mathbf{x}_2 = [1 \quad -1]^T$$

➔ stretching by 2(= λ_2) *times* to the direction of $\mathbf{x}_2=[1\ \ -1]^T$.

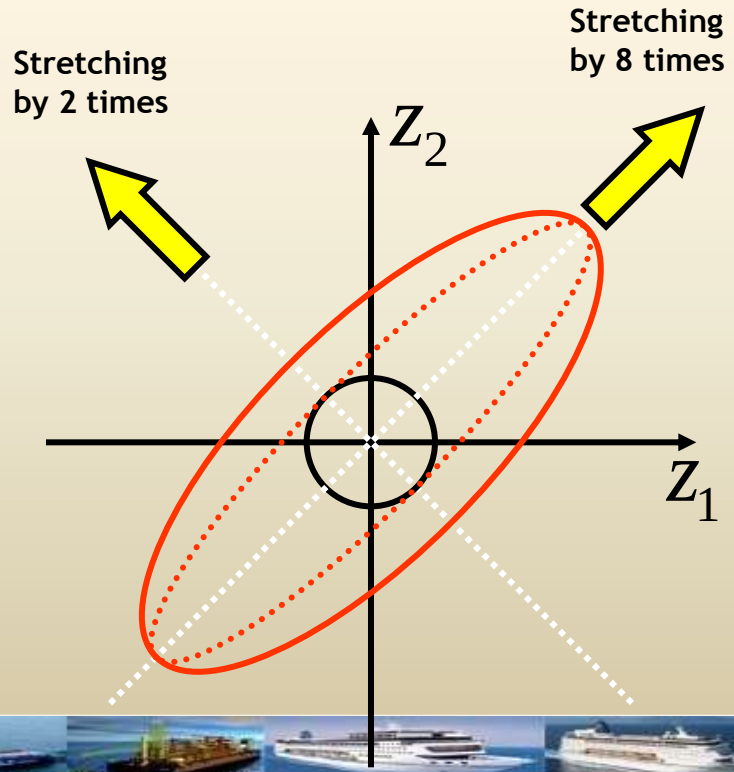
$$\mathbf{x}_1^T\mathbf{x}_2 = [1 \quad 1]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal

A real square matrix is orthogonal

if and only if column vectors $a_1, \ldots, a_n$ form an orthonormal system,

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^T\mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$



Stretching of an Elastic Membrane

So $\lambda_1 = 8, \mathbf{x}_1 = [1 \quad 1]^T$

➔ stretching by 8(= λ_1) *times* to the direction of $\mathbf{x}_1=[1 \ \ 1]^T$.

$\lambda_2 = 2, \mathbf{x}_2 = [1 \quad -1]^T$

➔ stretching by 2(= λ_2) *times* to the direction of $\mathbf{x}_2=[1 \ \ -1]^T$.

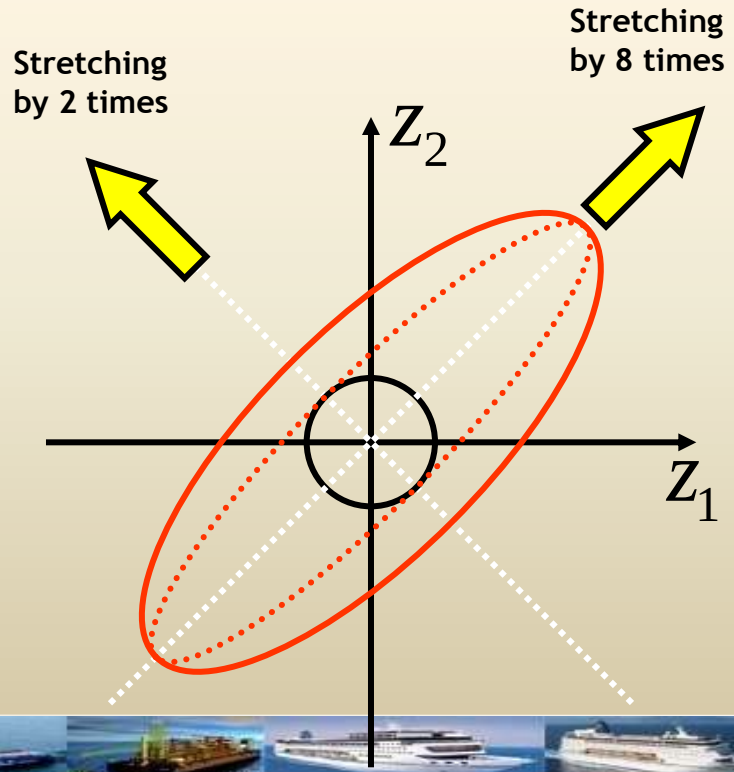
$$\mathbf{x}_1^T\mathbf{x}_2 = [1 \quad 1]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal

A real square matrix is orthogonal

if and only if column vectors $a_1, \ldots, a_n$ form an orthonormal system,

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^T\mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$



Stretching of an Elastic Membrane

So $\lambda_1 = 8, \mathbf{x}_1 = [1 \quad 1]^T$

➔ stretching by 8(= λ_1) *times* to the direction of $\mathbf{x}_1=[1 \ \ 1]^T$.

$\lambda_2 = 2, \mathbf{x}_2 = [1 \quad -1]^T$

➔ stretching by 2(= λ_2) *times* to the direction of $\mathbf{x}_2=[1 \ \ -1]^T$.

This is *principal directions*. This vector make 45° and 135° angles with the positive z_1-direction.

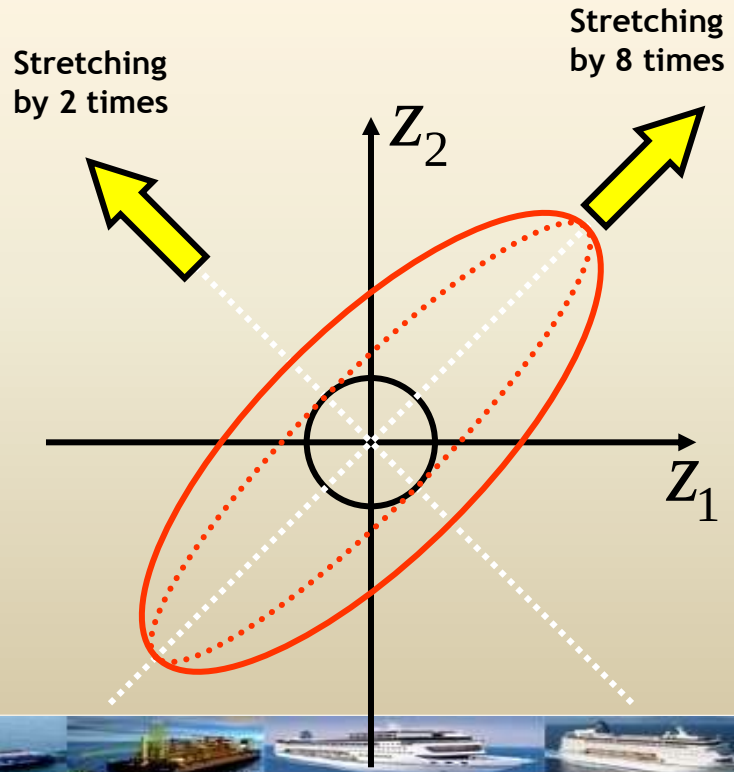
$$\mathbf{x}_1^T\mathbf{x}_2 = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 0$$

that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal

A real square matrix is orthogonal

if and only if column vectors $a_1, \ldots, a_n$ form an orthonormal system,

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^T\mathbf{a}_k = \begin{cases}0 & if \quad j \neq k\\ 1 & if \quad j = k\end{cases}$$



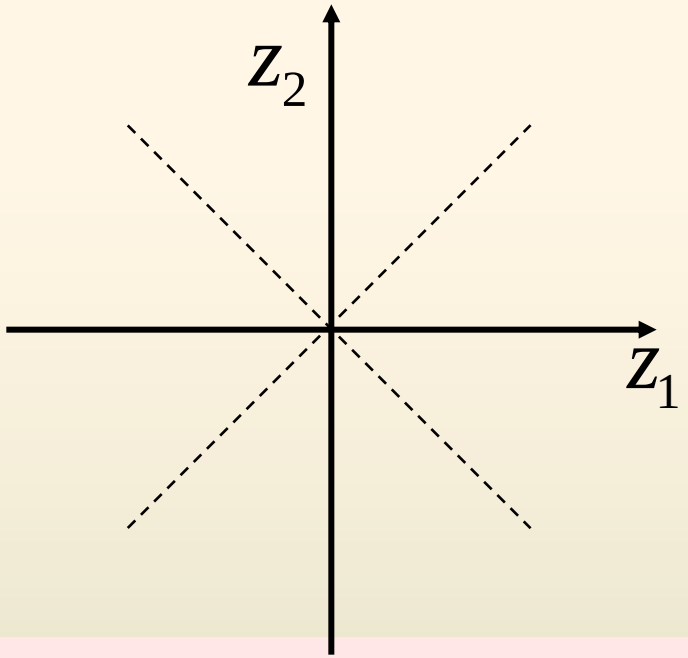
Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda\mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$



Denoting the corresponding eigenvalues of the matrix A by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{Ax}_j = \lambda_j \mathbf{x}_j$, so that we simply obtain

$$\begin{aligned} \mathbf{y} = \mathbf{Az} &= \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right) \\ &= c_1\mathbf{Ax}_1 + c_2\mathbf{Ax}_2 + \cdots + c_n\mathbf{Ax}_n \\ &= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n \end{aligned}$$

Stretching of an Elastic Membrane

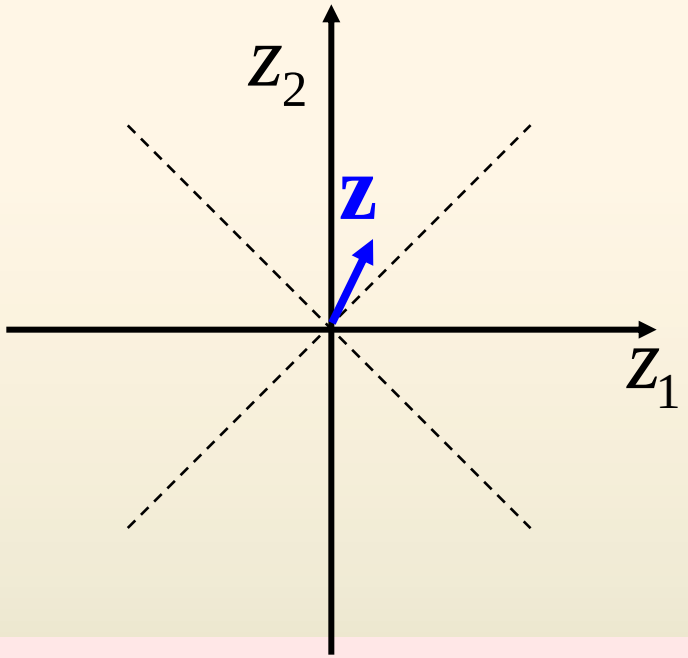
$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$



Denoting the corresponding eigenvalues of the matrix A by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{Ax}_j = \lambda_j \mathbf{x}_j$, so that we simply obtain

$$\begin{aligned} \mathbf{y} = \mathbf{Az} &= \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right) \\ &= c_1\mathbf{Ax}_1 + c_2\mathbf{Ax}_2 + \cdots + c_n\mathbf{Ax}_n \\ &= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n \end{aligned}$$

Stretching of an Elastic Membrane

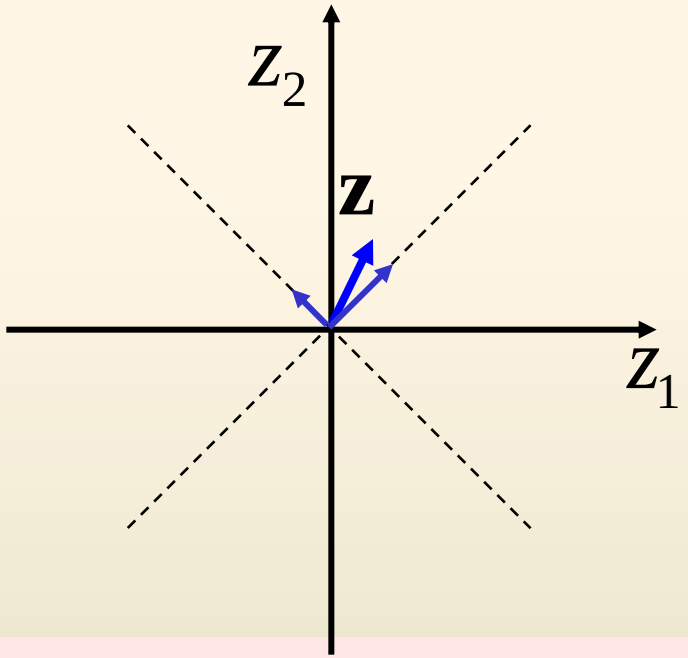
$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$



Denoting the corresponding eigenvalues of the matrix A by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{Ax}_j = \lambda_j \mathbf{x}_j$, so that we simply obtain

$$\begin{aligned} \mathbf{y} = \mathbf{Az} &= \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right) \\ &= c_1\mathbf{Ax}_1 + c_2\mathbf{Ax}_2 + \cdots + c_n\mathbf{Ax}_n \\ &= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n \end{aligned}$$

Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

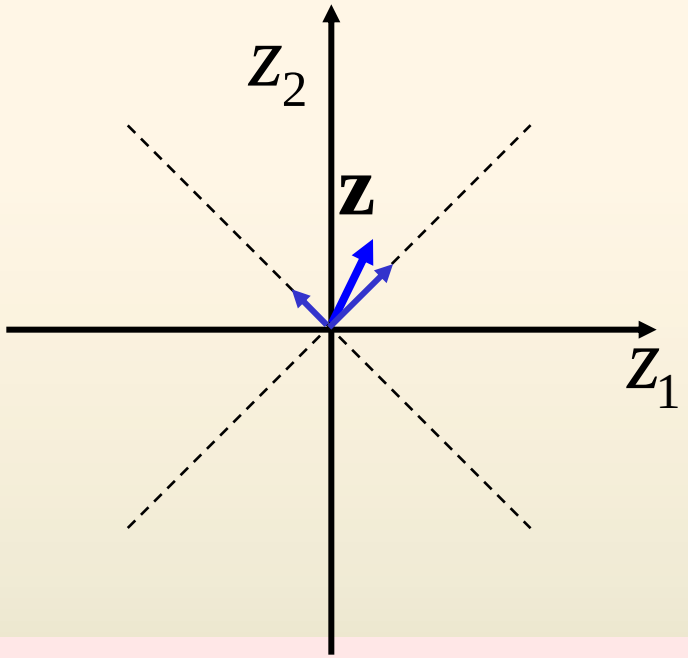
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\mathbf{Az} = \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)$$



Denoting the corresponding eigenvalues of the matrix A by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{Ax}_j = \lambda_j \mathbf{x}_j$, so that we simply obtain

$$\begin{aligned} \mathbf{y} = \mathbf{Az} &= \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right) \\ &= c_1\mathbf{Ax}_1 + c_2\mathbf{Ax}_2 + \cdots + c_n\mathbf{Ax}_n \\ &= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n \end{aligned}$$

Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

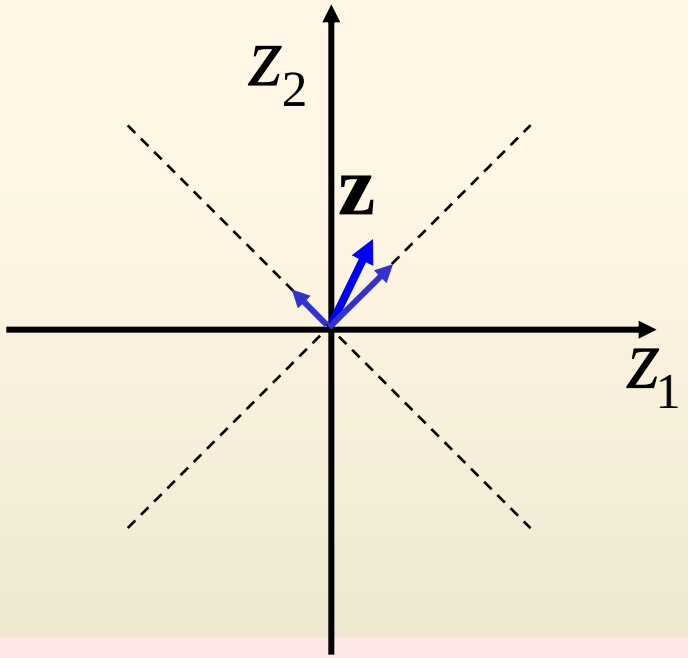
$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\mathbf{Az} = \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)$$

$$= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2$$



Denoting the corresponding eigenvalues of the matrix A by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{Ax}_j = \lambda_j \mathbf{x}_j$, so that we simply obtain

$$\mathbf{y} = \mathbf{Az} = \mathbf{A}\left(c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \cdots + c_n \mathbf{x}_n\right)$$

$$= c_1 \mathbf{Ax}_1 + c_2 \mathbf{Ax}_2 + \cdots + c_n \mathbf{Ax}_n$$

$$= c_1 \lambda_1 \mathbf{x}_1 + c_2 \lambda_2 \mathbf{x}_2 + \cdots + c_n \lambda_n \mathbf{x}_n$$

Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

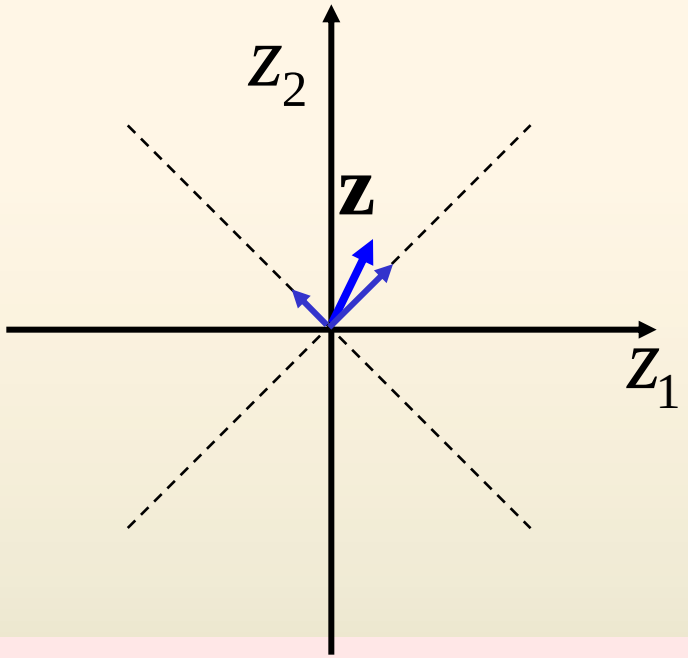
$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\mathbf{Az} = \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)$$

$$= \alpha \mathbf{A}\mathbf{x}_1 + \beta \mathbf{A}\mathbf{x}_2$$

$$= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2$$



Denoting the corresponding eigenvalues of the matrix A by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{Ax}_j = \lambda_j \mathbf{x}_j$, so that we simply obtain

$$\mathbf{y} = \mathbf{Az} = \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right)$$

$$= c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \cdots + c_n\mathbf{A}\mathbf{x}_n$$

$$= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n$$

Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

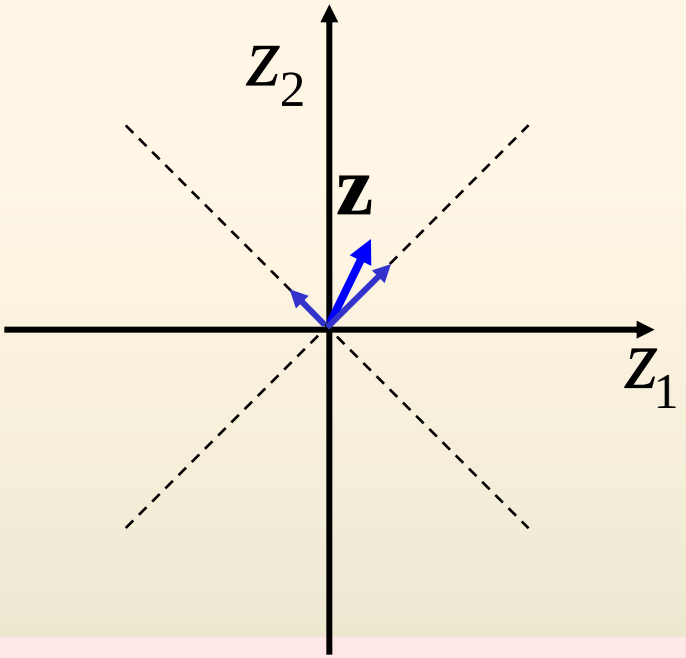
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A}\mathbf{x}_1 + \beta \mathbf{A}\mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$



Denoting the corresponding eigenvalues of the matrix A by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{Ax}_j = \lambda_j \mathbf{x}_j$, so that we simply obtain

$$\begin{aligned} \mathbf{y} = \mathbf{Az} &= \mathbf{A}\left(c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \cdots + c_n \mathbf{x}_n\right) \\ &= c_1 \mathbf{A}\mathbf{x}_1 + c_2 \mathbf{A}\mathbf{x}_2 + \cdots + c_n \mathbf{A}\mathbf{x}_n \\ &= c_1 \lambda_1 \mathbf{x}_1 + c_2 \lambda_2 \mathbf{x}_2 + \cdots + c_n \lambda_n \mathbf{x}_n \end{aligned}$$

Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

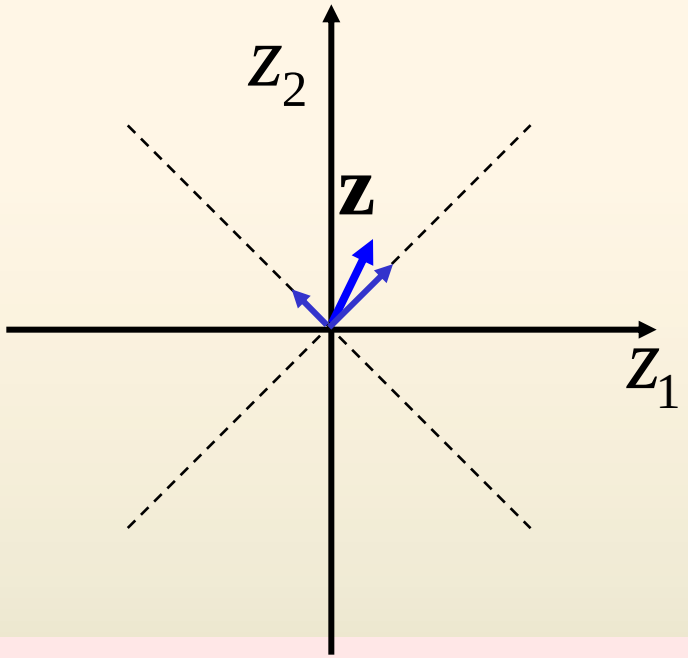
$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A}\mathbf{x}_1 + \beta \mathbf{A}\mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Denoting the corresponding eigenvalues of the matrix A by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{Ax}_j = \lambda_j \mathbf{x}_j$, so that we simply obtain

$$\begin{aligned} \mathbf{y} = \mathbf{Az} &= \mathbf{A}\left(c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \cdots + c_n \mathbf{x}_n\right) \\ &= c_1 \mathbf{A}\mathbf{x}_1 + c_2 \mathbf{A}\mathbf{x}_2 + \cdots + c_n \mathbf{A}\mathbf{x}_n \\ &= c_1 \lambda_1 \mathbf{x}_1 + c_2 \lambda_2 \mathbf{x}_2 + \cdots + c_n \lambda_n \mathbf{x}_n \end{aligned}$$

Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

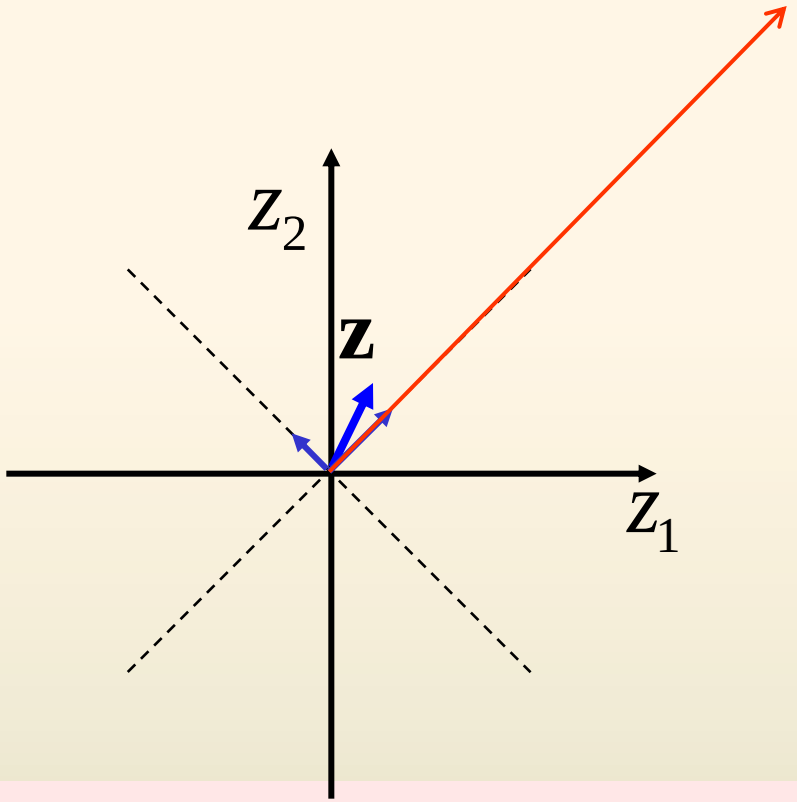
$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A}\mathbf{x}_1 + \beta \mathbf{A}\mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Denoting the corresponding eigenvalues of the matrix A by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{Ax}_j = \lambda_j \mathbf{x}_j$, so that we simply obtain

$$\begin{aligned} \mathbf{y} = \mathbf{Az} &= \mathbf{A}\left(c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \cdots + c_n \mathbf{x}_n\right) \\ &= c_1 \mathbf{A}\mathbf{x}_1 + c_2 \mathbf{A}\mathbf{x}_2 + \cdots + c_n \mathbf{A}\mathbf{x}_n \\ &= c_1 \lambda_1 \mathbf{x}_1 + c_2 \lambda_2 \mathbf{x}_2 + \cdots + c_n \lambda_n \mathbf{x}_n \end{aligned}$$

Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

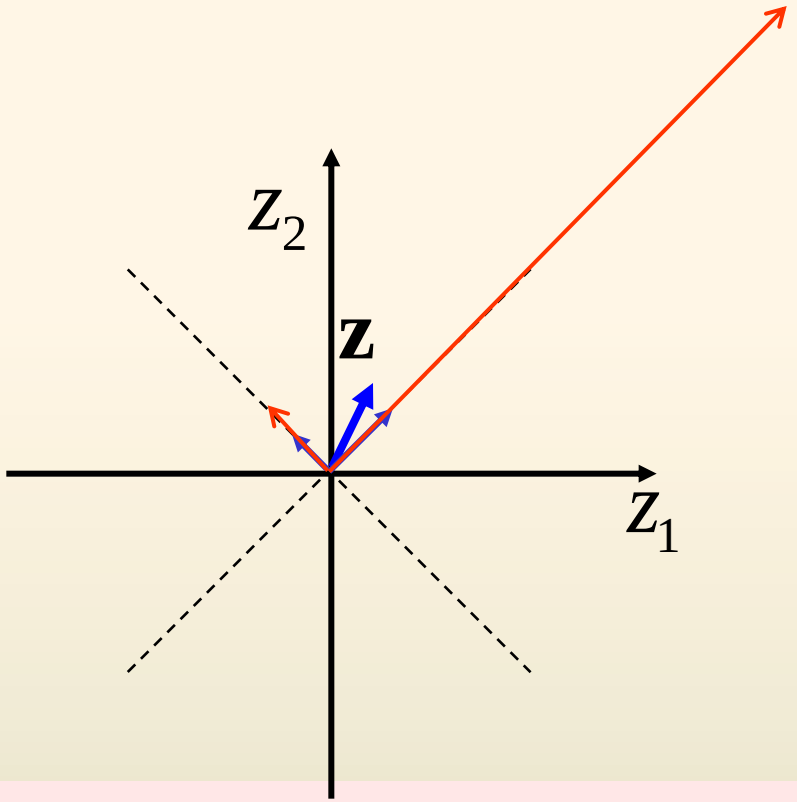
$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A}\mathbf{x}_1 + \beta \mathbf{A}\mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Denoting the corresponding eigenvalues of the matrix A by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{Ax}_j = \lambda_j \mathbf{x}_j$, so that we simply obtain

$$\begin{aligned} \mathbf{y} = \mathbf{Az} &= \mathbf{A}\left(c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \cdots + c_n \mathbf{x}_n\right) \\ &= c_1 \mathbf{A}\mathbf{x}_1 + c_2 \mathbf{A}\mathbf{x}_2 + \cdots + c_n \mathbf{A}\mathbf{x}_n \\ &= c_1 \lambda_1 \mathbf{x}_1 + c_2 \lambda_2 \mathbf{x}_2 + \cdots + c_n \lambda_n \mathbf{x}_n \end{aligned}$$

Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

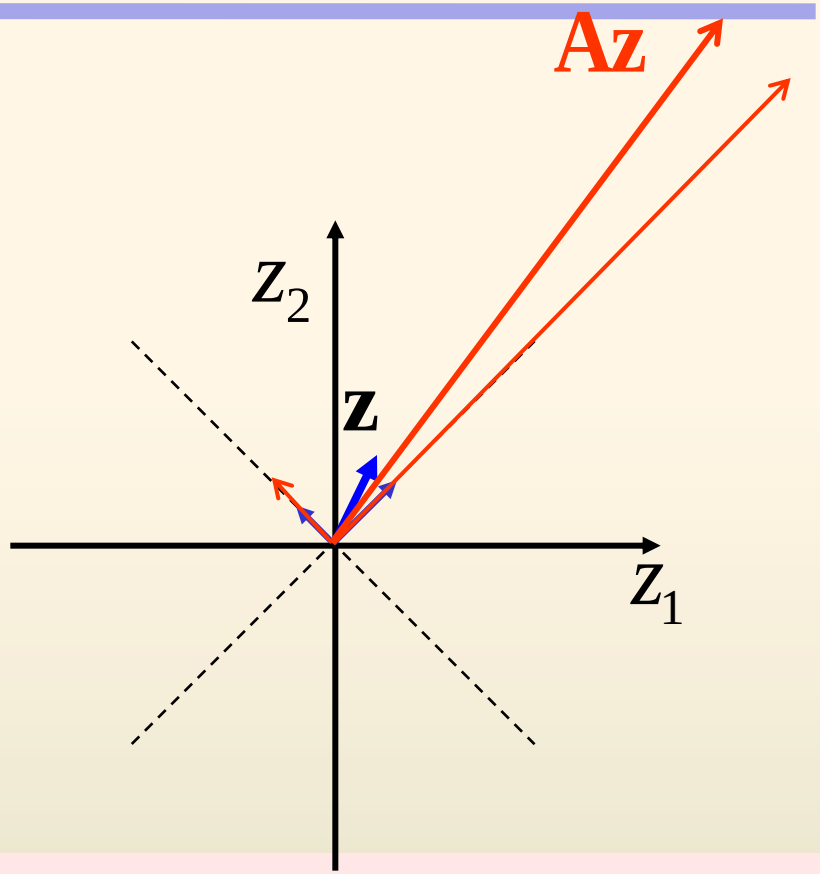
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Denoting the corresponding eigenvalues of the matrix A by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{Ax}_j = \lambda_j \mathbf{x}_j$, so that we simply obtain

$$\begin{aligned} \mathbf{y} = \mathbf{Az} &= \mathbf{A}\left(c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \cdots + c_n \mathbf{x}_n\right) \\ &= c_1 \mathbf{A} \mathbf{x}_1 + c_2 \mathbf{A} \mathbf{x}_2 + \cdots + c_n \mathbf{A} \mathbf{x}_n \\ &= c_1 \lambda_1 \mathbf{x}_1 + c_2 \lambda_2 \mathbf{x}_2 + \cdots + c_n \lambda_n \mathbf{x}_n \end{aligned}$$

Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

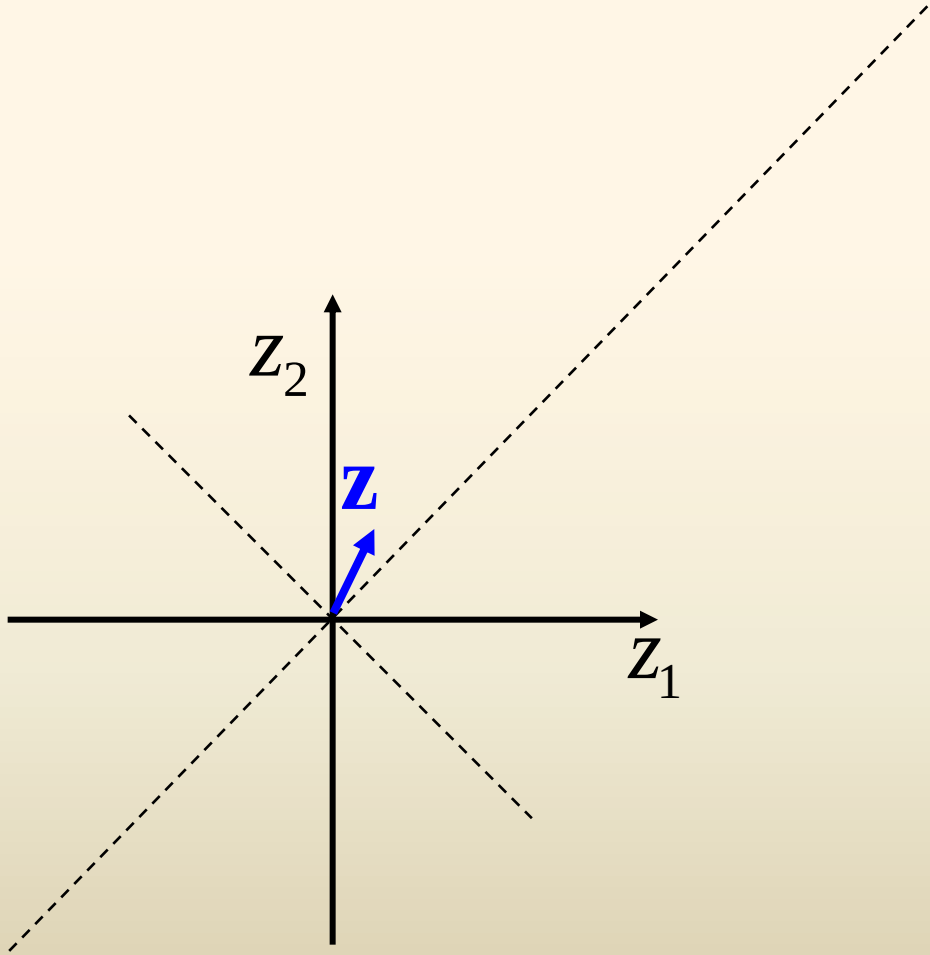
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

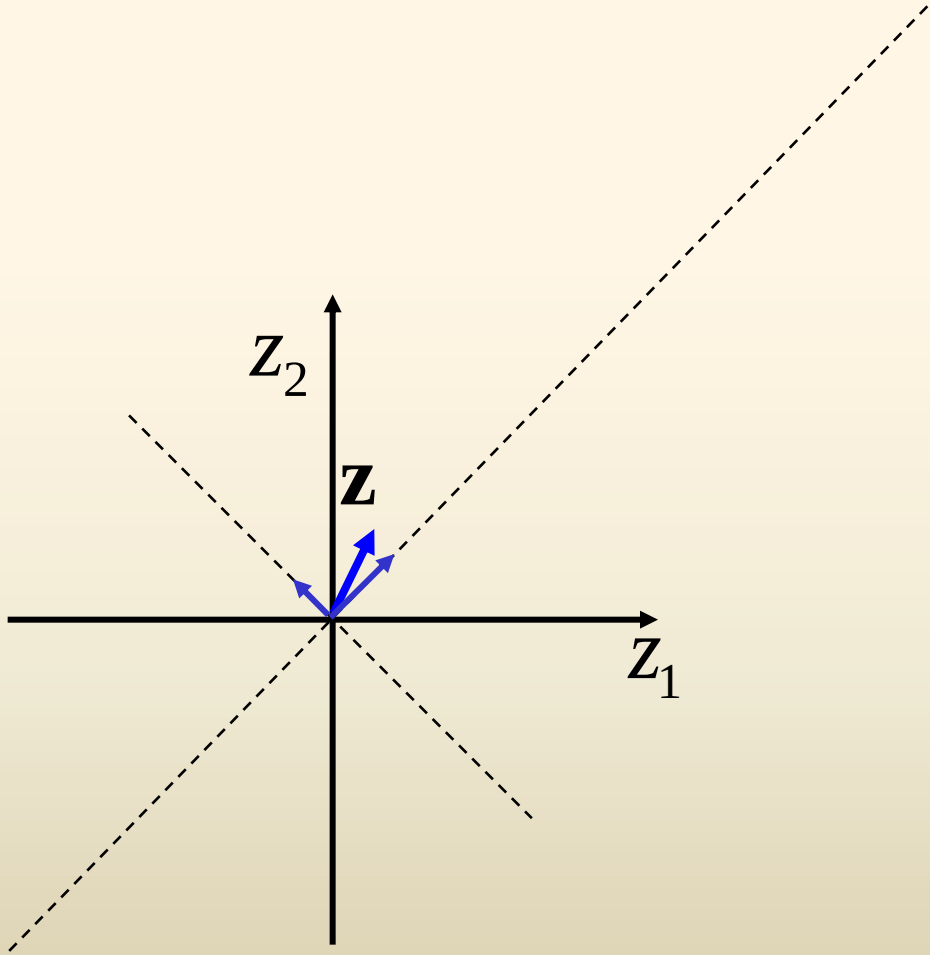
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

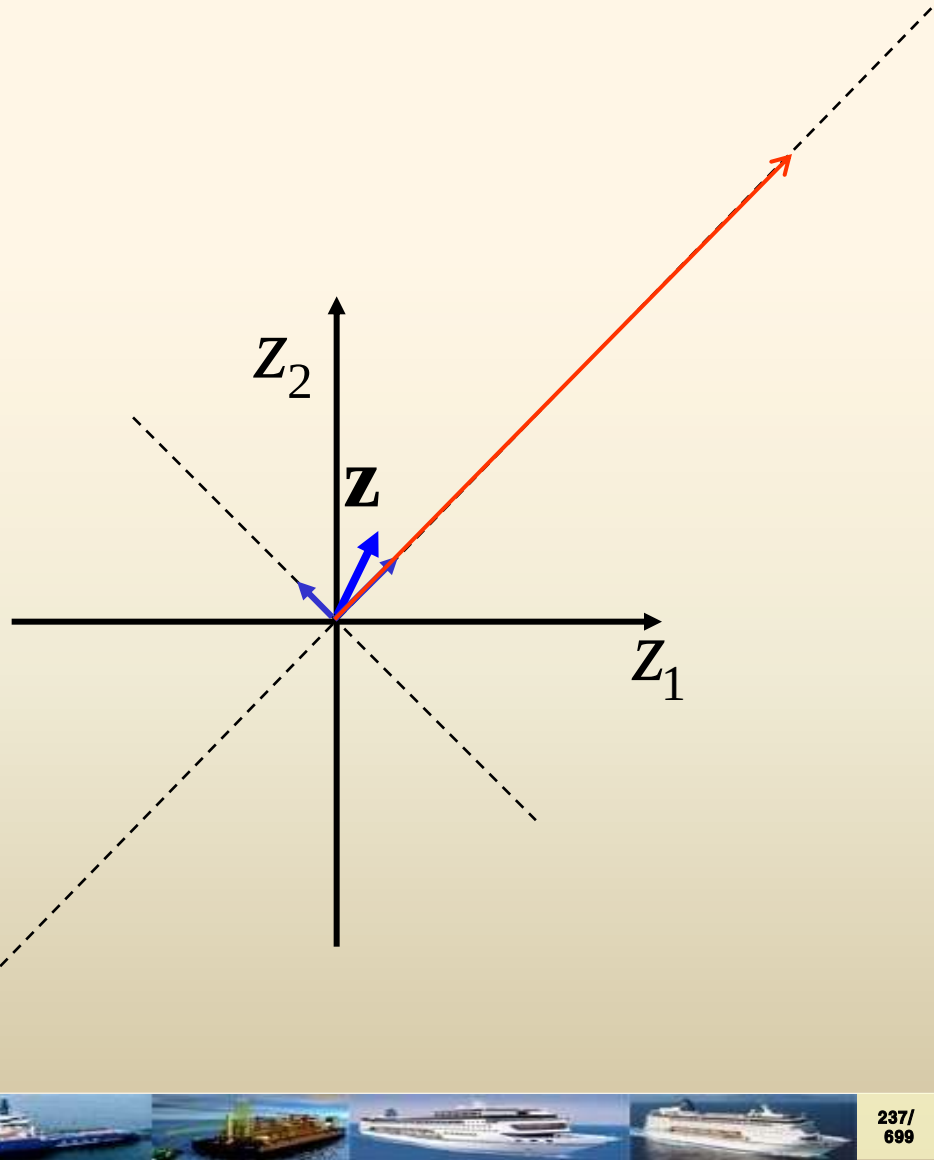
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A}\mathbf{x}_1 + \beta \mathbf{A}\mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

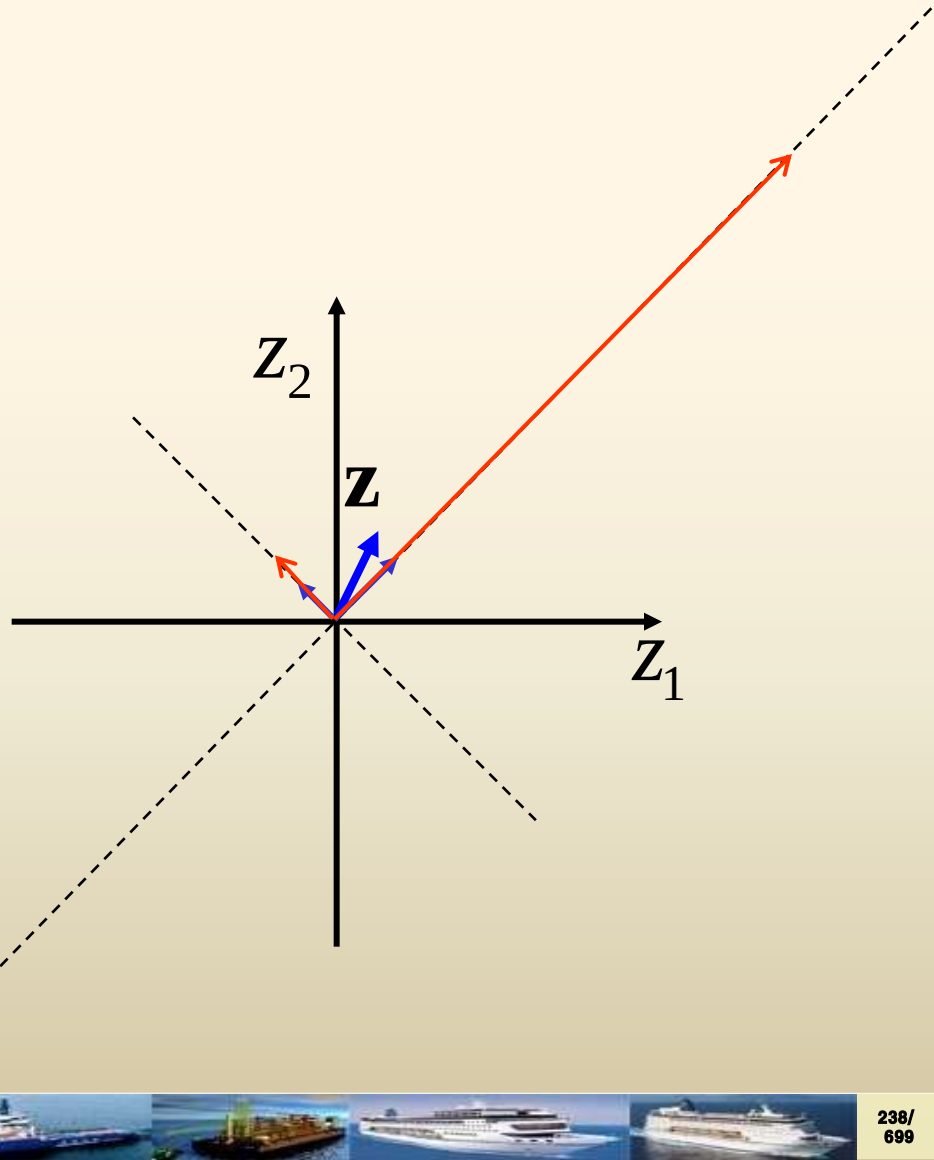
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A}\mathbf{x}_1 + \beta \mathbf{A}\mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

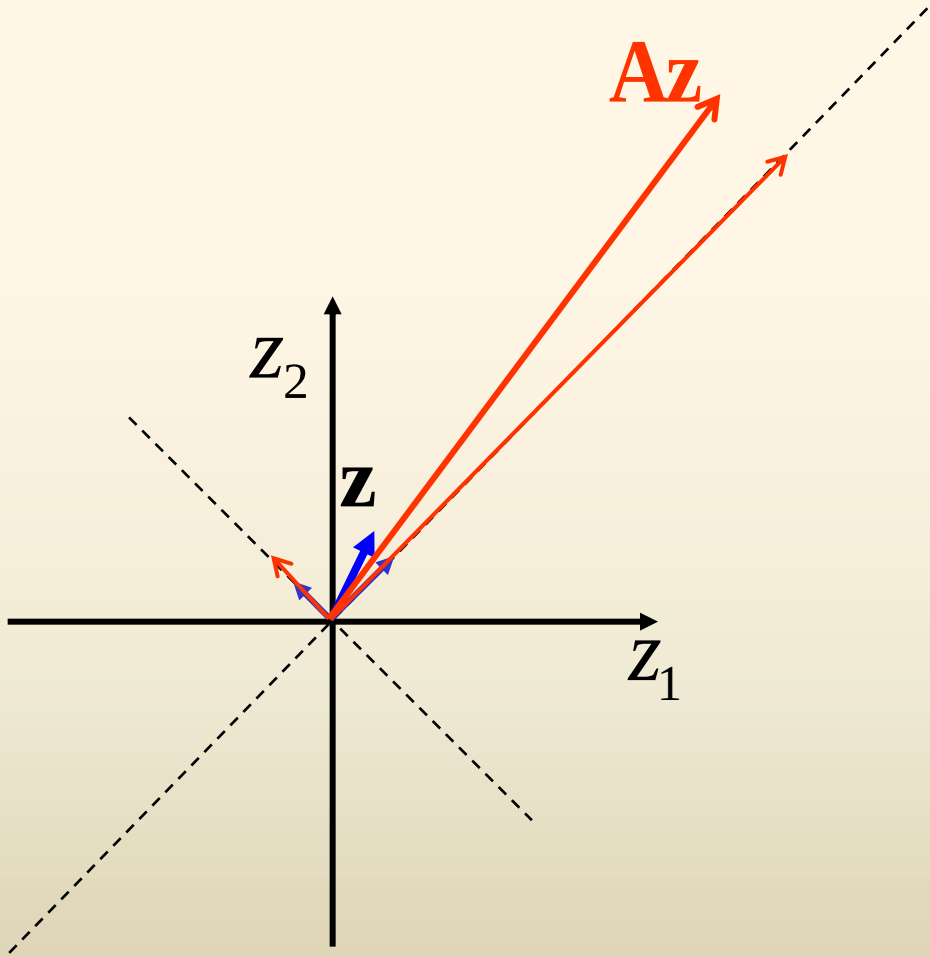
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

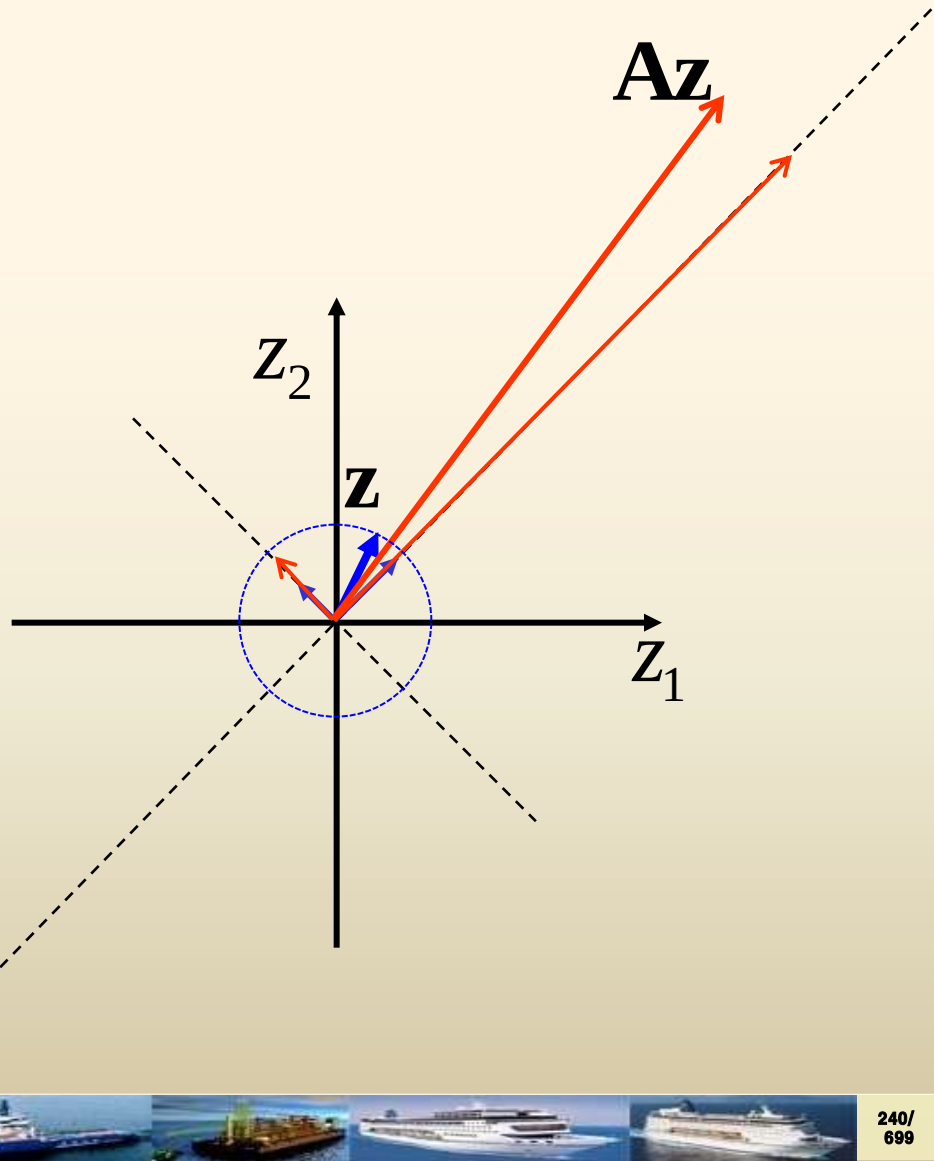
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

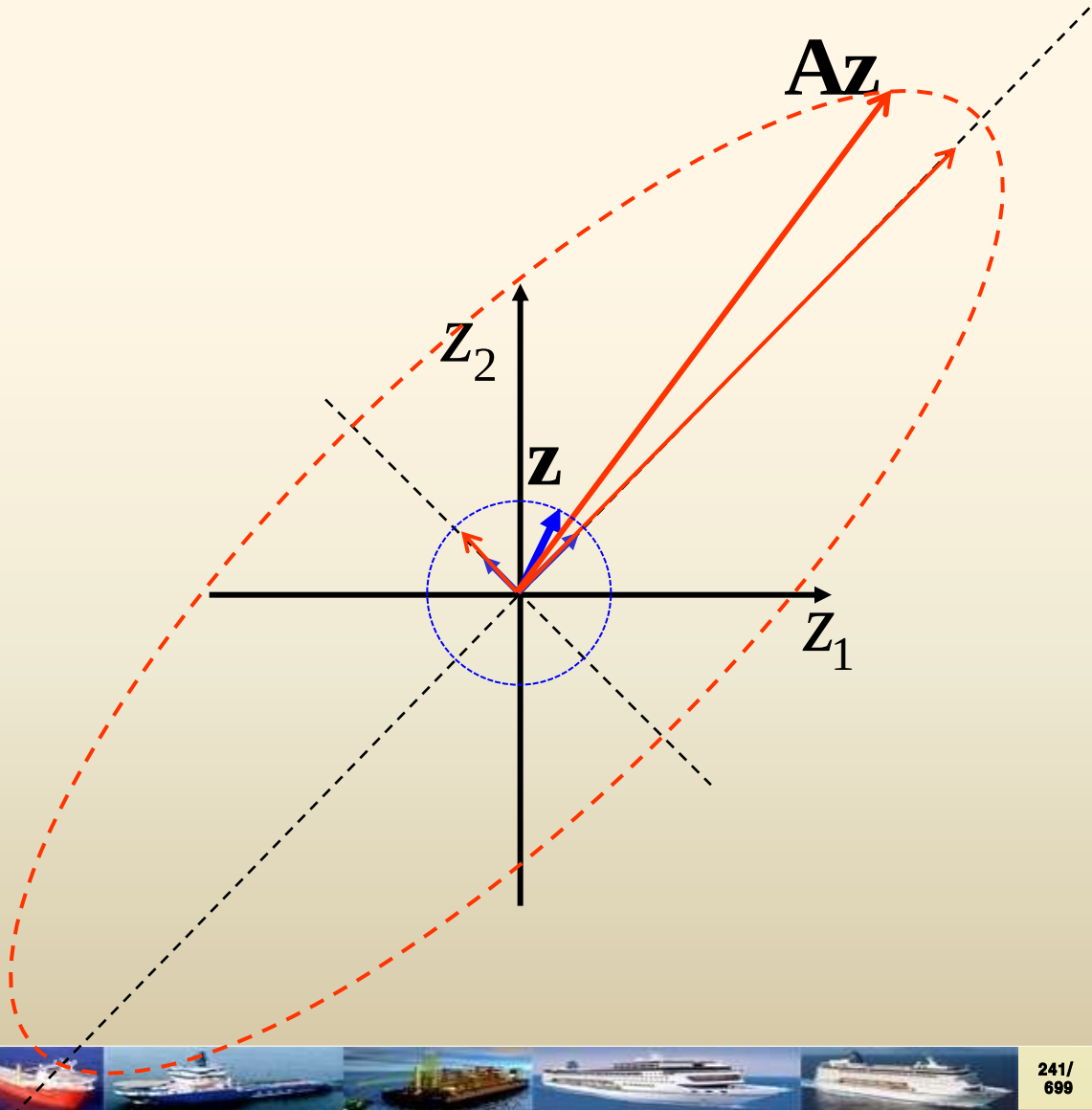
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

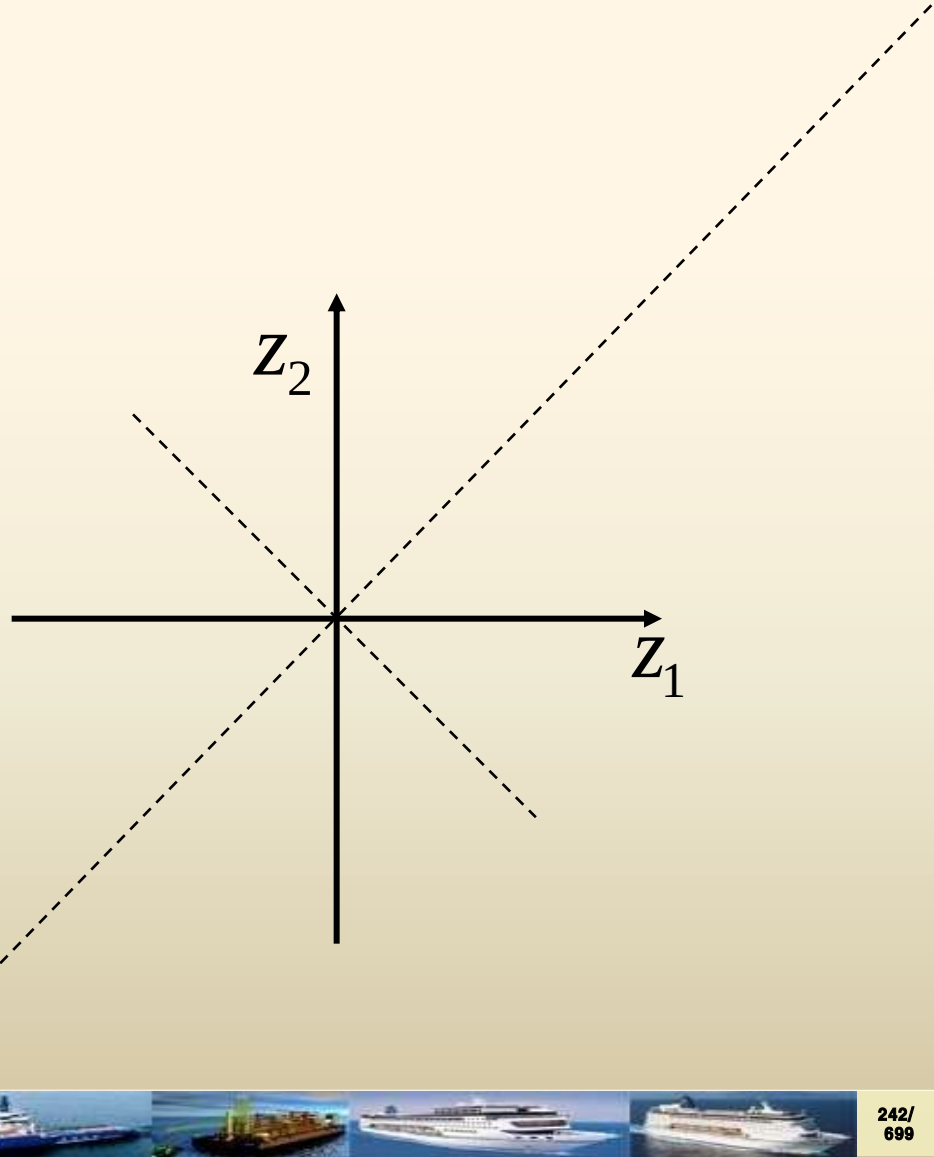
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

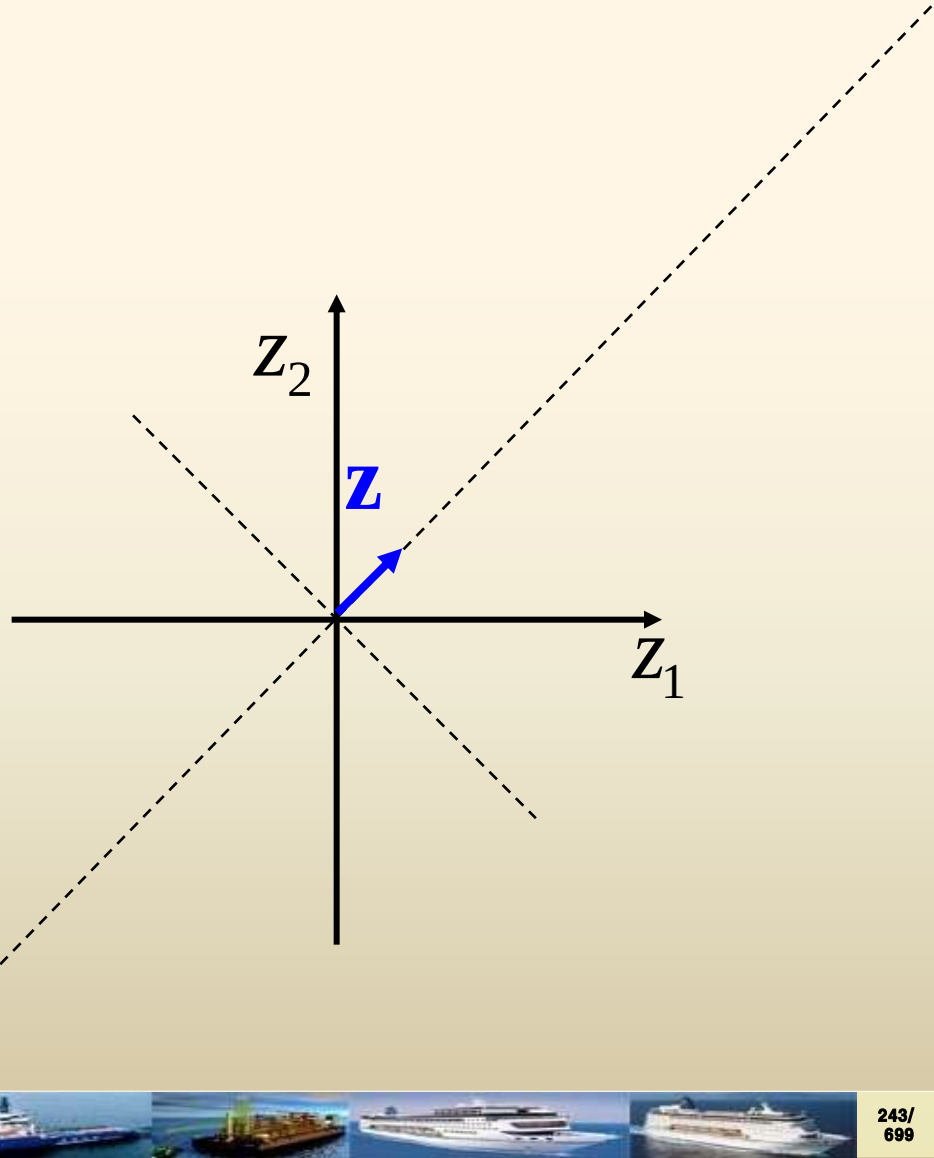
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

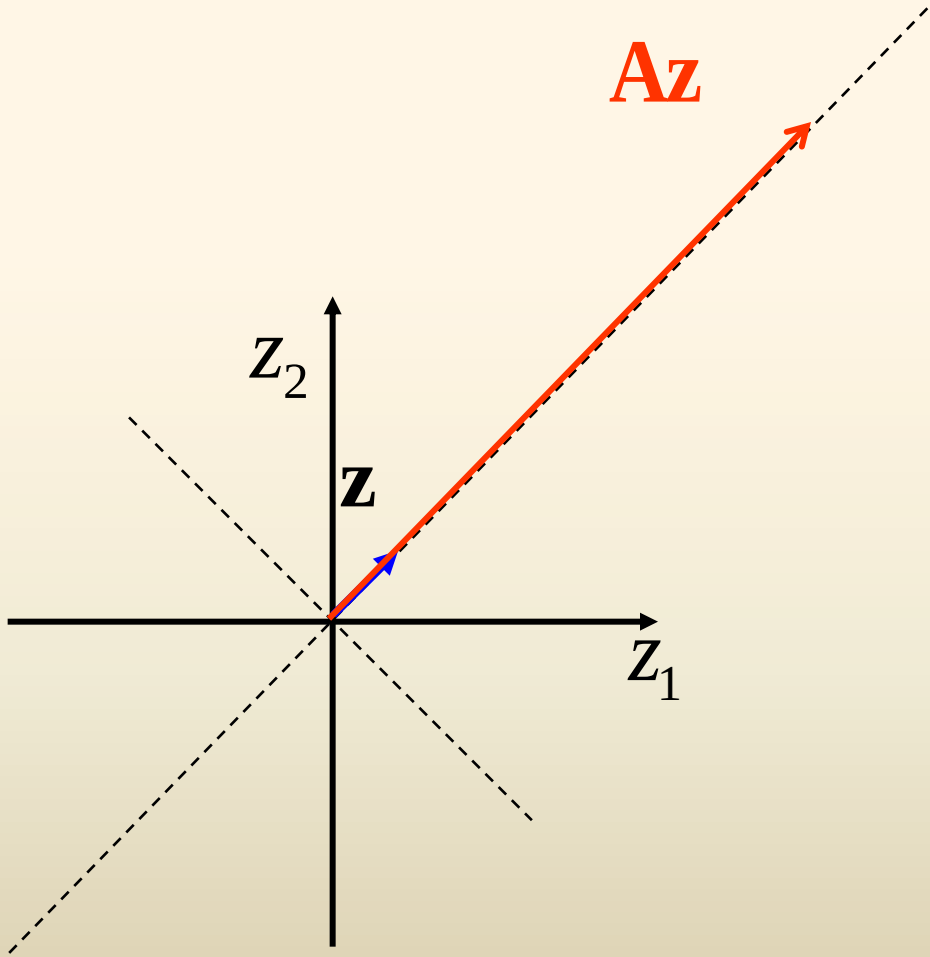
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

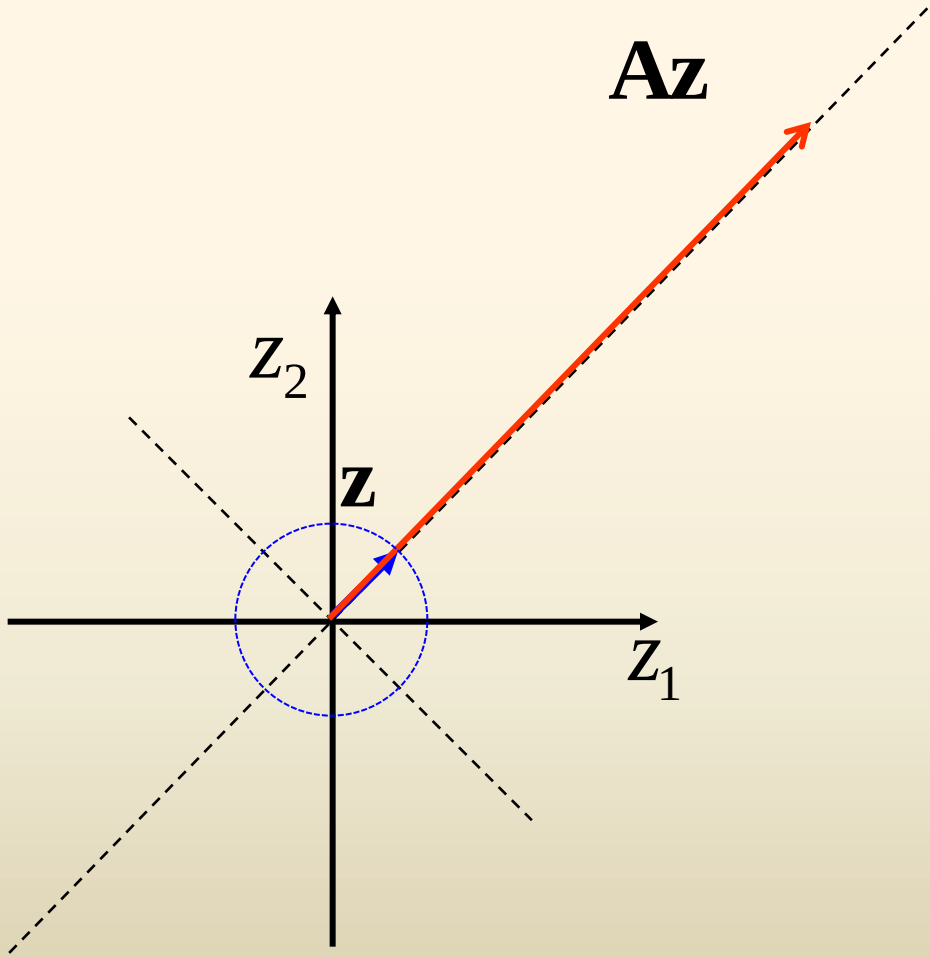
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

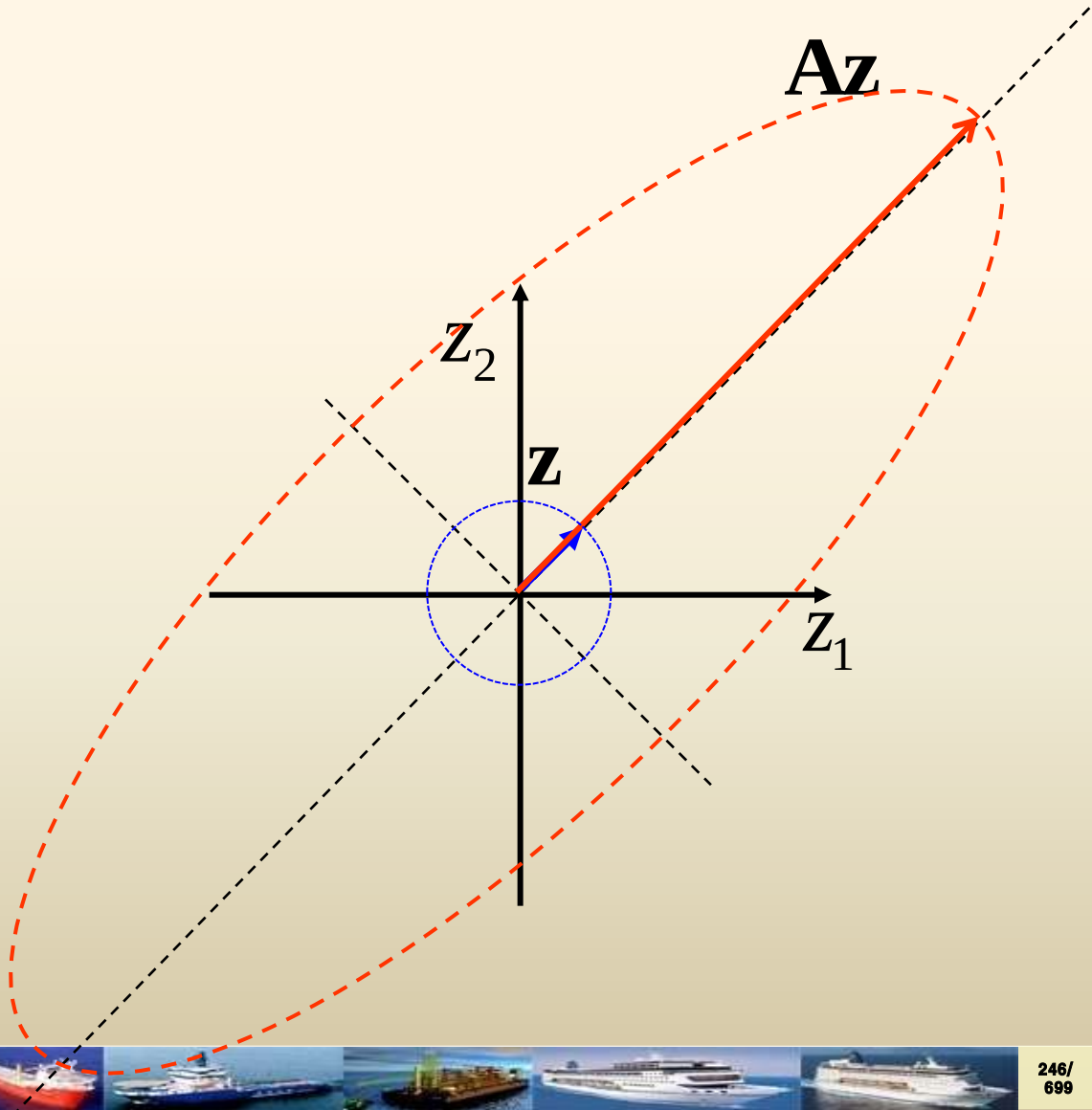
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

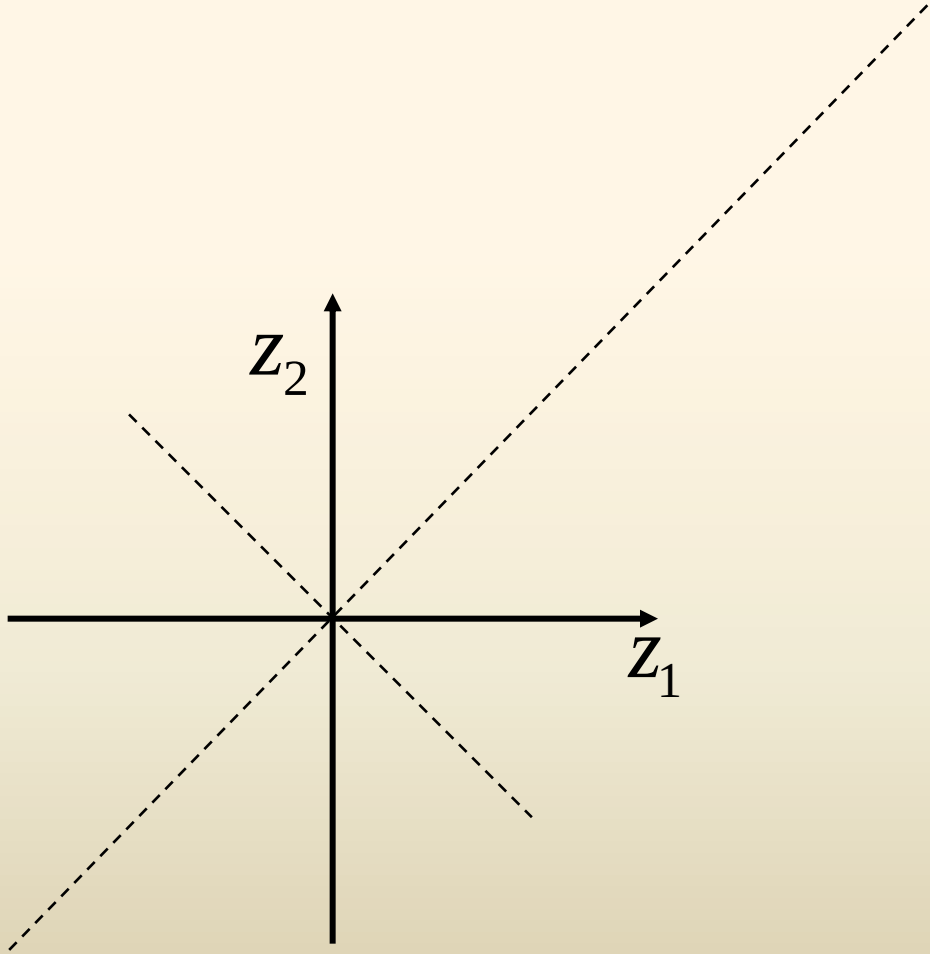
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

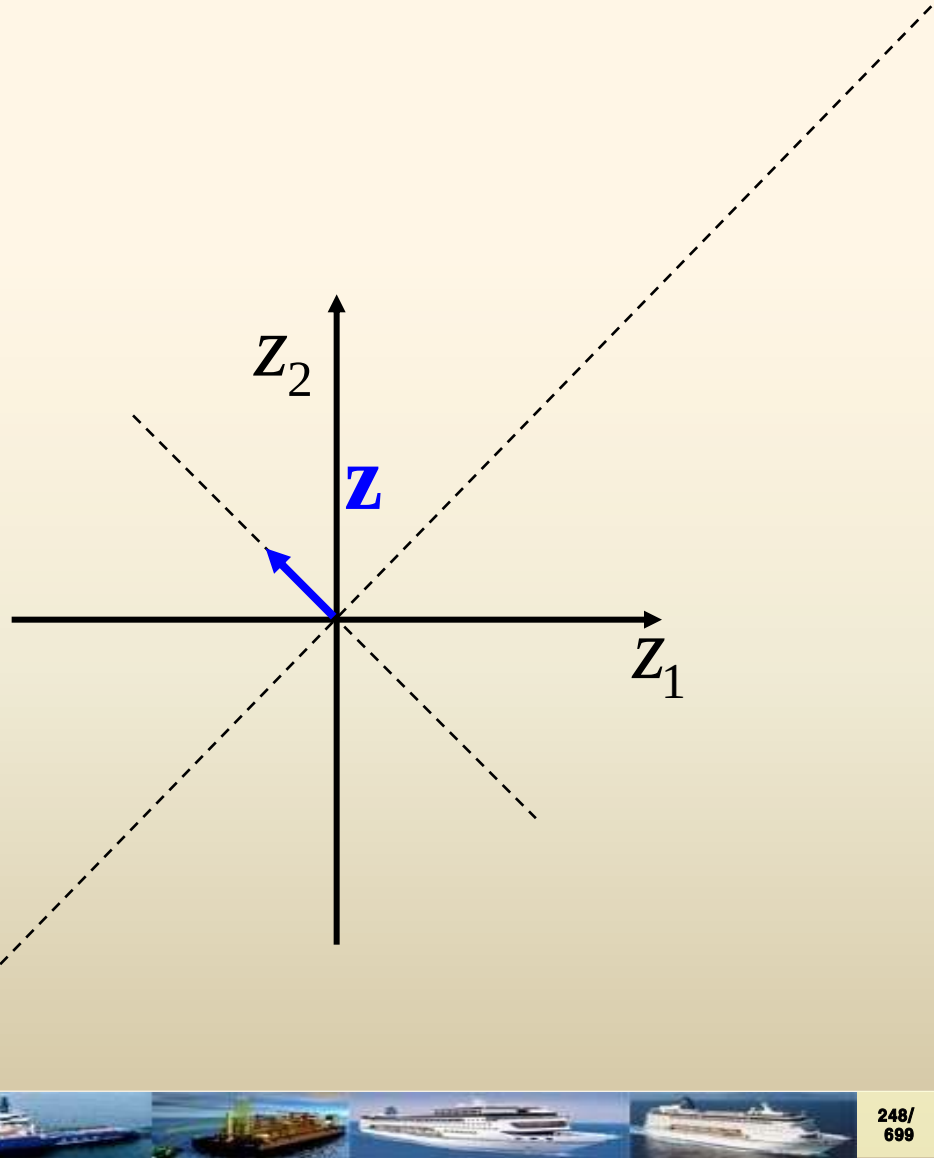
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

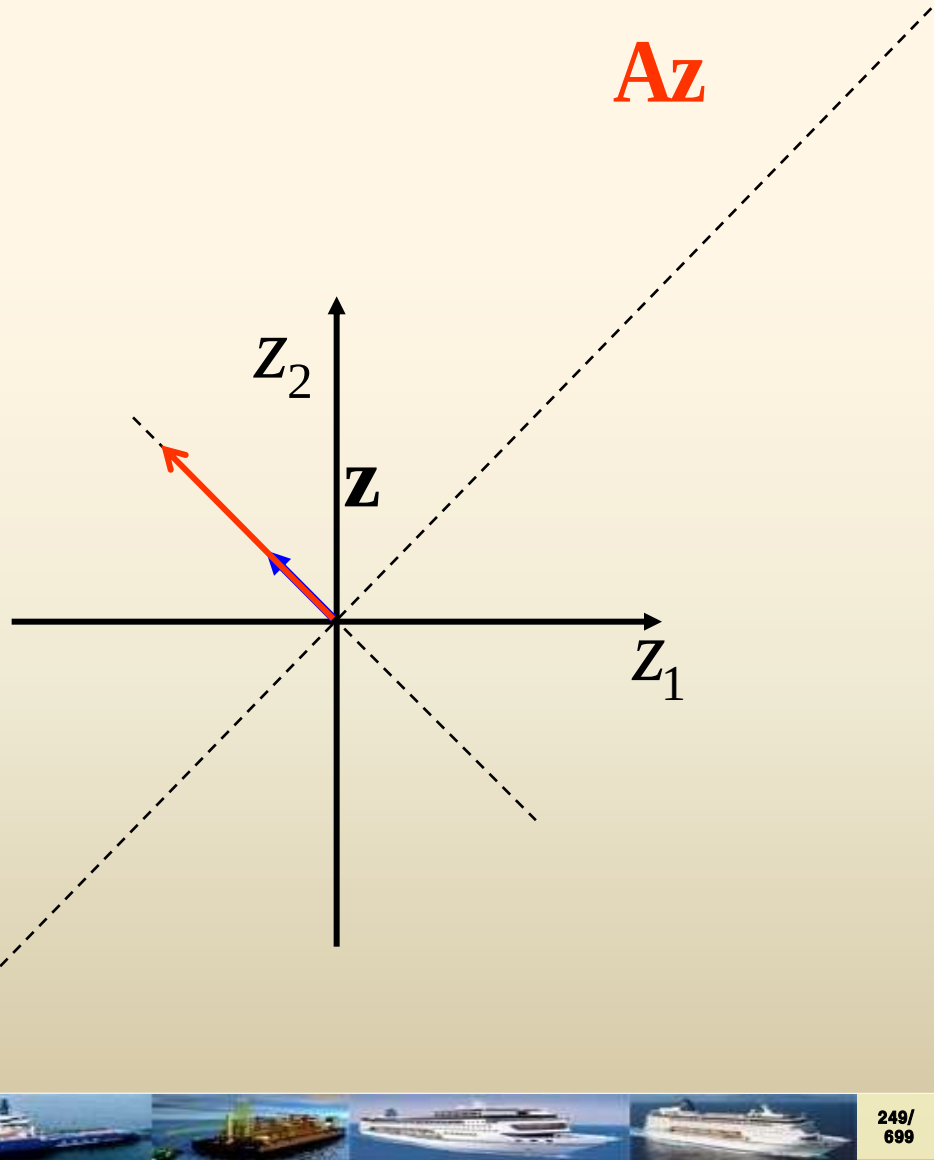
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

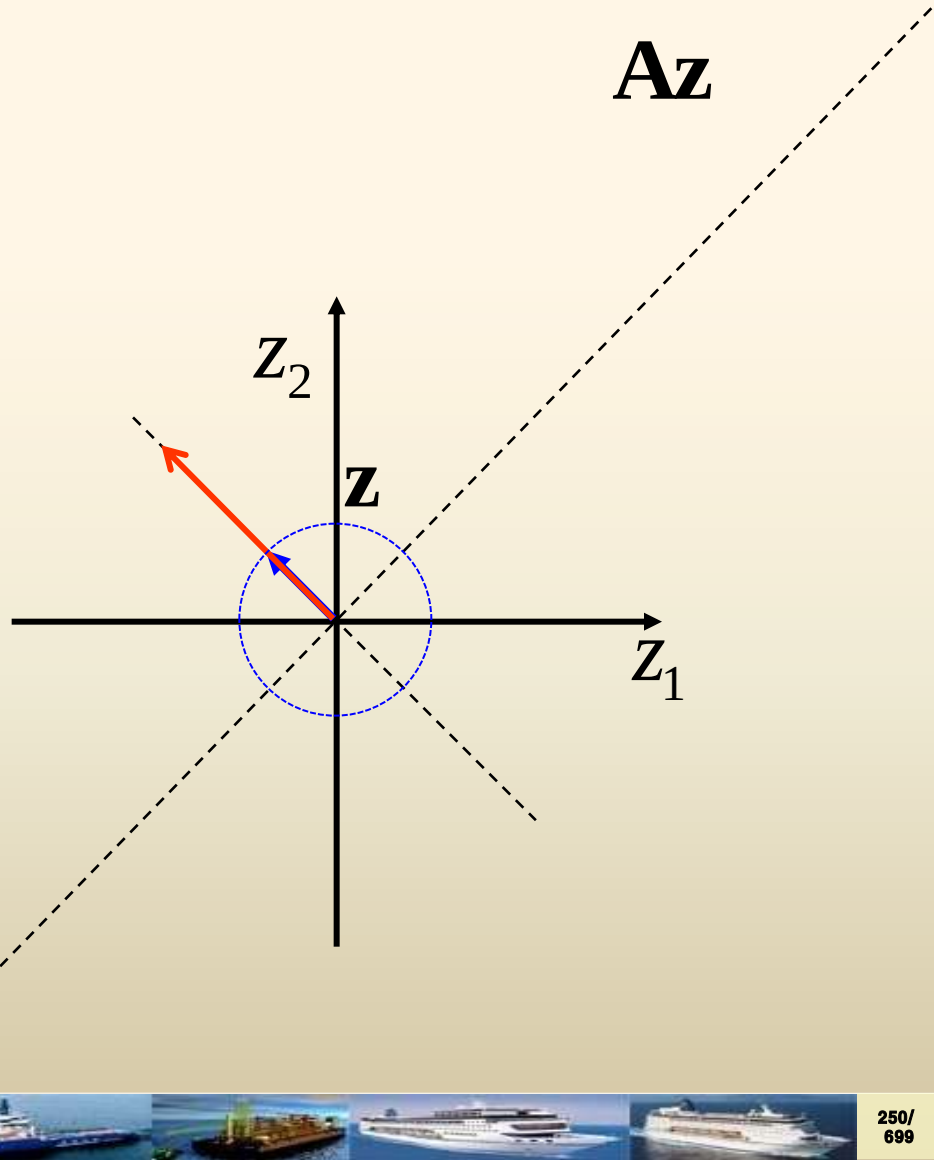
$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

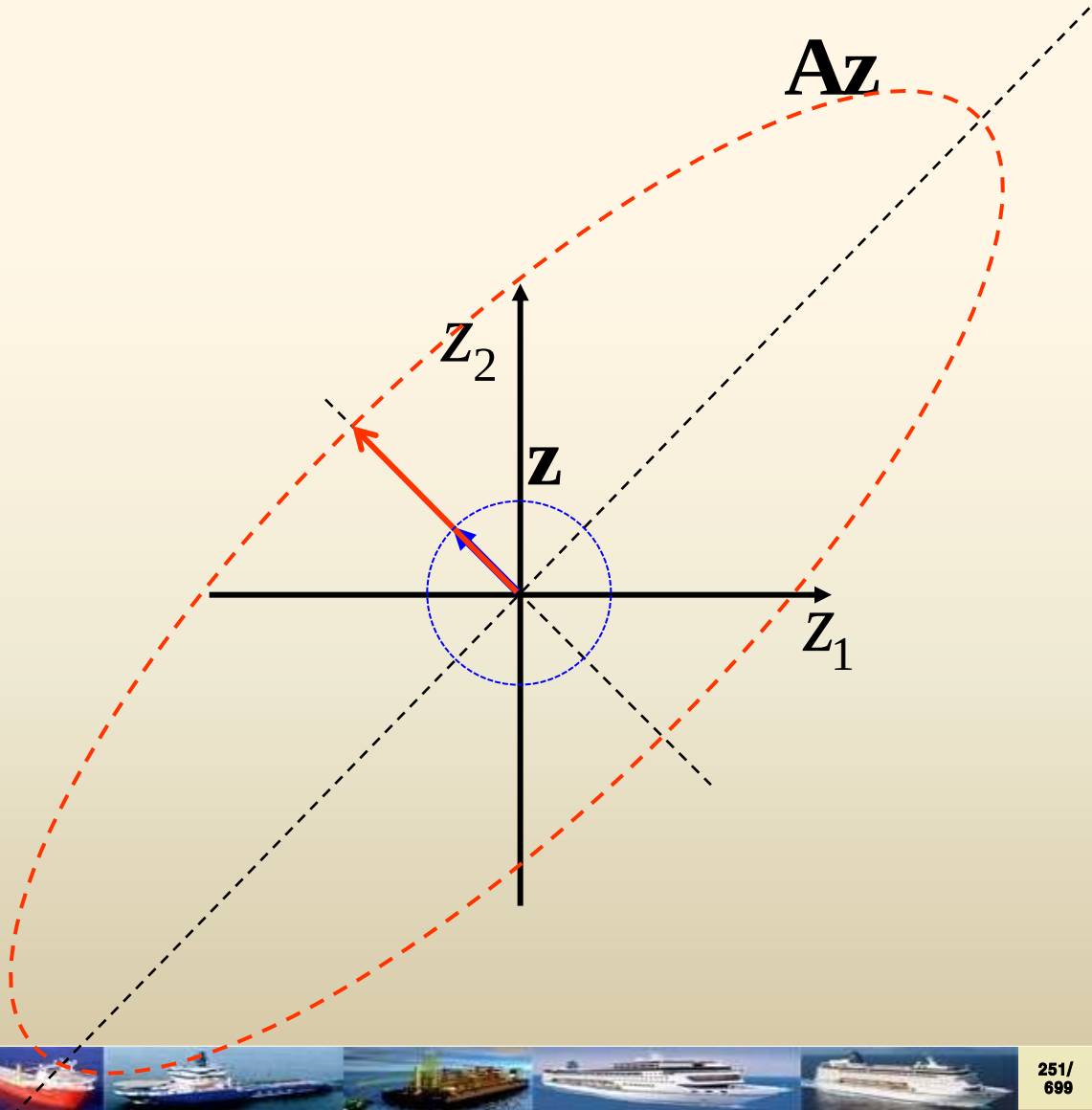
$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\begin{aligned} \mathbf{Az} &= \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \\ &= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2 \\ &= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2 \\ &= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2 \end{aligned}$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane (square)

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

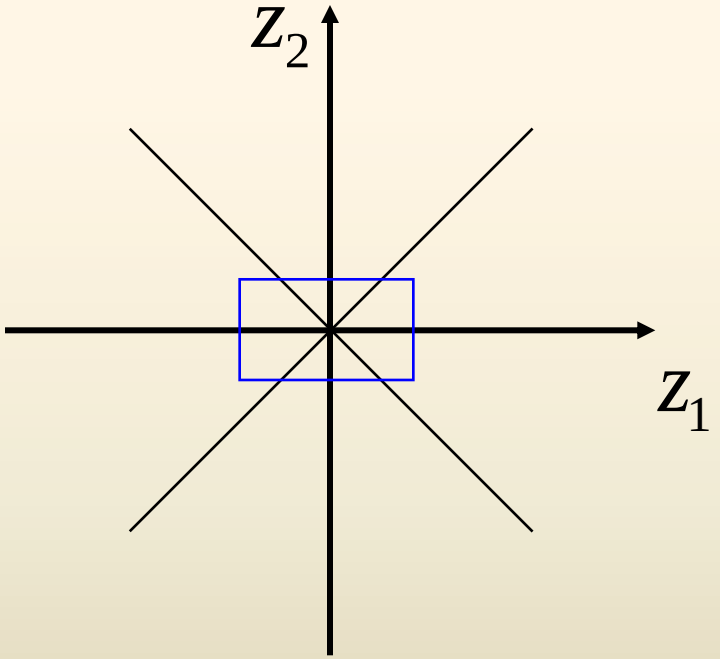
$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\mathbf{Az} = \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)$$

$$= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2$$

$$= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane (square)

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

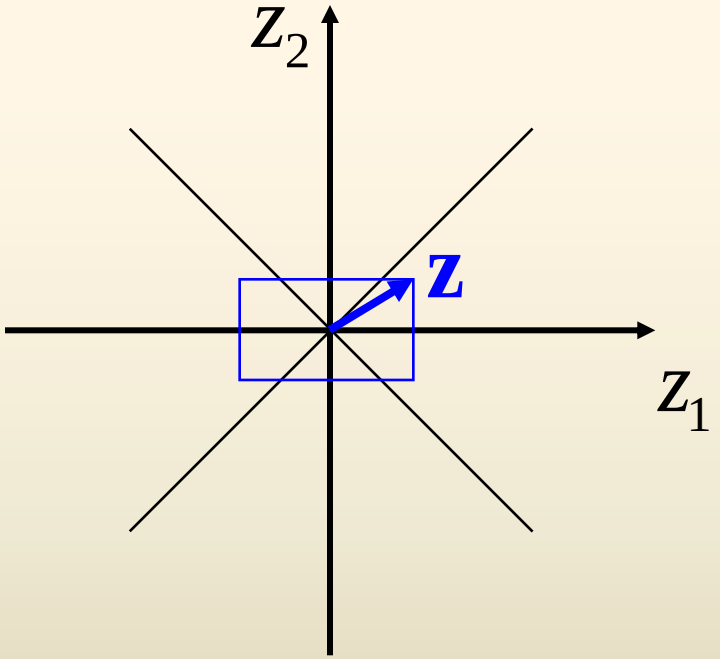
$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\mathbf{Az} = \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)$$

$$= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2$$

$$= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane (square)

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

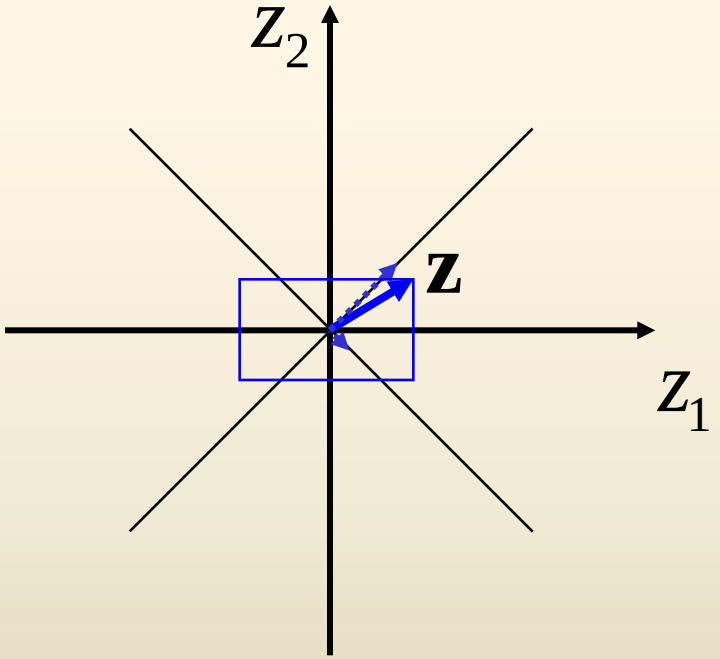
$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\mathbf{Az} = \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)$$

$$= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2$$

$$= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane (square)

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

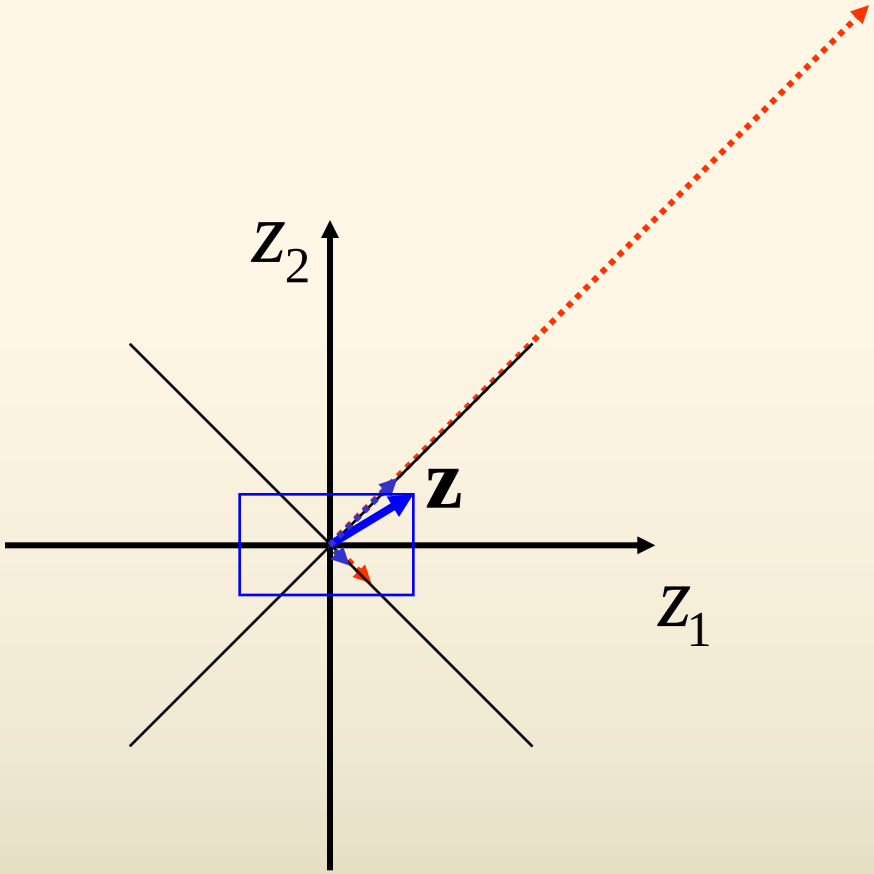
$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\mathbf{Az} = \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)$$

$$= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2$$

$$= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane (square)

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

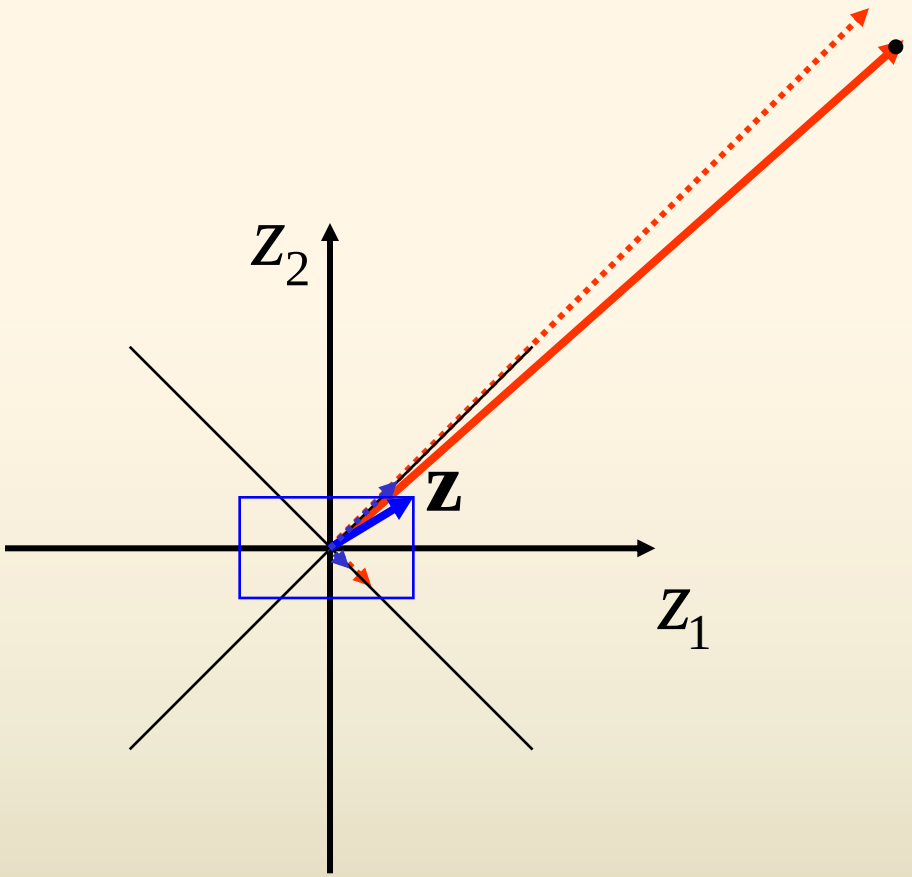
$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\mathbf{Az} = \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)$$

$$= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2$$

$$= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane (square)

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

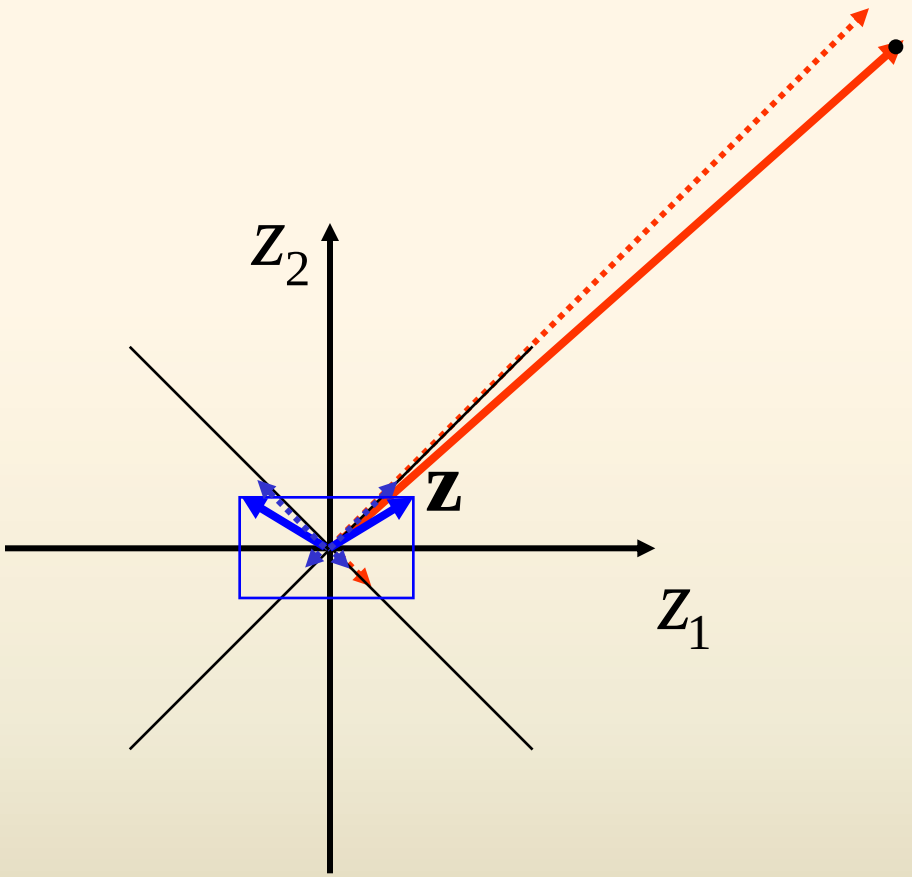
$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\mathbf{Az} = \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)$$

$$= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2$$

$$= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane (square)

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

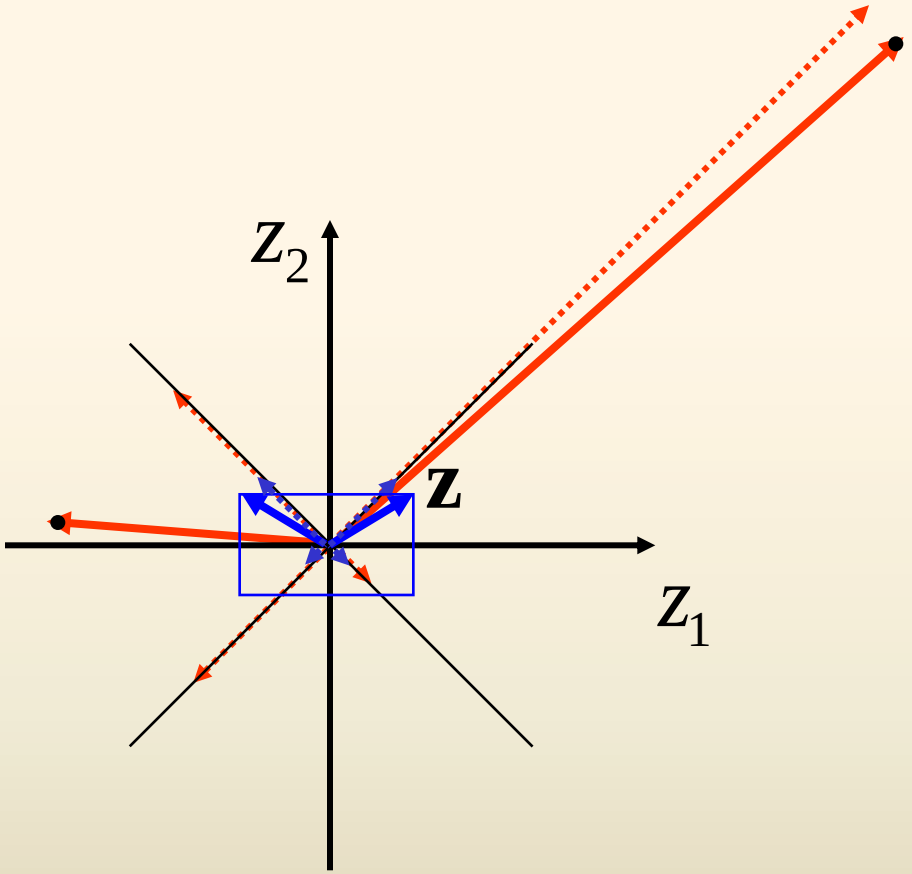
$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\mathbf{Az} = \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)$$

$$= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2$$

$$= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane (square)

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

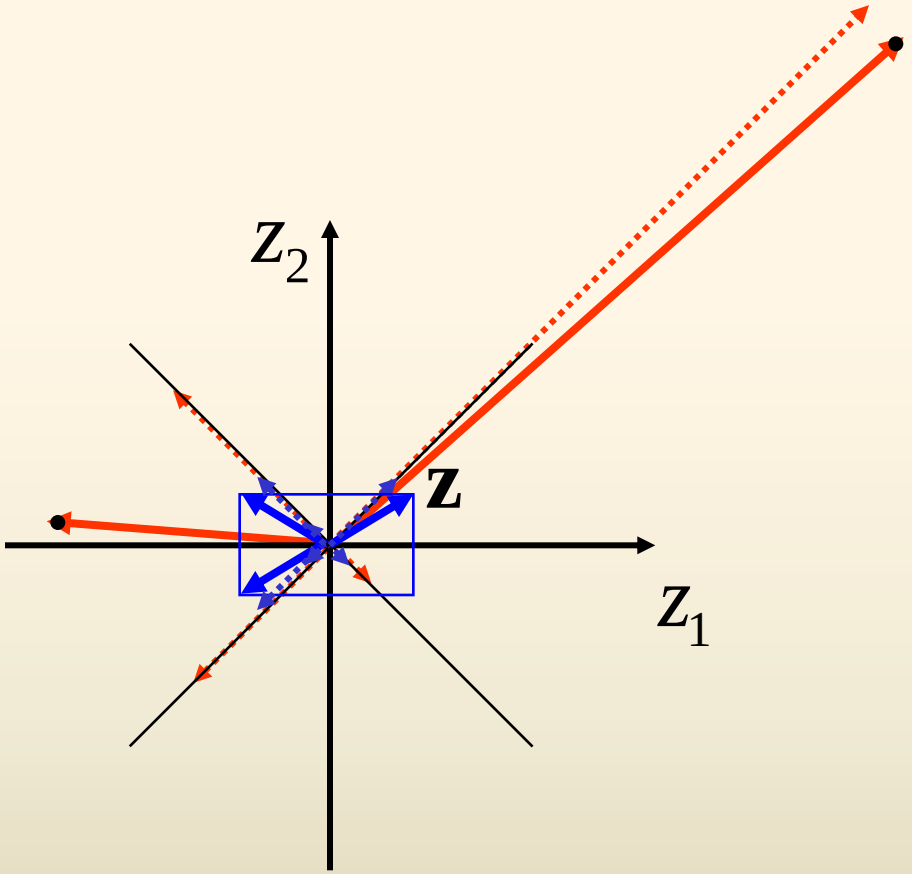
$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\mathbf{Az} = \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)$$

$$= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2$$

$$= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane (square)

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

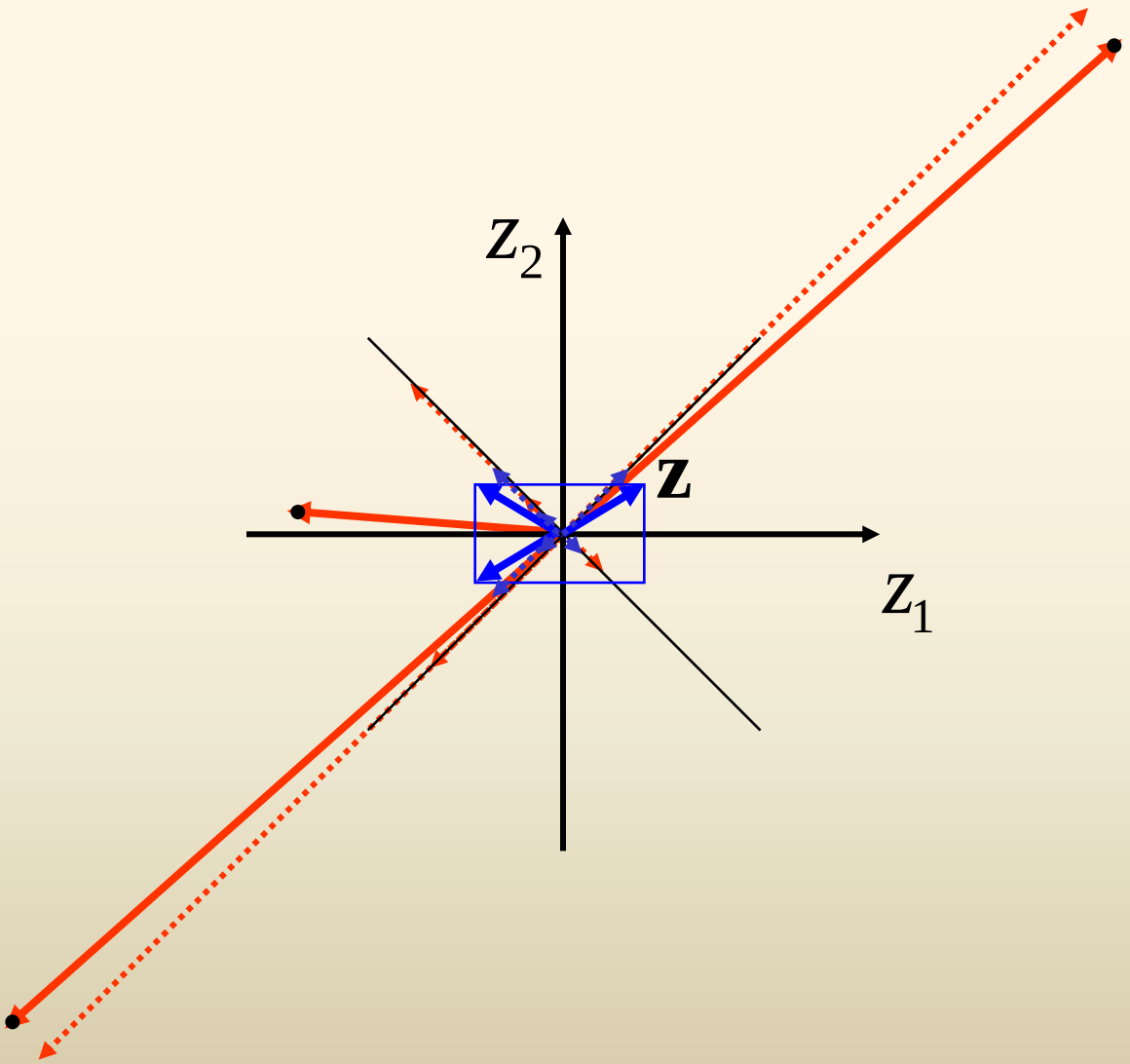
$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\mathbf{Az} = \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)$$

$$= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2$$

$$= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane (square)

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

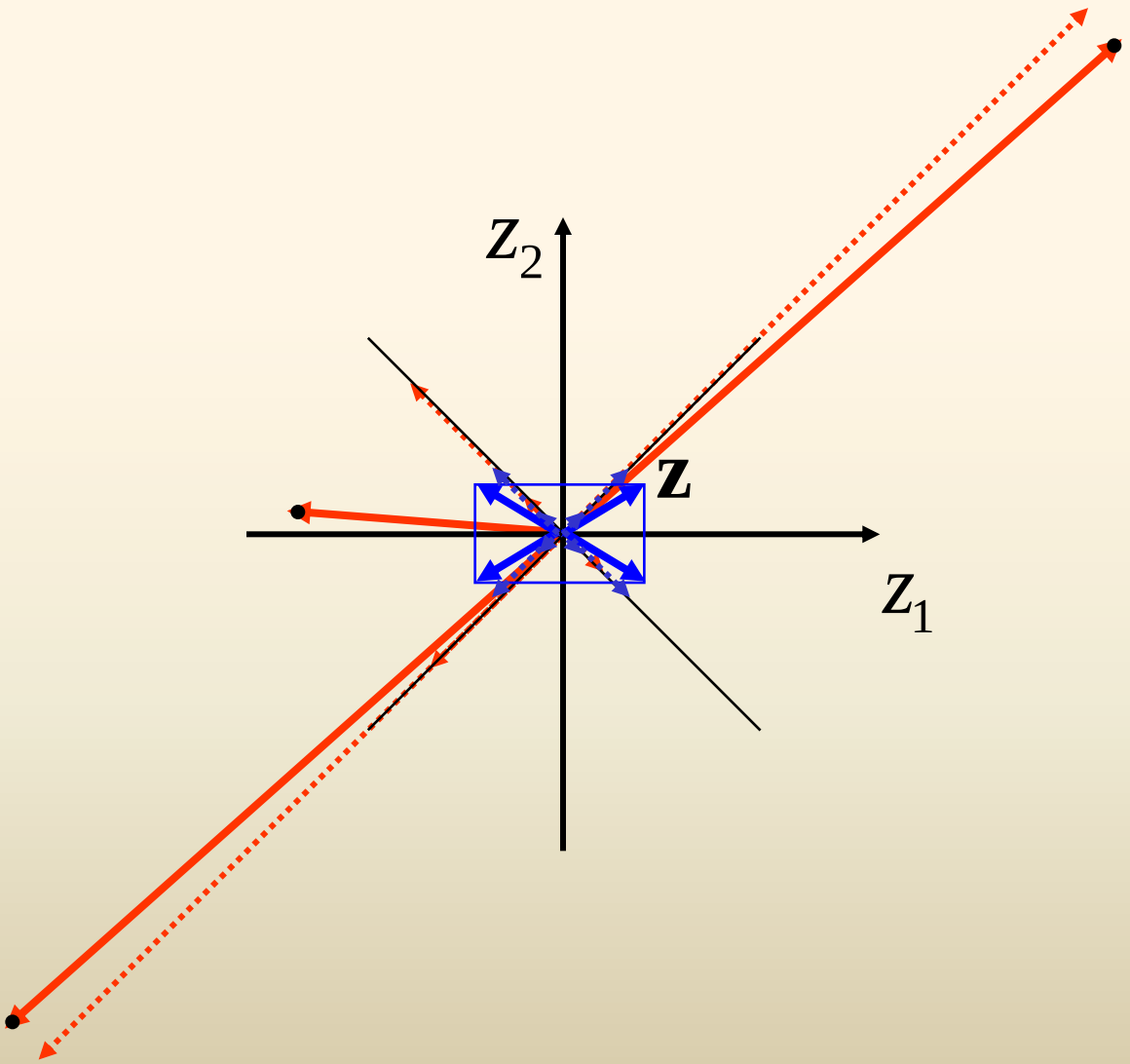
$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\mathbf{Az} = \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)$$

$$= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2$$

$$= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane (square)

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathbf{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathbf{T}}$$

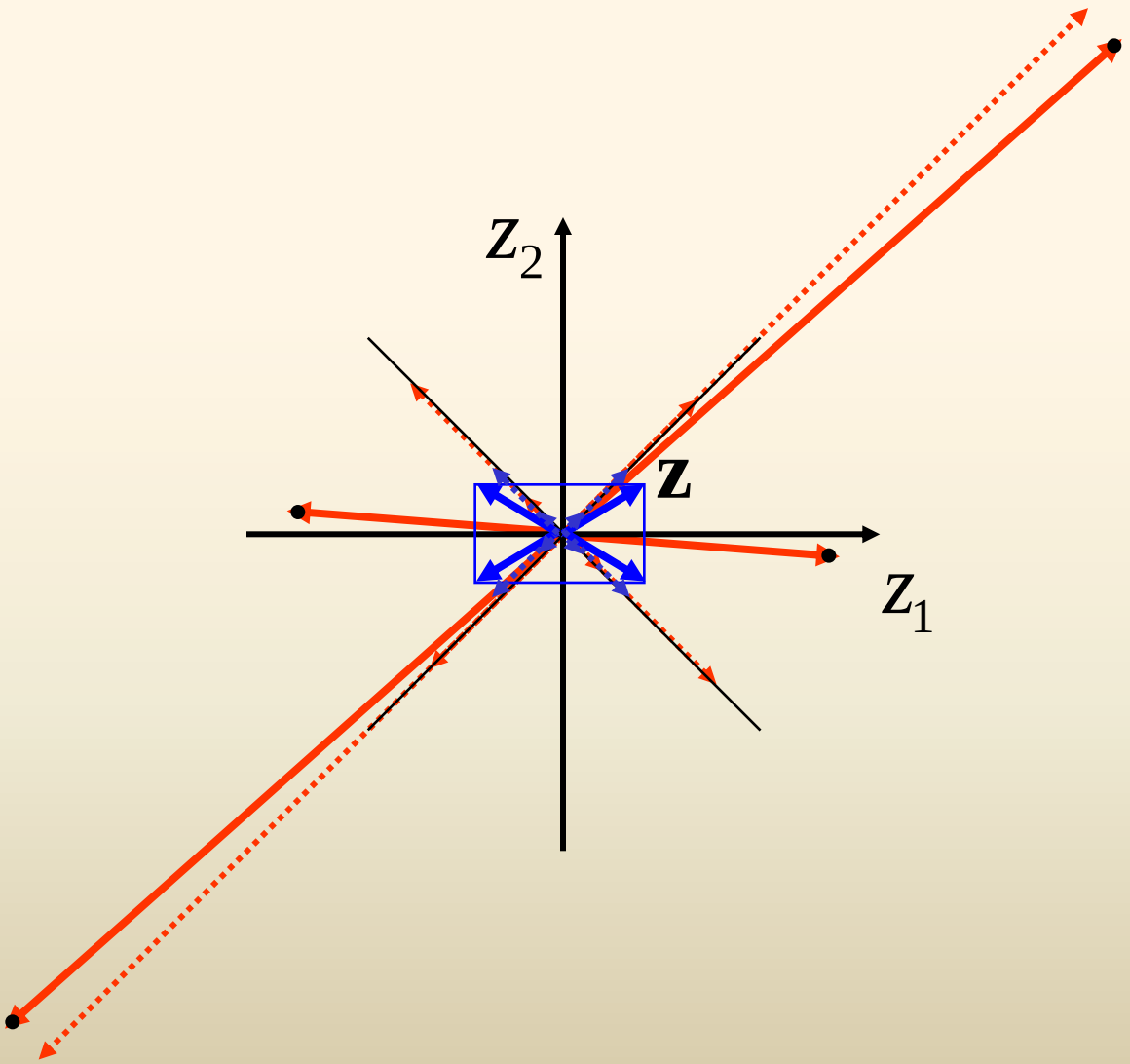
$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\mathbf{Az} = \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)$$

$$= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2$$

$$= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane (square)

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

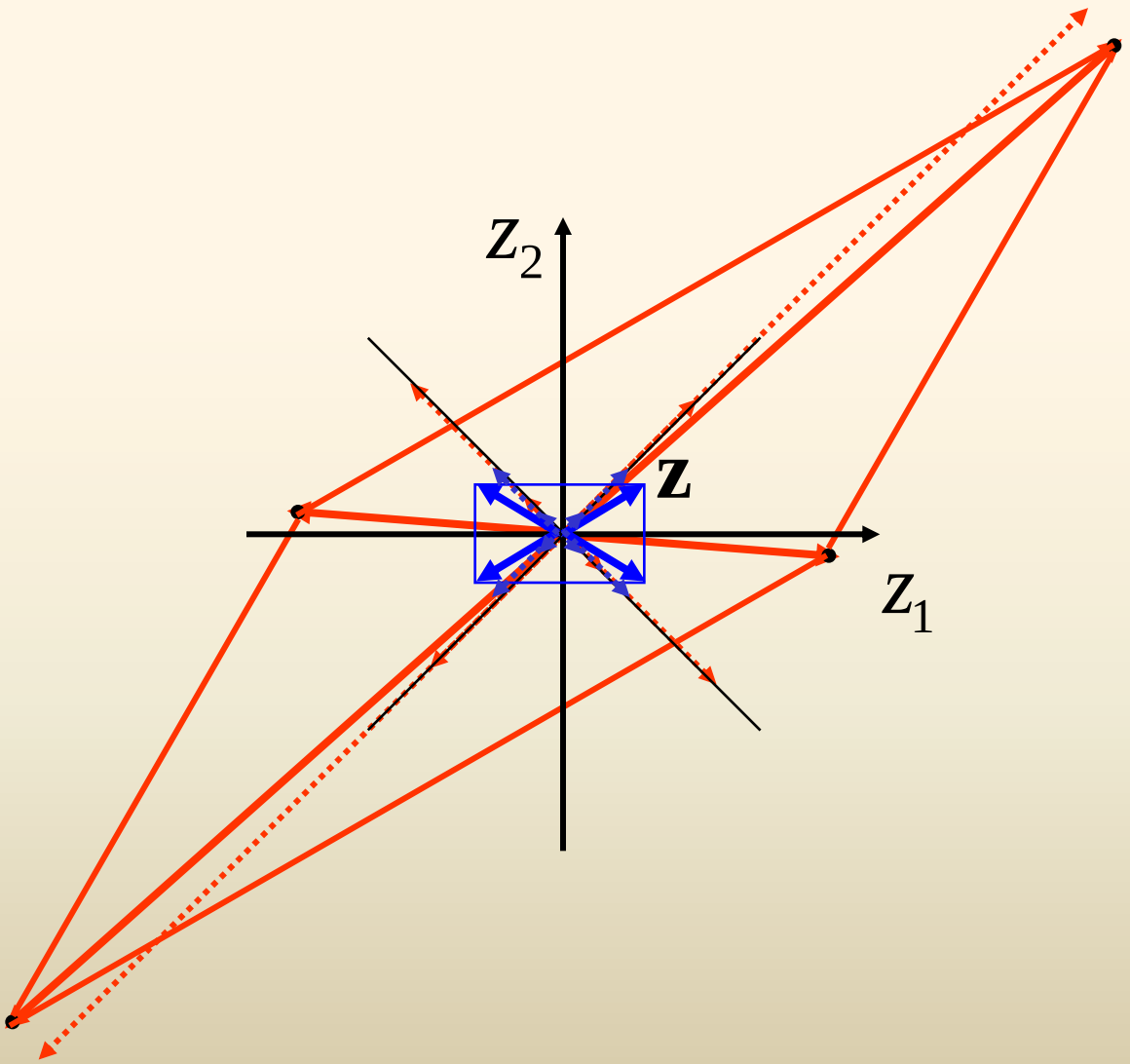
$$\mathbf{z} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\mathbf{Az} = \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)$$

$$= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2$$

$$= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$

$$\therefore \mathbf{Az} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Markov matrix

Linear equations Ax = b come from steady state problems. Eigenvalues have their greatest importance in dynamic problems. The solution of *d*u/*dt* = Au is changing with time - Growing or decaying or oscillating. We can't find it by elimination. This chapter enters a new part of linear algebra. All matrices in this chapter are square.

A good model comes from the powers $A, A^2, A^3, \ldots$ of a matrix.

Suppose you need the hundredth power A^{100}.

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \quad \begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix} \quad \begin{bmatrix} 0.650 & 0.525 \\ 0.350 & 0.475 \end{bmatrix} \quad \cdots \quad \begin{bmatrix} 0.6000 & 0.6000 \\ 0.4000 & 0.4000 \end{bmatrix}$$

$$\mathbf{A} \qquad \mathbf{A}^2 \qquad \mathbf{A}^3 \qquad\qquad \mathbf{A}^{100}$$

A^{100} was found by using the eigenvalues of A, not by multiplying 100 matrices.

Those eigenvalues are a new way to see into the heart of a matrix.

To explain eigenvalues, we first explain eigenvectors.

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

Almost all vectors change direction, when they are multiplied by A.

Certain exceptional vectors x are in the same direction as Ax. Those are the "eigenvectors". Multiply an eigenvector by A, and the vector Ax is a number λ times the original x. The basic equation is $Ax = \lambda x$.

$$\mathbf{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \text{ and } \mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \mathbf{x}_1$$

A vector — Multiply by A — Same direction

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = \frac{1}{2}\mathbf{x}_2$$

A vector — Multiply by A — Same direction

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

Almost all vectors change direction, when they are multiplied by A.

Certain exceptional vectors x are in the same direction as Ax. Those are the "eigenvectors". Multiply an eigenvector by A, and the vector Ax is a number λ times the original x. The basic equation is $Ax = \lambda x$.

$$\mathbf{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \quad \text{and} \quad \mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \mathbf{x}_1$$

Eigenvector

A vector — Multiply by A — Same direction

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = \frac{1}{2}\mathbf{x}_2$$

A vector — Multiply by A — Same direction

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

Almost all vectors change direction, when they are multiplied by A.

Certain exceptional vectors x are in the same direction as Ax. Those are the "eigenvectors". Multiply an eigenvector by A, and the vector Ax is a number λ times the original x. The basic equation is $\mathbf{Ax} = \lambda\mathbf{x}$.

$$\mathbf{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \quad \text{and} \quad \mathbf{Ax}_1 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \mathbf{x}_1 \qquad \therefore \lambda_1 = 1$$

A vector — Multiply by A — Same direction — Eigenvector — Eigenvalue

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{Ax}_2 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = \frac{1}{2}\mathbf{x}_2$$

A vector — Multiply by A — Same direction

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

Almost all vectors change direction, when they are multiplied by A.

Certain exceptional vectors x are in the same direction as Ax. Those are the "eigenvectors". Multiply an eigenvector by A, and the vector Ax is a number λ times the original x. The basic equation is $Ax = \lambda x$.

$$\mathbf{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \quad \text{and} \quad \mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \mathbf{x}_1 \qquad \therefore \lambda_1 = 1$$

A vector — Multiply by A — Same direction — Eigenvector — Eigenvalue

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = \frac{1}{2}\mathbf{x}_2$$

A vector — Multiply by A — Same direction — Eigenvector

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

Almost all vectors change direction, when they are multiplied by A.

Certain exceptional vectors x are in the same direction as Ax. Those are the "eigenvectors". Multiply an eigenvector by A, and the vector Ax is a number λ times the original x. The basic equation is $\mathbf{Ax} = \lambda\mathbf{x}$.

$$\mathbf{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \quad \text{and} \quad \mathbf{Ax}_1 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \mathbf{x}_1 \qquad \therefore \lambda_1 = 1$$

Eigenvector

A vector — Multiply by A — Same direction

Eigenvalue

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{Ax}_2 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = \frac{1}{2}\mathbf{x}_2 \qquad \therefore \lambda_2 = \frac{1}{2}$$

Eigenvector

A vector — Multiply by A — Same direction

Eigenvalue

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

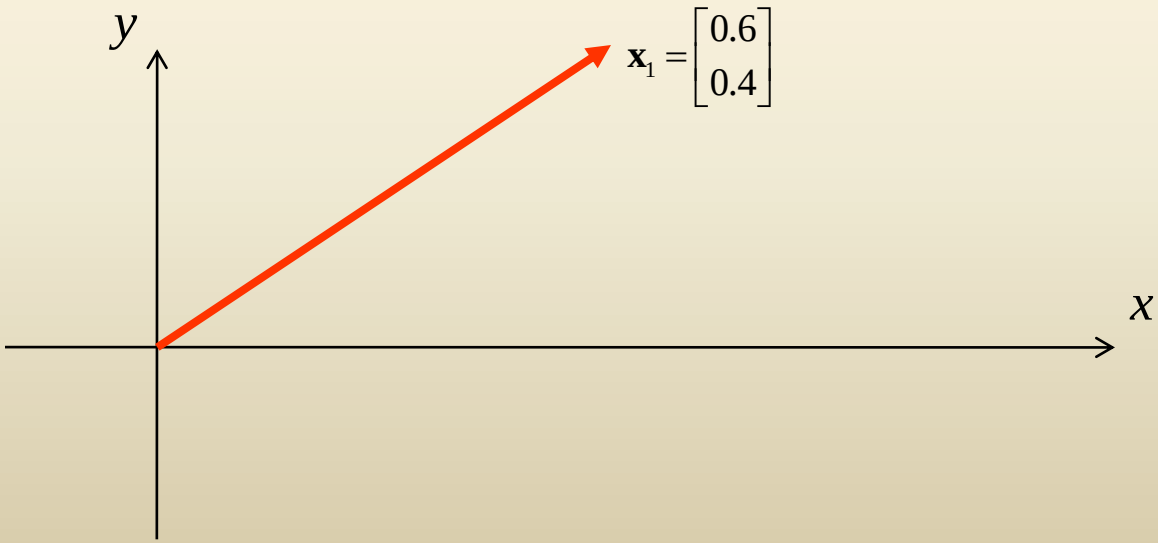
If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

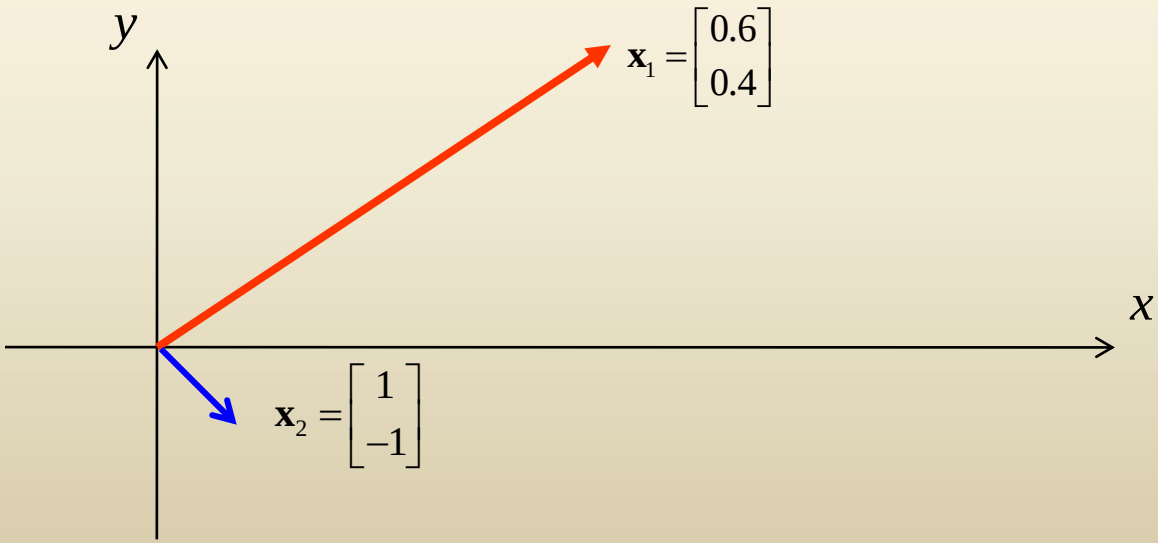


need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.



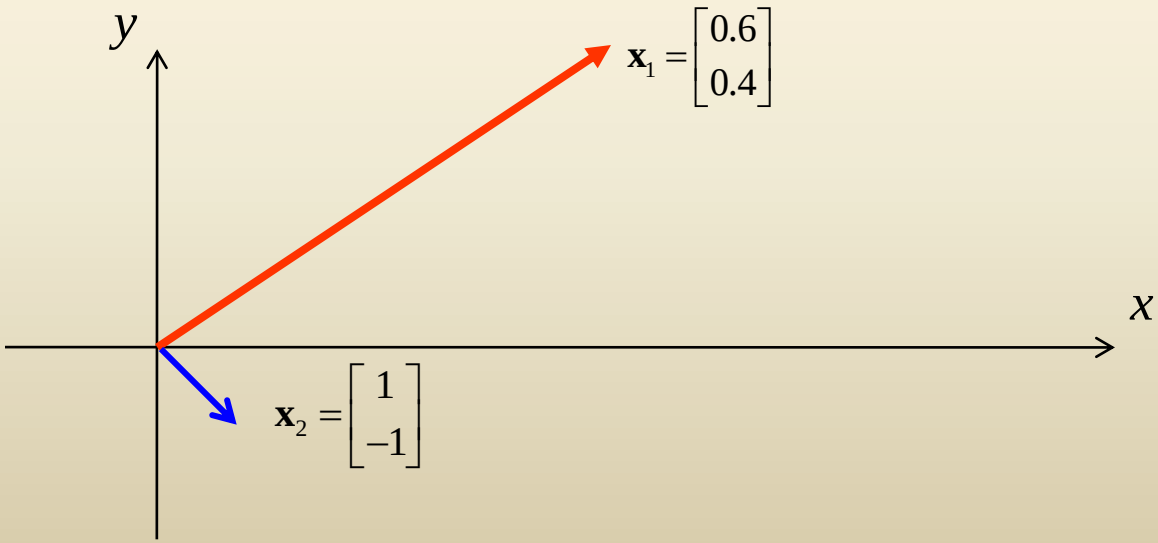
need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$



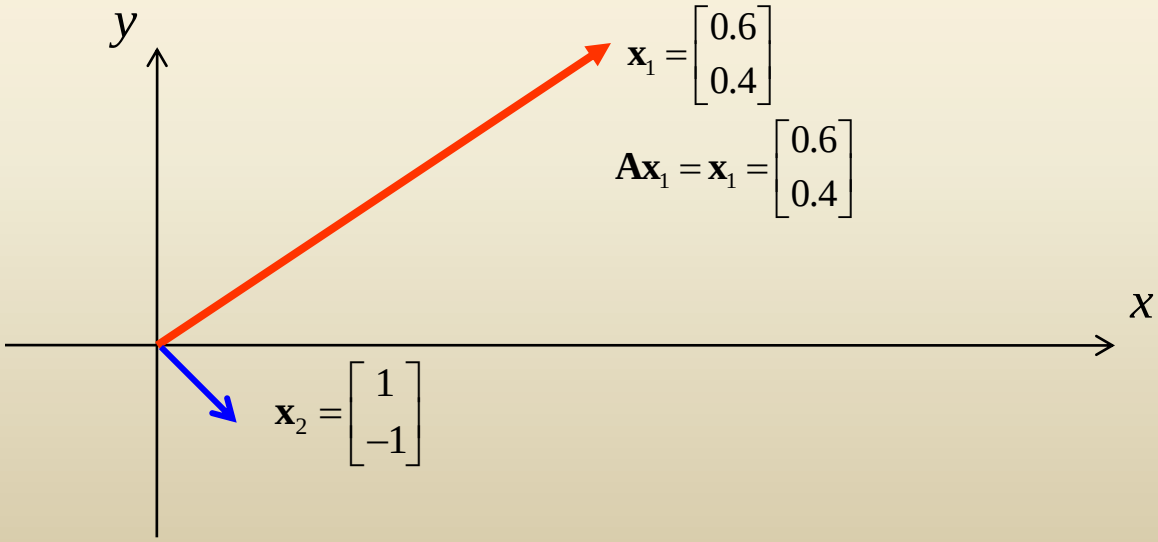
need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$



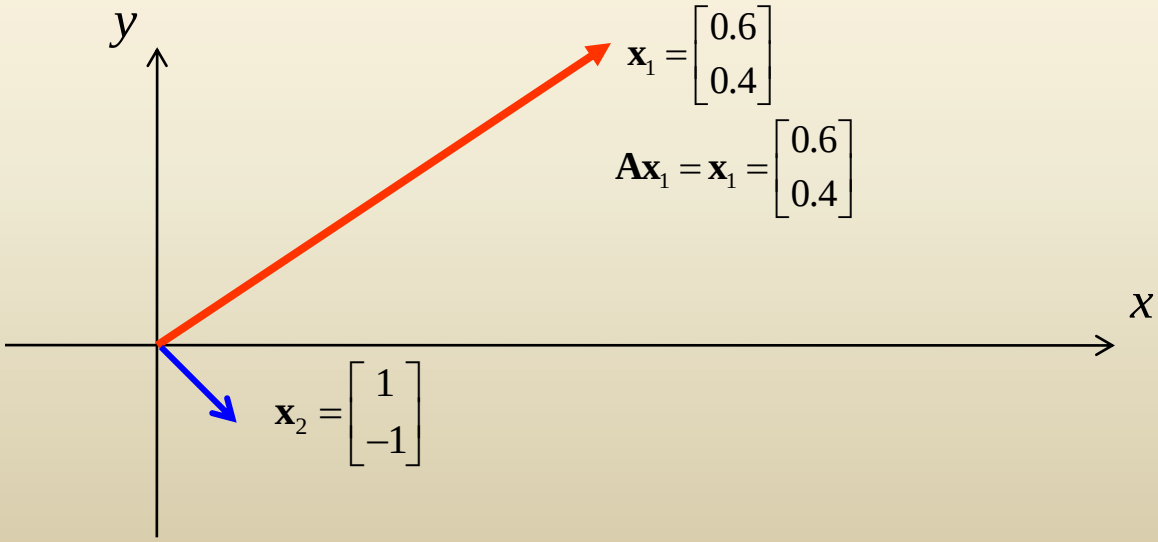
need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$



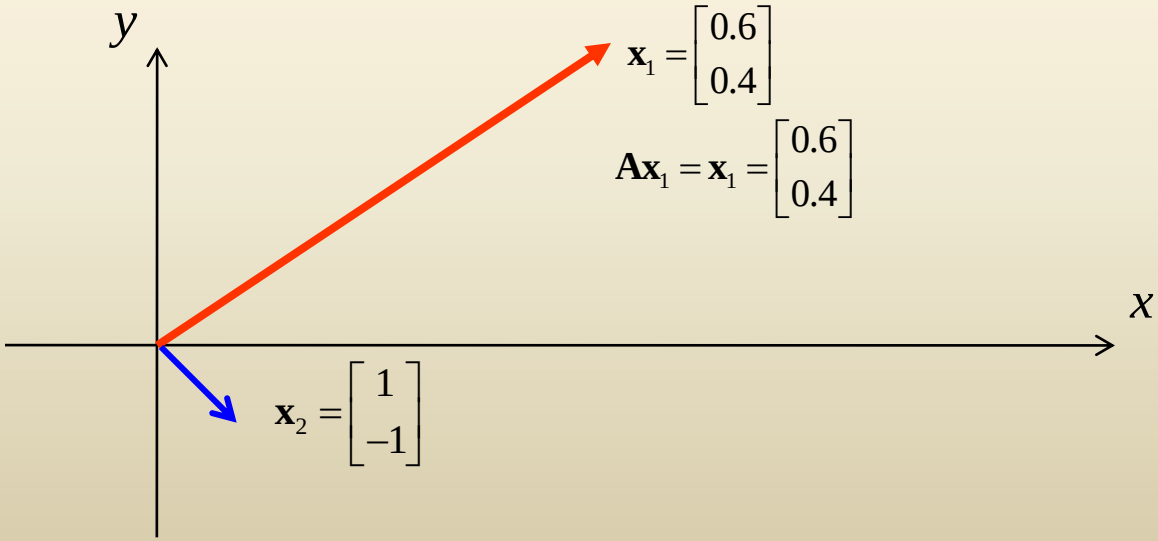
need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1, \qquad \mathbf{A}\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$$



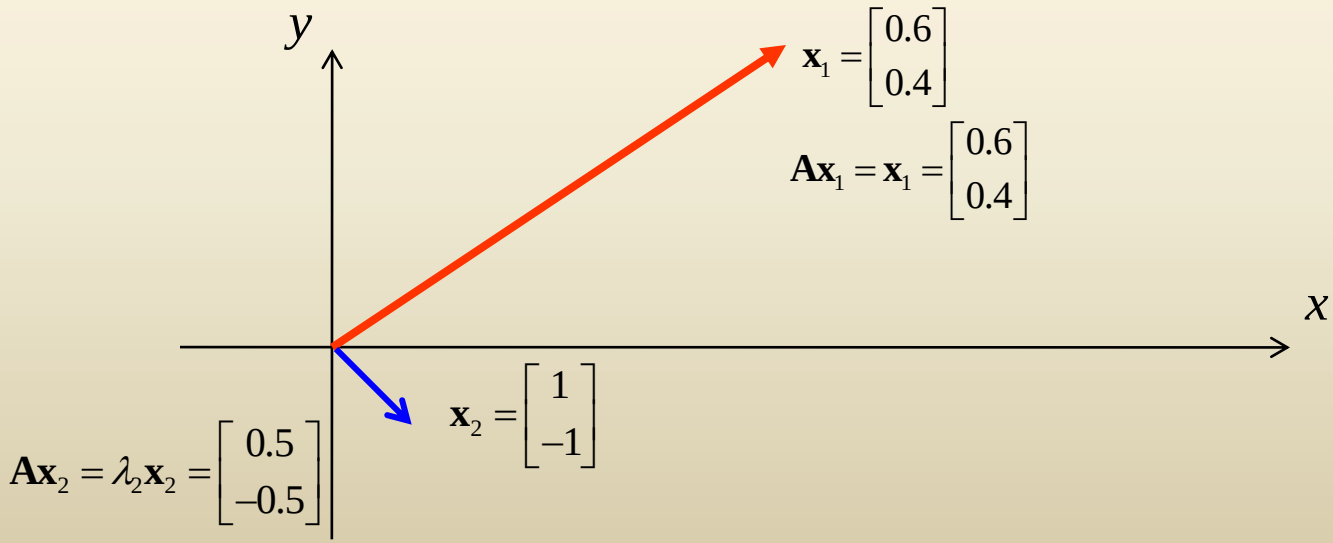
need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1, \qquad \mathbf{A}\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$$



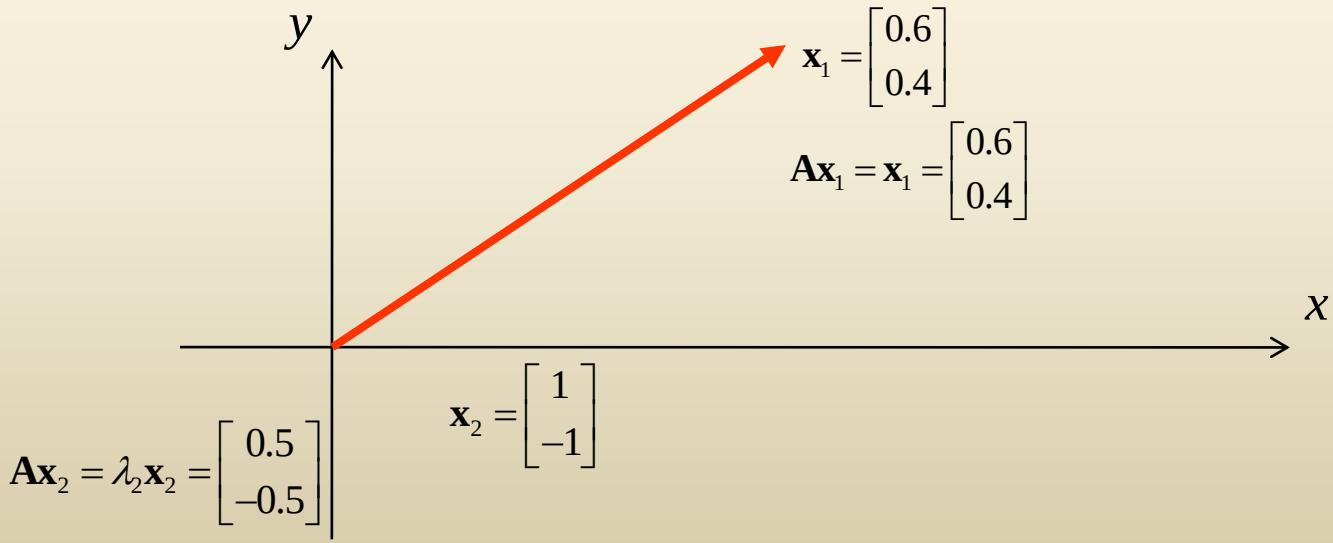
need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1, \qquad \mathbf{A}\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$$



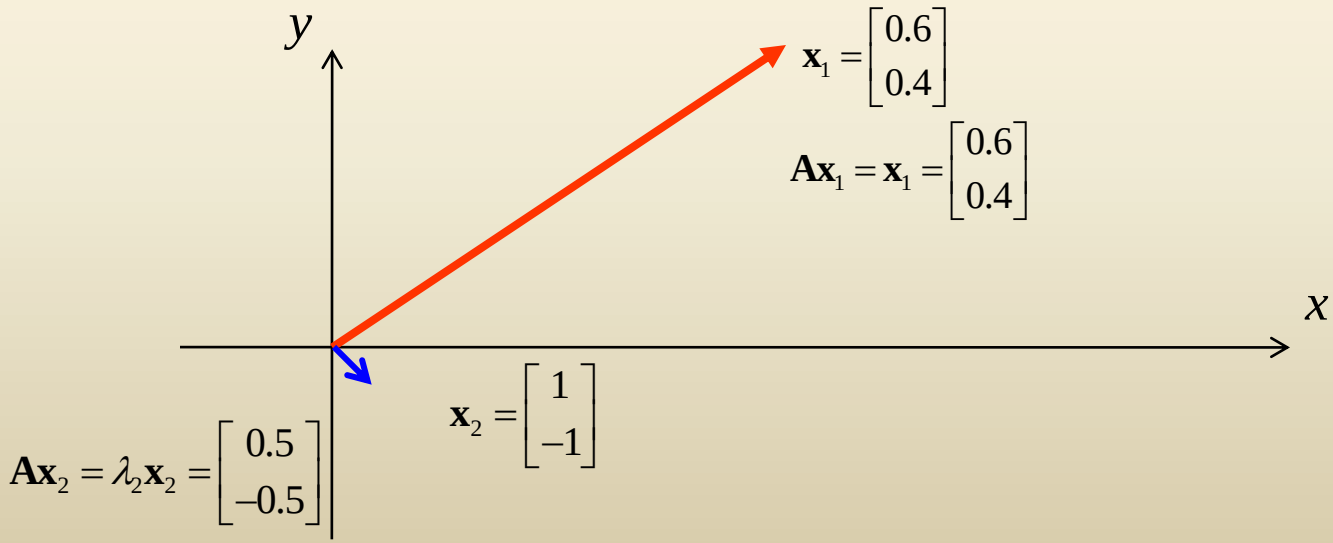
need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1, \qquad \mathbf{A}\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$$



need the hundredth power A^{100}

Markov matrix

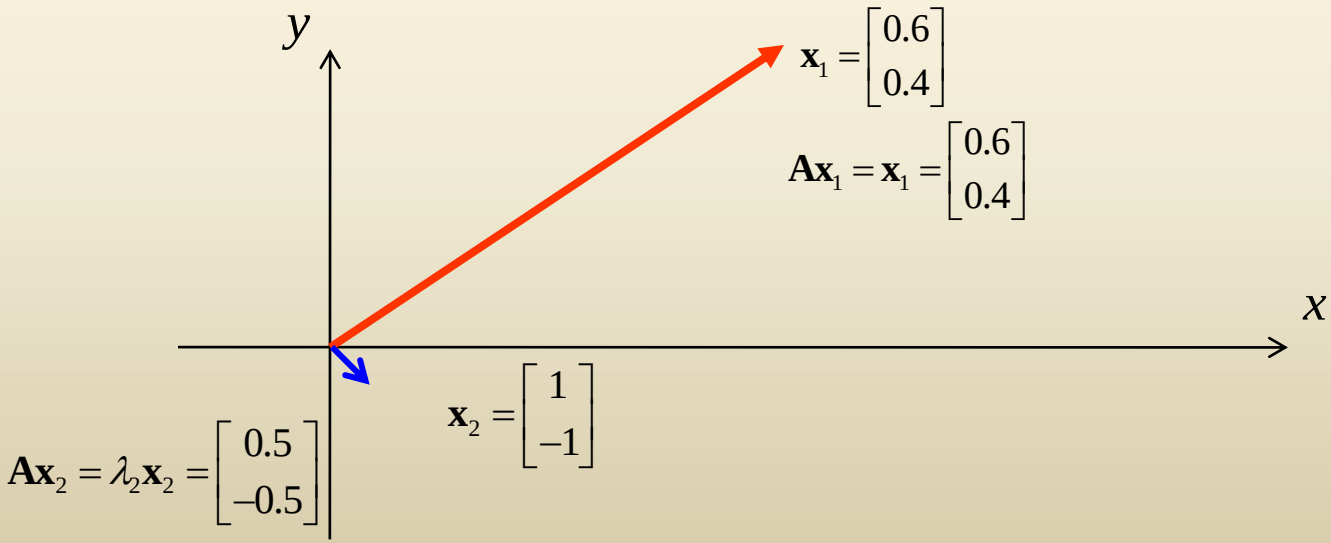
$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$

$$\mathbf{A}^2\mathbf{x}_1 = \mathbf{A}(\mathbf{A}\mathbf{x}_1) = \mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$

$$\mathbf{A}\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$$



need the hundredth power A^{100}

Markov matrix

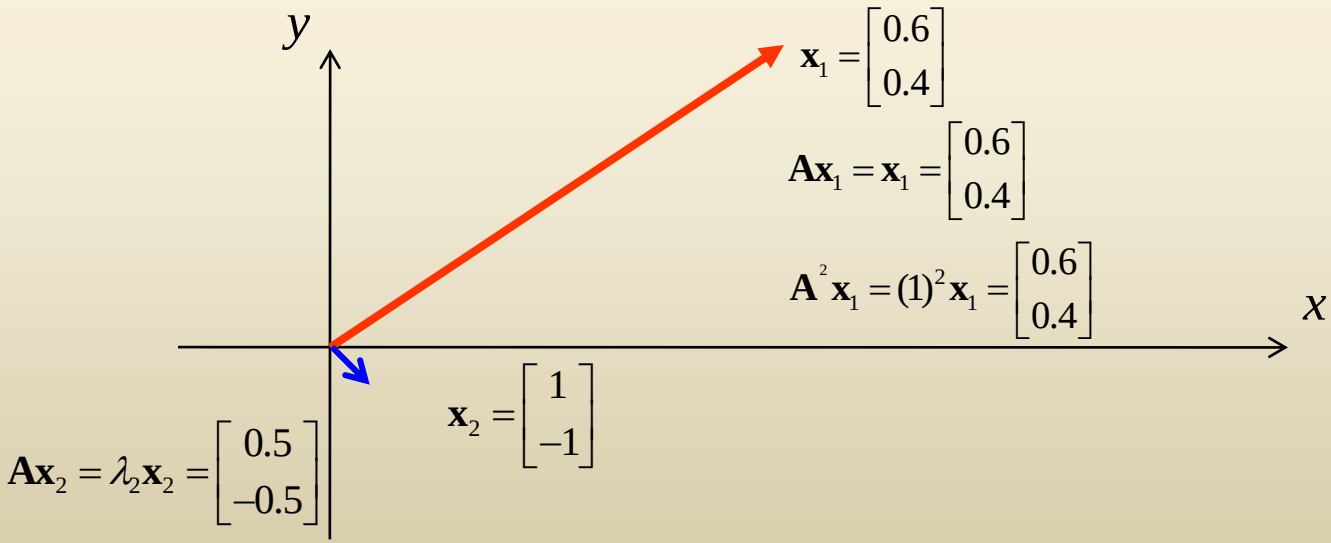
$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$

$$\mathbf{A}^2\mathbf{x}_1 = \mathbf{A}(\mathbf{A}\mathbf{x}_1) = \mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$

$$\mathbf{A}\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$$



need the hundredth power A^{100}

Markov matrix

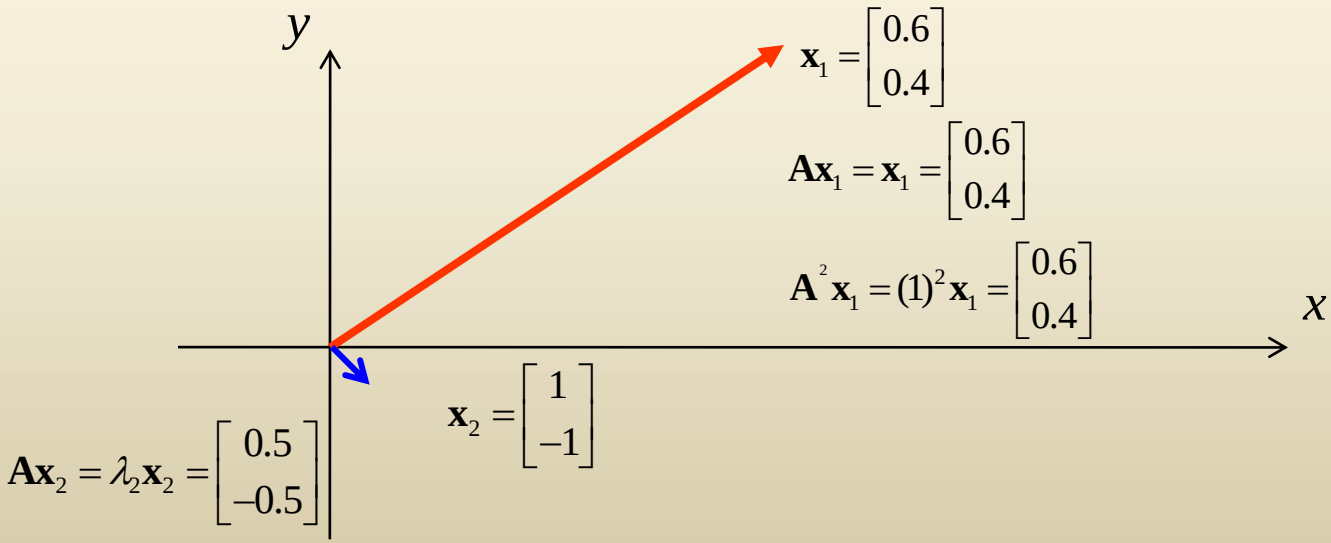
$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$

$$\mathbf{A}^2\mathbf{x}_1 = \mathbf{A}(\mathbf{A}\mathbf{x}_1) = \mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$

$$\mathbf{A}\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$$



need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

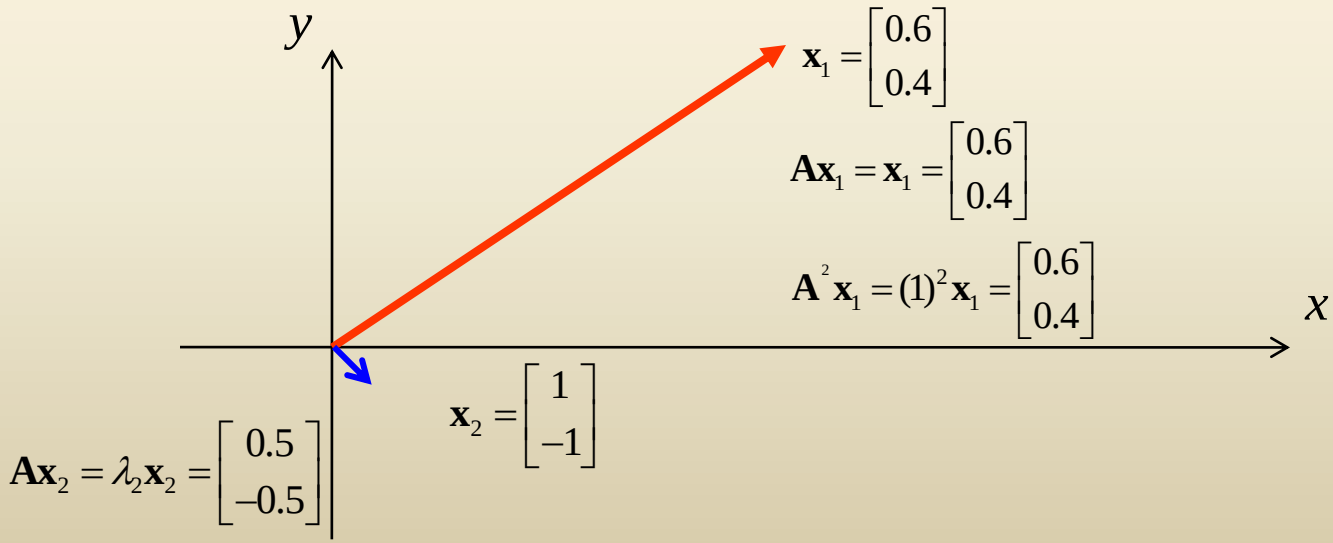
If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$

$$\mathbf{A}^2\mathbf{x}_1 = \mathbf{A}(\mathbf{A}\mathbf{x}_1) = \mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$

$$\mathbf{A}\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$$

$$\mathbf{A}^2\mathbf{x}_2 = \mathbf{A}(\mathbf{A}\mathbf{x}_2) = \mathbf{A}\left(\frac{1}{2}\mathbf{x}_2\right) = \frac{1}{2}\mathbf{A}\mathbf{x}_2 = \frac{1}{2}\frac{1}{2}\mathbf{x}_2 = \frac{1}{2}^2\,\mathbf{x}_2$$



need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

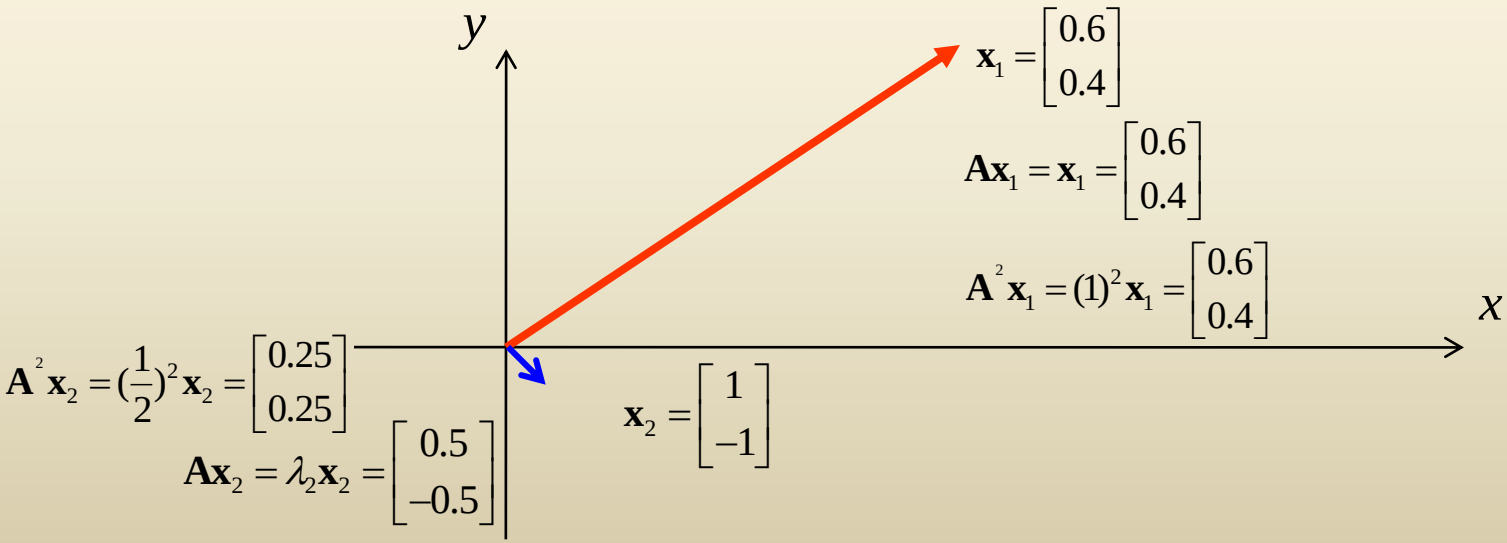
If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$

$$\mathbf{A}^2\mathbf{x}_1 = \mathbf{A}(\mathbf{A}\mathbf{x}_1) = \mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$

$$\mathbf{A}\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$$

$$\mathbf{A}^2\mathbf{x}_2 = \mathbf{A}(\mathbf{A}\mathbf{x}_2) = \mathbf{A}\left(\frac{1}{2}\mathbf{x}_2\right) = \frac{1}{2}\mathbf{A}\mathbf{x}_2 = \frac{1}{2}\frac{1}{2}\mathbf{x}_2 = \frac{1}{2}^2\,\mathbf{x}_2$$



need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

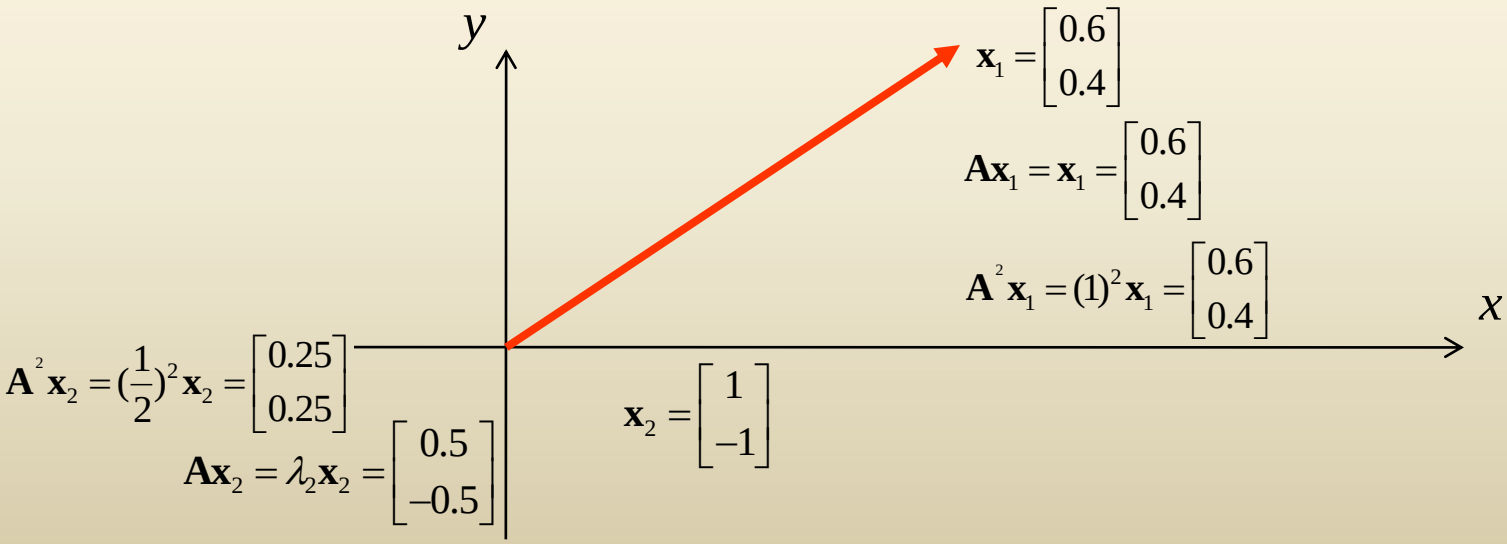
If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$

$$\mathbf{A}^2\mathbf{x}_1 = \mathbf{A}(\mathbf{A}\mathbf{x}_1) = \mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$

$$\mathbf{A}\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$$

$$\mathbf{A}^2\mathbf{x}_2 = \mathbf{A}(\mathbf{A}\mathbf{x}_2) = \mathbf{A}\left(\frac{1}{2}\mathbf{x}_2\right) = \frac{1}{2}\mathbf{A}\mathbf{x}_2 = \frac{1}{2}\frac{1}{2}\mathbf{x}_2 = \frac{1}{2}^2\,\mathbf{x}_2$$



need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

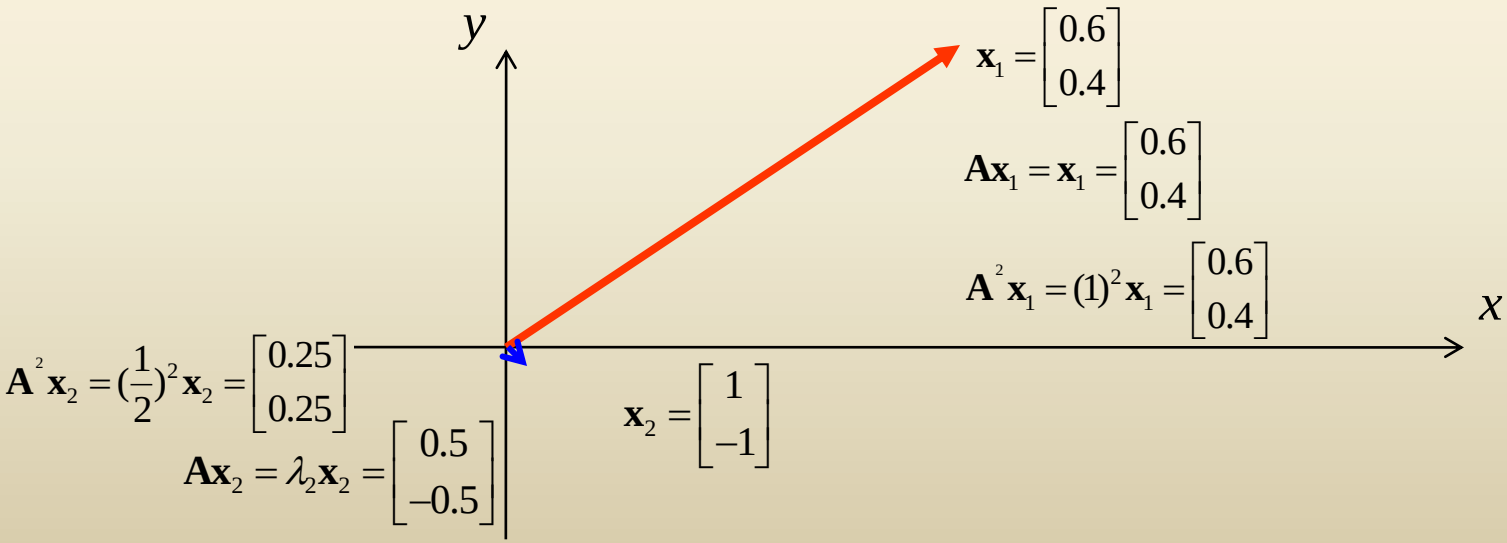
If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$

$$\mathbf{A}^2\mathbf{x}_1 = \mathbf{A}(\mathbf{A}\mathbf{x}_1) = \mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$

$$\mathbf{A}\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$$

$$\mathbf{A}^2\mathbf{x}_2 = \mathbf{A}(\mathbf{A}\mathbf{x}_2) = \mathbf{A}\left(\frac{1}{2}\mathbf{x}_2\right) = \frac{1}{2}\mathbf{A}\mathbf{x}_2 = \frac{1}{2}\frac{1}{2}\mathbf{x}_2 = \frac{1}{2}^2\mathbf{x}_2$$



need the hundredth power A^{100}

Markov matrix

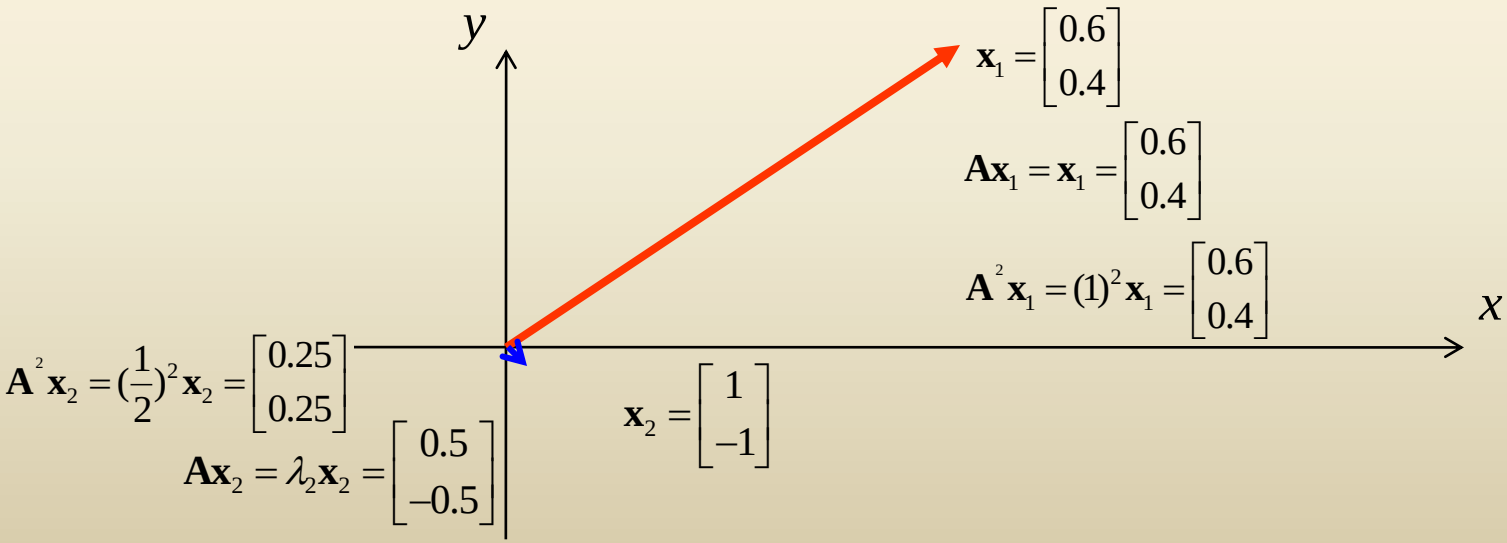
$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1, \qquad \mathbf{A}\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$$

$$\mathbf{A}^2\mathbf{x}_1 = \mathbf{A}(\mathbf{A}\mathbf{x}_1) = \mathbf{A}\mathbf{x}_1 = \mathbf{x}_1, \qquad \mathbf{A}^2\mathbf{x}_2 = \mathbf{A}(\mathbf{A}\mathbf{x}_2) = \mathbf{A}\left(\frac{1}{2}\mathbf{x}_2\right) = \frac{1}{2}\mathbf{A}\mathbf{x}_2 = \frac{1}{2}\frac{1}{2}\mathbf{x}_2 = \frac{1}{2}^2\,\mathbf{x}_2$$

The eigenvalues of A^{100} are $1^{100} = 1$ and $\frac{1}{2}^{100}$ = very small number.



need the hundredth power A^{100}

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\quad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $\mathbf{x}_1 + 0.2\mathbf{x}_2$:

$$\begin{bmatrix}0.8\\ 0.2\end{bmatrix}=\mathbf{x}_1+0.2\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.2\\ -0.2\end{bmatrix}$$

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\quad \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\qquad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $\mathbf{x}_1 + 0.2\mathbf{x}_2$:

$$\begin{bmatrix}0.8\\ 0.2\end{bmatrix}=\mathbf{x}_1+0.2\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.2\\ -0.2\end{bmatrix}$$

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\quad \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\qquad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $x_1 + 0.2x_2$:

$$\begin{bmatrix}0.8\\ 0.2\end{bmatrix}=\mathbf{x}_1+0.2\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.2\\ -0.2\end{bmatrix}$$

Multiplying by A gives the first column of A^2 . Do it separately for x_1 and $0.2x_2$.

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\quad \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\qquad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $\mathbf{x}_1+0.2\mathbf{x}_2$:

$$\begin{bmatrix}0.8\\ 0.2\end{bmatrix}=\mathbf{x}_1+0.2\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.2\\ -0.2\end{bmatrix}$$

Multiplying by A gives the first column of $\mathbf{A}^2$. Do it separately for $\mathbf{x}_1$ and $0.2\mathbf{x}_2$.

$$\mathbf{A}\begin{bmatrix}0.8\\ 0.2\end{bmatrix}=\mathbf{A}(\mathbf{x}_1+0.2\mathbf{x}_2)=\mathbf{A}\mathbf{x}_1+0.2\mathbf{A}\mathbf{x}_2=\mathbf{x}_1+0.2\cdot\tfrac{1}{2}\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.1\\ -0.1\end{bmatrix}$$

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\quad \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\qquad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of $\mathbf{A}$ is the combination $\mathbf{x}_1+0.2\mathbf{x}_2$:

$$\begin{bmatrix}0.8\\ 0.2\end{bmatrix}=\mathbf{x}_1+0.2\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.2\\ -0.2\end{bmatrix}$$

Multiplying by $\mathbf{A}$ gives the first column of $\mathbf{A}^2$. Do it separately for $\mathbf{x}_1$ and $0.2\mathbf{x}_2$.

$$\mathbf{A}\begin{bmatrix}0.8\\ 0.2\end{bmatrix}=\mathbf{A}(\mathbf{x}_1+0.2\mathbf{x}_2)=\mathbf{A}\mathbf{x}_1+0.2\mathbf{A}\mathbf{x}_2=\mathbf{x}_1+0.2\cdot\tfrac{1}{2}\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.1\\ -0.1\end{bmatrix}$$

Each eigenvector is multiplied by its eigenvalue, when we multiply by $\mathbf{A}$. We didn't need thse eigenvectors to find $\mathbf{A}^2$. But it is the good way to do 99 multiplications.

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\quad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $\mathbf{x}_1+0.2\mathbf{x}_2$:

$$\begin{bmatrix}0.8\\ 0.2\end{bmatrix}=\mathbf{x}_1+0.2\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.2\\ -0.2\end{bmatrix}$$

Multiplying by A gives the first column of A^2 . Do it separately for $\mathbf{x}_1$ and $0.2\mathbf{x}_2$.

$$\mathbf{A}\begin{bmatrix}0.8\\ 0.2\end{bmatrix}=\mathbf{A}(\mathbf{x}_1+0.2\mathbf{x}_2)=\mathbf{A}\mathbf{x}_1+0.2\mathbf{A}\mathbf{x}_2=\mathbf{x}_1+0.2\cdot\tfrac{1}{2}\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.1\\ -0.1\end{bmatrix}$$

Each eigenvector is multiplied by its eigenvalue, when we multiply by A. We didn't need thse eigenvectors to find A^2. But it is the good way to do 99 multiplications.

$$\mathbf{A}^{99}\begin{bmatrix}0.8\\ 0.2\end{bmatrix}=\mathbf{x}_1+0.2\cdot\left(\tfrac{1}{2}\right)^{99}\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}very\\ small\\ vector\end{bmatrix}$$

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\quad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $\mathbf{x}_1 + 0.2\mathbf{x}_2$:

$$\begin{bmatrix}0.3\\ 0.7\end{bmatrix}=\mathbf{x}_1-0.3\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.3\\ -0.3\end{bmatrix}$$

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\quad \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\qquad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $\mathbf{x}_1+0.2\mathbf{x}_2$:

$$\begin{bmatrix}0.3\\ 0.7\end{bmatrix}=\mathbf{x}_1-0.3\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.3\\ -0.3\end{bmatrix}$$

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\quad \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\qquad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of $\mathbf{A}$ is the combination $\mathbf{x}_1+0.2\mathbf{x}_2$:

$$\begin{bmatrix}0.3\\ 0.7\end{bmatrix}=\mathbf{x}_1-0.3\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.3\\ -0.3\end{bmatrix}$$

Multiplying by $\mathbf{A}$ gives the first column of $\mathbf{A}^2$. Do it separately for $\mathbf{x}_1$ and $0.2\mathbf{x}_2$.

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\quad \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\qquad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $\mathbf{x}_1+0.2\mathbf{x}_2$:

$$\begin{bmatrix}0.3\\ 0.7\end{bmatrix}=\mathbf{x}_1-0.3\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.3\\ -0.3\end{bmatrix}$$

Multiplying by A gives the first column of A^2 . Do it separately for $\mathbf{x}_1$ and $0.2\mathbf{x}_2$.

$$\mathbf{A}\begin{bmatrix}0.3\\ 0.7\end{bmatrix}=\mathbf{A}(\mathbf{x}_1-0.3\mathbf{x}_2)=\mathbf{A}\mathbf{x}_1-0.3\mathbf{A}\mathbf{x}_2=\mathbf{x}_1-0.3\cdot\tfrac{1}{2}\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.15\\ -0.15\end{bmatrix}$$

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\quad \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\qquad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of $\mathbf{A}$ is the combination $\mathbf{x}_1+0.2\mathbf{x}_2$:

$$\begin{bmatrix}0.3\\ 0.7\end{bmatrix}=\mathbf{x}_1-0.3\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.3\\ -0.3\end{bmatrix}$$

Multiplying by $\mathbf{A}$ gives the first column of $\mathbf{A}^2$. Do it separately for $\mathbf{x}_1$ and $0.2\mathbf{x}_2$.

$$\mathbf{A}\begin{bmatrix}0.3\\ 0.7\end{bmatrix}=\mathbf{A}(\mathbf{x}_1-0.3\mathbf{x}_2)=\mathbf{A}\mathbf{x}_1-0.3\mathbf{A}\mathbf{x}_2=\mathbf{x}_1-0.3\cdot\tfrac{1}{2}\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.15\\ -0.15\end{bmatrix}$$

Each eigenvector is multiplied by its eigenvalue, when we multiply by $\mathbf{A}$. We didn't need thse eigenvectors to find $\mathbf{A}^2$. But it is the good way to do 99 multiplications.

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\quad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $\mathbf{x}_1+0.2\mathbf{x}_2$:

$$\begin{bmatrix}0.3\\ 0.7\end{bmatrix}=\mathbf{x}_1-0.3\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.3\\ -0.3\end{bmatrix}$$

Multiplying by A gives the first column of A^2 . Do it separately for $\mathbf{x}_1$ and $0.2\mathbf{x}_2$.

$$\mathbf{A}\begin{bmatrix}0.3\\ 0.7\end{bmatrix}=\mathbf{A}(\mathbf{x}_1-0.3\mathbf{x}_2)=\mathbf{A}\mathbf{x}_1-0.3\mathbf{A}\mathbf{x}_2=\mathbf{x}_1-0.3\cdot\tfrac{1}{2}\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.15\\ -0.15\end{bmatrix}$$

Each eigenvector is multiplied by its eigenvalue, when we multiply by A. We didn't need thse eigenvectors to find A^2. But it is the good way to do 99 multiplications.

$$\mathbf{A}^{99}\begin{bmatrix}0.3\\ 0.7\end{bmatrix}=\mathbf{x}_1-0.3\cdot\left(\tfrac{1}{2}\right)^{99}\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}very\\ small\\ vector\end{bmatrix}$$

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\quad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

$$\mathbf{A}^{99}\begin{bmatrix}0.8\\ 0.2\end{bmatrix}=\mathbf{x}_1+0.2\cdot\left(\tfrac{1}{2}\right)^{99}\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}very\\ small\\ vector\end{bmatrix}$$

$$\mathbf{A}^{99}\begin{bmatrix}0.3\\ 0.7\end{bmatrix}=\mathbf{x}_1-0.3\cdot\left(\tfrac{1}{2}\right)^{99}\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}-\begin{bmatrix}very\\ small\\ vector\end{bmatrix}$$

$$\therefore \mathbf{A}^{99}\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix}=\begin{bmatrix}0.6 & 0.6\\ 0.4 & 0.4\end{bmatrix}$$

$$\therefore \mathbf{A}^{100}=\mathbf{A}^{99}\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix}=\begin{bmatrix}0.6 & 0.6\\ 0.4 & 0.4\end{bmatrix}$$

The eigenvector $\mathbf{x}_1$ is a "steady state" that doesn't change (because $\lambda_1 = 1$).

The eigenvector $\mathbf{x}_2$ is a "decaying mode" that virtually disappears (because $\lambda_2 = 0.5$).

The higher power of A, the closer its columns approach the steady state.

We mention that this particular A is a Markov matrix. Its entries are positive and every column adds to 1. Those facts guarantee that the largest eigenvalue is $\lambda = 1$. Its eigenvector $\mathbf{x}_1 = (0.6, 0.4)$ is the steady state - which all columns of $\mathbf{A}^k$ will approach.

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\qquad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,$$
$$\mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

$$\text{find}: \mathbf{A}^{100}\begin{bmatrix}3.8\\ -0.8\end{bmatrix}$$

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\quad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

$$\text{find}: \mathbf{A}^{100}\begin{bmatrix}3.8\\ -0.8\end{bmatrix}$$

$$\begin{bmatrix}3.8\\ -0.8\end{bmatrix}=3\mathbf{x}_1+2\mathbf{x}_2=3\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+2\begin{bmatrix}1\\ -1\end{bmatrix}$$

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\quad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

$$\text{find}: \mathbf{A}^{100}\begin{bmatrix}3.8\\ -0.8\end{bmatrix}$$

$$\begin{bmatrix}3.8\\ -0.8\end{bmatrix}=3\mathbf{x}_1+2\mathbf{x}_2=3\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+2\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}\begin{bmatrix}3.8\\ -0.8\end{bmatrix}=3\mathbf{A}\mathbf{x}_1+2\mathbf{A}\mathbf{x}_2=3\mathbf{x}_1+2\frac{1}{2}\mathbf{x}_2$$

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\quad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

$$\text{find}: \mathbf{A}^{100}\begin{bmatrix}3.8\\ -0.8\end{bmatrix}$$

$$\begin{bmatrix}3.8\\ -0.8\end{bmatrix}=3\mathbf{x}_1+2\mathbf{x}_2=3\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+2\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}\begin{bmatrix}3.8\\ -0.8\end{bmatrix}=3\mathbf{A}\mathbf{x}_1+2\mathbf{A}\mathbf{x}_2=3\mathbf{x}_1+2\frac{1}{2}\mathbf{x}_2$$

$$=3\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+2\frac{1}{2}\begin{bmatrix}1\\ -1\end{bmatrix}$$

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\quad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

$$\text{find}: \mathbf{A}^{100}\begin{bmatrix}3.8\\ -0.8\end{bmatrix}$$

$$\begin{bmatrix}3.8\\ -0.8\end{bmatrix}=3\mathbf{x}_1+2\mathbf{x}_2=3\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+2\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}\begin{bmatrix}3.8\\ -0.8\end{bmatrix}=3\mathbf{A}\mathbf{x}_1+2\mathbf{A}\mathbf{x}_2=3\mathbf{x}_1+2\frac{1}{2}\mathbf{x}_2$$

$$=3\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+2\frac{1}{2}\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^2\begin{bmatrix}3.8\\ -0.8\end{bmatrix}=3\mathbf{A}^2\mathbf{x}_1+2\mathbf{A}^2\mathbf{x}_2=3\mathbf{x}_1+2\left(\frac{1}{2}\right)^2\mathbf{x}_2$$

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\quad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

find : $\mathbf{A}^{100}\begin{bmatrix}3.8\\ -0.8\end{bmatrix}$

$$\begin{bmatrix}3.8\\ -0.8\end{bmatrix}=3\mathbf{x}_1+2\mathbf{x}_2=3\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+2\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}\begin{bmatrix}3.8\\ -0.8\end{bmatrix}=3\mathbf{A}\mathbf{x}_1+2\mathbf{A}\mathbf{x}_2=3\mathbf{x}_1+2\frac{1}{2}\mathbf{x}_2$$

$$=3\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+2\frac{1}{2}\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^2\begin{bmatrix}3.8\\ -0.8\end{bmatrix}=3\mathbf{A}^2\mathbf{x}_1+2\mathbf{A}^2\mathbf{x}_2=3\mathbf{x}_1+2\left(\frac{1}{2}\right)^2\mathbf{x}_2$$

$$=3\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+2\left(\frac{1}{2}\right)^2\begin{bmatrix}1\\ -1\end{bmatrix}$$

Markov Matrix

always $\lambda_1 = 1,\ |\lambda_2| < 1$

Given : Markov matrix A

$$\mathbf{A} = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}$$

$$0 \le a,\ b < 1$$

Find : The eigenvalues of A

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a-\lambda & b \\ 1-a & 1-b-\lambda \end{vmatrix}$$

$$= (a-\lambda)(1-b-\lambda) - b(1-a)$$

$$= \lambda^2 - (1+a-b)\lambda + a - ab - b + ab$$

$$= \lambda^2 - (1+a-b)\lambda + a - b$$

$$= \lambda^2 - (1+a-b)\lambda + 1\cdot(a-b)$$

$$= (\lambda - 1)(\lambda - (a-b))$$

$$(\lambda - 1)(\lambda - (a-b)) = 0$$

$$\therefore \lambda_1 = 1,\ \lambda_2 = a - b$$

$$0 < a < 1$$

$$0 < b < 1$$

$$0 - 1 = -1 < \lambda_2 = a - b < 1 = 1 - 0$$

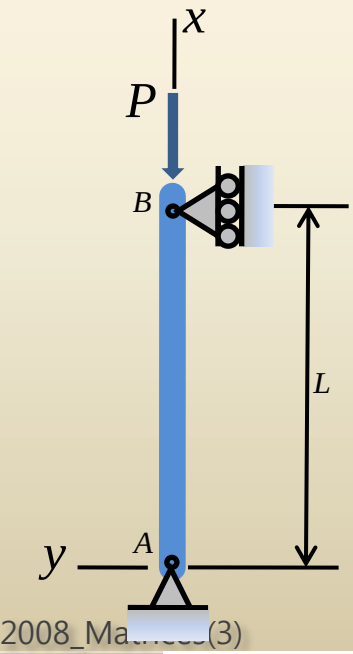
$$\therefore \lambda_1 = 1,\ |\lambda_2| < 1$$

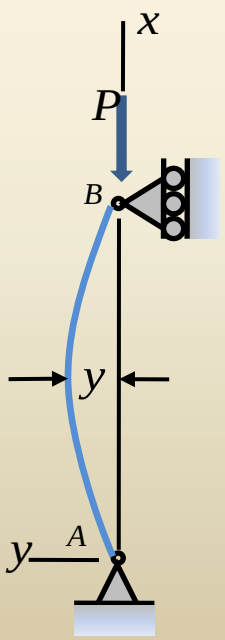
Buckling of a Thin Vertical Column*

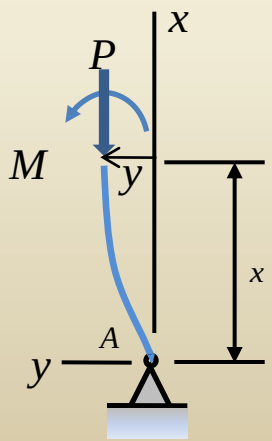
- ☑ 정의 : 기둥의 길이가 그 단면의 치수에 비해 클 때, 기둥의 끝단에 압축하중이 가해졌을 경우 하중이 어느 크기에 이르면 내력과 외력의 평형이 무너져 기둥이 휘는 현상.
- ☑ 좌굴된 기둥은 마치 보인 것 처럼 굽어지기 때문에, 보의 처짐곡선을 적용할 수 있다.
  - 보의 굽힘모멘트 방정식 : $EIy'' = M$
  - A점에 대한 모멘트의 평형 : $M + Py = 0$

$$\boxed{EIy'' + Py = 0}$$

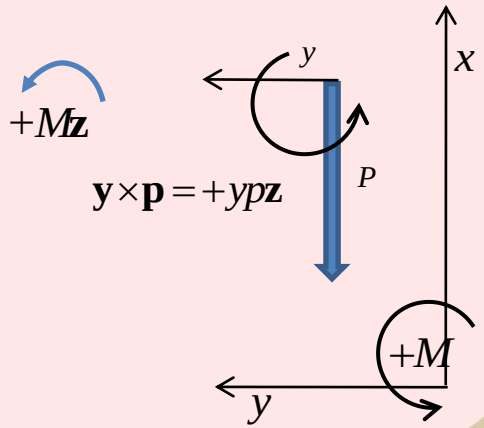
Bucking 대한 미분방정식







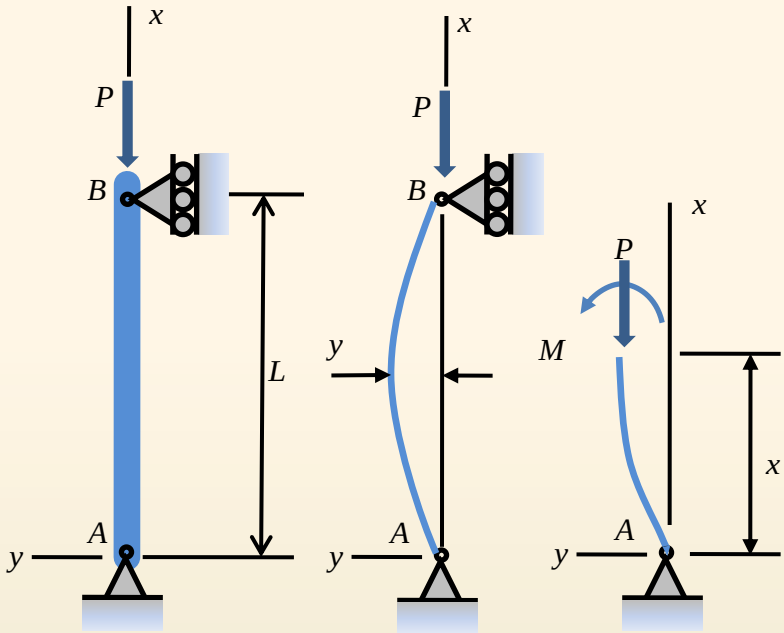
$$\sum M_z = M + py = 0$$



James M. Gere, Mechanics of Materials 6th Edition, Thomson, pp. 748-762
Zill D.G., Cullen M.R., Advanced Engineering Mathematics, 9th edition, Jones and Bartlett, 2006, p169

Buckling of a Thin Vertical Column

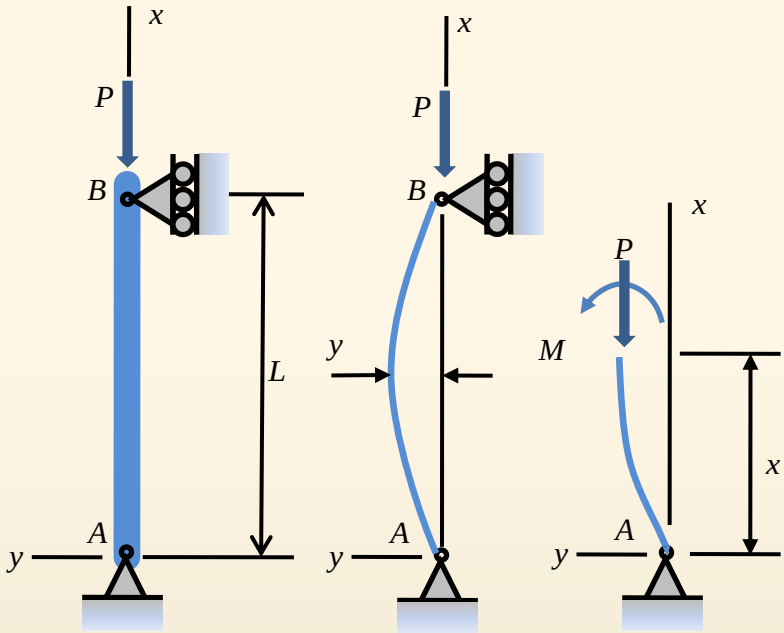
Buckling of a Thin Vertical Column



Buckling of a Thin Vertical Column

Buckling of a Thin Vertical Column

$$EI\frac{d^2y}{dx^2}+Py=0$$

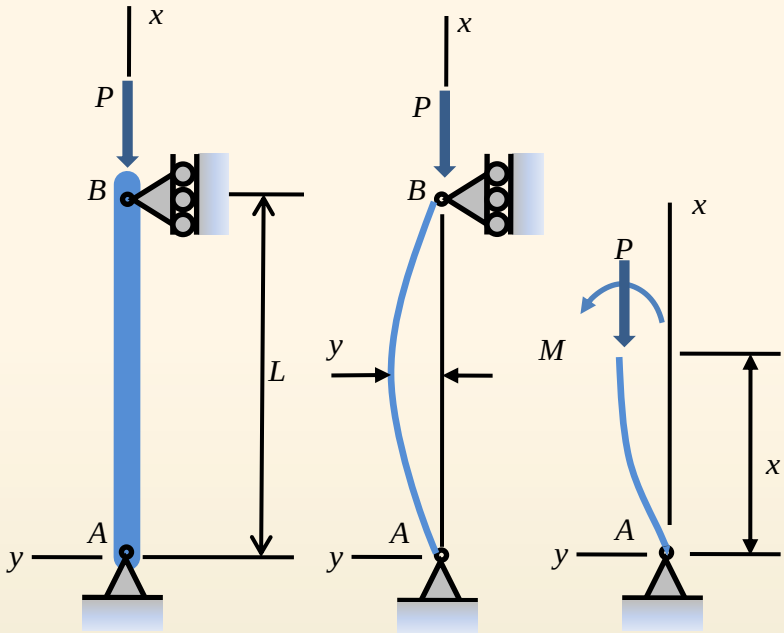


Buckling of a Thin Vertical Column

Buckling of a Thin Vertical Column

$$EI\frac{d^2y}{dx^2}+Py=0$$

When both end side of Vertical Column, Boundary condition is



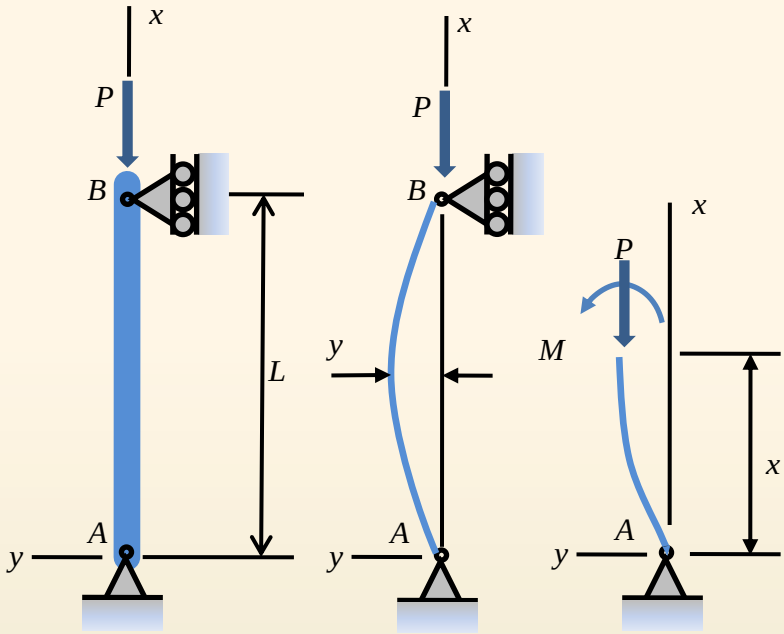
Buckling of a Thin Vertical Column

Buckling of a Thin Vertical Column

$$EI\frac{d^2y}{dx^2}+Py=0$$

When both end side of Vertical Column, Boundary condition is

$$y(0)=0,\quad y(L)=0$$



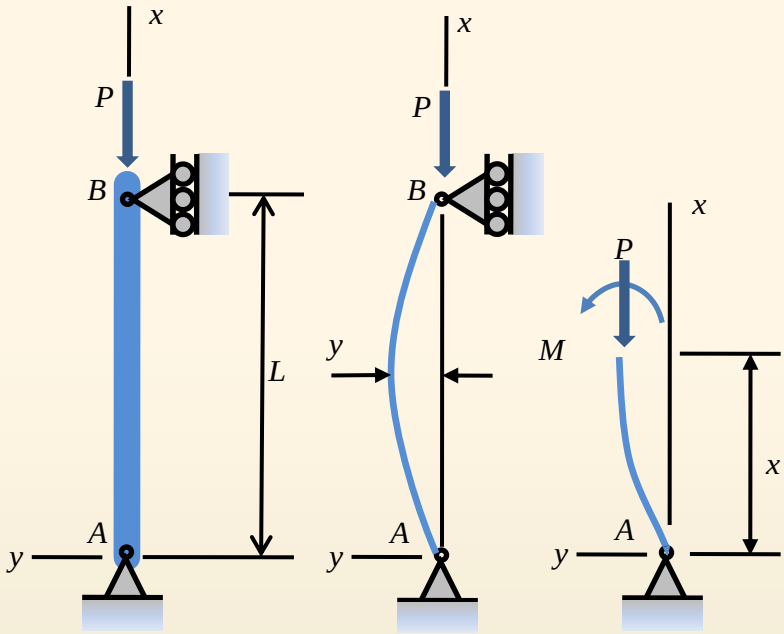
Buckling of a Thin Vertical Column

Buckling of a Thin Vertical Column

$$EI\frac{d^2y}{dx^2}+Py=0$$

When both end side of Vertical Column, Boundary condition is

$$y(0)=0,\quad y(L)=0$$



By writing

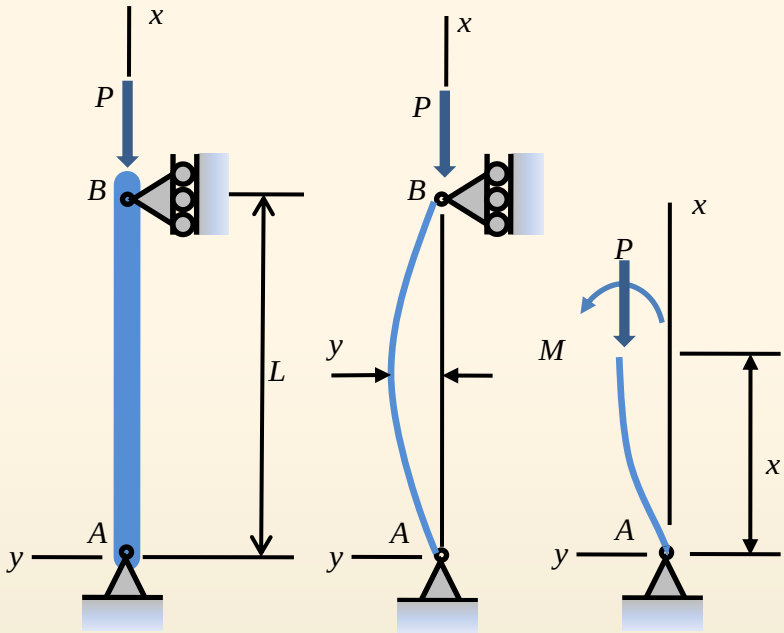
Buckling of a Thin Vertical Column

Buckling of a Thin Vertical Column

$$EI\frac{d^2y}{dx^2}+Py=0$$

When both end side of Vertical Column, Boundary condition is

$$y(0)=0,\quad y(L)=0$$



By writing $\lambda=k^2=\frac{P}{EI}$

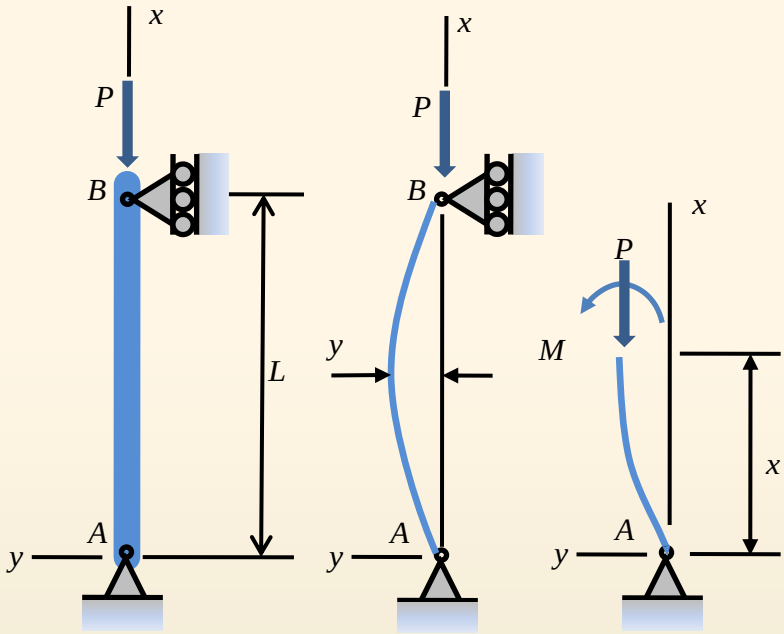
Buckling of a Thin Vertical Column

Buckling of a Thin Vertical Column

$$EI\frac{d^2y}{dx^2}+Py=0$$

When both end side of Vertical Column, Boundary condition is

$$y(0)=0,\quad y(L)=0$$



By writing $\lambda=k^2=\dfrac{P}{EI}$

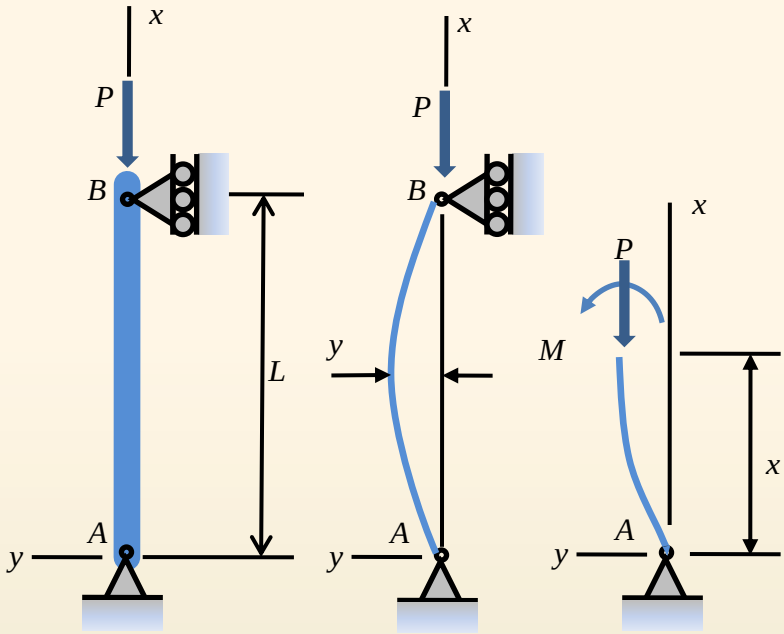
$$y''+\lambda y=0,\ y(0)=0,\ y(L)=0$$

$$y''+k^2y=0$$

Buckling of a Thin Vertical Column

Buckling of a Thin Vertical Column

$$EI\frac{d^2y}{dx^2}+Py=0$$



When both end side of Vertical Column, Boundary condition is

$$y(0)=0,\quad y(L)=0$$

By writing $\lambda=k^2=\dfrac{P}{EI}$

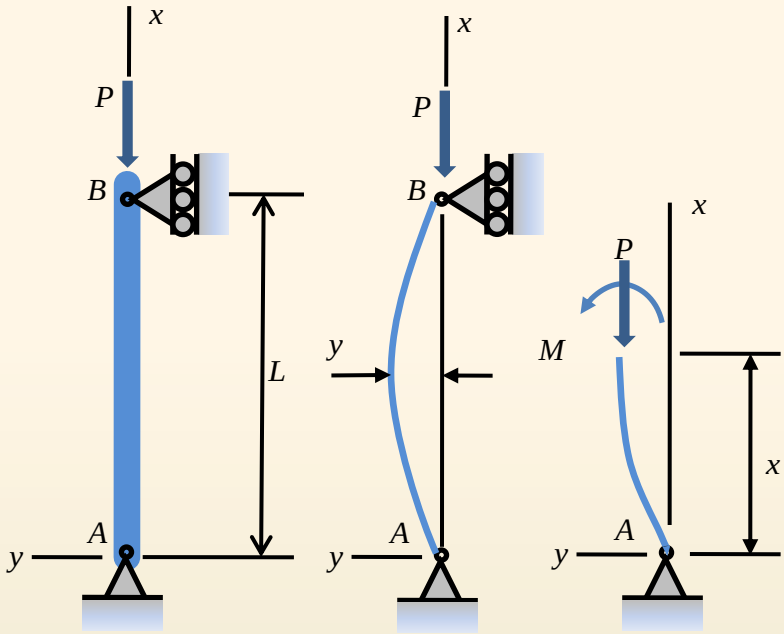
$$y''+\lambda y=0,\ y(0)=0,\ y(L)=0$$

$$y''+k^2y=0$$

Buckling of a Thin Vertical Column

Buckling of a Thin Vertical Column

$$EI\frac{d^2y}{dx^2}+Py=0$$



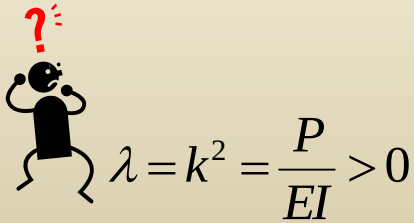
When both end side of Vertical Column, Boundary condition is

$$y(0)=0,\ \ y(L)=0$$

By writing $\lambda=k^2=\dfrac{P}{EI}$

$$y''+\lambda y=0,\ y(0)=0,\ \ y(L)=0$$

$$y''+k^2y=0$$



$$\lambda=k^2=\frac{P}{EI}>0$$

Buckling of a Thin Vertical Column

$$\lambda = k^2 = \frac{P}{EI} > 0 \qquad y(0) = 0,\ y(L) = 0$$

Case I : $\lambda = 0$

$$y'' = 0$$

$$y = c_1 x + c_2$$

integral

From boundary condition:

$$c_1 = 0,\ c_2 = 0$$

$$\therefore y = 0 \quad \text{(trivial solution)}$$

Case II : $\lambda < 0$ **Write** $\lambda = -k^2,\ k > 0$

$$y'' - k^2 y = 0$$

Let: $y = e^{mx}$

$$\rightarrow (m^2 - k^2)e^{mx} = 0$$

$$\rightarrow m_1 = k,\ m_2 = -k$$

$$y = c_1 e^{kx} + c_2 e^{-kx}$$

From boundary condition:

$$y(0) = c_1 + c_2 = 0, c_2 = -c_1$$

$$y(L) = c_1 e^{kL} + c_2 e^{-kL}$$

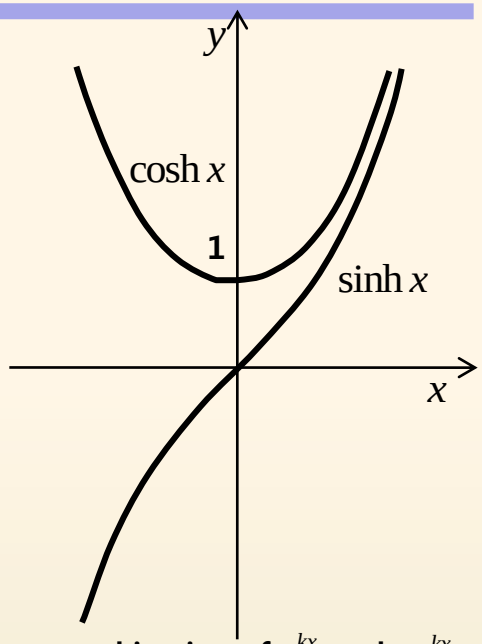
$$= c_1 (e^{kL} - e^{-kL}) = 0$$

If $(e^{kL} - e^{-kL}) = 0$

$$\rightarrow e^{2kL} = 1,$$

$$\rightarrow k = 0\ (\because L \neq 0)$$ ⇨ **Case I : trivial sol.**

If $c_1 = 0 \rightarrow c_2 = 0$ ⇨ **trivial sol.**



Linear combination of e^{kx} **and** e^{-kx}

(basis transformation)

$$\frac{e^{kx} + e^{-kx}}{2} = \cosh kx, \qquad \frac{e^{kx} - e^{-kx}}{2} = \sinh kx$$

$$y = c_1 \cosh kx + c_2 \sinh kx$$

From boundary condition:

$$y(0) = c_1 = 0$$

$$y(L) = c_2 \sinh kL = 0$$

If $k = 0\ (\because L \neq 0)$ ⇨ **Case I : trivial sol.**

If $c_2 = 0$ ⇨ **trivial sol.**

*James M. Gere, Mechanics of Materials 6th Edition, Thomson, pp. 748-762

Buckling of a Thin Vertical Column

$$\lambda = k^2 = \frac{P}{EI} > 0 \qquad y(0) = 0,\ y(L) = 0$$

Case III : $\lambda > 0$ Write $\lambda = k^2,\ k > 0$

$$y'' + k^2 y = 0$$

Let: $y = e^{mx}$

Then roots of auxiliary equation is

$$\rightarrow\ (m^2 + k^2)e^{mx} = 0$$

$$\rightarrow\ m_1 = ik,\ m_2 = -ik$$

$$\therefore y = c_1 \cos kx + c_2 \sin kx$$

$$y(0) = C_1 = 0$$

$$y(L) = C_2 \sin kL = 0$$

If $c_2 = 0 : y = 0$ 이므로 trivial solution.

$$\therefore c_2 \neq 0,\ \sin kL = 0$$

$$\therefore kL = 0,\ \pi,\ 2\pi, \ldots$$

If $kL = 0 : k = 0\ (\because L \neq 0)$ ⇨ Case I : trivial sol.

$$\therefore kL = n\pi,\ n = 1, 2, 3\ldots$$

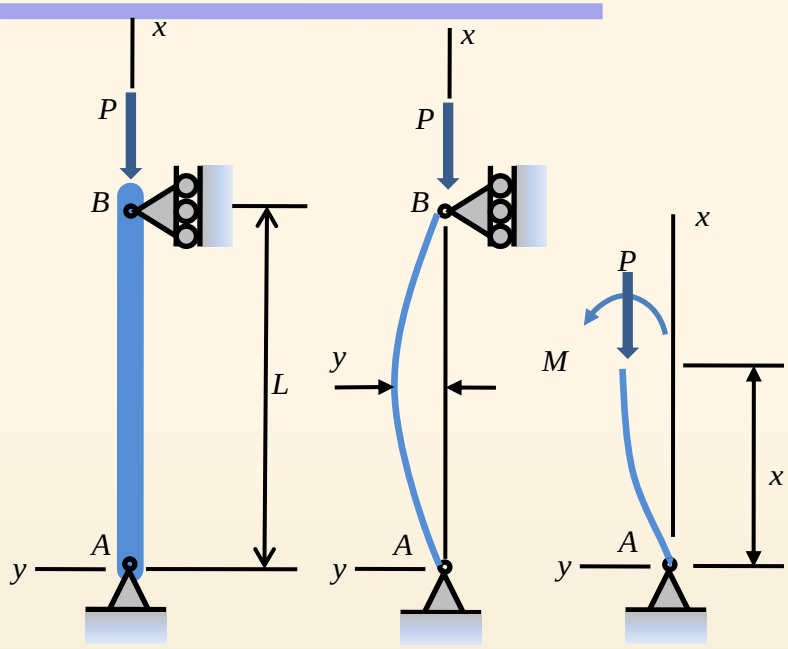
$$\therefore y_n = C_2 \sin kx = C_2 \sin \frac{n\pi x}{L},\ n = 1,\ 2,\ 3\ldots$$

*James M. Gere, Mechanics of Materials 6th Edition, Thomson, pp. 748-762

Buckling of a Thin Vertical Column

$$y'' + k^2 y = 0,\ y(0) = 0,\ y(L) = 0$$

$$\lambda = k^2 = \frac{P}{EI}$$



Homogeneous solution: $y = C_1 \sin kx + C_2 \cos kx$

Boundary conditions: $y(0) = 0,\ y(L) = 0$

$y(0) = C_2 = 0$

$y(L) = C_1 \sin kL = 0$

If $c_1 = 0 : y = 0$ 이므로 trivial solution.

$\therefore c_1 \neq 0, \sin kL = 0$

$\therefore kL = 0,\ \pi,\ 2\pi, \ldots$

If $kL = 0 : k = 0\ (\because L \neq 0)$ ⇨ **trivial sol.**

$\therefore kL = n\pi,\ n = 1, 2, 3\ldots$

$$\therefore y_n = C_1 \sin kx = C_1 \sin \frac{n\pi x}{L},\ n = 1,\ 2,\ 3\ldots$$ **:Deflection curve**

Eigenvalues:

$$\lambda_n = k_n^{\ 2} = \left(\frac{n\pi}{L}\right)^2$$

James M. Gere, Mechanics of Materials 6th Edition, Thomson, pp. 748-762

Buckling of a Thin Vertical Column

Buckling of a Thin Vertical Column

$$y_n(x) = c_1 \sin(\frac{n\pi x}{L})$$

Buckling of a Thin Vertical Column

$$y_n(x) = c_1 \sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{P_n}{EI} = \frac{n^2\pi^2}{L^2}$$

$$P_n = \frac{n^2\pi^2 EI}{L^2},\ n = 1, 2, 3, ..$$ **(Critical loads)**

Buckling of a Thin Vertical Column

$$y_n(x) = c_1 \sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{P_n}{EI} = \frac{n^2\pi^2}{L^2}$$

$$P_n = \frac{n^2\pi^2 EI}{L^2},\ n = 1, 2, 3, ..$$ **(Critical loads)**

When n=1 $P_1 = \frac{\pi^2 EI}{L^2}$ **(Euler load)**

Buckling of a Thin Vertical Column

$$y_n(x) = c_1 \sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{P_n}{EI} = \frac{n^2\pi^2}{L^2}$$

$$P_n = \frac{n^2\pi^2 EI}{L^2},\ n = 1, 2, 3, ..$$ **(Critical loads)**

When n=1 $P_1 = \frac{\pi^2 EI}{L^2}$ **(Euler load)**

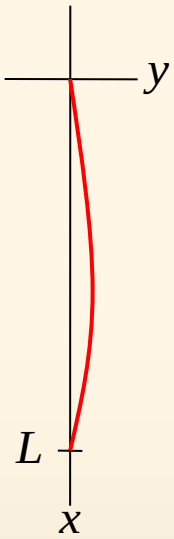
$$y_1(x) = c_1 \sin\left(\frac{\pi x}{L}\right)$$ **(first buckling mode)**

Buckling of a Thin Vertical Column

$$y_n(x) = c_1 \sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{P_n}{EI} = \frac{n^2\pi^2}{L^2}$$

$$P_n = \frac{n^2\pi^2 EI}{L^2},\ n = 1, 2, 3, ..$$ **(Critical loads)**



n=1

When n=1 $P_1 = \frac{\pi^2 EI}{L^2}$ **(Euler load)**

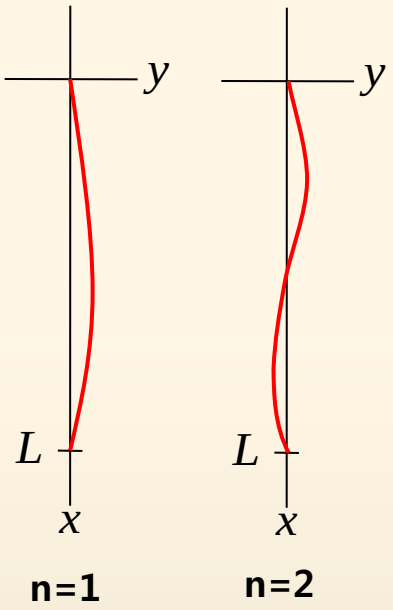
$$y_1(x) = c_1 \sin\left(\frac{\pi x}{L}\right)$$ **(first buckling mode)**

Buckling of a Thin Vertical Column

$$y_n(x) = c_1 \sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{P_n}{EI} = \frac{n^2\pi^2}{L^2}$$

$$P_n = \frac{n^2\pi^2 EI}{L^2},\ n = 1, 2, 3, ..$$ **(Critical loads)**



When n=1 $P_1 = \frac{\pi^2 EI}{L^2}$ **(Euler load)**

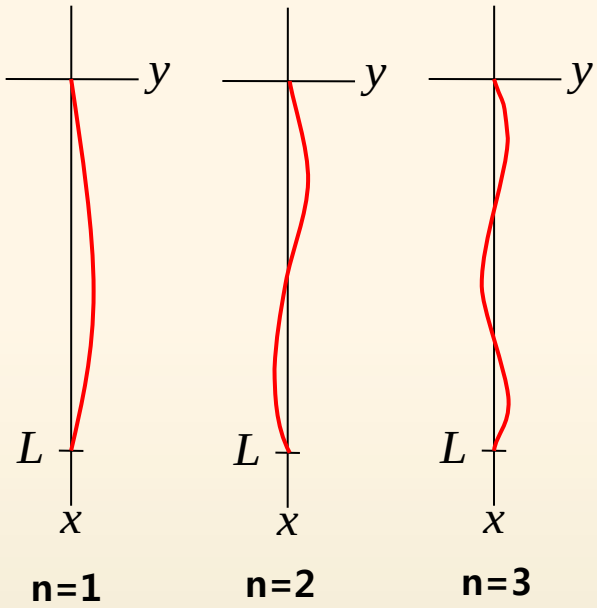
$$y_1(x) = c_1 \sin\left(\frac{\pi x}{L}\right)$$ **(first buckling mode)**

Buckling of a Thin Vertical Column

$$y_n(x) = c_1 \sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{P_n}{EI} = \frac{n^2\pi^2}{L^2}$$

$$P_n = \frac{n^2\pi^2 EI}{L^2},\ n = 1, 2, 3, ..$$ **(Critical loads)**

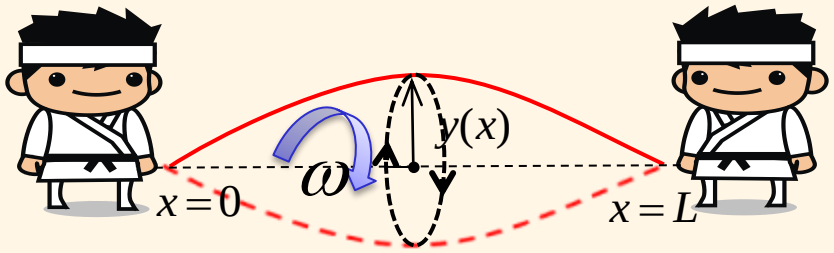


When n=1 $P_1 = \frac{\pi^2 EI}{L^2}$ **(Euler load)**

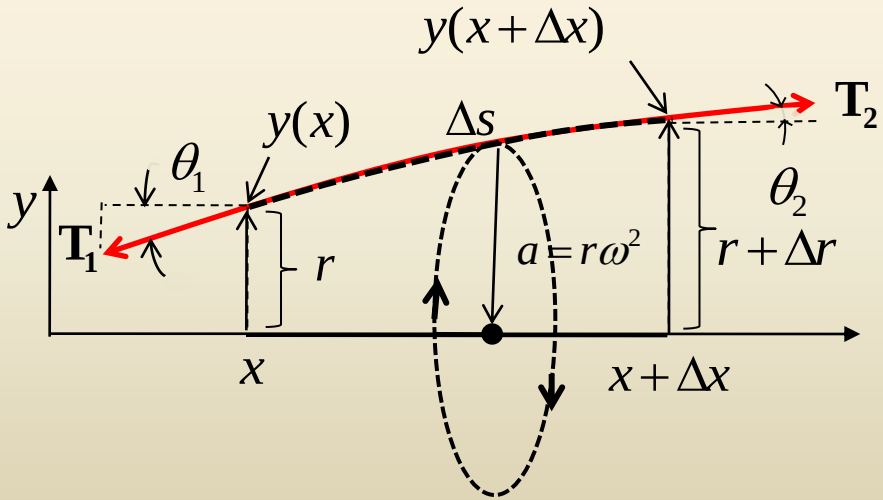
$$y_1(x) = c_1 \sin\left(\frac{\pi x}{L}\right)$$ **(first buckling mode)**

Rotating String

ρ : string density
ω : string angular velocity
T : magnitude of tension



ρ : string density
ω : string angular velocity
T : magnitude of tension



$$\sum F_x = \mathbf{T}_1\cos\theta_1 - \mathbf{T}_2\cos\theta_2 = 0$$

$$\mathbf{T}_1\cos\theta_1 = \mathbf{T}_2\cos\theta_2 = T$$

$$\sum F_y = \mathbf{T}_2\sin\theta_2 - \mathbf{T}_1\sin\theta_1$$

$$= T\frac{\sin\theta_2}{\cos\theta_2} - T\frac{\sin\theta_1}{\cos\theta_1} \because \mathbf{T}_1\cos\theta_1 = \mathbf{T}_2\cos\theta_2 = T$$

$$= T\tan\theta_2 - T\tan\theta_1$$

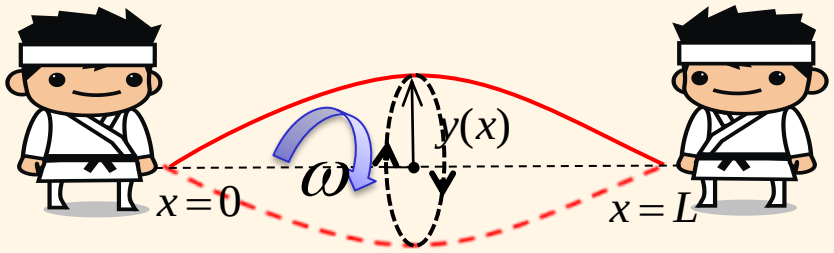
$$= T[y'(x+\Delta x) - y'(x)]$$

$$\because \tan\theta_1 = \left.\frac{dy}{dx}\right|_x, \tan\theta_2 = \left.\frac{dy}{dx}\right|_{x+\Delta x}$$

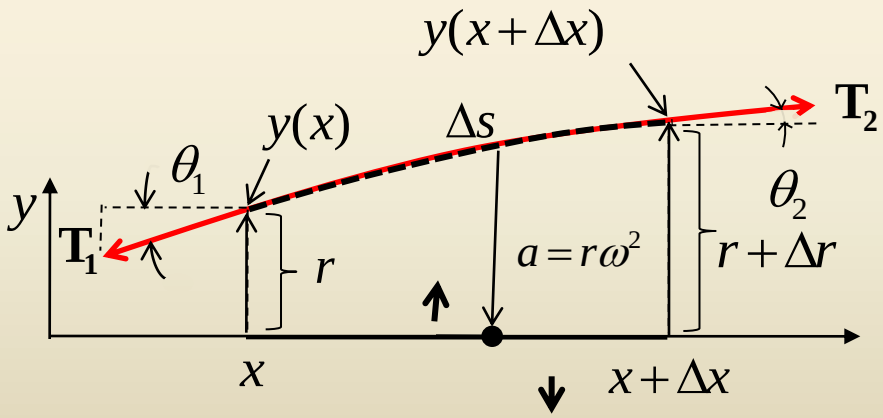
Zill D.G., Cullen M.R., Advanced Engineering Mathematics, 9th edition, Jones and Bartlett, 2006, p170

Rotating String

ρ : string density
ω : string angular velocity
T : magnitude of tension



ρ : string density
ω : string angular velocity
T : magnitude of tension



$$\sum F_y = T[y'(x+\Delta x)-y'(x)]$$

assum.: $\Delta x \ll 1$ linearization

Acceleration point in the direction opposite to the positive y direction

Mass: $m=\rho\Delta s \approx \rho\Delta x$

Centripetal acceleration: $a=\ominus r\omega^2$

When Δx is small,

$$r+\Delta r = r + r'\Delta x + \cdots = y + y'\Delta x + \cdots$$

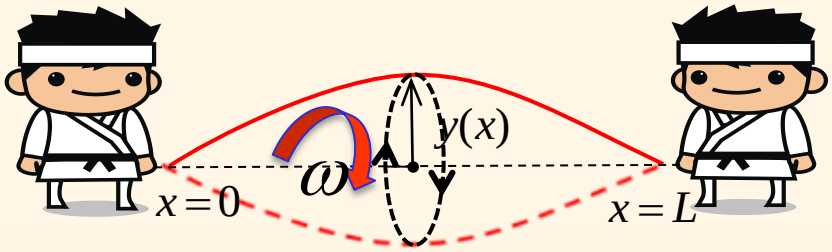
linearization

$$r+\Delta r \approx r = y\,,\; a=-r\omega^2=-y\omega^2$$

$$\therefore \sum F_y = ma \approx -(\rho\Delta x)y\omega^2$$

Rotating String

ρ : string density
ω : string angular velocity
T : magnitude of tension



$$\sum F_y = T[y'(x+\Delta x) - y'(x)]$$
$$\sum F_y = ma \approx -(\rho \Delta x) y \omega^2$$

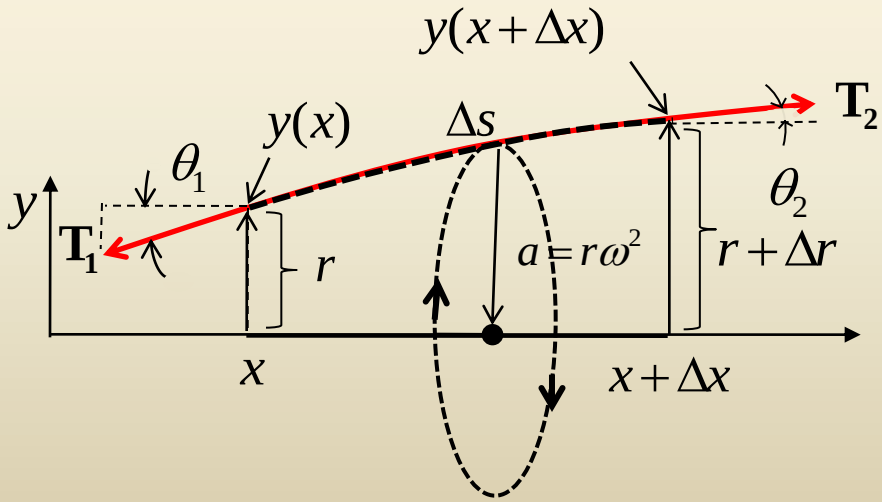
$$\therefore T[y'(x+\Delta x) - y'(x)] = -(\rho \Delta x) y \omega^2$$

$$T\frac{y'(x+\Delta x) - y'(x)}{\Delta x} + \rho\omega^2 y = 0$$

$$\frac{y'(x+\Delta x) - y'(x)}{\Delta x} \approx \frac{d^2y}{dx^2}$$

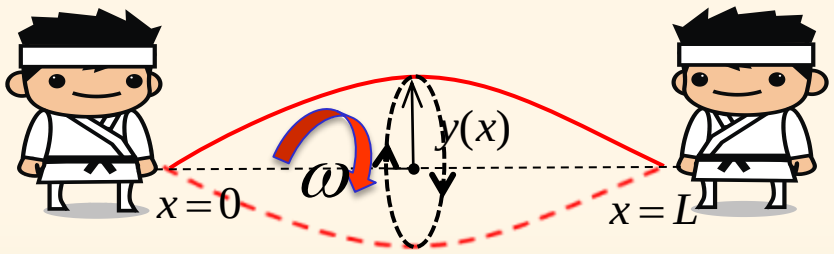
$$\therefore T\frac{d^2y}{dx^2} + \rho\omega^2 y = 0$$

ρ : string density
ω : string angular velocity
T : magnitude of tension

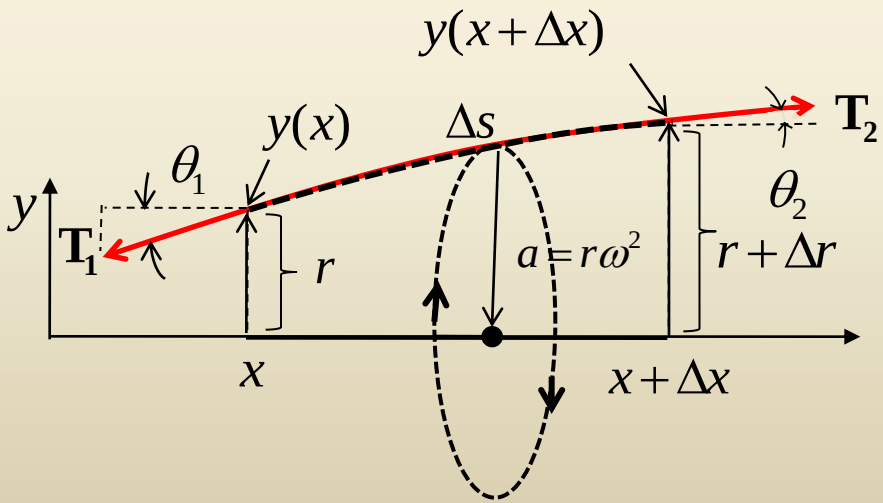


Rotating String

ρ : string density
ω : string angular velocity
T : magnitude of tension



ρ : string density
ω : string angular velocity
T : magnitude of tension



$$\sum F_y = T[y'(x+\Delta x) - y'(x)] \ , ma \approx -(\rho \Delta x) y \omega^2$$

$$\therefore T\frac{d^2y}{dx^2} + \rho\omega^2 y = 0$$

What if ..

$$r + \Delta r = r + r'\Delta x + \cdots = y + y'\Delta x + \cdots$$

$$then, a = -(r+\Delta r)\omega^2 = -(y + y'\Delta x)\omega^2$$

$$\therefore T[y'(x+\Delta x) - y'(x)] = -(\rho\Delta x)(y + y'\Delta x)\omega^2$$

$$\Rightarrow T\frac{d^2y}{dx^2} + \rho\omega^2 \Delta x y' + \rho\omega^2 y = 0$$

Not a form of $\frac{dy}{dx}$ or polynomial of x

$\Delta x y' \approx 0 \because \Delta x \ll 1, \Delta s \approx \Delta x$ means y'(=dy/dx) is small too

$$\therefore T\frac{d^2y}{dx^2} + \rho\omega^2 y = 0$$

Linear Models: Boundary-Value Problems

Eigenvalues and Eigenfunctions

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

Case I : $\lambda = 0$

$$y'' = 0$$

integral

$$y = c_1 x + c_2$$

From boundary condition:

$$c_1 = 0,\ c_2 = 0$$

$$\therefore y = 0 \text{ (trivial solution)}$$

Linear Models: Boundary-Value Problems

Eigenvalues and Eigenfunctions

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

Case II : $\lambda < 0$ **Write** $\lambda = -\alpha^2, \ \alpha > 0$

$$y'' - \alpha^2 y = 0$$

Let: $y = e^{mx}$

$$\rightarrow (m^2 - \alpha^2)e^{mx} = 0 \rightarrow m_1 = \alpha, \ m_2 = -\alpha$$

$$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

From boundary condition:

$$y(0) = c_1 + c_2 = 0, c_2 = -c_1$$

$$y(L) = c_1 e^{\alpha L} + c_2 e^{-\alpha L}$$

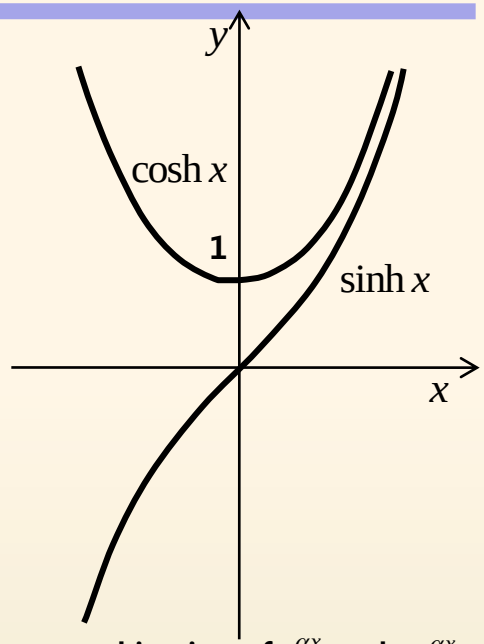
$$= c_1(e^{\alpha L} - e^{-\alpha L}) = 0$$

If $(e^{\alpha L} - e^{-\alpha L}) = 0$

$$\rightarrow e^{2\alpha L} = 1,$$

$$\rightarrow \alpha = 0 \ (\because L \neq 0)$$ ⇨ **Case I : trivial sol.**

If $c_1 = 0 \rightarrow c_2 = 0$ ⇨ **trivial sol.**



Linear combination of $e^{\alpha x}$ **and** $e^{-\alpha x}$

(basis transformation)

$$\frac{e^{\alpha x} + e^{-\alpha x}}{2} = \cosh \alpha x, \quad \frac{e^{\alpha x} - e^{-\alpha x}}{2} = \sinh \alpha x$$

$$y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$$

From boundary condition:

$$y(0) = c_1 = 0$$

$$y(L) = c_2 \sinh \alpha L = 0$$

If $\alpha = 0 \ (\because L \neq 0)$ ⇨ **Case I : trivial sol.**

If $c_2 = 0$ ⇨ **trivial sol.**

Linear Models: Boundary-Value Problems

Eigenvalues and Eigenfunctions

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

Case III : $\lambda > 0$ Write $\lambda = \alpha^2, \ \alpha > 0$

$$y'' + \alpha^2 y = 0$$

Let: $y = e^{mx}$

Then roots of auxiliary equation is

$$\rightarrow \ (m^2 + \alpha^2)e^{mx} = 0$$

$$\rightarrow \ m_1 = i\alpha, \ m_2 = -i\alpha$$

$$\therefore y = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$y(0) = C_1 = 0$$

$$y(L) = C_2 \sin \alpha L = 0$$

If $c_2 = 0 : y = 0$ 이므로 trivial solution.

$$\therefore c_2 \neq 0 \ , \sin \alpha L = 0$$

$$\therefore \alpha L = 0, \ \pi, \ 2\pi, \ldots$$

If $\alpha L = 0 : \alpha = 0 \ (\because L \neq 0)$ ⇨ Case I : trivial sol.

$$\therefore \alpha L = n\pi, \ n = 1, 2, 3 \ldots$$

$$\therefore y_n = C_2 \sin \alpha x = C_2 \sin \frac{n\pi x}{L}, \ n = 1, \ 2, \ 3 \ldots$$

Linear Models: Boundary-Value Problems

Eigenvalues and Eigenfunctions

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

Case III : $\lambda > 0$ **Write** $\lambda = \alpha^2, \ \alpha > 0$

$$y'' + \alpha^2 y = 0$$

Then roots of auxiliary equation is

Let: $y = e^{mx} \longrightarrow (m^2 + \alpha^2)e^{mx} = 0 \longrightarrow m_1 = i\alpha, \ m_2 = -i\alpha$

$$\therefore y = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$y(0) = 0 \Rightarrow c_1 = 0$$

$$y(L) = 0 \Rightarrow c_2 = 0 \ : \textbf{Trivial}$$

$$\therefore \alpha L = n\pi, \quad \lambda_n = \alpha_n{}^2 = (\frac{n\pi}{L})^2, y_n = c_2 \sin \sqrt{\frac{n\pi}{L}} x$$

$$y_1 = \sin \frac{\pi}{L}, \ y_2 = \sin \frac{2\pi}{L}, \ y_3 = \sin \frac{3\pi}{L}, \cdots$$ **(nontrivial solution)**

Inverse of a matrix.
Gauss-Jordan Elimination

Notation of inverse matrix

In this inverse section, only *square matrices* are considered exclusively.

Notation of inverse of an $n \times n$ matrix $A=[a_{jk}]$: A^{-1}

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$, where I is the $n \times n$ unit matrix.

Nonsingular matrix : A matrix that has an inverse.

(If a matrix has an inverse, the inverse is unique)

Singular matrix : A matrix that has no inverse.

Proof of uniqueness of inverse matrix

If B and C are inverses of A $(\mathbf{AB} = \mathbf{I} \ \& \ \mathbf{CA} = \mathbf{I})$,

We obtain $$\mathbf{B} = \mathbf{IB} = (\mathbf{CA})\mathbf{B} = \mathbf{C}(\mathbf{AB}) = \mathbf{CI} = \mathbf{C}$$

(the uniqueness of inverse)

Inverse by the Gauss-Jordan Method

For Practical determination of the inverse A^{-1} of a nonsingular $n \times n$ matrix A, Gauss elimination can be used.
: This method is called Gauss-Jordan elimination

Inverse by the Gauss-Jordan Method

For Practical determination of the inverse $\mathbf{A}^{-1}$ of a nonsingular $n \times n$ matrix $\mathbf{A}$, Gauss elimination can be used.
: This method is called Gauss-Jordan elimination

Step 1. Make augmented matrix. $\tilde{\mathbf{A}} = [\mathbf{A}\ \mathbf{I}]$

Inverse by the Gauss-Jordan Method

For Practical determination of the inverse $\mathbf{A}^{-1}$ of a nonsingular $n \times n$ matrix $\mathbf{A}$, Gauss elimination can be used.
: This method is called Gauss-Jordan elimination

Step 1. Make augmented matrix. $\tilde{\mathbf{A}} = [\mathbf{A}\ \mathbf{I}]$

Step 2. Make Multiplication of $\mathbf{AX}=\mathbf{I}$ by $\mathbf{A}^{-1}$
(by applying Gauss elimination to $\tilde{\mathbf{A}} = [\mathbf{A}\ \mathbf{I}]$**)**

Inverse by the Gauss-Jordan Method

For Practical determination of the inverse A^{-1} of a nonsingular $n \times n$ matrix A, Gauss elimination can be used.
: This method is called Gauss-Jordan elimination

Step 1. Make augmented matrix. $\tilde{A} = [A\ I]$

Step 2. Make Multiplication of $AX=I$ by A^{-1}
(by applying Gauss elimination to $\tilde{A} = [A\ I]$**)**

⟶ **This gives a matrix of the form $[U\ \ H]$**

Inverse by the Gauss-Jordan Method

For Practical determination of the inverse A^{-1} of a nonsingular $n \times n$ matrix A, Gauss elimination can be used.
: This method is called Gauss-Jordan elimination

Step 1. Make augmented matrix. $\tilde{A} = [A\ I]$

Step 2. Make Multiplication of $AX=I$ by A^{-1}
(by applying Gauss elimination to $\tilde{A} = [A\ I]$**)**

⟶ This gives a matrix of the form $[U\ H]$

Step 3. Reduce U by further elementary row operations to diagonal form.
(Eliminate the entries of U above the main diagonal and making the diagonal entries all 1 by multiplication. See the example next page.)

Inverse of a Matrix. Gauss-Jordan elimination.

Determine the inverse $\mathbf{A}^{-1}$ of

$$\mathbf{A}=\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

Inverse of a Matrix. Gauss-Jordan elimination.

Determine the inverse $\mathbf{A}^{-1}$ of

$$\mathbf{A}=\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

Step 1. Make augmented matrix.

$$[\mathbf{A}\ \mathbf{I}]=\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Inverse of a Matrix. Gauss-Jordan elimination.

Determine the inverse A^{-1} of

$$\mathbf{A}=\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

Step 1. Make augmented matrix.

$$[\mathbf{A}\ \mathbf{I}]=\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Step 2. Make Multiplication of $AX=I$ by A^{-1} by applying Gauss elimination to

Inverse of a Matrix. Gauss-Jordan elimination

$$\mathbf{AX}=\mathbf{I} \;\Rightarrow\; \mathbf{A}^{-1}\mathbf{AX}=\mathbf{A}^{-1}\mathbf{I} \;\Rightarrow\; \mathbf{IX}=\mathbf{A}^{-1}$$

$$[\mathbf{A}\,|\,\mathbf{I}] \;\Rightarrow\; \text{Gauss-Jordan Elimination} \;\Rightarrow\; [\mathbf{I}\,|\,\mathbf{A}^{-1}]$$

$$[\mathbf{A}\ \mathbf{I}]=\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Inverse of a Matrix. Gauss-Jordan elimination

$$\mathbf{AX}=\mathbf{I} \;\Rightarrow\; \mathbf{A}^{-1}\mathbf{AX}=\mathbf{A}^{-1}\mathbf{I} \;\Rightarrow\; \mathbf{IX}=\mathbf{A}^{-1}$$

$$[\mathbf{A}\,|\,\mathbf{I}] \;\Rightarrow\; \text{Gauss-Jordan Elimination} \;\Rightarrow\; [\mathbf{I}\,|\,\mathbf{A}^{-1}]$$

$$[\mathbf{A}\ \mathbf{I}]=\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

Row2 + 3Row1

Row3 − 3Row1

Inverse of a Matrix. Gauss-Jordan elimination

$$\mathbf{AX}=\mathbf{I} \Rightarrow \mathbf{A}^{-1}\mathbf{AX}=\mathbf{A}^{-1}\mathbf{I} \Rightarrow \mathbf{IX}=\mathbf{A}^{-1}$$

$$[\mathbf{A}\,|\,\mathbf{I}] \Rightarrow \text{Gauss-Jordan Elimination} \Rightarrow [\mathbf{I}\,|\,\mathbf{A}^{-1}]$$

$$[\mathbf{A}\ \mathbf{I}]=\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

Row2 + 3Row1

Row3 − 3Row1

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

Row3 − Row2

Inverse of a Matrix. Gauss-Jordan elimination

$$\mathbf{AX}=\mathbf{I} \Rightarrow \mathbf{A}^{-1}\mathbf{AX}=\mathbf{A}^{-1}\mathbf{I} \Rightarrow \mathbf{IX}=\mathbf{A}^{-1}$$

$$[\mathbf{A}\,|\,\mathbf{I}] \Rightarrow \text{Gauss-Jordan Elimination} \Rightarrow [\mathbf{I}\,|\,\mathbf{A}^{-1}]$$

$$[\mathbf{A}\ \mathbf{I}]=\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

Row2 + 3Row1

Row3 − 3Row1

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

Row3 − Row2

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & -0.6 & -0.4 & 0.4 \\ 0 & 2 & 0 & -2.6 & -0.4 & 1.4 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

Row1 +0.4Row3

Row2 +1.4Row3

Inverse of a Matrix. Gauss-Jordan elimination

$$\mathbf{AX}=\mathbf{I} \Rightarrow \mathbf{A}^{-1}\mathbf{AX}=\mathbf{A}^{-1}\mathbf{I} \Rightarrow \mathbf{IX}=\mathbf{A}^{-1}$$

$$[\mathbf{A}\,|\,\mathbf{I}] \Rightarrow \text{Gauss-Jordan Elimination} \Rightarrow [\mathbf{I}\,|\,\mathbf{A}^{-1}]$$

$$[\mathbf{A}\ \mathbf{I}]=\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Row2 + 3Row1
Row3 − 3Row1

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

Row3 − Row2

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

Row1 +0.4Row3

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & -0.6 & -0.4 & 0.4 \\ 0 & 2 & 0 & -2.6 & -0.4 & 1.4 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

Row2 +1.4Row3

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 0.7 & -0.2 & -0.3 \\ 0 & 2 & 0 & -2.6 & -0.4 & 1.4 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

Row1 -0.5Row2

Inverse of a Matrix. Gauss-Jordan elimination

$$AX = I \;\Rightarrow\; A^{-1}AX = A^{-1}I \;\Rightarrow\; IX = A^{-1}$$

$$[A \mid I] \;\Rightarrow\; \text{Gauss-Jordan Elimination} \;\Rightarrow\; [I \mid A^{-1}]$$

$$[A\ I] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Row2 + 3Row1

Row3 − 3Row1

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

Row3 − Row2

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

Row1 +0.4Row3

Row2 +1.4Row3

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & -0.6 & -0.4 & 0.4 \\ 0 & 2 & 0 & -2.6 & -0.4 & 1.4 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

Row1 -0.5Row2

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 0.7 & -0.2 & -0.3 \\ 0 & 2 & 0 & -2.6 & -0.4 & 1.4 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

diagonal matrix

Inverse of a Matrix. Gauss-Jordan elimination

$$AX = I \Rightarrow A^{-1}AX = A^{-1}I \Rightarrow IX = A^{-1}$$

$$\Updownarrow \qquad \Updownarrow \qquad \Updownarrow$$

$$[A \mid I] \Rightarrow \text{Gauss-Jordan Elimination} \Rightarrow [I \mid A^{-1}]$$

$$[A\ I] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Row2 + 3Row1

Row3 − 3Row1

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

Row3 − Row2

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

Row1 +0.4Row3

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & -0.6 & -0.4 & 0.4 \\ 0 & 2 & 0 & -2.6 & -0.4 & 1.4 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

Row2 +1.4Row3

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 0.7 & -0.2 & -0.3 \\ 0 & 2 & 0 & -2.6 & -0.4 & 1.4 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

diagonal matrix

Row1 -0.5Row2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

-Row1

0.5Row2

-0.2Row3

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$$\mathbf{A}^{-1}$$

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$$\mathbf{A}^{-1}$$

Check the result.

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$$\mathbf{A}^{-1}$$

Check the result.

Let
$\mathbf{AA^{-1} = B}$

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$$\mathbf{A}^{-1}$$

Check the result.

Let $\mathbf{AA^{-1} = B}$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$\mathbf{A}^{-1}$

Check the result.

Let $\mathbf{AA^{-1} = B}$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$\mathbf{A}^{-1}$

$$b_{11} = (-1)\times(-0.7) + 1\times(-1.3) + 2\times 0.8 = 1$$

Check the result.

Let $\mathbf{AA^{-1} = B}$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$\mathbf{A}^{-1}$

$$b_{11} = (-1)\times(-0.7) + 1\times(-1.3) + 2\times 0.8 = 1$$
$$b_{12} = (-1)\times(0.2) + 1\times(-0.2) + 2\times 0.2 = 0$$

Check the result.

Let $\mathbf{AA^{-1} = B}$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$\mathbf{A}^{-1}$

$$b_{11} = (-1)\times(-0.7) + 1\times(-1.3) + 2\times 0.8 = 1$$
$$b_{12} = (-1)\times(0.2) + 1\times(-0.2) + 2\times 0.2 = 0$$
$$b_{13} = (-1)\times(0.3) + 1\times(0.7) + 2\times(-0.2) = 0$$

Check the result.

Let $\mathbf{AA^{-1} = B}$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$\mathbf{A}^{-1}$

$$b_{11} = (-1)\times(-0.7) + 1\times(-1.3) + 2\times 0.8 = 1$$
$$b_{12} = (-1)\times(0.2) + 1\times(-0.2) + 2\times 0.2 = 0$$
$$b_{13} = (-1)\times(0.3) + 1\times(0.7) + 2\times(-0.2) = 0$$
$$b_{21} = (3)\times(-0.7) + (-1)\times(-1.3) + 1\times(0.8) = 0$$

Check the result.

Let $AA^{-1} = B$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$\mathbf{A}^{-1}$

Check the result.

Let $\mathbf{AA^{-1} = B}$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

$$b_{11} = (-1)\times(-0.7)+1\times(-1.3)+2\times 0.8 = 1$$
$$b_{12} = (-1)\times(0.2)+1\times(-0.2)+2\times 0.2 = 0$$
$$b_{13} = (-1)\times(0.3)+1\times(0.7)+2\times(-0.2) = 0$$
$$b_{21} = (3)\times(-0.7)+(-1)\times(-1.3)+1\times(0.8) = 0$$
$$b_{22} = (3)\times(0.2)+(-1)\times(-0.2)+1\times(0.2) = 1$$

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$\mathbf{A}^{-1}$

Check the result.

Let $\mathbf{AA^{-1} = B}$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

$$b_{11} = (-1)\times(-0.7)+1\times(-1.3)+2\times 0.8 = 1$$
$$b_{12} = (-1)\times(0.2)+1\times(-0.2)+2\times 0.2 = 0$$
$$b_{13} = (-1)\times(0.3)+1\times(0.7)+2\times(-0.2) = 0$$
$$b_{21} = (3)\times(-0.7)+(-1)\times(-1.3)+1\times(0.8) = 0$$
$$b_{22} = (3)\times(0.2)+(-1)\times(-0.2)+1\times(0.2) = 1$$
$$b_{23} = (3)\times(0.3)+(-1)\times(0.7)+1\times(-0.2) = 0$$

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$\mathbf{A}^{-1}$

Check the result.

Let $\mathbf{AA^{-1} = B}$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

$$b_{11} = (-1)\times(-0.7)+1\times(-1.3)+2\times 0.8 = 1$$
$$b_{12} = (-1)\times(0.2)+1\times(-0.2)+2\times 0.2 = 0$$
$$b_{13} = (-1)\times(0.3)+1\times(0.7)+2\times(-0.2) = 0$$
$$b_{21} = (3)\times(-0.7)+(-1)\times(-1.3)+1\times(0.8) = 0$$
$$b_{22} = (3)\times(0.2)+(-1)\times(-0.2)+1\times(0.2) = 1$$
$$b_{23} = (3)\times(0.3)+(-1)\times(0.7)+1\times(-0.2) = 0$$
$$b_{31} = (-1)\times(-0.7)+(3)\times(-1.3)+4\times(0.8) = 0$$

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$\mathbf{A}^{-1}$

Check the result.

Let $\mathbf{AA^{-1} = B}$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

$$b_{11} = (-1)\times(-0.7)+1\times(-1.3)+2\times 0.8 = 1$$
$$b_{12} = (-1)\times(0.2)+1\times(-0.2)+2\times 0.2 = 0$$
$$b_{13} = (-1)\times(0.3)+1\times(0.7)+2\times(-0.2) = 0$$
$$b_{21} = (3)\times(-0.7)+(-1)\times(-1.3)+1\times(0.8) = 0$$
$$b_{22} = (3)\times(0.2)+(-1)\times(-0.2)+1\times(0.2) = 1$$
$$b_{23} = (3)\times(0.3)+(-1)\times(0.7)+1\times(-0.2) = 0$$
$$b_{31} = (-1)\times(-0.7)+(3)\times(-1.3)+4\times(0.8) = 0$$
$$b_{32} = (-1)\times(0.2)+(3)\times(-0.2)+4\times(0.2) = 0$$

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$\mathbf{A}^{-1}$

Check the result.

Let $\mathbf{AA^{-1} = B}$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

$$b_{11} = (-1)\times(-0.7)+1\times(-1.3)+2\times 0.8 = 1$$
$$b_{12} = (-1)\times(0.2)+1\times(-0.2)+2\times 0.2 = 0$$
$$b_{13} = (-1)\times(0.3)+1\times(0.7)+2\times(-0.2) = 0$$
$$b_{21} = (3)\times(-0.7)+(-1)\times(-1.3)+1\times(0.8) = 0$$
$$b_{22} = (3)\times(0.2)+(-1)\times(-0.2)+1\times(0.2) = 1$$
$$b_{23} = (3)\times(0.3)+(-1)\times(0.7)+1\times(-0.2) = 0$$
$$b_{31} = (-1)\times(-0.7)+(3)\times(-1.3)+4\times(0.8) = 0$$
$$b_{32} = (-1)\times(0.2)+(3)\times(-0.2)+4\times(0.2) = 0$$
$$b_{33} = (-1)\times(0.3)+(3)\times(0.7)+4\times(-0.2) = 1$$

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$\mathbf{A}^{-1}$

Check the result.

Let $\mathbf{AA^{-1} = B}$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

$$b_{11} = (-1)\times(-0.7)+1\times(-1.3)+2\times 0.8 = 1$$
$$b_{12} = (-1)\times(0.2)+1\times(-0.2)+2\times 0.2 = 0$$
$$b_{13} = (-1)\times(0.3)+1\times(0.7)+2\times(-0.2) = 0$$
$$b_{21} = (3)\times(-0.7)+(-1)\times(-1.3)+1\times(0.8) = 0$$
$$b_{22} = (3)\times(0.2)+(-1)\times(-0.2)+1\times(0.2) = 1$$
$$b_{23} = (3)\times(0.3)+(-1)\times(0.7)+1\times(-0.2) = 0$$
$$b_{31} = (-1)\times(-0.7)+(3)\times(-1.3)+4\times(0.8) = 0$$
$$b_{32} = (-1)\times(0.2)+(3)\times(-0.2)+4\times(0.2) = 0$$
$$b_{33} = (-1)\times(0.3)+(3)\times(0.7)+4\times(-0.2) = 1$$

$$\therefore \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Useful formulas for Inverses
Theorem 2. Inverse of a Matrix

cofactor: $C_{jk} = (-1)^{j+k} M_{jk}$

M_{jk} : a determinant of order $n-1$

The inverse of a nonsingular $n \times 2n$ matrix $\mathbf{A}=[a_{jk}]$ is given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} [\mathbf{C}_{jk}]^{\mathsf{T}} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \cdots (4)$$

Where C_{jk} is *the cofactor of* a_{jk} in det A

CAUTION! Note well that in $\mathbf{A}^{-1}$, the cofactor C_{jk} occupies the same place as a_{kj} (not a_{jk}) does in A.)

Useful formulas for Inverses
Theorem 2. Inverse of a Matrix

cofactor: $C_{jk} = (-1)^{j+k} M_{jk}$

M_{jk} : a determinant of order $n-1$

The inverse of a nonsingular $n \times 2n$ matrix $\mathbf{A}=[a_{jk}]$ is given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}[\mathbf{C}_{jk}]^{\mathsf{T}} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \cdots (4)$$

Where C_{jk} is *the cofactor of* a_{jk} in $\det \mathbf{A}$

CAUTION! Note well that in $\mathbf{A}^{-1}$, the cofactor C_{jk} occupies the same place as a_{kj} (not a_{jk}) does in $\mathbf{A}$.)

Proof)

Useful formulas for Inverses
Theorem 2. Inverse of a Matrix

cofactor: $C_{jk} = (-1)^{j+k} M_{jk}$

M_{jk} : a determinant of order $n-1$

The inverse of a nonsingular $n \times 2n$ matrix $A=[a_{jk}]$ is given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} [\mathbf{C}_{jk}]^{\mathsf{T}} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ . & . & . & . \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \cdots (4)$$

Where C_{jk} is *the cofactor of* a_{jk} in det A

CAUTION! Note well that in A^{-1}, the cofactor C_{jk} occupies the same place as a_{kj} (not a_{jk}) does in A.)

Proof)

Let $$\mathbf{B} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ . & . & . & . \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Useful formulas for Inverses
Theorem 2. Inverse of a Matrix

cofactor: $C_{jk} = (-1)^{j+k} M_{jk}$

M_{jk} : a determinant of order $n-1$

The inverse of a nonsingular $n \times 2n$ matrix $A=[a_{jk}]$ is given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} [\mathbf{C}_{jk}]^{\mathsf{T}} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ . & . & . & . \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \cdots (4)$$

Where C_{jk} is *the cofactor of* a_{jk} in det A

CAUTION! Note well that in A^{-1}, the cofactor C_{jk} occupies the same place as a_{kj} (not a_{jk}) does in A.)

Proof)

Let $$\mathbf{B} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ . & . & . & . \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$ **and show that BA=I**

Useful formulas for Inverses
Theorem 2. Inverse of a Matrix

$(k = l),\ D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \cdots (9)$

$(k \neq l),\ a_{1l}C_{1k} + a_{2l}C_{2k} + \cdots + a_{nl}C_{nk} = 0 \cdots (10)$

Let

$$\mathbf{BA = G = [}g_{kl}\mathbf{]} \ldots (5)$$

Useful formulas for Inverses
Theorem 2. Inverse of a Matrix

$$(k = l),\ D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \cdots (9)$$

$$(k \neq l),\ a_{1l}C_{1k} + a_{2l}C_{2k} + \cdots + a_{nl}C_{nk} = 0 \cdots (10)$$

Let

$$\mathbf{BA = G = [}g_{kl}\mathbf{]} \ldots (5)$$

and then show that G = I.

Useful formulas for Inverses
Theorem 2. Inverse of a Matrix

$$(k = l),\ D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \cdots (9)$$

$$(k \neq l),\ a_{1l}C_{1k} + a_{2l}C_{2k} + \cdots + a_{nl}C_{nk} = 0 \cdots (10)$$

Let

$$\mathbf{BA = G = [}g_{kl}\mathbf{]} \ldots (5)$$

and then show that G = I.

By definition of matrix multiplication and because of the form of B in (4)

Useful formulas for Inverses
Theorem 2. Inverse of a Matrix

$$(k = l),\ D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \cdots (9)$$

$$(k \neq l),\ a_{1l}C_{1k} + a_{2l}C_{2k} + \cdots + a_{nl}C_{nk} = 0 \cdots (10)$$

Let

$$\mathbf{BA} = \mathbf{G} = [g_{kl}] \ldots (5)$$

and then show that G = I.

By definition of matrix multiplication and because of the form of B in (4)

$$g_{kl} = \sum_{s=1}^{n} \frac{C_{sk}}{\det \mathbf{A}} a_{sl}$$

$$= \frac{1}{\det \mathbf{A}} \left(a_{1l}C_{1k} + \cdots + a_{nl}C_{nk} \right) \cdots (6)$$

Useful formulas for Inverses
Theorem 2. Inverse of a Matrix

Sec 7.7

$(k = l),\ D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \cdots (9)$

$(k \neq l),\ a_{1l}C_{1k} + a_{2l}C_{2k} + \cdots + a_{nl}C_{nk} = 0 \cdots (10)$

Let

$$\mathbf{BA} = \mathbf{G} = [g_{kl}] \ldots (5)$$

and then show that G = I.

By definition of matrix multiplication and because of the form of B in (4)

$$g_{kl} = \sum_{s=1}^{n} \frac{C_{sk}}{\det \mathbf{A}} a_{sl}$$

$$= \frac{1}{\det \mathbf{A}} \left(a_{1l}C_{1k} + \cdots + a_{nl}C_{nk} \right) \cdots (6)$$

In Sec 7.7 (9) and (10),

If $l = k$, $a_{1l}C_{1k} + \cdots + a_{nl}C_{nk} = \det \mathbf{A}$

$l \neq k$, $a_{1l}C_{1k} + \cdots + a_{nl}C_{nk} = 0$

Useful formulas for Inverses
Theorem 2. Inverse of a Matrix

Sec 7.7

$(k = l),\ D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \cdots (9)$

$(k \neq l),\ a_{1l}C_{1k} + a_{2l}C_{2k} + \cdots + a_{nl}C_{nk} = 0 \cdots (10)$

Let

$$\mathbf{BA} = \mathbf{G} = [g_{kl}] \ldots (5)$$

and then show that G = I.

By definition of matrix multiplication and because of the form of B in (4)

$$g_{kl} = \sum_{s=1}^{n} \frac{C_{sk}}{\det \mathbf{A}} a_{sl}$$

$$= \frac{1}{\det \mathbf{A}} \left(a_{1l}C_{1k} + \cdots + a_{nl}C_{nk} \right) \cdots (6)$$

In Sec 7.7 (9) and (10),

If $l = k$, $a_{1l}C_{1k} + \cdots + a_{nl}C_{nk} = \det \mathbf{A}$

$l \neq k$, $a_{1l}C_{1k} + \cdots + a_{nl}C_{nk} = 0$

Hence,

$$g_{kk} = \frac{1}{\det \mathbf{A}} \det \mathbf{A} = 1$$

$$g_{lk} = 0$$

Useful formulas for Inverses
Theorem 2. Inverse of a Matrix

Sec 7.7

$(k = l),\ D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \cdots (9)$

$(k \neq l),\ a_{1l}C_{1k} + a_{2l}C_{2k} + \cdots + a_{nl}C_{nk} = 0 \cdots (10)$

Let

$$\mathbf{BA} = \mathbf{G} = [g_{kl}] \ldots (5)$$

and then show that G = I.

By definition of matrix multiplication and because of the form of B in (4)

$$g_{kl} = \sum_{s=1}^{n} \frac{C_{sk}}{\det \mathbf{A}} a_{sl}$$

$$= \frac{1}{\det \mathbf{A}} \left(a_{1l}C_{1k} + \cdots + a_{nl}C_{nk} \right) \cdots (6)$$

In Sec 7.7 (9) and (10),

If $l = k$, $a_{1l}C_{1k} + \cdots + a_{nl}C_{nk} = \det \mathbf{A}$

$l \neq k$, $a_{1l}C_{1k} + \cdots + a_{nl}C_{nk} = 0$

Hence,

$$g_{kk} = \frac{1}{\det \mathbf{A}} \det \mathbf{A} = 1$$

$$g_{lk} = 0$$

g_{kk} are the entries of main diagonal of matrix G. and that means only entries of main diagonal is 1.

$\therefore$ G = I

$\therefore$ B = A^{-1}

Inverse of 2 x 2 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Ex 2)

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Inverse of 2 x 2 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Ex 2)

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$\det \mathbf{A} = 3 \cdot 4 - 1 \cdot 2 = 10$$

Inverse of 2 x 2 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Ex 2)

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\det \mathbf{A} = 3 \cdot 4 - 1 \cdot 2 = 10$$

$$C_{11} = 4,$$

Inverse of 2 x 2 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Ex 2)

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$\det \mathbf{A} = 3 \cdot 4 - 1 \cdot 2 = 10$$

$$C_{11} = 4,$$

$$C_{12} = -2,$$

Inverse of 2 x 2 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Ex 2)

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$\det \mathbf{A} = 3 \cdot 4 - 1 \cdot 2 = 10$$

$$C_{11} = 4, \qquad C_{21} = -1,$$

$$C_{12} = -2,$$

Inverse of 2 x 2 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Ex 2)

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$\det \mathbf{A} = 3 \cdot 4 - 1 \cdot 2 = 10$$

$$C_{11} = 4, \quad C_{21} = -1,$$

$$C_{12} = -2, \quad C_{22} = 3,$$

Inverse of 2 x 2 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Ex 2)

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$\det \mathbf{A} = 3 \cdot 4 - 1 \cdot 2 = 10$$

$$C_{11} = 4, \qquad C_{21} = -1,$$

$$C_{12} = -2, \qquad C_{22} = 3,$$

$$\therefore \mathbf{A}^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$

Inverse of 2 x 2 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Ex 2)

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$\det \mathbf{A} = 3 \cdot 4 - 1 \cdot 2 = 10$$

$$C_{11} = 4, \qquad C_{21} = -1,$$

$$C_{12} = -2, \qquad C_{22} = 3,$$

$$\therefore \mathbf{A}^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

Inverse of 3 x 3 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} \ ?$$

Inverse of 3 x 3 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} \ ?$$

$$\det \mathbf{A} = -1 \cdot \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

Inverse of 3 x 3 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} \ ?$$

$$\det \mathbf{A} = -1 \cdot \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= 7 - 13 + 16 = 10$$

Inverse of 3 x 3 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} \ ?$$

$$\det \mathbf{A} = -1 \cdot \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= 7 - 13 + 16 = 10$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7,$$

Inverse of 3 x 3 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} \ ?$$

$$\det \mathbf{A} = -1 \cdot \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= 7 - 13 + 16 = 10$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13,$$

Inverse of 3 x 3 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} \ ?$$

$$\det \mathbf{A} = -1 \cdot \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= 7 - 13 + 16 = 10$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13,$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8,$$

Inverse of 3 x 3 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} \ ?$$

$$\det \mathbf{A} = -1 \cdot \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= 7 - 13 + 16 = 10$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \qquad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13,$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8,$$

Inverse of 3 x 3 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} \ ?$$

$$\det \mathbf{A} = -1 \cdot \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= 7 - 13 + 16 = 10$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \qquad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \qquad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2,$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8,$$

Inverse of 3 x 3 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} \ ?$$

$$\det \mathbf{A} = -1 \cdot \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= 7 - 13 + 16 = 10$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \qquad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \qquad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2,$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8, \qquad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2,$$

Inverse of 3 x 3 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} \ ?$$

$$C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3,$$

$$\det \mathbf{A} = -1 \cdot \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= 7 - 13 + 16 = 10$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \qquad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \qquad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2,$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8, \qquad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2,$$

Inverse of 3 x 3 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} \ ?$$

$$\det \mathbf{A} = -1 \cdot \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= 7 - 13 + 16 = 10$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \qquad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \qquad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2,$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8, \qquad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2,$$

$$C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7,$$

Inverse of 3 x 3 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} \ ?$$

$$\det \mathbf{A} = -1 \cdot \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= 7 - 13 + 16 = 10$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \qquad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \qquad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2,$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8, \qquad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2,$$

$$C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7,$$

$$C_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2,$$

Inverse of 3 x 3 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} \ ?$$

$$\det \mathbf{A} = -1 \cdot \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= 7 - 13 + 16 = 10$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \qquad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \qquad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2,$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8, \qquad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2,$$

$$C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7,$$

$$C_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2,$$

$$\therefore \mathbf{A}^{-1} = \frac{1}{10} \begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}$$

Inverse of 3 x 3 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} \ ?$$

$$\det \mathbf{A} = -1 \cdot \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= 7 - 13 + 16 = 10$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \qquad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \qquad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2,$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8, \qquad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2,$$

$$C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7,$$

$$C_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2,$$

$$\therefore \mathbf{A}^{-1} = \frac{1}{10} \begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Inverse of Diagonal matrices

$A=[a_{jk}]$, $a_{jk}=0$ when $j \neq k$, have an inverse if and only if all $a_{jj} \neq 0$. Then A^{-1} is diagonal, too, with entries $1/a_{11}, \cdots, 1/a_{nn}$.

Proof) For a diagonal matrix we have in (4)

Inverse of Diagonal matrices

$A=[a_{jk}]$, $a_{jk}=0$ when $j \neq k$, have an inverse if and only if all $a_{jj} \neq 0$. Then A^{-1} is diagonal, too, with entries $1/a_{11}, \cdots, 1/a_{nn}$.

Proof) For a diagonal matrix we have in (4)

$$\frac{C_{11}}{D} = \frac{a_{22} \cdots a_{nn}}{a_{11} a_{22} \cdots a_{nn}} = \frac{1}{a_{11}}, \quad etc.$$

Inverse of Diagonal matrices

$\mathbf{A}=[a_{jk}]$, $a_{jk}=0$ when $j\neq k$, have an inverse if and only if all $a_{jj}\neq 0$. Then $\mathbf{A}^{-1}$ is diagonal, too, with entries $1/a_{11},\cdots, 1/a_{nn}$.

Proof) For a diagonal matrix we have in (4)

$$\frac{C_{11}}{D}=\frac{a_{22}\cdots a_{nn}}{a_{11}a_{22}\cdots a_{nn}}=\frac{1}{a_{11}},\quad etc.$$

Ex 4) Inverse of Diagonal Matrix

$$\mathbf{A}=\begin{bmatrix}-0.5 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix},$$

Inverse of Diagonal matrices

$\mathbf{A}=[a_{jk}]$, $a_{jk}=0$ when $j\neq k$, have an inverse if and only if all $a_{jj}\neq 0$. Then $\mathbf{A}^{-1}$ is diagonal, too, with entries $1/a_{11},\cdots, 1/a_{nn}$.

Proof) For a diagonal matrix we have in (4)

$$\frac{C_{11}}{D}=\frac{a_{22}\cdots a_{nn}}{a_{11}a_{22}\cdots a_{nn}}=\frac{1}{a_{11}},\quad etc.$$

Ex 4) Inverse of Diagonal Matrix

$$\mathbf{A}=\begin{bmatrix}-0.5 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix},$$

$$\therefore \mathbf{A}^{-1}=\begin{bmatrix}-\frac{1}{0.5} & 0 & 0\\ 0 & \frac{1}{4} & 0\\ 0 & 0 & \frac{1}{1}\end{bmatrix}=\begin{bmatrix}-2 & 0 & 0\\ 0 & 0.25 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Inverse of Products

Products can be inverted by taking the inverse of each factor and multiplying these inverses in reverse order,

Hence for more than two factors, $(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}$

$$(\mathbf{AC}\cdots\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}\cdots\mathbf{C}^{-1}\mathbf{A}^{-1}$$

$$(\mathbf{AC})(\mathbf{C}^{-1}\mathbf{A}^{-1}) = \mathbf{ACC}^{-1}\mathbf{A}^{-1} = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$$

$$(\mathbf{C}^{-1}\mathbf{A}^{-1})(\mathbf{AC}) = \mathbf{C}^{-1}\mathbf{A}^{-1}\mathbf{AC} = \mathbf{C}^{-1}\mathbf{IC} = \mathbf{C}^{-1}\mathbf{C} = \mathbf{I}$$

Determinants of Matrix Products

For any n x n matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det\mathbf{A} \cdot \det\mathbf{B}$$

If $\mathbf{A}$ or $\mathbf{B}$ is singular, so are $\mathbf{AB}$ and $\mathbf{BA}$

Now $\mathbf{A}$ and $\mathbf{B}$ be nonsingular. Then we can reduce $\mathbf{A}$ to a diagonal matrix $\hat{\mathbf{A}} = [a_{jk}]$ by Gauss-Jordan steps.

Determinants of Matrix Products

For any n x n matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det\mathbf{A} \cdot \det\mathbf{B}$$

If $\mathbf{A}$ or $\mathbf{B}$ is singular, so are $\mathbf{AB}$ and $\mathbf{BA}$

Now $\mathbf{A}$ and $\mathbf{B}$ be nonsingular. Then we can reduce $\mathbf{A}$ to a diagonal matrix $\hat{\mathbf{A}} = [a_{jk}]$ by Gauss-Jordan steps.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \hat{\mathbf{A}} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix}.$$

Determinants of Matrix Products

For any n x n matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det\mathbf{A} \cdot \det\mathbf{B}$$

If $\mathbf{A}$ or $\mathbf{B}$ is singular, so are $\mathbf{AB}$ and $\mathbf{BA}$

Now $\mathbf{A}$ and $\mathbf{B}$ be nonsingular. Then we can reduce $\mathbf{A}$ to a diagonal matrix $\hat{\mathbf{A}} = [a_{jk}]$ by Gauss-Jordan steps.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \hat{\mathbf{A}} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix}.$$

$\mathbf{A}$ and $\hat{\mathbf{A}}$ are row-equivalent matrices.

Determinants of Matrix Products

$\mathbf{A}$ and $\hat{\mathbf{A}}$ are row-equivalent matrices.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \hat{\mathbf{A}} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix}.$$

Determinants of Matrix Products

A and $\hat{\mathrm{A}}$ are row-equivalent matrices.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \hat{\mathbf{A}} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix}.$$

$$\mathbf{Ax} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_n \end{bmatrix} = \mathbf{b}$$

Determinants of Matrix Products

A and $\hat{\mathrm{A}}$ are row-equivalent matrices.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \hat{\mathbf{A}} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix}.$$

$$\mathbf{Ax} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_n \end{bmatrix} = \mathbf{b}$$

$$\hat{\mathbf{A}}\mathbf{x} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ . \\ \hat{b}_n \end{bmatrix} = \hat{\mathbf{b}}$$

Determinants of Matrix Products

A and $\hat{\mathbf{A}}$ are row-equivalent matrices.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \hat{\mathbf{A}} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix}.$$

$$\mathbf{Ax} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_n \end{bmatrix} = \mathbf{b}$$

$$\hat{\mathbf{A}}\mathbf{x} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ . \\ \hat{b}_n \end{bmatrix} = \hat{\mathbf{b}}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \hat{\mathbf{A}}^{-1}\hat{\mathbf{b}}$$

Determinants of Matrix Products

$\mathbf{A}$ and $\hat{\mathbf{A}}$ are row-equivalent matrices.

Addition of a multiple of a row to another row does not alter the value of the determinant.

Determinants of Matrix Products

$\mathbf{A}$ and $\hat{\mathbf{A}}$ are row-equivalent matrices.

Addition of a multiple of a row to another row does not alter the value of the determinant.

$$\therefore \det(\mathbf{AB}) = \det(\hat{\mathbf{A}}\mathbf{B})$$

Determinants of Matrix Products

$\mathbf{A}$ and $\hat{\mathbf{A}}$ are row-equivalent matrices.

Addition of a multiple of a row to another row does not alter the value of the determinant.

$$\therefore \det(\mathbf{AB}) = \det(\hat{\mathbf{A}}\mathbf{B})$$

$$\hat{\mathbf{A}}\mathbf{B} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

Determinants of Matrix Products

$\mathbf{A}$ and $\hat{\mathbf{A}}$ are row-equivalent matrices.

Addition of a multiple of a row to another row does not alter the value of the determinant.

$$\therefore \det(\mathbf{AB}) = \det(\hat{\mathbf{A}}\mathbf{B})$$

$$\hat{\mathbf{A}}\mathbf{B} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} \hat{a}_{11}b_{11} & \hat{a}_{11}b_{12} & \cdots & \hat{a}_{11}b_{1n} \\ \hat{a}_{22}b_{21} & \hat{a}_{22}b_{22} & \cdots & \hat{a}_{22}b_{2n} \\ & & \vdots & \\ \hat{a}_{nn}b_{n1} & \hat{a}_{nn}b_{n2} & \cdots & \hat{a}_{nn}b_{nn} \end{bmatrix}$$

Determinants of Matrix Products

$\mathbf{A}$ and $\hat{\mathbf{A}}$ are row-equivalent matrices.

Addition of a multiple of a row to another row does not alter the value of the determinant.

$$\therefore \det(\mathbf{AB}) = \det(\hat{\mathbf{A}}\mathbf{B})$$

$$\hat{\mathbf{A}}\mathbf{B} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} \hat{a}_{11}b_{11} & \hat{a}_{11}b_{12} & \cdots & \hat{a}_{11}b_{1n} \\ \hat{a}_{22}b_{21} & \hat{a}_{22}b_{22} & \cdots & \hat{a}_{22}b_{2n} \\ & & \vdots & \\ \hat{a}_{nn}b_{n1} & \hat{a}_{nn}b_{n2} & \cdots & \hat{a}_{nn}b_{nn} \end{bmatrix}$$

$$\det(\hat{\mathbf{A}}\mathbf{B}) = \begin{vmatrix} \hat{a}_{11}b_{11} & \hat{a}_{11}b_{12} & \cdots & \hat{a}_{11}b_{1n} \\ \hat{a}_{22}b_{21} & \hat{a}_{22}b_{22} & \cdots & \hat{a}_{22}b_{2n} \\ & & \vdots & \\ \hat{a}_{nn}b_{n1} & \hat{a}_{nn}b_{n2} & \cdots & \hat{a}_{nn}b_{nn} \end{vmatrix} = \hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn} \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

Determinants of Matrix Products

A and $\hat{\mathrm{A}}$ are row-equivalent matrices.

$$\therefore \det(\mathbf{AB}) = \det(\hat{\mathbf{A}}\mathbf{B})$$

$$\det(\hat{\mathbf{A}}\mathbf{B}) = \begin{vmatrix} \hat{a}_{11}b_{11} & \hat{a}_{11}b_{12} & \cdots & \hat{a}_{11}b_{1n} \\ \hat{a}_{22}b_{21} & \hat{a}_{22}b_{22} & \cdots & \hat{a}_{22}b_{2n} \\ & & \vdots & \\ \hat{a}_{nn}b_{n1} & \hat{a}_{nn}b_{n2} & \cdots & \hat{a}_{nn}b_{nn} \end{vmatrix} = \hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn} \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

Determinants of Matrix Products

A and $\hat{\mathrm{A}}$ are row-equivalent matrices.

$$\therefore \det(\mathbf{AB}) = \det(\hat{\mathbf{A}}\mathbf{B})$$

$$\det(\hat{\mathbf{A}}\mathbf{B}) = \begin{vmatrix} \hat{a}_{11}b_{11} & \hat{a}_{11}b_{12} & \cdots & \hat{a}_{11}b_{1n} \\ \hat{a}_{22}b_{21} & \hat{a}_{22}b_{22} & \cdots & \hat{a}_{22}b_{2n} \\ & & \vdots & \\ \hat{a}_{nn}b_{n1} & \hat{a}_{nn}b_{n2} & \cdots & \hat{a}_{nn}b_{nn} \end{vmatrix} = \hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn} \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

$$= \hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn} \cdot \det(\mathbf{B})$$

Determinants of Matrix Products

A and $\hat{\mathrm{A}}$ are row-equivalent matrices.

$$\therefore \det(\mathbf{AB}) = \det(\hat{\mathbf{A}}\mathbf{B})$$

$$\det(\hat{\mathbf{A}}\mathbf{B}) = \begin{vmatrix} \hat{a}_{11}b_{11} & \hat{a}_{11}b_{12} & \cdots & \hat{a}_{11}b_{1n} \\ \hat{a}_{22}b_{21} & \hat{a}_{22}b_{22} & \cdots & \hat{a}_{22}b_{2n} \\ & & \vdots & \\ \hat{a}_{nn}b_{n1} & \hat{a}_{nn}b_{n2} & \cdots & \hat{a}_{nn}b_{nn} \end{vmatrix} = \hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn} \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

$$= \hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn} \cdot \det(\mathbf{B})$$

$$= \det(\mathbf{A}) \cdot \det(\mathbf{B})$$

Determinants of Matrix Products

A and $\hat{\mathbf{A}}$ are row-equivalent matrices.

$$\therefore \det(\mathbf{AB}) = \det(\hat{\mathbf{A}}\mathbf{B})$$

$$\det(\hat{\mathbf{A}}\mathbf{B}) = \begin{vmatrix} \hat{a}_{11}b_{11} & \hat{a}_{11}b_{12} & \cdots & \hat{a}_{11}b_{1n} \\ \hat{a}_{22}b_{21} & \hat{a}_{22}b_{22} & \cdots & \hat{a}_{22}b_{2n} \\ & & \vdots & \\ \hat{a}_{nn}b_{n1} & \hat{a}_{nn}b_{n2} & \cdots & \hat{a}_{nn}b_{nn} \end{vmatrix} = \hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn} \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

$$= \hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn} \cdot \det(\mathbf{B})$$

$$= \det(\mathbf{A}) \cdot \det(\mathbf{B})$$

$$\therefore \det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

Symmetric, Skew-Symmetric, and Orthogonal Matrices

A *real* square matrix $\mathbf{A} = [a_{jk}]$ is called

Symmetric, Skew-Symmetric, and Orthogonal Matrices

A *real* square matrix $\mathbf{A} = [a_{jk}]$ is called

symmetric if transposition leaves it unchanged

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}, \qquad \text{thus} \quad a_{kj} = a_{jk}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

A *real* square matrix $\mathbf{A} = [a_{jk}]$ is called

symmetric if transposition leaves it unchanged

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}, \qquad \text{thus} \quad a_{kj} = a_{jk}$$

skew-symmetric if transposition gives the negative of $\mathbf{A}$

$$\mathbf{A}^{\mathrm{T}} = -\mathbf{A}, \qquad \text{thus} \quad a_{kj} = -a_{jk}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

A *real* square matrix $\mathbf{A} = [a_{jk}]$ is called

symmetric if transposition leaves it unchanged

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}, \qquad \text{thus} \quad a_{kj} = a_{jk}$$

skew-symmetric if transposition gives the negative of $\mathbf{A}$

$$\mathbf{A}^{\mathrm{T}} = -\mathbf{A}, \qquad \text{thus} \quad a_{kj} = -a_{jk}$$

orthogonal if transposition gives the inverse of $\mathbf{A}$

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

orthogonal

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}, \qquad \mathbf{A}^{\mathbf{T}} = -\mathbf{A}, \qquad \mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix} \qquad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}, \qquad \mathbf{A}^{\mathbf{T}} = -\mathbf{A}, \qquad \mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix} \qquad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric ***skew-symmetric***

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}, \qquad \mathbf{A}^{\mathbf{T}} = -\mathbf{A}, \qquad \mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix} \qquad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

orthogonal

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

orthogonal

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

---

symmetric

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}, \qquad \mathbf{A}^{\mathbf{T}} = -\mathbf{A}, \qquad \mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix} \qquad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$a_{kj} = -a_{jk}$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}, \qquad \mathbf{A}^{\mathbf{T}} = -\mathbf{A}, \qquad \mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix} \qquad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$a_{kj} = -a_{jk}$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric ***skew-symmetric***

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}, \qquad \mathbf{A}^{\mathbf{T}} = -\mathbf{A}, \qquad \mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix} \qquad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$a_{kj} = -a_{jk}$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

orthogonal

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$a_{kj} = -a_{jk}$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

orthogonal

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

---

skew-symmetric $a_{kj} = -a_{jk}$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

orthogonal

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

---

skew-symmetric $a_{kj} = -a_{jk}$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\mathbf{A}^{\mathrm{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

orthogonal

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

---

skew-symmetric $a_{kj} = -a_{jk}$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

If $j = k$, then

$$a_{jj} = -a_{jj}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

orthogonal

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

---

skew-symmetric $a_{kj} = -a_{jk}$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

If $j = k$, then

$$a_{jj} = -a_{jj} \qquad \therefore a_{jj} = 0$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

$$\mathbf{A}^T = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

$$\mathbf{A}^T = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

$$\mathbf{A}^T = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}, \qquad \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A}, \qquad \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1} \qquad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

orthogonal

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

orthogonal

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

---

orthogonal

$$\mathbf{A} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

orthogonal

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

orthogonal

$$\mathbf{A} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \quad \mathbf{A}^{\mathbf{T}} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

orthogonal

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

orthogonal

$$\mathbf{A} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \quad \mathbf{A}^{\mathbf{T}} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\mathbf{A} \cdot \mathbf{A}^{\mathbf{T}} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

orthogonal

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

orthogonal

$$\mathbf{A} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \quad \mathbf{A}^{\mathbf{T}} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\mathbf{A} \cdot \mathbf{A}^{\mathbf{T}} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

symmetric

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A},$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\mathbf{A}^{\mathbf{T}} = -\mathbf{A},$$

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

orthogonal

$$\mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

orthogonal

$$\mathbf{A} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \quad \mathbf{A}^{\mathbf{T}} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\mathbf{A} \cdot \mathbf{A}^{\mathbf{T}} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \mathbf{A}^{\mathbf{T}} = \mathbf{A}^{-1}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

Any real square matrix $\mathbf{A}$ may be written as the sum of a symmetric matrix $\mathbf{R}$ and a skew-symmetric matrix $\mathbf{S}$, where

$$\mathbf{R} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathbf{T}}\right), \qquad \mathbf{S} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathbf{T}}\right)$$

$$\mathbf{A} = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

Any real square matrix $\mathbf{A}$ may be written as the sum of a symmetric matrix $\mathbf{R}$ and a skew-symmetric matrix $\mathbf{S}$, where

$$\mathbf{R} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathbf{T}}\right), \qquad \mathbf{S} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathbf{T}}\right)$$

$$\mathbf{A} = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix}$$

$$\mathbf{A}^{\mathbf{T}} = \begin{bmatrix} 9 & 2 & 5 \\ 5 & 3 & 4 \\ 2 & -8 & 3 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

Any real square matrix $\mathbf{A}$ may be written as the sum of a symmetric matrix $\mathbf{R}$ and a skew-symmetric matrix $\mathbf{S}$, where

$$\mathbf{R}=\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^{\mathbf{T}}\right), \qquad \mathbf{S}=\frac{1}{2}\left(\mathbf{A}-\mathbf{A}^{\mathbf{T}}\right)$$

$$\mathbf{A}=\begin{bmatrix}9 & 5 & 2\\ 2 & 3 & -8\\ 5 & 4 & 3\end{bmatrix} \qquad \mathbf{R}=\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^{\mathbf{T}}\right)$$

$$\mathbf{A}^{\mathbf{T}}=\begin{bmatrix}9 & 2 & 5\\ 5 & 3 & 4\\ 2 & -8 & 3\end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

Any real square matrix $\mathbf{A}$ may be written as the sum of a symmetric matrix $\mathbf{R}$ and a skew-symmetric matrix $\mathbf{S}$, where

$$\mathbf{R} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathbf{T}}\right), \qquad \mathbf{S} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathbf{T}}\right)$$

$$\mathbf{A} = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix} \qquad \mathbf{R} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathbf{T}}\right) = \begin{bmatrix} 9.0 & 3.5 & 3.5 \\ 3.5 & 3.0 & -2.0 \\ 3.5 & -2.0 & 3.0 \end{bmatrix}$$

$$\mathbf{A}^{\mathbf{T}} = \begin{bmatrix} 9 & 2 & 5 \\ 5 & 3 & 4 \\ 2 & -8 & 3 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

Any real square matrix $\mathbf{A}$ may be written as the sum of a symmetric matrix $\mathbf{R}$ and a skew-symmetric matrix $\mathbf{S}$, where

$$\mathbf{R} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathbf{T}}\right), \qquad \mathbf{S} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathbf{T}}\right)$$

$$\mathbf{A} = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix} \qquad \mathbf{R} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathbf{T}}\right) = \begin{bmatrix} 9.0 & 3.5 & 3.5 \\ 3.5 & 3.0 & -2.0 \\ 3.5 & -2.0 & 3.0 \end{bmatrix}$$

$$\mathbf{A}^{\mathbf{T}} = \begin{bmatrix} 9 & 2 & 5 \\ 5 & 3 & 4 \\ 2 & -8 & 3 \end{bmatrix} \qquad \mathbf{S} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathbf{T}}\right)$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

Any real square matrix $\mathbf{A}$ may be written as the sum of a symmetric matrix $\mathbf{R}$ and a skew-symmetric matrix $\mathbf{S}$, where

$$\mathbf{R} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right), \qquad \mathbf{S} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathrm{T}}\right)$$

$$\mathbf{A} = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix} \qquad \mathbf{R} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right) = \begin{bmatrix} 9.0 & 3.5 & 3.5 \\ 3.5 & 3.0 & -2.0 \\ 3.5 & -2.0 & 3.0 \end{bmatrix}$$

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 9 & 2 & 5 \\ 5 & 3 & 4 \\ 2 & -8 & 3 \end{bmatrix} \qquad \mathbf{S} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathrm{T}}\right) = \begin{bmatrix} 0 & 1.5 & -1.5 \\ -1.5 & 0 & -6.0 \\ 1.5 & 6.0 & 0 \end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

Any real square matrix $\mathbf{A}$ may be written as the sum of a symmetric matrix $\mathbf{R}$ and a skew-symmetric matrix $\mathbf{S}$, where

$$\mathbf{R}=\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^{\mathrm{T}}\right), \qquad \mathbf{S}=\frac{1}{2}\left(\mathbf{A}-\mathbf{A}^{\mathrm{T}}\right)$$

$$\mathbf{A}=\begin{bmatrix}9 & 5 & 2\\ 2 & 3 & -8\\ 5 & 4 & 3\end{bmatrix} \qquad \mathbf{R}=\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^{\mathrm{T}}\right)=\begin{bmatrix}9.0 & 3.5 & 3.5\\ 3.5 & 3.0 & -2.0\\ 3.5 & -2.0 & 3.0\end{bmatrix}$$

$$\mathbf{A}^{\mathrm{T}}=\begin{bmatrix}9 & 2 & 5\\ 5 & 3 & 4\\ 2 & -8 & 3\end{bmatrix} \qquad \mathbf{S}=\frac{1}{2}\left(\mathbf{A}-\mathbf{A}^{\mathrm{T}}\right)=\begin{bmatrix}0 & 1.5 & -1.5\\ -1.5 & 0 & -6.0\\ 1.5 & 6.0 & 0\end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix}9 & 5 & 2\\ 2 & 3 & -8\\ 5 & 4 & 3\end{bmatrix}$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

Any real square matrix $\mathbf{A}$ may be written as the sum of a symmetric matrix $\mathbf{R}$ and a skew-symmetric matrix $\mathbf{S}$, where

$$\mathbf{R}=\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^{\mathrm{T}}\right), \qquad \mathbf{S}=\frac{1}{2}\left(\mathbf{A}-\mathbf{A}^{\mathrm{T}}\right)$$

$$\mathbf{A}=\begin{bmatrix}9 & 5 & 2\\ 2 & 3 & -8\\ 5 & 4 & 3\end{bmatrix} \qquad \mathbf{R}=\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^{\mathrm{T}}\right)=\begin{bmatrix}9.0 & 3.5 & 3.5\\ 3.5 & 3.0 & -2.0\\ 3.5 & -2.0 & 3.0\end{bmatrix}$$

$$\mathbf{A}^{\mathrm{T}}=\begin{bmatrix}9 & 2 & 5\\ 5 & 3 & 4\\ 2 & -8 & 3\end{bmatrix} \qquad \mathbf{S}=\frac{1}{2}\left(\mathbf{A}-\mathbf{A}^{\mathrm{T}}\right)=\begin{bmatrix}0 & 1.5 & -1.5\\ -1.5 & 0 & -6.0\\ 1.5 & 6.0 & 0\end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix}9 & 5 & 2\\ 2 & 3 & -8\\ 5 & 4 & 3\end{bmatrix}=\mathbf{R}+\mathbf{S}=\begin{bmatrix}9.0 & 3.5 & 3.5\\ 3.5 & 3.0 & -2.0\\ 3.5 & -2.0 & 3.0\end{bmatrix}+\begin{bmatrix}0 & 1.5 & -1.5\\ -1.5 & 0 & -6.0\\ 1.5 & 6.0 & 0\end{bmatrix}$$

Orthogonal Transformations and Orthogonal Matrices

Orthogonal Transformations are transformations

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$ where **A** is an orthogonal matrix

With each vector $\mathbf{x}$ in R^n such a transformation assigns a vector $\mathbf{y}$ in R^n.

Ex) the plane rotation through an angle θ.

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is an orthogonal transformation. It can be shown that any orthogonal transformation in the plane or in three-dimensional space is a *rotation*.

Orthogonal Transformations and Orthogonal Matrices

Invariance of Inner Product

Theorem 8.3.2 Invariance of Inner Product

An orthogonal transformation preserves the value of the inner product of vectors $\mathbf{a}$ and $\mathbf{b}$ in R^n, defined by

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{a}^{\mathrm{T}}\mathbf{b}=\begin{bmatrix}a_1 & \cdots & a_n\end{bmatrix}\begin{bmatrix}b_1\\ \vdots\\ b_n\end{bmatrix}$$

That is, for any $\mathbf{a}$ and $\mathbf{b}$ in R^n, orthogonal n x n matrix $\mathbf{A}$, and $\mathbf{u} = \mathbf{Aa}$, $\mathbf{v} = \mathbf{Ab}$ we have $\mathbf{u}\cdot\mathbf{v}=\mathbf{a}\cdot\mathbf{b}$.

Hence the transformation also preserves the *length* or *norm* of any vector $\mathbf{a}$ in R^n given by

$$\|\mathbf{a}\|=\sqrt{\mathbf{a}\cdot\mathbf{a}}=\sqrt{\mathbf{a}^{\mathrm{T}}\mathbf{a}}$$

Orthogonal Transformations and Orthogonal Matrices

Invariance of Inner Product

Theorem 8.3.2 Invariance of Inner Product

An orthogonal transformation preserves the value of the inner product of vectors **a** and **b** in R^n

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

orthogonal n x n matrix A

$$\mathbf{u} = A\mathbf{a}$$

$$\mathbf{v} = A\mathbf{b}$$

$$\mathbf{u}\bullet\mathbf{v} = \mathbf{a}\bullet\mathbf{b}$$

Orthogonal Transformations and Orthogonal Matrices

Invariance of Inner Product

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Theorem 8.3.2 Invariance of Inner Product

An orthogonal transformation preserves the value of the inner product of vectors **a** and **b** in R^n

orthogonal n x n matrix A

$$\mathbf{u} = A\mathbf{a}$$

$$\mathbf{v} = A\mathbf{b}$$

$$\mathbf{u}\bullet\mathbf{v} = \mathbf{a}\bullet\mathbf{b}$$

$$(A\mathbf{a})^T = \mathbf{a}^T A^T$$

Orthogonal Transformations and Orthogonal Matrices

Invariance of Inner Product

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Theorem 8.3.2 Invariance of Inner Product

An orthogonal transformation preserves the value of the inner product of vectors **a** and **b** in R^n

orthogonal n x n matrix A

$$\mathbf{u} = A\mathbf{a}$$

$$\mathbf{v} = A\mathbf{b}$$

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{a}\cdot\mathbf{b}$$

$$(A\mathbf{a})^T = \mathbf{a}^T A^T$$

$$A^T A = A^{-1}A = I$$

➔ A is orthogonal.

Orthogonal Transformations and Orthogonal Matrices

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Invariance of Inner Product

Theorem 8.3.2 Invariance of Inner Product

An orthogonal transformation preserves the value of the inner product of vectors **a** and **b** in R^n

orthogonal n x n matrix A

$$\mathbf{u}=A\mathbf{a}$$
$$\mathbf{v}=A\mathbf{b}$$

$$\mathbf{u}\bullet\mathbf{v}=\mathbf{a}\bullet\mathbf{b}$$

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}=(A\mathbf{a})^T A\mathbf{b}$$

$$(A\mathbf{a})^T=\mathbf{a}^T A^T$$

$$A^T A=A^{-1}A=I$$

➔ A is orthogonal.

Orthogonal Transformations and Orthogonal Matrices

Invariance of Inner Product

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Theorem 8.3.2 Invariance of Inner Product

An orthogonal transformation preserves the value of the inner product of vectors a and b in R^n

orthogonal n x n matrix A

$$\mathbf{u}=\mathbf{Aa}$$
$$\mathbf{v}=\mathbf{Ab}$$

$$\mathbf{u}\bullet\mathbf{v}=\mathbf{a}\bullet\mathbf{b}$$

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}=(\mathbf{Aa})^T\mathbf{Ab}=\mathbf{a}^T\mathbf{A}^T\mathbf{Ab}$$

$$(\mathbf{Aa})^T=\mathbf{a}^T\mathbf{A}^T$$

$$\mathbf{A}^T\mathbf{A}=\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}$$

➔ A is orthogonal.

Orthogonal Transformations and Orthogonal Matrices

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Invariance of Inner Product

Theorem 8.3.2 Invariance of Inner Product

An orthogonal transformation preserves the value of the inner product of vectors a and b in R^n

orthogonal n x n matrix A

$$\mathbf{u}=A\mathbf{a}$$
$$\mathbf{v}=A\mathbf{b}$$

$$\mathbf{u}\bullet\mathbf{v}=\mathbf{a}\bullet\mathbf{b}$$

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}=(A\mathbf{a})^TA\mathbf{b}=\mathbf{a}^TA^TA\mathbf{b}$$
$$=\mathbf{a}^TI\mathbf{b}$$

$$(A\mathbf{a})^T=\mathbf{a}^TA^T$$

$$A^TA=A^{-1}A=I$$

➔ A is orthogonal.

Orthogonal Transformations and Orthogonal Matrices

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Invariance of Inner Product

Theorem 8.3.2 Invariance of Inner Product

An orthogonal transformation preserves the value of the inner product of vectors a and b in R^n

orthogonal n x n matrix A

$$\mathbf{u}=A\mathbf{a}$$
$$\mathbf{v}=A\mathbf{b}$$

$$\mathbf{u}\bullet\mathbf{v}=\mathbf{a}\bullet\mathbf{b}$$

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}=(A\mathbf{a})^TA\mathbf{b}=\mathbf{a}^TA^TA\mathbf{b}$$
$$=\mathbf{a}^TI\mathbf{b}$$
$$=\mathbf{a}^T\mathbf{b}$$

$$(A\mathbf{a})^T=\mathbf{a}^TA^T$$
$$A^TA=A^{-1}A=I$$

➔ A is orthogonal.

Orthogonal Transformations and Orthogonal Matrices

Invariance of Inner Product

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Theorem 8.3.2 Invariance of Inner Product

An orthogonal transformation preserves the value of the inner product of vectors **a** and **b** in R^n

orthogonal n x n matrix A

$$\mathbf{u}=A\mathbf{a}$$
$$\mathbf{v}=A\mathbf{b}$$

$$\mathbf{u}\bullet\mathbf{v}=\mathbf{a}\bullet\mathbf{b}$$

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}=(A\mathbf{a})^TA\mathbf{b}=\mathbf{a}^TA^TA\mathbf{b}$$
$$=\mathbf{a}^TI\mathbf{b}$$
$$=\mathbf{a}^T\mathbf{b}$$
$$=\mathbf{a}\cdot\mathbf{b}$$

$$(A\mathbf{a})^T=\mathbf{a}^TA^T$$
$$A^TA=A^{-1}A=I$$

➔ A is orthogonal.

Orthonormality of Column and Row Vectors

Theorem 8.3.3

A real square matrix is orthogonal if and only if column vectors $a_1, \dots, a_n$ (and also its row vectors) form an orthonormal system, that is,

orthogonal (orthonomal) vectors

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j{}^{\mathrm{T}} \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{x}_1{}^{\mathrm{T}} \mathbf{x}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

Let A be orthogonal. Then $A^{-1}A = A^{T}A = I$, in terms of column vector $a_1, \dots, a_n$,

orthogonal matrix

$$\mathbf{I} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{\mathrm{T}}\mathbf{A}$$

Orthonormality of Column and Row Vectors

Theorem 8.3.3

A real square matrix is orthogonal if and only if column vectors $a_1, \dots, a_n$ (and also its row vectors) form an orthonormal system, that is,

orthogonal (orthonomal) vectors

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j{}^{\mathrm{T}} \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{x}_1{}^{\mathrm{T}} \mathbf{x}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

Let $\mathbf{A}$ be orthogonal. Then $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}$, in terms of column vector $\mathbf{a}_1, \dots, \mathbf{a}_n$,

orthogonal matrix

$$\mathbf{I} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} \mathbf{a}_1{}^{\mathrm{T}} \\ \vdots \\ \mathbf{a}_n{}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$$

Orthonormality of Column and Row Vectors

Theorem 8.3.3

A real square matrix is orthogonal if and only if column vectors $a_1, \dots, a_n$ (and also its row vectors) form an orthonormal system, that is,

orthogonal (orthonomal) vectors

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j{}^{\mathrm{T}} \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{x}_1{}^{\mathrm{T}} \mathbf{x}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

Let A be orthogonal. Then $A^{-1}A = A^{T}A = I$, in terms of column vector $a_1, \dots, a_n$,

orthogonal matrix

$$\mathbf{I} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} \mathbf{a}_1{}^{\mathrm{T}} \\ \vdots \\ \mathbf{a}_n{}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1{}^{\mathrm{T}}\mathbf{a}_1 & \mathbf{a}_1{}^{\mathrm{T}}\mathbf{a}_2 & \cdots & \mathbf{a}_1{}^{\mathrm{T}}\mathbf{a}_n \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{a}_n{}^{\mathrm{T}}\mathbf{a}_1 & \mathbf{a}_n{}^{\mathrm{T}}\mathbf{a}_2 & \cdots & \mathbf{a}_n{}^{\mathrm{T}}\mathbf{a}_n \end{bmatrix}$$

Determinant of an Orthogonal Matrix

Theorem 8.3.4 Determinant of an Orthogonal Matrix

The determinant of an orthogonal matrix has the value +1 or -1.

Proof

From $\det \mathbf{AB} = \det \mathbf{A} \cdot \det \mathbf{B}$(Sec.7.8, Theorem 4)* and $\det \mathbf{A}^T = \det \mathbf{A}$ (Sec. 7.7, Theorem 2d)*, we get for an orthogonal matrix

$$1 = \det \mathbf{I}$$

*Kreyszig E., Advanced Engineering Mathematics, 9th edition, Wiley, 2006

Determinant of an Orthogonal Matrix

Theorem 8.3.4 Determinant of an Orthogonal Matrix

The determinant of an orthogonal matrix has the value +1 or -1.

Proof

From $\det \mathbf{AB} = \det \mathbf{A} \cdot \det \mathbf{B}$ (Sec. 7.8, Theorem 4)* and $\det \mathbf{A}^T = \det \mathbf{A}$ (Sec. 7.7, Theorem 2d)*, we get for an orthogonal matrix

$$1 = \det \mathbf{I} = \det\left(\mathbf{A}\mathbf{A}^{-1}\right)$$

*Kreyszig E., Advanced Engineering Mathematics, 9th edition, Wiley, 2006

Determinant of an Orthogonal Matrix

Theorem 8.3.4 Determinant of an Orthogonal Matrix

The determinant of an orthogonal matrix has the value +1 or -1.

Proof

From $\det \mathbf{AB} = \det \mathbf{A} \cdot \det \mathbf{B}$ (Sec. 7.8, Theorem 4)* and $\det \mathbf{A}^{\mathrm{T}} = \det \mathbf{A}$ (Sec. 7.7, Theorem 2d)*, we get for an orthogonal matrix

$$1 = \det \mathbf{I} = \det\left(\mathbf{A}\mathbf{A}^{-1}\right) = \det\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right)$$

*Kreyszig E., Advanced Engineering Mathematics, 9th edition, Wiley, 2006

Determinant of an Orthogonal Matrix

Theorem 8.3.4 Determinant of an Orthogonal Matrix

The determinant of an orthogonal matrix has the value +1 or -1.

Proof

From $\det \mathbf{AB} = \det \mathbf{A} \cdot \det \mathbf{B}$ (Sec. 7.8, Theorem 4)* and $\det \mathbf{A}^T = \det \mathbf{A}$ (Sec. 7.7, Theorem 2d)*, we get for an orthogonal matrix

$$1 = \det \mathbf{I} = \det\left(\mathbf{A}\mathbf{A}^{-1}\right) = \det\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right)$$

$$= \det \mathbf{A} \cdot \det \mathbf{A}^{\mathrm{T}}$$

*Kreyszig E., Advanced Engineering Mathematics, 9th edition, Wiley, 2006

Determinant of an Orthogonal Matrix

Theorem 8.3.4 Determinant of an Orthogonal Matrix

The determinant of an orthogonal matrix has the value +1 or -1.

Proof

From $\det \mathbf{AB} = \det \mathbf{A} \cdot \det \mathbf{B}$ (Sec. 7.8, Theorem 4)* and $\det \mathbf{A}^T = \det \mathbf{A}$ (Sec. 7.7, Theorem 2d)*, we get for an orthogonal matrix

$$1 = \det \mathbf{I} = \det\left(\mathbf{A}\mathbf{A}^{-1}\right) = \det\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right)$$

$$= \det \mathbf{A} \cdot \det \mathbf{A}^{\mathrm{T}}$$

$$= \det \mathbf{A} \cdot \det \mathbf{A}$$

*Kreyszig E., Advanced Engineering Mathematics, 9th edition, Wiley, 2006

Determinant of an Orthogonal Matrix

Theorem 8.3.4 Determinant of an Orthogonal Matrix

The determinant of an orthogonal matrix has the value +1 or -1.

Proof

From $\det \mathbf{AB} = \det \mathbf{A} \cdot \det \mathbf{B}$ (Sec. 7.8, Theorem 4)* and $\det \mathbf{A}^T = \det \mathbf{A}$ (Sec. 7.7, Theorem 2d)*, we get for an orthogonal matrix

$$1 = \det \mathbf{I} = \det\left(\mathbf{A}\mathbf{A}^{-1}\right) = \det\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right)$$

$$= \det \mathbf{A} \cdot \det \mathbf{A}^{\mathrm{T}}$$

$$= \det \mathbf{A} \cdot \det \mathbf{A}$$

$$= (\det \mathbf{A})^2$$

*Kreyszig E., Advanced Engineering Mathematics, 9th edition, Wiley, 2006

Determinant of an Orthogonal Matrix

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

From example 1

$$\mathbf{A}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

➔ orthogonal

From example 3

$$\mathbf{A}=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

➔ orthogonal

Determinant of an Orthogonal Matrix

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

From example 1

$$\mathbf{A}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

➔ orthogonal

$$\begin{aligned}\det \mathbf{A} &= \frac{2}{3}\left[\left(\frac{2}{3}\right)\cdot\left(-\frac{2}{3}\right)-\left(\frac{1}{3}\right)\cdot\left(\frac{2}{3}\right)\right] \\ &\quad -\frac{1}{3}\left[\left(-\frac{2}{3}\right)\cdot\left(-\frac{2}{3}\right)-\left(\frac{1}{3}\right)\cdot\left(\frac{1}{3}\right)\right] \\ &\quad +\frac{2}{3}\left[\left(-\frac{2}{3}\right)\cdot\left(\frac{2}{3}\right)-\left(\frac{2}{3}\right)\cdot\left(\frac{1}{3}\right)\right] \\ &= -1\end{aligned}$$

From example 3

$$\mathbf{A}=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

➔ orthogonal

Determinant of an Orthogonal Matrix

Symmetric: $\mathbf{A}^T=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^T=-\mathbf{A}$
Orthogonal: $\mathbf{A}^T=\mathbf{A}^{-1}$

From example 1

$$\mathbf{A}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

➔ orthogonal

$$\begin{aligned}\det\mathbf{A} &= \frac{2}{3}\left[\left(\frac{2}{3}\right)\cdot\left(-\frac{2}{3}\right)-\left(\frac{1}{3}\right)\cdot\left(\frac{2}{3}\right)\right] \\ &\quad -\frac{1}{3}\left[\left(-\frac{2}{3}\right)\cdot\left(-\frac{2}{3}\right)-\left(\frac{1}{3}\right)\cdot\left(\frac{1}{3}\right)\right] \\ &\quad +\frac{2}{3}\left[\left(-\frac{2}{3}\right)\cdot\left(\frac{2}{3}\right)-\left(\frac{2}{3}\right)\cdot\left(\frac{1}{3}\right)\right] \\ &= -1\end{aligned}$$

From example 3

$$\mathbf{A}=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

➔ orthogonal

$$\begin{aligned}\det\mathbf{A} &= \cos^2\theta+\sin^2\theta \\ &= 1\end{aligned}$$

Eigenvalues of an Orthogonal Matrix

Theorem 8.3.5 Eigenvalues of an Orthogonal Matrix

The eigenvalues of an orthogonal matrix A are *real or complex* conjugated in pairs and *have absolute value 1*.

From example 1

$$A = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

➔ **orthogonal**

Eigenvalues of an Orthogonal Matrix

Theorem 8.3.5 Eigenvalues of an Orthogonal Matrix

The eigenvalues of an orthogonal matrix A are *real or complex* conjugated in pairs and *have absolute value 1*.

From example 1

$$\mathbf{A}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} \frac{2}{3}-\lambda & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3}-\lambda & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3}-\lambda \end{vmatrix}=0$$

➔ **orthogonal**

Eigenvalues of an Orthogonal Matrix

Theorem 8.3.5 Eigenvalues of an Orthogonal Matrix

The eigenvalues of an orthogonal matrix A are *real or complex* conjugated in pairs and *have absolute value 1*.

From example 1

$$\mathbf{A}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \qquad \det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} \frac{2}{3}-\lambda & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3}-\lambda & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3}-\lambda \end{vmatrix}=0$$

➔ **orthogonal**

$$\left(\frac{2}{3}-\lambda\right)\left[\left(\frac{2}{3}-\lambda\right)\left(-\frac{2}{3}-\lambda\right)-\frac{1}{3}\cdot\frac{2}{3}\right]-\frac{1}{3}\left[-\frac{2}{3}\left(-\frac{2}{3}-\lambda\right)-\frac{1}{3}\cdot\frac{1}{3}\right]+\frac{2}{3}\left[-\frac{2}{3}\cdot\frac{2}{3}-\left(\frac{2}{3}-\lambda\right)\frac{1}{3}\right]=0$$

Eigenvalues of an Orthogonal Matrix

Theorem 8.3.5 Eigenvalues of an Orthogonal Matrix

The eigenvalues of an orthogonal matrix A are *real or complex* conjugated in pairs and *have absolute value 1*.

From example 1

$$\mathbf{A}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \qquad \det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} \frac{2}{3}-\lambda & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3}-\lambda & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3}-\lambda \end{vmatrix}=0$$

➔ **orthogonal**

$$\left(\frac{2}{3}-\lambda\right)\left[\left(\frac{2}{3}-\lambda\right)\left(-\frac{2}{3}-\lambda\right)-\frac{1}{3}\cdot\frac{2}{3}\right]-\frac{1}{3}\left[-\frac{2}{3}\left(-\frac{2}{3}-\lambda\right)-\frac{1}{3}\cdot\frac{1}{3}\right]+\frac{2}{3}\left[-\frac{2}{3}\cdot\frac{2}{3}-\left(\frac{2}{3}-\lambda\right)\frac{1}{3}\right]=0$$

$$-\lambda^3+\frac{2}{3}\lambda^2+\frac{2}{3}\lambda-1=0$$

Eigenvalues of an Orthogonal Matrix

Theorem 8.3.5 Eigenvalues of an Orthogonal Matrix

The eigenvalues of an orthogonal matrix A are *real or complex* conjugated in pairs and *have absolute value 1*.

From example 1

$$A = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

➔ **orthogonal**

Eigenvalues of an Orthogonal Matrix

Theorem 8.3.5 Eigenvalues of an Orthogonal Matrix

The eigenvalues of an orthogonal matrix A are *real or complex* conjugated in pairs and *have absolute value 1*.

From example 1

$$\mathbf{A}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} \frac{2}{3}-\lambda & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3}-\lambda & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3}-\lambda \end{vmatrix}=0$$

➔ **orthogonal**

Eigenvalues of an Orthogonal Matrix

Theorem 8.3.5 Eigenvalues of an Orthogonal Matrix

The eigenvalues of an orthogonal matrix A are *real or complex* conjugated in pairs and *have absolute value 1*.

From example 1

$$\mathbf{A}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} \frac{2}{3}-\lambda & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3}-\lambda & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3}-\lambda \end{vmatrix}=0$$

➔ **orthogonal**

$$-\lambda^3+\frac{2}{3}\lambda^2+\frac{2}{3}\lambda-1=0$$

Eigenvalues of an Orthogonal Matrix

Theorem 8.3.5 Eigenvalues of an Orthogonal Matrix

The eigenvalues of an orthogonal matrix A are *real or complex* conjugated in pairs and *have absolute value 1*.

From example 1

$$\mathbf{A}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \qquad \det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} \frac{2}{3}-\lambda & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3}-\lambda & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3}-\lambda \end{vmatrix}=0$$

➔ **orthogonal**

$$-\lambda^3+\frac{2}{3}\lambda^2+\frac{2}{3}\lambda-1=0$$

$$(\lambda+1)\left(\lambda^2-\frac{5}{3}\lambda+1\right)=0$$

Eigenvalues of an Orthogonal Matrix

Theorem 8.3.5 Eigenvalues of an Orthogonal Matrix

The eigenvalues of an orthogonal matrix A are *real or complex* conjugated in pairs and *have absolute value 1*.

From example 1

$$\mathbf{A}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \qquad \det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} \frac{2}{3}-\lambda & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3}-\lambda & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3}-\lambda \end{vmatrix}=0$$

➔ **orthogonal**

$$-\lambda^3+\frac{2}{3}\lambda^2+\frac{2}{3}\lambda-1=0$$

$$(\lambda+1)\left(\lambda^2-\frac{5}{3}\lambda+1\right)=0$$

$$\therefore \lambda=-1,\ \frac{5\pm i\sqrt{11}}{6}$$

Eigenvalues of an Orthogonal Matrix

Theorem 8.3.5 Eigenvalues of an Orthogonal Matrix

The eigenvalues of an orthogonal matrix A are *real or complex* conjugated in pairs and *have absolute value 1*.

From example 1

$$\mathbf{A}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \qquad \det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} \frac{2}{3}-\lambda & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3}-\lambda & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3}-\lambda \end{vmatrix}=0$$

➔ **orthogonal**

$$-\lambda^3+\frac{2}{3}\lambda^2+\frac{2}{3}\lambda-1=0$$

$$(\lambda+1)\left(\lambda^2-\frac{5}{3}\lambda+1\right)=0$$

$$\therefore \lambda=-1,\ \frac{5\pm i\sqrt{11}}{6} \qquad \therefore |\lambda|=1$$

Eigenvalues of an Orthogonal Matrix

eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal. Give an example.

Symmetric: $\mathbf{A}^T=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^T=-\mathbf{A}$
Orthogonal: $\mathbf{A}^T=\mathbf{A}^{-1}$

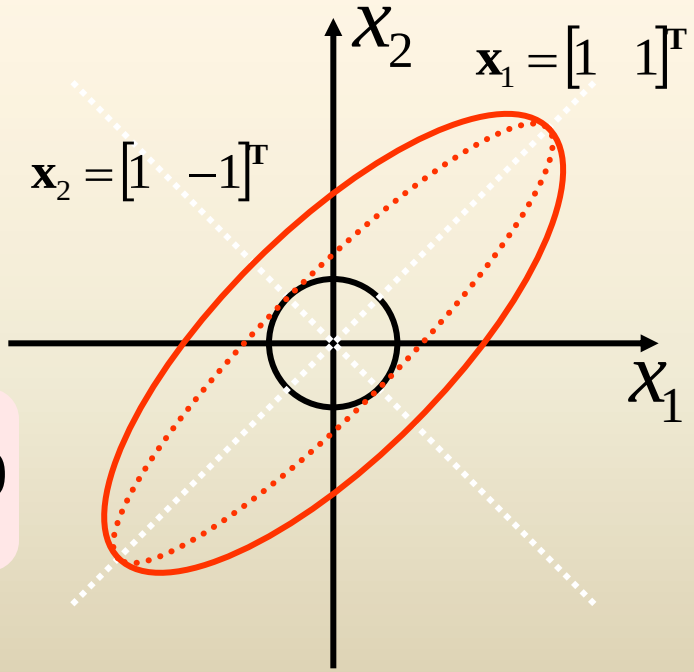
example 8.2-1

$$\mathbf{A}=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix},$$

$$\lambda_1=8,\ \mathbf{x}_1=\begin{bmatrix}1 & 1\end{bmatrix}^T$$

$$\lambda_2=2,\ \mathbf{x}_2=\begin{bmatrix}1 & -1\end{bmatrix}^T$$

$$\mathbf{x}_1^T\mathbf{x}_2=\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}=0$$



A real square matrix is orthogonal if and only if column vectors $a_1, \dots, a_n$ form an orthonormal system,

$$\mathbf{a}_j\cdot\mathbf{a}_k=\mathbf{a}_j^T\mathbf{a}_k=\begin{cases}0 & if\ \ j\neq k\\ 1 & if\ \ j=k\end{cases}$$

Eigenvalues of an Orthogonal Matrix

Show that the inverse of a skew-symmetric matrix is skew-symmetric matrix.

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Eigenvalues of an Orthogonal Matrix

Show that the inverse of a skew-symmetric matrix is skew-symmetric matrix.

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Let A is a skew-symmetric matrix, and $B = A^{-1}$ then,

Eigenvalues of an Orthogonal Matrix

Show that the inverse of a skew-symmetric matrix is skew-symmetric matrix.

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Let A is a skew-symmetric matrix, and $B = A^{-1}$ then,

$$B^T = B^T I$$

Eigenvalues of an Orthogonal Matrix

Show that the inverse of a skew-symmetric matrix is skew-symmetric matrix.

Symmetric: $\mathbf{A}^T=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^T=-\mathbf{A}$
Orthogonal: $\mathbf{A}^T=\mathbf{A}^{-1}$

Let $\mathbf{A}$ is a skew-symmetric matrix, and $\mathbf{B} = \mathbf{A}^{-1}$ then,

$$\mathbf{B}^T = \mathbf{B}^T\mathbf{I} = \mathbf{B}^T(\mathbf{AB})$$

Eigenvalues of an Orthogonal Matrix

Show that the inverse of a skew-symmetric matrix is skew-symmetric matrix.

Symmetric: $\mathbf{A}^T=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^T=-\mathbf{A}$
Orthogonal: $\mathbf{A}^T=\mathbf{A}^{-1}$

Let $\mathbf{A}$ is a skew-symmetric matrix, and $\mathbf{B} = \mathbf{A}^{-1}$ then,

$$\mathbf{B}^T = \mathbf{B}^T\mathbf{I} = \mathbf{B}^T(\mathbf{AB}) = (\mathbf{B}^T\mathbf{A})\mathbf{B}$$

Eigenvalues of an Orthogonal Matrix

Show that the inverse of a skew-symmetric matrix is skew-symmetric matrix.

Symmetric: $\mathbf{A}^T=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^T=-\mathbf{A}$
Orthogonal: $\mathbf{A}^T=\mathbf{A}^{-1}$

Let $\mathbf{A}$ is a skew-symmetric matrix, and $\mathbf{B} = \mathbf{A}^{-1}$ then,

$$\mathbf{B}^T = \mathbf{B}^T\mathbf{I} = \mathbf{B}^T(\mathbf{AB}) = (\mathbf{B}^T\mathbf{A})\mathbf{B} = (\mathbf{A}^T\mathbf{B})^T\mathbf{B}$$

Eigenvalues of an Orthogonal Matrix

Show that the inverse of a skew-symmetric matrix is skew-symmetric matrix.

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Let A is a skew-symmetric matrix, and $B = A^{-1}$ then,

$$B^T = B^T I = B^T(AB) = (B^T A)B = (A^T B)^T B$$

$$= (-AB)^T B, \quad (\because A \text{ is a skew-symmetric matrix.})$$

Eigenvalues of an Orthogonal Matrix

Show that the inverse of a skew-symmetric matrix is skew-symmetric matrix.

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Let A is a skew-symmetric matrix, and $B = A^{-1}$ then,

$$B^T = B^T I = B^T(AB) = (B^T A)B = (A^T B)^T B$$

$$= (-AB)^T B, \quad (\because A \text{ is a skew-symmetric matrix.})$$

$$= (-I)^T B$$

Eigenvalues of an Orthogonal Matrix

Show that the inverse of a skew-symmetric matrix is skew-symmetric matrix.

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Let A is a skew-symmetric matrix, and $B = A^{-1}$ then,

$$B^T = B^T I = B^T(AB) = (B^T A)B = (A^T B)^T B$$

$$= (-AB)^T B, \quad (\because A \text{ is a skew-symmetric matrix.})$$

$$= (-I)^T B = -IB$$

Eigenvalues of an Orthogonal Matrix

Show that the inverse of a skew-symmetric matrix is skew-symmetric matrix.

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Let A is a skew-symmetric matrix, and $B = A^{-1}$ then,

$$B^T = B^T I = B^T(AB) = (B^T A)B = (A^T B)^T B$$

$$= (-AB)^T B, \quad (\because A \text{ is a skew-symmetric matrix.})$$

$$= (-I)^T B = -IB = -B$$

Eigenvalues of an Orthogonal Matrix

Show that the inverse of a skew-symmetric matrix is skew-symmetric matrix.

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Let A is a skew-symmetric matrix, and $B = A^{-1}$ then,

$$B^T = B^T I = B^T(AB) = (B^T A)B = (A^T B)^T B$$

$$= (-AB)^T B, \quad (\because A \text{ is a skew-symmetric matrix.})$$

$$= (-I)^T B = -IB = -B$$

So, B is also a skew-symmetric matrix.

Eigenbases. Diagonalization. Quadratic Form

Eigenbasis

$\mathbf{y} = \mathbf{A}\mathbf{z}$: a transformation

$$\mathbf{z} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

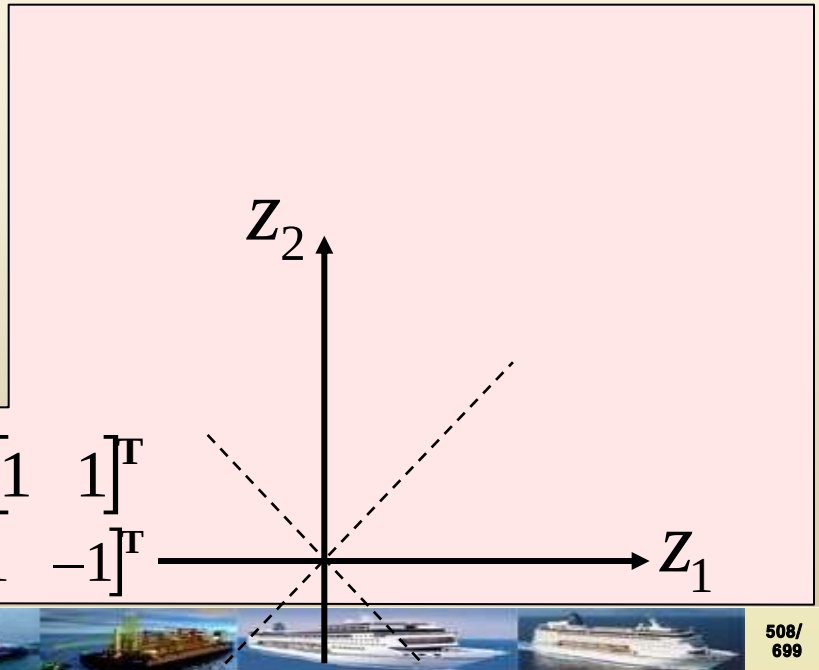
If we are interested in a transformation $\mathbf{y} = \mathbf{A}\mathbf{z}$, such an "*eigenbasis*" (basis of eigenvectors) - if it exists - is of great advantage because then we can represent any $\mathbf{z}$ in R^n uniquely as *a linear combination of the eigenvectors* $\mathbf{x}_1, \ldots, \mathbf{x}_n$, say,

And denoting the corresponding eigenvalues of the matrix $\mathbf{A}$ by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$, so that we simply obtain

$$\mathbf{y} = \mathbf{A}\mathbf{z} = \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right)$$

Ex.)

$$\mathbf{y} = \mathbf{A}\mathbf{z} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \lambda_1 = 8,\ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$
$$\lambda_2 = 2,\ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$



Eigenbasis

$\mathbf{y} = \mathbf{A}\mathbf{z}$: a transformation

$$\mathbf{z} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

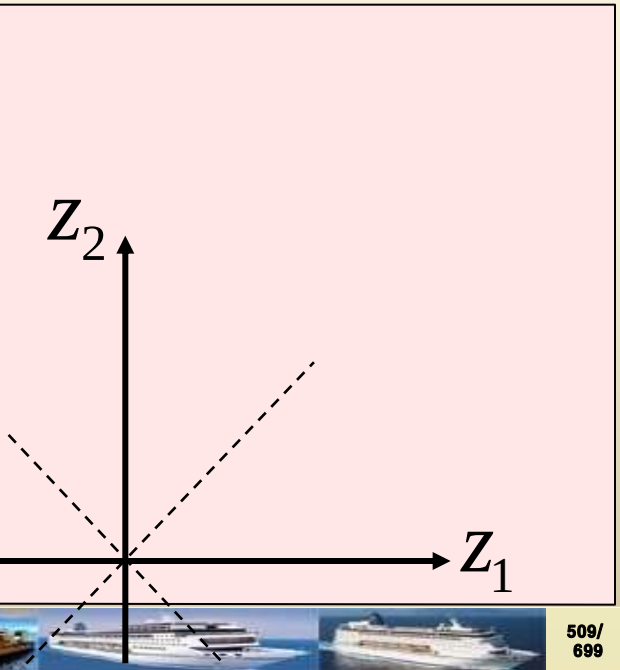
If we are interested in a transformation $\mathbf{y} = \mathbf{A}\mathbf{z}$, such an "*eigenbasis*" (basis of eigenvectors) - if it exists - is of great advantage because then we can represent any $\mathbf{z}$ in R^n uniquely as *a linear combination of the eigenvectors* $\mathbf{x}_1, \ldots, \mathbf{x}_n$, say,

And denoting the corresponding eigenvalues of the matrix $\mathbf{A}$ by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$, so that we simply obtain

$$\mathbf{y} = \mathbf{A}\mathbf{z} = \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right)$$
$$= c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \cdots + c_n\mathbf{A}\mathbf{x}_n$$

Ex.)

$$\mathbf{y} = \mathbf{A}\mathbf{z} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \lambda_1 = 8,\ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$
$$\lambda_2 = 2,\ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$



Eigenbasis

$\mathbf{y} = \mathbf{A}\mathbf{z}$: a transformation

$$\mathbf{z} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

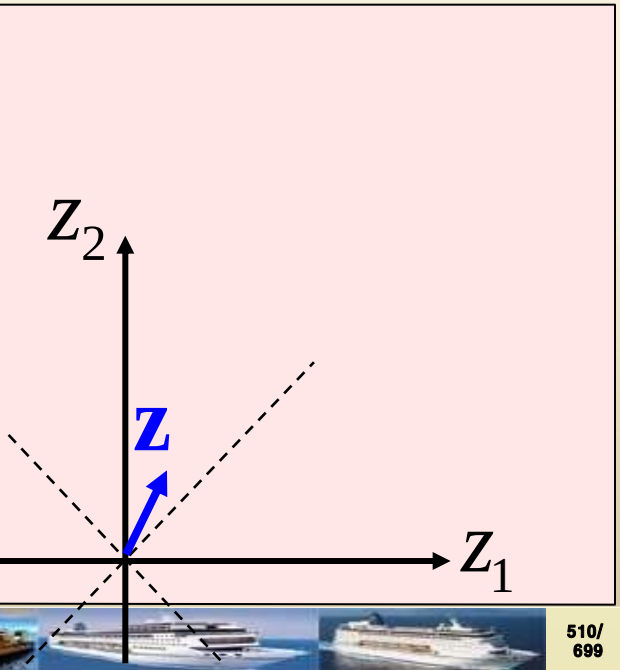
If we are interested in a transformation $\mathbf{y} = \mathbf{A}\mathbf{z}$, such an "*eigenbasis*" (basis of eigenvectors) - if it exists - is of great advantage because then we can represent any $\mathbf{z}$ in R^n uniquely as *a linear combination of the eigenvectors* $\mathbf{x}_1, \ldots, \mathbf{x}_n$, say,

And denoting the corresponding eigenvalues of the matrix $\mathbf{A}$ by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$, so that we simply obtain

$$\begin{aligned}\mathbf{y} = \mathbf{A}\mathbf{z} &= \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right) \\ &= c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \cdots + c_n\mathbf{A}\mathbf{x}_n \\ &= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n\end{aligned}$$

Ex.)

$$\mathbf{y} = \mathbf{A}\mathbf{z} = \begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} \quad \lambda_1 = 8,\ \mathbf{x}_1 = \begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}}$$
$$\lambda_2 = 2,\ \mathbf{x}_2 = \begin{bmatrix}1 & -1\end{bmatrix}^{\mathrm{T}}$$



Eigenbasis

$\mathbf{y} = \mathbf{A}\mathbf{z}$: a transformation

$$\mathbf{z} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

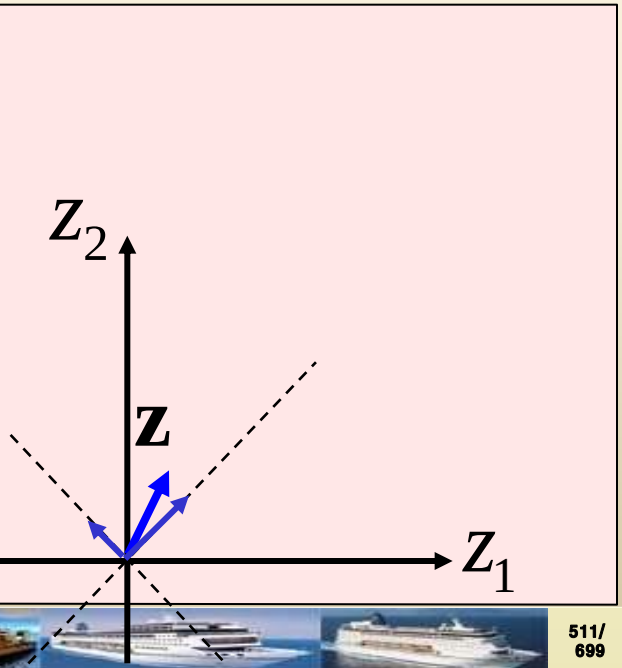
If we are interested in a transformation $\mathbf{y} = \mathbf{A}\mathbf{z}$, such an "*eigenbasis*" (basis of eigenvectors) - if it exists - is of great advantage because then we can represent any $\mathbf{z}$ in R^n uniquely as *a linear combination of the eigenvectors* $\mathbf{x}_1, \ldots, \mathbf{x}_n$, say,

And denoting the corresponding eigenvalues of the matrix $\mathbf{A}$ by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$, so that we simply obtain

$$\begin{aligned}\mathbf{y} = \mathbf{A}\mathbf{z} &= \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right) \\ &= c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \cdots + c_n\mathbf{A}\mathbf{x}_n \\ &= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n\end{aligned}$$

Ex.)

$$\mathbf{y} = \mathbf{A}\mathbf{z} = \begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} \quad \lambda_1 = 8,\ \mathbf{x}_1 = \begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}}$$
$$\lambda_2 = 2,\ \mathbf{x}_2 = \begin{bmatrix}1 & -1\end{bmatrix}^{\mathrm{T}}$$



Eigenbasis

$\mathbf{y} = \mathbf{A}\mathbf{z}$: a transformation

$$\mathbf{z} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

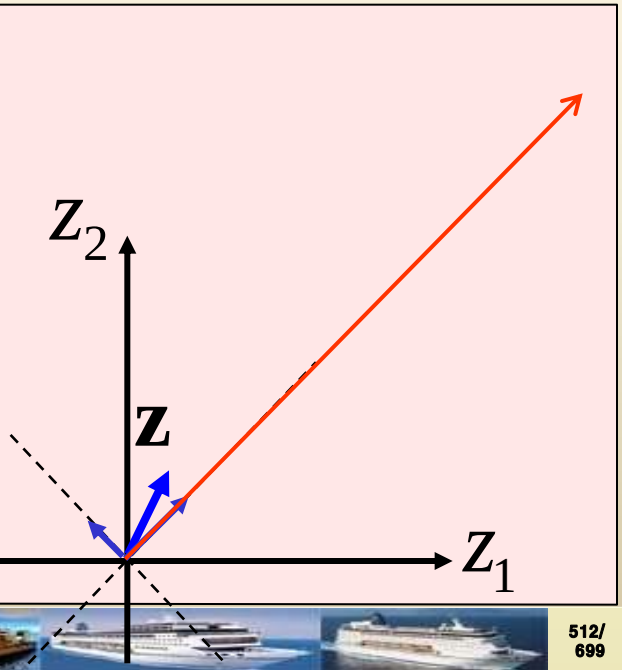
If we are interested in a transformation $\mathbf{y} = \mathbf{A}\mathbf{z}$, such an "*eigenbasis*" (basis of eigenvectors) - if it exists - is of great advantage because then we can represent any $\mathbf{z}$ in R^n uniquely as *a linear combination of the eigenvectors* $\mathbf{x}_1, \ldots, \mathbf{x}_n$, say,

And denoting the corresponding eigenvalues of the matrix $\mathbf{A}$ by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$, so that we simply obtain

$$\begin{aligned}\mathbf{y} = \mathbf{A}\mathbf{z} &= \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right) \\ &= c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \cdots + c_n\mathbf{A}\mathbf{x}_n \\ &= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n\end{aligned}$$

Ex.)

$$\mathbf{y} = \mathbf{A}\mathbf{z} = \begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} \quad \lambda_1 = 8,\ \mathbf{x}_1 = \begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}}$$
$$\lambda_2 = 2,\ \mathbf{x}_2 = \begin{bmatrix}1 & -1\end{bmatrix}^{\mathrm{T}}$$



Eigenbasis

$\mathbf{y} = \mathbf{A}\mathbf{z}$: a transformation

$$\mathbf{z} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

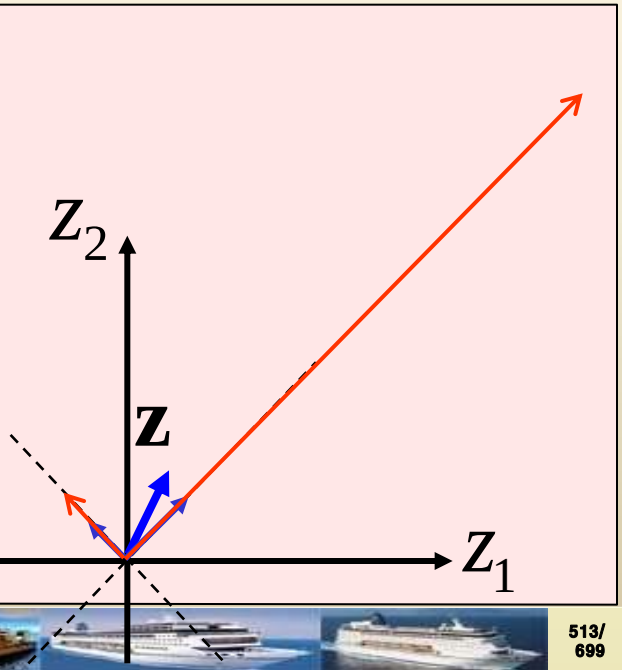
If we are interested in a transformation $\mathbf{y} = \mathbf{A}\mathbf{z}$, such an "*eigenbasis*" (basis of eigenvectors) - if it exists - is of great advantage because then we can represent any $\mathbf{z}$ in R^n uniquely as *a linear combination of the eigenvectors* $\mathbf{x}_1, \ldots, \mathbf{x}_n$, say,

And denoting the corresponding eigenvalues of the matrix $\mathbf{A}$ by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$, so that we simply obtain

$$\begin{aligned}\mathbf{y} = \mathbf{A}\mathbf{z} &= \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right) \\ &= c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \cdots + c_n\mathbf{A}\mathbf{x}_n \\ &= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n\end{aligned}$$

Ex.)

$$\mathbf{y} = \mathbf{A}\mathbf{z} = \begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} \quad \lambda_1 = 8,\ \mathbf{x}_1 = \begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}}$$
$$\lambda_2 = 2,\ \mathbf{x}_2 = \begin{bmatrix}1 & -1\end{bmatrix}^{\mathrm{T}}$$



Eigenbasis

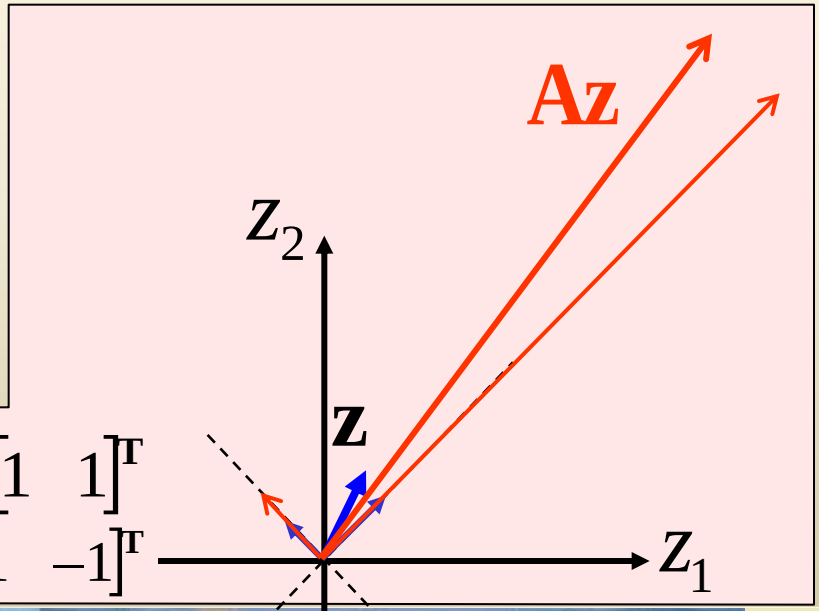
$\mathbf{y} = \mathbf{A}\mathbf{z}$: a transformation

$$\mathbf{z} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

If we are interested in a transformation $\mathbf{y} = \mathbf{A}\mathbf{z}$, such an "*eigenbasis*" (basis of eigenvectors) - if it exists - is of great advantage because then we can represent any $\mathbf{z}$ in R^n uniquely as *a linear combination of the eigenvectors* $\mathbf{x}_1, \ldots, \mathbf{x}_n$, say,

And denoting the corresponding eigenvalues of the matrix $\mathbf{A}$ by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$, so that we simply obtain

$$\begin{aligned}\mathbf{y} = \mathbf{A}\mathbf{z} &= \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right) \\ &= c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \cdots + c_n\mathbf{A}\mathbf{x}_n \\ &= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n\end{aligned}$$



Ex.)

$$\mathbf{y} = \mathbf{A}\mathbf{z} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \lambda_1 = 8,\ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}} \quad \lambda_2 = 2,\ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

Basis of Eigenvectors

Theorem 8.4.1 Basis of Eigenvectors

If an n x n matrix $\mathbf{A}$ has n distinct eigenvalues, then

$\mathbf{A}$ has a basis of eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for R^n.

Proof

Basis of Eigenvectors

Theorem 8.4.1 Basis of Eigenvectors

If an n x n matrix $\mathbf{A}$ has n distinct eigenvalues, then

$\mathbf{A}$ has a basis of eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for R^n.

Proof

All we have to show is that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent.

Suppose they are not.

Basis of Eigenvectors

Theorem 8.4.1 Basis of Eigenvectors

If an n x n matrix $\mathbf{A}$ has n distinct eigenvalues, then

$\mathbf{A}$ has a basis of eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for R^n.

Proof

All we have to show is that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent.

Suppose they are not.

⇨ 1 to r : independent
r is the largest integer that is a linearly independent

Basis of Eigenvectors

Theorem 8.4.1 Basis of Eigenvectors

If an n x n matrix $\mathbf{A}$ has n distinct eigenvalues, then

$\mathbf{A}$ has a basis of eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for R^n.

Proof

All we have to show is that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent.

Suppose they are not.

⇨ 1 to r : independent
r is the largest integer that is a linearly independent

⇨ $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$: independent

$\{\mathbf{x}_1, \cdots, \mathbf{x}_{r+1}\}$: dependent

Basis of Eigenvectors

Theorem 8.4.1 Basis of Eigenvectors

If an n x n matrix A has n distinct eigenvalues, then

A has a basis of eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for R^n.

Proof

All we have to show is that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent.

Suppose they are not.

⇨ 1 to r : independent
r is the largest integer that is a linearly independent

⇨ $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$: independent
$\{\mathbf{x}_1, \cdots, \mathbf{x}_{r+1}\}$: dependent

⇨ $c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0$

Basis of Eigenvectors (Proof)

$$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0 \qquad \text{➔①}$$

Basis of Eigenvectors (Proof)

$$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0 \qquad \rightarrow ①$$

Multiply both sides by $\mathbf{A}$

$$c_1\mathbf{A}\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{A}\mathbf{x}_{r+1} = 0$$

Basis of Eigenvectors (Proof)

$$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0 \qquad \rightarrow ①$$

Multiply both sides by $\mathbf{A}$

$$c_1\mathbf{A}\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{A}\mathbf{x}_{r+1} = 0$$

Use $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$

$$c_1\lambda_1\mathbf{x}_1 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = 0$$

Basis of Eigenvectors (Proof)

$$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0 \qquad \rightarrow ①$$

Multiply both sides by $\mathbf{A}$

$$c_1\mathbf{A}\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{A}\mathbf{x}_{r+1} = 0$$

Use $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$

$$c_1\lambda_1\mathbf{x}_1 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = 0 \rightarrow ②$$

Basis of Eigenvectors (Proof)

$$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0 \qquad \rightarrow ①$$

Multiply both sides by A

$$c_1\mathbf{A}\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{A}\mathbf{x}_{r+1} = 0$$

Use $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$

$$c_1\lambda_1\mathbf{x}_1 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = 0 \rightarrow ②$$

② - λ_{r+1} x ①:

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{x}_1 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = 0$$

Basis of Eigenvectors (Proof)

$$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0 \qquad \rightarrow ①$$

Multiply both sides by $\mathbf{A}$

$$c_1\mathbf{A}\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{A}\mathbf{x}_{r+1} = 0$$

Use $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$

$$c_1\lambda_1\mathbf{x}_1 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = 0 \rightarrow ②$$

② - λ_{r+1} x ①:

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{x}_1 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = 0$$

$\mathbf{x}_1$, … , $\mathbf{x}_r$ is linearly independent.

Basis of Eigenvectors (Proof)

$$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0 \qquad \rightarrow ①$$

Multiply both sides by A

$$c_1\mathbf{A}\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{A}\mathbf{x}_{r+1} = 0$$

Use $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$

$$c_1\lambda_1\mathbf{x}_1 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = 0 \rightarrow ②$$

② - λ_{r+1} x ①:

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{x}_1 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = 0$$

$\mathbf{x}_1$, … , $\mathbf{x}_r$ is linearly independent.

$$c_1(\lambda_1 - \lambda_{r+1}) = \cdots = c_r(\lambda_r - \lambda_{r+1}) = 0$$

Basis of Eigenvectors (Proof)

$$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0 \quad \rightarrow ①$$

Multiply both sides by A

$$c_1\mathbf{A}\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{A}\mathbf{x}_{r+1} = 0$$

Use $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$

$$c_1\lambda_1\mathbf{x}_1 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = 0 \rightarrow ②$$

② - λ_{r+1} x ①:

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{x}_1 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = 0$$

$\mathbf{x}_1$, … , $\mathbf{x}_r$ is linearly independent.

$$c_1(\lambda_1 - \lambda_{r+1}) = \cdots = c_r(\lambda_r - \lambda_{r+1}) = 0$$

All the eigenvalues are distinct.

$$\lambda_1 - \lambda_{r+1} \neq 0$$
$$\vdots$$
$$\lambda_r - \lambda_{r+1} \neq 0$$

Basis of Eigenvectors (Proof)

$$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0 \quad \rightarrow ①$$

Multiply both sides by A

$$c_1\mathbf{A}\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{A}\mathbf{x}_{r+1} = 0$$

Use $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$

$$c_1\lambda_1\mathbf{x}_1 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = 0 \rightarrow ②$$

② - λ_{r+1} x ①:

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{x}_1 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = 0$$

$\mathbf{x}_1$, … , $\mathbf{x}_r$ is linearly independent.

$$c_1(\lambda_1 - \lambda_{r+1}) = \cdots = c_r(\lambda_r - \lambda_{r+1}) = 0$$

All the eigenvalues are distinct.

$$\lambda_1 - \lambda_{r+1} \neq 0$$
$$\vdots$$
$$\lambda_r - \lambda_{r+1} \neq 0$$
$$\therefore c_1 = \cdots = c_r = 0$$

Basis of Eigenvectors (Proof)

$$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0 \quad \rightarrow ①$$

Multiply both sides by A

$$c_1\mathbf{A}\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{A}\mathbf{x}_{r+1} = 0$$

Use $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$

$$c_1\lambda_1\mathbf{x}_1 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = 0 \rightarrow ②$$

② - λ_{r+1} x ①:

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{x}_1 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = 0$$

$\mathbf{x}_1$, … , $\mathbf{x}_r$ is linearly independent.

$$c_1(\lambda_1 - \lambda_{r+1}) = \cdots = c_r(\lambda_r - \lambda_{r+1}) = 0$$

All the eigenvalues are distinct.

$$\lambda_1 - \lambda_{r+1} \neq 0$$
$$\vdots$$
$$\lambda_r - \lambda_{r+1} \neq 0$$
$$\therefore c_1 = \cdots = c_r = 0$$

With this, ① reduces to

$$c_{r+1}\mathbf{x}_{r+1} = 0$$

Basis of Eigenvectors (Proof)

$$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0 \quad \rightarrow ①$$

Multiply both sides by A

$$c_1\mathbf{A}\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{A}\mathbf{x}_{r+1} = 0$$

Use $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$

$$c_1\lambda_1\mathbf{x}_1 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = 0 \rightarrow ②$$

② - λ_{r+1} x ①:

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{x}_1 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = 0$$

$\mathbf{x}_1$, … , $\mathbf{x}_r$ is linearly independent.

$$c_1(\lambda_1 - \lambda_{r+1}) = \cdots = c_r(\lambda_r - \lambda_{r+1}) = 0$$

All the eigenvalues are distinct.

$$\lambda_1 - \lambda_{r+1} \neq 0$$
$$\vdots$$
$$\lambda_r - \lambda_{r+1} \neq 0$$
$$\therefore c_1 = \cdots = c_r = 0$$

With this, ① reduces to

$$c_{r+1}\mathbf{x}_{r+1} = 0$$
$$\therefore c_{r+1} = 0$$

Basis of Eigenvectors (Proof)

$$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0 \quad \rightarrow ①$$

Multiply both sides by $\mathbf{A}$

$$c_1\mathbf{A}\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{A}\mathbf{x}_{r+1} = 0$$

Use $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$

$$c_1\lambda_1\mathbf{x}_1 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = 0 \rightarrow ②$$

② - λ_{r+1} x ①:

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{x}_1 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = 0$$

$\mathbf{x}_1, \ldots, \mathbf{x}_r$ is linearly independent.

$$c_1(\lambda_1 - \lambda_{r+1}) = \cdots = c_r(\lambda_r - \lambda_{r+1}) = 0$$

All the eigenvalues are distinct.

$$\lambda_1 - \lambda_{r+1} \neq 0$$
$$\vdots$$
$$\lambda_r - \lambda_{r+1} \neq 0$$
$$\therefore c_1 = \cdots = c_r = 0$$

With this, ① reduces to

$$c_{r+1}\mathbf{x}_{r+1} = 0$$
$$\therefore c_{r+1} = 0$$

This contradicts the fact that not all scalars in ① are zero.

Basis of Eigenvectors

Example (Eigenbasis. Nondistinct Eigenvalues. Nonexistence)

$$\mathbf{A}=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{vmatrix}=(5-\lambda)^2-9=0$$

$$\lambda_1=8,\ \lambda_2=2$$

(1) $\lambda=\lambda_1=8$

$$-3x_1+3x_2=0$$
$$3x_1-3x_2=0$$
$$\therefore x_1=x_2$$
$$x_1=x_2=1$$
$$\mathbf{x}_1=\begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}}$$

(2) $\lambda=\lambda_2=2$

$$3x_1+3x_2=0$$
$$3x_1+3x_2=0$$
$$\therefore x_1=-x_2$$
$$x_1=1,\ x_2=-1$$
$$\mathbf{x}_2=\begin{bmatrix}1 & -1\end{bmatrix}^{\mathrm{T}}$$

➔ **eigenbasis for R^n**

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-\lambda & 1\\ 0 & -\lambda\end{vmatrix}=(-\lambda)^2=0$$

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0\quad,\lambda=0$$

$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$0\cdot x_1+1\cdot x_2=0\qquad\therefore\ \mathbf{x}=\begin{bmatrix}k\\ 0\end{bmatrix}$$

Matrix A may *not have enough linearly independent eigenvectors* to make up a basis.

Symmetric Matrices

Theorem 8.4.2 Symmetric Matrices

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal

A real square matrix is orthogonal

if and only if column vectors $a_1, \ldots, a_n$ form an orthonormal system,

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j{}^{\mathrm{T}} \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

From example 8.4.1

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

normalize

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

normalize

Symmetric Matrices

Theorem 8.4.2 Symmetric Matrices

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal

A real square matrix is orthogonal

if and only if column vectors $a_1, \dots, a_n$ form an orthonormal system,

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j{}^{\mathrm{T}} \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

From example 8.4.1

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

normalize $\rightarrow \mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathrm{T}}$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

normalize

Symmetric Matrices

Theorem 8.4.2 Symmetric Matrices

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal

A real square matrix is orthogonal

if and only if column vectors $a_1, \ldots, a_n$ form an orthonormal system,

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j{}^{\mathrm{T}} \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

From example 8.4.1

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

normalize $\rightarrow \mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathrm{T}}$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

normalize $\rightarrow \mathbf{x}_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^{\mathrm{T}}$

Symmetric Matrices

Theorem 8.4.2 Symmetric Matrices

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal

A real square matrix is orthogonal

if and only if column vectors $a_1, \ldots, a_n$ form an orthonormal system,

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j{}^T \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

From example 8.4.1

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$$

$$\text{normalize} \rightarrow \mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$$

$$\text{normalize} \rightarrow \mathbf{x}_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T$$

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_1{}^T \mathbf{x}_2$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = 0$$

Symmetric Matrices

Theorem 8.4.2 Symmetric Matrices

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal

A real square matrix is orthogonal

if and only if column vectors $a_1, \dots, a_n$ form an orthonormal system,

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j{}^T \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

From example 8.4.1

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$$

$$\text{normalize} \rightarrow \mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$$

$$\text{normalize} \rightarrow \mathbf{x}_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T$$

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_1{}^T \mathbf{x}_2$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = 0$$

So $\mathbf{x}_1$, $\mathbf{x}_2$ is an orthonormal basis of eigenvectors.

Similar Matrices. Similarity Transformation

Definition. Similar Matrices. Similarity Transformation

An n x n matrix $\hat{\mathbf{A}}$ is called similar to an n x n matrix $\mathbf{A}$ if

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

for some (nonsingular!) n x n matrix $\mathbf{P}$. This transformation, which gives $\hat{\mathbf{A}}$ from $\mathbf{A}$, is called *a similarity transformation*.

Eigenvalues and Eigenvectors of Similar Matrices

Theorem 8.4.3 Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ is similar to $\mathbf{A}$, then $\hat{\mathbf{A}}$ has the same eigenvalues as $\mathbf{A}$. Furthermore, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to the same eigenvalue.

Proof

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}(\lambda\mathbf{x})$$

Eigenvalues and Eigenvectors of Similar Matrices

Theorem 8.4.3 Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ is similar to $\mathbf{A}$, then $\hat{\mathbf{A}}$ has the same eigenvalues as $\mathbf{A}$. Furthermore, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to the same eigenvalue.

Proof

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}(\lambda\mathbf{x}) = \lambda\mathbf{P}^{-1}\mathbf{x} \quad \rightarrow \ ①$$

Eigenvalues and Eigenvectors of Similar Matrices

Theorem 8.4.3 Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ is similar to $\mathbf{A}$, then $\hat{\mathbf{A}}$ has the same eigenvalues as $\mathbf{A}$. Furthermore, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to the same eigenvalue.

Proof

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}(\lambda\mathbf{x}) = \lambda\mathbf{P}^{-1}\mathbf{x} \quad \rightarrow \ ①$$

Use $\mathbf{I} = \mathbf{PP}^{-1}$

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\mathbf{AIx}$$

Eigenvalues and Eigenvectors of Similar Matrices

Theorem 8.4.3 Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ is similar to $\mathbf{A}$, then $\hat{\mathbf{A}}$ has the same eigenvalues as $\mathbf{A}$. Furthermore, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to the same eigenvalue.

Proof

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}(\lambda\mathbf{x}) = \lambda\mathbf{P}^{-1}\mathbf{x} \quad \rightarrow \ ①$$

Use $\mathbf{I} = \mathbf{PP}^{-1}$

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\mathbf{AIx} = \mathbf{P}^{-1}\mathbf{A}(\mathbf{PP}^{-1})\mathbf{x}$$

Eigenvalues and Eigenvectors of Similar Matrices

Theorem 8.4.3 Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ is similar to $\mathbf{A}$, then $\hat{\mathbf{A}}$ has the same eigenvalues as $\mathbf{A}$. Furthermore, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to the same eigenvalue.

Proof

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}(\lambda\mathbf{x}) = \lambda\mathbf{P}^{-1}\mathbf{x} \quad \rightarrow \ ①$$

Use $\mathbf{I} = \mathbf{PP}^{-1}$

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\mathbf{AIx} = \underbrace{\mathbf{P}^{-1}\mathbf{A}(\mathbf{PP}^{-1})}_{\hat{\mathbf{A}}}\mathbf{x}$$

Eigenvalues and Eigenvectors of Similar Matrices

Theorem 8.4.3 Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ is similar to $\mathbf{A}$, then $\hat{\mathbf{A}}$ has the same eigenvalues as $\mathbf{A}$. Furthermore, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to the same eigenvalue.

Proof

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}(\lambda\mathbf{x}) = \lambda\mathbf{P}^{-1}\mathbf{x} \quad \rightarrow \text{①}$$

Use $\mathbf{I} = \mathbf{PP}^{-1}$

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\mathbf{AIx} = \underbrace{\mathbf{P}^{-1}\mathbf{A}(\mathbf{PP}^{-1})}_{\hat{\mathbf{A}}}\mathbf{x}$$

$$= \hat{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} \quad \rightarrow \text{②}$$

Eigenvalues and Eigenvectors of Similar Matrices

$\mathbf{Ax} = \lambda\mathbf{x}$

Theorem 8.4.3 Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ is similar to $\mathbf{A}$, then $\hat{\mathbf{A}}$ has the same eigenvalues as $\mathbf{A}$. Furthermore, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to the same eigenvalue.

Proof

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}(\lambda\mathbf{x}) = \lambda\mathbf{P}^{-1}\mathbf{x} \rightarrow ①$$

Use $\mathbf{I} = \mathbf{PP}^{-1}$

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\mathbf{AIx} = \underbrace{\mathbf{P}^{-1}\mathbf{A}(\mathbf{PP}^{-1})}_{\hat{\mathbf{A}}}\mathbf{x} = \hat{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} \rightarrow ②$$

From ①, ②

$$\hat{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} = \lambda\mathbf{P}^{-1}\mathbf{x}$$

Eigenvalues and Eigenvectors of Similar Matrices

$\mathbf{Ax} = \lambda\mathbf{x}$

Theorem 8.4.3 Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ is similar to $\mathbf{A}$, then $\hat{\mathbf{A}}$ has the same eigenvalues as $\mathbf{A}$. Furthermore, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to the same eigenvalue.

Proof

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}(\lambda\mathbf{x}) = \lambda\mathbf{P}^{-1}\mathbf{x} \quad \rightarrow \text{①}$$

Use $\mathbf{I} = \mathbf{PP}^{-1}$

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\mathbf{AIx} = \underbrace{\mathbf{P}^{-1}\mathbf{A}(\mathbf{PP}^{-1})}_{\hat{\mathbf{A}}}\mathbf{x} = \hat{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} \quad \rightarrow \text{②}$$

From ①, ②

$$\hat{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} = \lambda\mathbf{P}^{-1}\mathbf{x}$$

If $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$, then

$$\hat{\mathbf{A}}\mathbf{y} = \lambda\mathbf{y}$$

→ $\mathbf{y}$ is an eigenvector of $\hat{\mathbf{A}}$

Eigenvalues and Eigenvectors of Similar Matrices

$\mathbf{Ax} = \lambda\mathbf{x}$

Theorem 8.4.3 Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ is similar to $\mathbf{A}$, then $\hat{\mathbf{A}}$ has the same eigenvalues as $\mathbf{A}$. Furthermore, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to the same eigenvalue.

Proof

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}(\lambda\mathbf{x}) = \lambda\mathbf{P}^{-1}\mathbf{x} \quad \rightarrow \text{①}$$

Use $\mathbf{I} = \mathbf{PP}^{-1}$

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\mathbf{AIx} = \underbrace{\mathbf{P}^{-1}\mathbf{A}(\mathbf{PP}^{-1})}_{\hat{\mathbf{A}}}\mathbf{x} = \hat{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} \quad \rightarrow \text{②}$$

From ①, ②

$$\hat{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} = \lambda\mathbf{P}^{-1}\mathbf{x}$$

If $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$, then

$$\hat{\mathbf{A}}\mathbf{y} = \lambda\mathbf{y}$$

→ $\mathbf{y}$ is an eigenvector of $\hat{\mathbf{A}}$

Indeed, $\mathbf{P}^{-1}\mathbf{x} = 0$ would give

$$\mathbf{x} = \mathbf{Ix} = \mathbf{PP}^{-1}\mathbf{x}$$

Eigenvalues and Eigenvectors of Similar Matrices

$\mathbf{Ax} = \lambda\mathbf{x}$

Theorem 8.4.3 Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ is similar to $\mathbf{A}$, then $\hat{\mathbf{A}}$ has the same eigenvalues as $\mathbf{A}$. Furthermore, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to the same eigenvalue.

Proof

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}(\lambda\mathbf{x}) = \lambda\mathbf{P}^{-1}\mathbf{x} \quad \rightarrow ①$$

Use $\mathbf{I} = \mathbf{PP}^{-1}$

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\mathbf{AIx} = \underbrace{\mathbf{P}^{-1}\mathbf{A}(\mathbf{PP}^{-1})}_{\hat{\mathbf{A}}}\mathbf{x} = \hat{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} \quad \rightarrow ②$$

From ①, ②

$$\hat{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} = \lambda\mathbf{P}^{-1}\mathbf{x}$$

If $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$, then

$$\hat{\mathbf{A}}\mathbf{y} = \lambda\mathbf{y}$$

→ $\mathbf{y}$ is an eigenvector of $\hat{\mathbf{A}}$

Indeed, $\mathbf{P}^{-1}\mathbf{x} = 0$ would give

$$\mathbf{x} = \mathbf{Ix} = \mathbf{PP}^{-1}\mathbf{x} = \mathbf{P}\cdot 0 = 0$$

Eigenvalues and Eigenvectors of Similar Matrices

$\mathbf{Ax} = \lambda\mathbf{x}$

Theorem 8.4.3 Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ is similar to $\mathbf{A}$, then $\hat{\mathbf{A}}$ has the same eigenvalues as $\mathbf{A}$. Furthermore, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to the same eigenvalue.

Proof

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}(\lambda\mathbf{x}) = \lambda\mathbf{P}^{-1}\mathbf{x} \quad \rightarrow ①$$

Use $\mathbf{I} = \mathbf{PP}^{-1}$

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\mathbf{AIx} = \underbrace{\mathbf{P}^{-1}\mathbf{A}(\mathbf{PP}^{-1})}_{\hat{\mathbf{A}}}\mathbf{x} = \hat{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} \quad \rightarrow ②$$

From ①, ②

$$\hat{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} = \lambda\mathbf{P}^{-1}\mathbf{x}$$

If $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$, then

$$\hat{\mathbf{A}}\mathbf{y} = \lambda\mathbf{y}$$

→ $\mathbf{y}$ is an eigenvector of $\hat{\mathbf{A}}$

Indeed, $\mathbf{P}^{-1}\mathbf{x} = 0$ would give

$$\mathbf{x} = \mathbf{Ix} = \mathbf{PP}^{-1}\mathbf{x} = \mathbf{P}\cdot 0 = 0$$

→ contradicting $\mathbf{x} \neq 0$

$$\therefore \mathbf{y} = \mathbf{P}^{-1}\mathbf{x} \neq 0$$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\mathbf{P}^{-1}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\mathbf{P}^{-1}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

Get similar matrix $\hat{\mathbf{A}}$.

$$\hat{\mathbf{A}}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

$$=\begin{bmatrix}4 & -3\\ -1 & 4\end{bmatrix}\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\mathbf{P}^{-1}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

Get similar matrix $\hat{\mathbf{A}}$.

$$\hat{\mathbf{A}}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

$$=\begin{bmatrix}4 & -3\\ -1 & 4\end{bmatrix}\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}6-\lambda & -3\\ 4 & -1-\lambda\end{vmatrix}=0$$

$$\mathbf{P}^{-1}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

Get similar matrix $\hat{\mathbf{A}}$.

$$\hat{\mathbf{A}}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

$$=\begin{bmatrix}4 & -3\\ -1 & 4\end{bmatrix}\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}6-\lambda & -3\\ 4 & -1-\lambda\end{vmatrix}=0$$

$$(6-\lambda)(-1-\lambda)+12=0$$

$$\mathbf{P}^{-1}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

Get similar matrix $\hat{\mathbf{A}}$.

$$\hat{\mathbf{A}}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

$$=\begin{bmatrix}4 & -3\\ -1 & 4\end{bmatrix}\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}6-\lambda & -3\\ 4 & -1-\lambda\end{vmatrix}=0$$

$$\mathbf{P}^{-1}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

$$(6-\lambda)(-1-\lambda)+12=0$$

$$\lambda^2-5\lambda+6=0$$

Get similar matrix $\hat{\mathbf{A}}$.

$$\hat{\mathbf{A}}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

$$=\begin{bmatrix}4 & -3\\ -1 & 4\end{bmatrix}\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}6-\lambda & -3\\ 4 & -1-\lambda\end{vmatrix}=0$$

$$\mathbf{P}^{-1}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

$$(6-\lambda)(-1-\lambda)+12=0$$

$$\lambda^2-5\lambda+6=0 \quad\Rightarrow\quad \lambda_1=3,\ \lambda_2=2$$

Get similar matrix $\hat{\mathbf{A}}$.

$$\hat{\mathbf{A}}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

$$=\begin{bmatrix}4 & -3\\ -1 & 4\end{bmatrix}\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\mathbf{P}^{-1}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

Get similar matrix $\hat{\mathbf{A}}$.

$$\hat{\mathbf{A}}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

$$=\begin{bmatrix}4 & -3\\ -1 & 4\end{bmatrix}\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}6-\lambda & -3\\ 4 & -1-\lambda\end{vmatrix}=0$$

$$(6-\lambda)(-1-\lambda)+12=0$$

$$\lambda^2-5\lambda+6=0 \quad\Rightarrow\quad \lambda_1=3,\ \lambda_2=2$$

$$\det(\hat{\mathbf{A}}-\lambda\mathbf{I})=\begin{vmatrix}3-\lambda & 0\\ 0 & 2-\lambda\end{vmatrix}=0$$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\mathbf{P}^{-1}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

Get similar matrix $\hat{\mathbf{A}}$.

$$\hat{\mathbf{A}}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

$$=\begin{bmatrix}4 & -3\\ -1 & 4\end{bmatrix}\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}6-\lambda & -3\\ 4 & -1-\lambda\end{vmatrix}=0$$

$$(6-\lambda)(-1-\lambda)+12=0$$

$$\lambda^2-5\lambda+6=0 \quad\Rightarrow\quad \lambda_1=3,\ \lambda_2=2$$

$$\det(\hat{\mathbf{A}}-\lambda\mathbf{I})=\begin{vmatrix}3-\lambda & 0\\ 0 & 2-\lambda\end{vmatrix}=0$$

$$(3-\lambda)(2-\lambda)=0$$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\mathbf{P}^{-1}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

Get similar matrix $\hat{\mathbf{A}}$.

$$\hat{\mathbf{A}}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

$$=\begin{bmatrix}4 & -3\\ -1 & 4\end{bmatrix}\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}6-\lambda & -3\\ 4 & -1-\lambda\end{vmatrix}=0$$

$$(6-\lambda)(-1-\lambda)+12=0$$

$$\lambda^2-5\lambda+6=0 \Rightarrow \lambda_1=3,\ \lambda_2=2$$

$$\det(\hat{\mathbf{A}}-\lambda\mathbf{I})=\begin{vmatrix}3-\lambda & 0\\ 0 & 2-\lambda\end{vmatrix}=0$$

$$(3-\lambda)(2-\lambda)=0 \Rightarrow \lambda_1=3,\ \lambda_2=2$$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\mathbf{P}^{-1}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

Get similar matrix $\hat{\mathbf{A}}$.

$$\hat{\mathbf{A}}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

$$=\begin{bmatrix}4 & -3\\ -1 & 4\end{bmatrix}\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}6-\lambda & -3\\ 4 & -1-\lambda\end{vmatrix}=0$$

$$(6-\lambda)(-1-\lambda)+12=0$$

$$\lambda^2-5\lambda+6=0 \Rightarrow \lambda_1=3,\ \lambda_2=2$$

$$\det(\hat{\mathbf{A}}-\lambda\mathbf{I})=\begin{vmatrix}3-\lambda & 0\\ 0 & 2-\lambda\end{vmatrix}=0$$

$$(3-\lambda)(2-\lambda)=0 \Rightarrow \lambda_1=3,\ \lambda_2=2$$

So $\mathbf{A}$ and $\hat{\mathbf{A}}$ has the same eigenvalues.

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\hat{\mathbf{A}}=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix},\qquad \lambda_1=3,\quad \lambda_2=2$$

$$\mathbf{P}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

(1) $\lambda=\lambda_1=3$

(2) $\lambda=\lambda_2=2$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{P}^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}, \qquad \mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \qquad \lambda_1 = 3, \quad \lambda_2 = 2$$

(1) $\lambda = \lambda_1 = 3$

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix}$$

(2) $\lambda = \lambda_2 = 2$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{P}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\hat{\mathbf{A}}=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix},\qquad \lambda_1=3,\quad \lambda_2=2$$

(1) $\lambda=\lambda_1=3$

$$\mathbf{A}-\lambda_1\mathbf{I}=\begin{bmatrix}3 & -3\\ 4 & -4\end{bmatrix}$$

$$3x_1-3x_2=0$$

(2) $\lambda=\lambda_2=2$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{P}^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}, \qquad \mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \qquad \lambda_1 = 3, \quad \lambda_2 = 2$$

(1) $\lambda = \lambda_1 = 3$

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix}$$

Let $x_1 = 1$,

$$\therefore \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$3x_1 - 3x_2 = 0$$

(2) $\lambda = \lambda_2 = 2$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\mathbf{P}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

$$\hat{\mathbf{A}}=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix},\qquad \lambda_1=3,\quad \lambda_2=2$$

(1) $\lambda=\lambda_1=3$

$$\mathbf{A}-\lambda_1\mathbf{I}=\begin{bmatrix}3 & -3\\ 4 & -4\end{bmatrix}$$

Let $x_1 = 1$,

$$\therefore \mathbf{x}_1=\begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}}$$

$$3x_1-3x_2=0$$

(2) $\lambda=\lambda_2=2$

$$\mathbf{A}-\lambda_2\mathbf{I}=\begin{bmatrix}4 & -3\\ 4 & -3\end{bmatrix}$$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{P}^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}, \qquad \mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \qquad \lambda_1 = 3, \quad \lambda_2 = 2$$

(1) $\lambda = \lambda_1 = 3$

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix}$$

Let $x_1 = 1$,

$$\therefore \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$3x_1 - 3x_2 = 0$$

(2) $\lambda = \lambda_2 = 2$

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix}$$

$$4x_1 - 3x_2 = 0$$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{P}^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}, \qquad \mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \qquad \lambda_1 = 3, \quad \lambda_2 = 2$$

(1) $\lambda = \lambda_1 = 3$

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix}$$

$$3x_1 - 3x_2 = 0$$

Let $x_1 = 1$,

$$\therefore \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

(2) $\lambda = \lambda_2 = 2$

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix}$$

$$4x_1 - 3x_2 = 0$$

Let $x_1 = 3$,

$$\therefore \mathbf{x}_2 = \begin{bmatrix} 3 & 4 \end{bmatrix}^{\mathrm{T}}$$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{P}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\hat{\mathbf{A}}=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix},\qquad \lambda_1=3,\quad \lambda_2=2$$

(1) $\lambda=\lambda_1=3$

$$\mathbf{A}-\lambda_1\mathbf{I}=\begin{bmatrix}3 & -3\\ 4 & -4\end{bmatrix}$$

$$3x_1-3x_2=0$$

Let $x_1 = 1$,

$$\therefore \mathbf{x}_1=\begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}}$$

(2) $\lambda=\lambda_2=2$

$$\mathbf{A}-\lambda_2\mathbf{I}=\begin{bmatrix}4 & -3\\ 4 & -3\end{bmatrix}$$

$$4x_1-3x_2=0$$

Let $x_1 = 3$,

$$\therefore \mathbf{x}_2=\begin{bmatrix}3 & 4\end{bmatrix}^{\mathrm{T}}$$

Eigenvectors of $\hat{\mathbf{A}}$ is

$$\mathbf{y}_1=\mathbf{P}^{-1}\mathbf{x}_1=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}$$

Eigenvalues and Eigenvectors of Similar Matrices

Example 8.4-3 (Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\qquad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\hat{\mathbf{A}}=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix},\qquad \lambda_1=3,\quad \lambda_2=2$$

$$\mathbf{P}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

(1) $\lambda=\lambda_1=3$

$$\mathbf{A}-\lambda_1\mathbf{I}=\begin{bmatrix}3 & -3\\ 4 & -4\end{bmatrix}$$

$$3x_1-3x_2=0$$

Let $x_1 = 1$,

$$\therefore \mathbf{x}_1=\begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}}$$

(2) $\lambda=\lambda_2=2$

$$\mathbf{A}-\lambda_2\mathbf{I}=\begin{bmatrix}4 & -3\\ 4 & -3\end{bmatrix}$$

$$4x_1-3x_2=0$$

Let $x_1 = 3$,

$$\therefore \mathbf{x}_2=\begin{bmatrix}3 & 4\end{bmatrix}^{\mathrm{T}}$$

Eigenvectors of $\hat{\mathbf{A}}$ is

$$\mathbf{y}_1=\mathbf{P}^{-1}\mathbf{x}_1=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$\mathbf{y}_2=\mathbf{P}^{-1}\mathbf{x}_2=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}\begin{bmatrix}3\\ 4\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}$$

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

Diagonalization of a Matrix

Theorem 8.4.4 Diagonalization of Matrix

If an n x n matrix $\mathbf{A}$ has a basis of eigenvectors, then

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

is *diagonal*, with the eigenvalues of $\mathbf{A}$ as the entries on the main diagonal.

Here $\mathbf{X}$ is the matrix with these eigenvectors as column vectors. Also

$$\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X}$$

Diagonalization of a Matrix (Proof)

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$
$$\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X}$$

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ constitute a basis of eigenvectors of $\mathbf{A}$ for R^n. Let the corresponding eigenvalues of A be $\lambda_1, \ldots, \lambda_n$.

Diagonalization of a Matrix (Proof)

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$
$$\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X}$$

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ constitute a basis of eigenvectors of $\mathbf{A}$ for R^n. Let the corresponding eigenvalues of A be $\lambda_1, \ldots, \lambda_n$.

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$$
$$\vdots$$
$$\mathbf{A}\mathbf{x}_n = \lambda_n\mathbf{x}_n$$

Diagonalization of a Matrix (Proof)

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$
$$\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X}$$

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ constitute a basis of eigenvectors of $\mathbf{A}$ for R^n. Let the corresponding eigenvalues of A be $\lambda_1, \ldots, \lambda_n$.

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$$
$$\vdots$$
$$\mathbf{A}\mathbf{x}_n = \lambda_n\mathbf{x}_n$$

Then $\mathbf{X} = [\, \mathbf{x}_1 \ \ldots \ \mathbf{x}_n \,]$ has rank n,

Hence $\mathbf{X}^{-1}$ exists.

Diagonalization of a Matrix (Proof)

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$
$$\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X}$$

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ constitute a basis of eigenvectors of $\mathbf{A}$ for R^n. Let the corresponding eigenvalues of A be $\lambda_1, \ldots, \lambda_n$.

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$$
$$\vdots$$
$$\mathbf{A}\mathbf{x}_n = \lambda_n\mathbf{x}_n$$

Then $\mathbf{X} = [\, \mathbf{x}_1 \ \ldots \ \mathbf{x}_n \,]$ has rank n,

Hence $\mathbf{X}^{-1}$ exists.

$$\mathbf{A}\mathbf{X} = \mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}$$

Diagonalization of a Matrix (Proof)

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$
$$\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X}$$

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ constitute a basis of eigenvectors of $\mathbf{A}$ for R^n. Let the corresponding eigenvalues of A be $\lambda_1, \ldots, \lambda_n$.

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$$
$$\vdots$$
$$\mathbf{A}\mathbf{x}_n = \lambda_n\mathbf{x}_n$$

Then $\mathbf{X} = [\, \mathbf{x}_1 \ \ldots \ \mathbf{x}_n \,]$ has rank n,

Hence $\mathbf{X}^{-1}$ exists.

$$\mathbf{A}\mathbf{X} = \mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{A}\mathbf{x}_1 & \cdots & \mathbf{A}\mathbf{x}_n\end{bmatrix}$$

Diagonalization of a Matrix (Proof)

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ constitute a basis of eigenvectors of A for R^n. Let the corresponding eigenvalues of A be $\lambda_1, \ldots, \lambda_n$.

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$$
$$\vdots$$
$$\mathbf{A}\mathbf{x}_n = \lambda_n\mathbf{x}_n$$

Then $\mathbf{X} = [\, \mathbf{x}_1 \ \ldots \ \mathbf{x}_n \,]$ has rank n,

Hence $\mathbf{X}^{-1}$ exists.

$$\mathbf{A}\mathbf{X} = \mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{A}\mathbf{x}_1 & \cdots & \mathbf{A}\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\lambda_1\mathbf{x}_1 & \cdots & \lambda_n\mathbf{x}_n\end{bmatrix}$$

Diagonalization of a Matrix (Proof)

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ constitute a basis of eigenvectors of A for R^n. Let the corresponding eigenvalues of A be $\lambda_1, \ldots, \lambda_n$.

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$$
$$\vdots$$
$$\mathbf{A}\mathbf{x}_n = \lambda_n\mathbf{x}_n$$

Then $\mathbf{X} = [\, \mathbf{x}_1 \ \ldots \ \mathbf{x}_n \,]$ has rank n,

Hence $\mathbf{X}^{-1}$ exists.

$$\mathbf{A}\mathbf{X} = \mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{A}\mathbf{x}_1 & \cdots & \mathbf{A}\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\lambda_1\mathbf{x}_1 & \cdots & \lambda_n\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}$$

Diagonalization of a Matrix (Proof)

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ constitute a basis of eigenvectors of A for R^n. Let the corresponding eigenvalues of A be $\lambda_1, \ldots, \lambda_n$.

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$$
$$\vdots$$
$$\mathbf{A}\mathbf{x}_n = \lambda_n\mathbf{x}_n$$

Then $\mathbf{X} = [\, \mathbf{x}_1 \ \ldots \ \mathbf{x}_n \,]$ has rank n,

Hence $\mathbf{X}^{-1}$ exists.

$$\mathbf{A}\mathbf{X} = \mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{A}\mathbf{x}_1 & \cdots & \mathbf{A}\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\lambda_1\mathbf{x}_1 & \cdots & \lambda_n\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}\underbrace{\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}}_{\mathbf{D}}$$

Diagonalization of a Matrix (Proof)

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ constitute a basis of eigenvectors of A for R^n. Let the corresponding eigenvalues of A be $\lambda_1, \ldots, \lambda_n$.

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$$
$$\vdots$$
$$\mathbf{A}\mathbf{x}_n = \lambda_n\mathbf{x}_n$$

Then $\mathbf{X} = [\, \mathbf{x}_1 \ \ldots \ \mathbf{x}_n \,]$ has rank n,

Hence $\mathbf{X}^{-1}$ exists.

$$\mathbf{AX} = \mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{A}\mathbf{x}_1 & \cdots & \mathbf{A}\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\lambda_1\mathbf{x}_1 & \cdots & \lambda_n\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}\underbrace{\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}}_{\mathbf{D}}$$
$$= \mathbf{XD}$$

Diagonalization of a Matrix (Proof)

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ constitute a basis of eigenvectors of A for R^n. Let the corresponding eigenvalues of A be $\lambda_1, \ldots, \lambda_n$.

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$$
$$\vdots$$
$$\mathbf{A}\mathbf{x}_n = \lambda_n\mathbf{x}_n$$

Then $\mathbf{X} = [\, \mathbf{x}_1 \ \ldots \ \mathbf{x}_n \,]$ has rank n,

Hence $\mathbf{X}^{-1}$ exists.

$$\mathbf{AX} = \mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{A}\mathbf{x}_1 & \cdots & \mathbf{A}\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\lambda_1\mathbf{x}_1 & \cdots & \lambda_n\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}\underbrace{\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}}_{\mathbf{D}}$$
$$= \mathbf{XD}$$

$$\mathbf{AX} = \mathbf{XD}$$

Diagonalization of a Matrix (Proof)

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ constitute a basis of eigenvectors of A for R^n. Let the corresponding eigenvalues of A be $\lambda_1, \ldots, \lambda_n$.

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$$
$$\vdots$$
$$\mathbf{A}\mathbf{x}_n = \lambda_n\mathbf{x}_n$$

Then $\mathbf{X} = [\,\mathbf{x}_1 \ \ldots \ \mathbf{x}_n\,]$ has rank n,

Hence $\mathbf{X}^{-1}$ exists.

$$\mathbf{AX} = \mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{A}\mathbf{x}_1 & \cdots & \mathbf{A}\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\lambda_1\mathbf{x}_1 & \cdots & \lambda_n\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}\underbrace{\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}}_{\mathbf{D}}$$
$$= \mathbf{XD}$$

$$\mathbf{AX} = \mathbf{XD}$$
$$\therefore \mathbf{D} = \mathbf{X}^{-1}\mathbf{AX}$$

Diagonalization of a Matrix (Proof)

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ constitute a basis of eigenvectors of A for R^n. Let the corresponding eigenvalues of A be $\lambda_1, \ldots, \lambda_n$.

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$$
$$\vdots$$
$$\mathbf{A}\mathbf{x}_n = \lambda_n\mathbf{x}_n$$

Then $\mathbf{X} = [\,\mathbf{x}_1 \ \ldots \ \mathbf{x}_n\,]$ has rank n,

Hence $\mathbf{X}^{-1}$ exists.

$$\mathbf{AX} = \mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{A}\mathbf{x}_1 & \cdots & \mathbf{A}\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\lambda_1\mathbf{x}_1 & \cdots & \lambda_n\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}\underbrace{\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}}_{\mathbf{D}}$$
$$= \mathbf{XD}$$

$$\mathbf{AX} = \mathbf{XD}$$
$$\therefore \mathbf{D} = \mathbf{X}^{-1}\mathbf{AX}$$
$$\mathbf{D}^2 = \mathbf{DD} = \mathbf{X}^{-1}\mathbf{AXX}^{-1}\mathbf{AX}$$

Diagonalization of a Matrix (Proof)

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$
$$\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X}$$

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ constitute a basis of eigenvectors of A for R^n. Let the corresponding eigenvalues of A be $\lambda_1, \ldots, \lambda_n$.

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$$
$$\vdots$$
$$\mathbf{A}\mathbf{x}_n = \lambda_n\mathbf{x}_n$$

Then $\mathbf{X} = [\, \mathbf{x}_1 \ \ldots \ \mathbf{x}_n \,]$ has rank n,

Hence $\mathbf{X}^{-1}$ exists.

$$\mathbf{AX} = \mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{A}\mathbf{x}_1 & \cdots & \mathbf{A}\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\lambda_1\mathbf{x}_1 & \cdots & \lambda_n\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}\underbrace{\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}}_{\mathbf{D}}$$
$$= \mathbf{XD}$$

$$\mathbf{AX} = \mathbf{XD}$$
$$\therefore \mathbf{D} = \mathbf{X}^{-1}\mathbf{AX}$$

$$\mathbf{D}^2 = \mathbf{DD} = \mathbf{X}^{-1}\mathbf{AXX}^{-1}\mathbf{AX}$$
$$= \mathbf{X}^{-1}\mathbf{AIAX}$$

Diagonalization of a Matrix (Proof)

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ constitute a basis of eigenvectors of A for R^n. Let the corresponding eigenvalues of A be $\lambda_1, \ldots, \lambda_n$.

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$$
$$\vdots$$
$$\mathbf{A}\mathbf{x}_n = \lambda_n\mathbf{x}_n$$

Then $\mathbf{X} = [\, \mathbf{x}_1 \ \ldots \ \mathbf{x}_n \,]$ has rank n,

Hence $\mathbf{X}^{-1}$ exists.

$$\mathbf{AX} = \mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{A}\mathbf{x}_1 & \cdots & \mathbf{A}\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\lambda_1\mathbf{x}_1 & \cdots & \lambda_n\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}\underbrace{\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}}_{\mathbf{D}}$$
$$= \mathbf{XD}$$

$$\mathbf{AX} = \mathbf{XD}$$
$$\therefore \mathbf{D} = \mathbf{X}^{-1}\mathbf{AX}$$

$$\mathbf{D}^2 = \mathbf{DD} = \mathbf{X}^{-1}\mathbf{AXX}^{-1}\mathbf{AX}$$
$$= \mathbf{X}^{-1}\mathbf{AIAX} = \mathbf{X}^{-1}\mathbf{A}^2\mathbf{X}$$

Diagonalization of a Matrix

$$\mathbf{AX} = \mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix} = \begin{bmatrix}\mathbf{Ax}_1 & \cdots & \mathbf{Ax}_n\end{bmatrix}$$

Let $n = 2$.

$$\mathbf{x}_1 = \begin{bmatrix}x_{11} & x_{12}\end{bmatrix}^{\mathrm{T}},\ \mathbf{x}_2 = \begin{bmatrix}x_{21} & x_{22}\end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{AX} = \mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2\end{bmatrix}$$

$$= \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_{11} & x_{21}\\ x_{12} & x_{22}\end{bmatrix}$$

$$= \begin{bmatrix}a_{11}x_{11}+a_{12}x_{21} & a_{11}x_{12}+a_{12}x_{22}\\ a_{21}x_{11}+a_{22}x_{21} & a_{21}x_{12}+a_{22}x_{22}\end{bmatrix}$$

$$= \begin{bmatrix}\mathbf{Ax}_1 & \mathbf{Ax}_2\end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$$

Diagonalization of a Matrix

Example (Diagonalization)

Diagonalize

$$\mathbf{A}=\begin{bmatrix}-6 & -6 & 10\\ -5 & -5 & 5\\ -9 & -9 & 13\end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix}-6-\lambda & -6 & 10\\ -5 & -5-\lambda & 5\\ -9 & -9 & 13-\lambda\end{vmatrix}$$

$$=(-6-\lambda)\left[(-5-\lambda)(13-\lambda)-5\cdot(-9)\right]$$
$$-(-6)\left[(-5)(13-\lambda)-5\cdot(-9)\right]$$
$$+10\left[(-5)(-9)-(-5-\lambda)\cdot(-9)\right]$$
$$=\lambda^3-2\lambda^2-8\lambda=0$$

$$\lambda=0,-2,4$$

$\lambda=0$:

$$\begin{bmatrix}-6-\lambda & -6 & 10\\ -5 & -5-\lambda & 5\\ -9 & -9 & 13-\lambda\end{bmatrix}=\begin{bmatrix}-6 & -6 & 10\\ -5 & -5 & 5\\ -9 & -9 & 13\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & -5/3\\ 0 & 0 & -2/3\\ 0 & 0 & -8/3\end{bmatrix}\Leftarrow\begin{bmatrix}-3 & -3 & 5\\ -1 & -1 & 1\\ 0 & 0 & -8/3\end{bmatrix}\Leftarrow\begin{bmatrix}-3 & -3 & 5\\ -1 & -1 & 1\\ -3 & -3 & 13/3\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & -5/3\\ 0 & 0 & -2/3\\ 0 & 0 & -8/3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\longrightarrow x_1+x_2=0, x_3=0$$

Choose $x_1=1\longrightarrow x_2=-1, x_3=0$

$$\mathbf{x}_1=[1,-1,0]^{\mathbf{T}}$$

Diagonalization of a Matrix

Example (Diagonalization)

Diagonalize

$$\mathbf{A}=\begin{bmatrix}-6 & -6 & 10\\ -5 & -5 & 5\\ -9 & -9 & 13\end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix}-6-\lambda & -6 & 10\\ -5 & -5-\lambda & 5\\ -9 & -9 & 13-\lambda\end{vmatrix}$$

$$=(-6-\lambda)\left[(-5-\lambda)(13-\lambda)-5\cdot(-9)\right]$$
$$-(-6)\left[(-5)(13-\lambda)-5\cdot(-9)\right]$$
$$+10\left[(-5)(-9)-(-5-\lambda)\cdot(-9)\right]$$
$$=\lambda^3-2\lambda^2-8\lambda=0$$

$$\lambda=0,-2,4$$

$\lambda=-2$:

$$\begin{bmatrix}-6-\lambda & -6 & 10\\ -5 & -5-\lambda & 5\\ -9 & -9 & 13-\lambda\end{bmatrix}=\begin{bmatrix}-4 & -6 & 10\\ -5 & -3 & 5\\ -9 & -9 & 15\end{bmatrix}$$

$$\begin{bmatrix}-2 & -3 & 5\\ -3 & 0 & 0\\ -1 & 0 & 0\end{bmatrix}\Leftarrow\begin{bmatrix}-2 & -3 & 5\\ -5 & -3 & 5\\ -3 & -3 & 5\end{bmatrix}$$

$$\begin{bmatrix}-2 & -3 & 5\\ -3 & 0 & 0\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\longrightarrow x_1=0,\ -3x_2+5x_3=0$$

Choose $x_2=5\longrightarrow x_3=3, x_1=0$

$$\mathbf{x}_2=[0,5,3]^{\mathbf{T}}$$

$$\mathbf{x}_1=[1,-1,0]^{\mathbf{T}}$$

Diagonalization of a Matrix

Example (Diagonalization)

Diagonalize

$$\mathbf{A}=\begin{bmatrix}-6 & -6 & 10\\ -5 & -5 & 5\\ -9 & -9 & 13\end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix}-6-\lambda & -6 & 10\\ -5 & -5-\lambda & 5\\ -9 & -9 & 13-\lambda\end{vmatrix}$$

$$=(-6-\lambda)\left[(-5-\lambda)(13-\lambda)-5\cdot(-9)\right]$$
$$-(-6)\left[(-5)(13-\lambda)-5\cdot(-9)\right]$$
$$+10\left[(-5)(-9)-(-5-\lambda)\cdot(-9)\right]$$
$$=\lambda^3-2\lambda^2-8\lambda=0$$

$$\lambda=0,-2,4$$

$\lambda=4$:

$$\begin{bmatrix}-6-\lambda & -6 & 10\\ -5 & -5-\lambda & 5\\ -9 & -9 & 13-\lambda\end{bmatrix}=\begin{bmatrix}-10 & -6 & 10\\ -5 & -9 & 5\\ -9 & -9 & 9\end{bmatrix}$$

$$\begin{bmatrix}0 & 2 & 0\\ 0 & -4 & 0\\ -1 & -1 & 1\end{bmatrix}\Leftarrow\begin{bmatrix}-5 & -3 & 5\\ -5 & -9 & 5\\ -1 & -1 & 1\end{bmatrix}$$

$$\begin{bmatrix}0 & 2 & 0\\ 0 & -4 & 0\\ -1 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\longrightarrow\quad x_2=0,-x_1+x_3=0$$

Choose $x_1=1\longrightarrow\quad x_3=1,x_2=0$

$$\mathbf{x}_3=[1,0,1]^{\mathbf{T}}$$

$$\mathbf{x}_1=[1,-1,0]^{\mathbf{T}}$$
$$\mathbf{x}_2=[0,5,3]^{\mathbf{T}}$$

Diagonalization of a Matrix

Example (Diagonalization)

Diagonalize

$$\mathbf{A}=\begin{bmatrix}-6 & -6 & 10\\ -5 & -5 & 5\\ -9 & -9 & 13\end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix}-6-\lambda & -6 & 10\\ -5 & -5-\lambda & 5\\ -9 & -9 & 13-\lambda\end{vmatrix}$$

$$=(-6-\lambda)\left[(-5-\lambda)(13-\lambda)-5\cdot(-9)\right]$$
$$-(-6)\left[(-5)(13-\lambda)-5\cdot(-9)\right]$$
$$+10\left[(-5)(-9)-(-5-\lambda)\cdot(-9)\right]$$
$$=\lambda^3-2\lambda^2-8\lambda=0$$

$$\lambda=0,-2,4$$

$$\mathbf{x}_1=[1,-1,0]^T\quad \mathbf{x}_2=[0,5,3]^T\quad \mathbf{x}_3=[1,0,1]^T$$

$$\therefore \mathbf{X}=\begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3\\ 1 & 0 & 1\\ -1 & 5 & 0\\ 0 & 3 & 1\end{bmatrix}$$

$$\therefore[\mathbf{X}\,|\,\mathbf{I}]=\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0\\ -1 & 5 & 0 & 0 & 1 & 0\\ 0 & 3 & 1 & 0 & 0 & 1\end{array}\right]$$

⇩ Gauss-Jordan Elimination

$$\therefore[\mathbf{I}\,|\,\mathbf{X}^{-1}]=\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/2 & 3/2 & -5/2\\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2\\ 0 & 0 & 1 & -3/2 & -3/2 & 5/2\end{array}\right]$$

Diagonalization of a Matrix

Example (Diagonalization)

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

$$= \frac{1}{2}\begin{bmatrix} 5 & 3 & -5 \\ 1 & 1 & -1 \\ -3 & -3 & 5 \end{bmatrix}\begin{bmatrix} -6 & -6 & 10 \\ -5 & -5 & 5 \\ -9 & -9 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 5 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -6 & -6 & 10 \\ -5 & -5 & 5 \\ -9 & -9 & 13 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 5 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$\mathbf{X}^{-1} = \frac{1}{2}\begin{bmatrix} 5 & 3 & -5 \\ 1 & 1 & -1 \\ -3 & -3 & 5 \end{bmatrix}$$

Similar Matrices & Singular Value Decomposition (SVD)

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

***n* x *n* matrix A**

***n* x *n* matrix P**

A and $\hat{\mathrm{A}}$ has the same eigenvalues

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

***n* x *n* matrix A**

X is the matrix with these eigenvectors of A as column vectors (not always orthogonal)

***D* : with the eigenvalues of A as the entries on the main diagonal.**

SVD

$$\underset{[m\times n]}{\mathbf{A}} = \underset{[m\times m]}{\mathbf{U}}\underset{[m\times n]}{\mathbf{\Sigma}}\underset{[n\times n]}{\mathbf{V}^{\mathrm{T}}} = (\text{orthogonal})(\text{diagonal})(\text{orthogonal})$$

$\mathbf{u}_1, \cdots, \mathbf{u}_m$ in $\mathbf{U}$ are eigenvectors of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$,

$\mathbf{v}_1, \cdots, \mathbf{v}_n$ in $\mathbf{V}$ are eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

$\sigma_1, \ldots, \sigma_r$: **Square root of eigenvalues of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$**

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ & & \sigma_r \\ 0 & & 0 \end{bmatrix}$$

$\text{rank } \mathbf{A} = r \le \min(n, m)$

Singular Value Decomposition (SVD)

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}} = (\text{orthogonal})(\text{diagonal})(\text{orthogonal})$$
$$[m\times n] \qquad [m\times m] \quad [m\times n] \quad [n\times n]$$

$\mathbf{u}_1,\cdots,\mathbf{u}_m$ in $\mathbf{U}$ are eigenvectors of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$,

$\mathbf{v}_1,\cdots,\mathbf{v}_n$ in $\mathbf{V}$ are eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

$\sigma_1,\ldots,\sigma_r$: **Square root of Eigenvalues of A^TA**

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ & & \sigma_r \\ 0 & & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\text{rank } \mathbf{A} = r \le \min(n,m)$$

$$\mathbf{A}\mathbf{v}_i = \sigma_i\mathbf{u}_i$$

Singular Value Decomposition (SVD)

$$\underset{[m\times n]}{\mathbf{A}} = \underset{[m\times m]}{\mathbf{U}}\underset{[m\times n]}{\mathbf{\Sigma}}\underset{[n\times n]}{\mathbf{V}^{\mathrm{T}}} = (\text{orthogonal})(\text{diagonal})(\text{orthogonal})$$

$\mathbf{u}_1,\cdots,\mathbf{u}_m$ in $\mathbf{U}$ are eigenvectors of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$,

$\mathbf{v}_1,\cdots,\mathbf{v}_n$ in $\mathbf{V}$ are eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

$\sigma_1,\ldots,\sigma_r$: **Square root of Eigenvalues of A^TA**

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ & & \sigma_r \\ 0 & & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\text{rank } \mathbf{A} = r \le \min(n,m)$$

$$\mathbf{A}\mathbf{v}_i = \sigma_i\mathbf{u}_i$$

$$\mathbf{U} = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_r\mathbf{u}_{r+1}\cdots\mathbf{u}_m] : \text{orthonormal}$$

$$\mathbf{V} = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_r\ \mathbf{v}_{r+1}\cdots\mathbf{v}_n] : \text{orthonormal}$$

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \\ & & \end{bmatrix} : \text{diagonal}(m\times n)$$

Singular Value Decomposition (SVD)

$$\underset{[m\times n]}{\mathbf{A}} = \underset{[m\times m]}{\mathbf{U}}\underset{[m\times n]}{\mathbf{\Sigma}}\underset{[n\times n]}{\mathbf{V}^{\mathrm{T}}} = (\text{orthogonal})(\text{diagonal})(\text{orthogonal})$$

$\mathbf{u}_1,\cdots,\mathbf{u}_m$ in $\mathbf{U}$ are eigenvectors of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$,

$\mathbf{v}_1,\cdots,\mathbf{v}_n$ in $\mathbf{V}$ are eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

$\sigma_1,\ldots,\sigma_r$: **Square root of Eigenvalues of A^TA**

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ & & \sigma_r \\ 0 & & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$\text{rank } \mathbf{A} = r \le \min(n,m)$

$$\mathbf{A}\mathbf{v}_i = \sigma_i\mathbf{u}_i$$

$$\mathbf{A}\mathbf{V} = \mathbf{U}\mathbf{\Sigma}$$

$\mathbf{U} = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_r\mathbf{u}_{r+1}\cdots\mathbf{u}_m]$:orthonormal

$\mathbf{V} = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_r\ \mathbf{v}_{r+1}\cdots\mathbf{v}_n]$:orthonormal

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \\ & & \end{bmatrix} : \text{diagonal}(m\times n)$$

Singular Value Decomposition (SVD)

$$\underset{[m\times n]}{\mathbf{A}} = \underset{[m\times m]}{\mathbf{U}}\underset{[m\times n]}{\mathbf{\Sigma}}\underset{[n\times n]}{\mathbf{V}^{\mathrm{T}}} = (\text{orthogonal})(\text{diagonal})(\text{orthogonal})$$

$\mathbf{u}_1,\cdots,\mathbf{u}_m$ in $\mathbf{U}$ are eigenvectors of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$,

$\mathbf{v}_1,\cdots,\mathbf{v}_n$ in $\mathbf{V}$ are eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

$\sigma_1,\ldots,\sigma_r$: **Square root of Eigenvalues of A^TA**

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ 0 & & & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$\text{rank } \mathbf{A} = r \le \min(n,m)$

$$\mathbf{A}\mathbf{v}_i = \sigma_i\mathbf{u}_i$$

$$\mathbf{A}\mathbf{V} = \mathbf{U}\mathbf{\Sigma}$$

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{-1}$$

$$= \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}}$$

$cf.\ \mathbf{A}\mathbf{x} = \lambda\mathbf{x}$

$\mathbf{U} = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_r\mathbf{u}_{r+1}\cdots\mathbf{u}_m]$:orthonormal

$\mathbf{V} = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_r\ \mathbf{v}_{r+1}\cdots\mathbf{v}_n]$:orthonormal

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \\ & & \end{bmatrix} : \text{diagonal}(m\times n)$$

Singular Value Decomposition (SVD)

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$

$\mathbf{U}$ are eigenvectors of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$,

$\mathbf{V}$ are eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

$\sigma_1, \ldots, \sigma_r$:

Square root of Eigenvalues of A^TA

Example) Find the singular value decomposition of $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$

Compute $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ **and its eigenvectors and then make them unit vectors:**

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\det(\mathbf{A}^{\mathrm{T}}\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 25 - 9 = \lambda^2 - 10\lambda + 16 = (\lambda - 8)(\lambda - 2) = 0$$

$$\therefore \lambda_1 = 8,\ \lambda_2 = 2$$

$$(\mathbf{A}^{\mathrm{T}}\mathbf{A} - \lambda_1)\mathbf{v}_1 = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \sigma_1 = \sqrt{8} = 2\sqrt{2},\ \sigma_2 = \sqrt{2}$$

$$(\mathbf{A}^{\mathrm{T}}\mathbf{A} - \lambda_2)\mathbf{v}_2 = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\mathbf{A}^{\mathrm{T}}\mathbf{A}$ **has unit eigenvectors** $\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Singular Value Decomposition (SVD)

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$

$\mathbf{U}$ are eigenvectors of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$,

$\mathbf{V}$ are eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

$\sigma_1, \ldots, \sigma_r$:

Square root of Eigenvalues of A^TA

Example) Find the singular value decomposition of $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$

Compute $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ and its eigenvectors and then make them unit vectors:

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\det(\mathbf{A}\mathbf{A}^{\mathrm{T}} - \lambda\mathbf{I}) = \begin{vmatrix} 8-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (\lambda-2)(\lambda-8) = 0 \quad \therefore \lambda_1 = 8,\ \lambda_2 = 2$$

$$(\mathbf{A}\mathbf{A}^{\mathrm{T}} - \lambda_1)\mathbf{u}_1 = \begin{bmatrix} 0 & 0 \\ 0 & -6 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(\mathbf{A}\mathbf{A}^{\mathrm{T}} - \lambda_2)\mathbf{u}_2 = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\mathbf{A}\mathbf{A}^{\mathrm{T}}$ has unit eigenvectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Singular Value Decomposition (SVD)

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$

$\mathbf{U}$ are eigenvectors of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$,

$\mathbf{V}$ are eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

$\sigma_1, \ldots, \sigma_r$:

Square root of Eigenvalues of A^TA

Example) Find the singular value decomposition of $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$

$$\sigma_1 = \sqrt{8} = 2\sqrt{2}, \quad \sigma_2 = \sqrt{2}$$

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Singular Value Decomposition (SVD)

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$

$\mathbf{U}$ are eigenvectors of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$,

$\mathbf{V}$ are eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

$\sigma_1, \ldots, \sigma_r$:

Square root of Eigenvalues of A^TA

Example) Find the singular value decomposition of $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$

$$\sigma_1 = \sqrt{8} = 2\sqrt{2}, \quad \sigma_2 = \sqrt{2}$$

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Singular Value Decomposition (SVD)

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$

$\mathbf{U}$ are eigenvectors of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$,

$\mathbf{V}$ are eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

$\sigma_1, \ldots, \sigma_r$:

Square root of Eigenvalues of A^TA

Example) Find the singular value decomposition of $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\sigma_1 = \sqrt{8} = 2\sqrt{2}, \quad \sigma_2 = \sqrt{2}$$

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Singular Value Decomposition

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
$$= (\text{orthonormal})\,(\text{diagonal})\,(\text{orthonormal})$$

$$\sigma\mathbf{u}\mathbf{v}^{\mathrm{T}} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix}$$

1) 시스템(행렬 A)에서 주요한 부분과 주요하지 않은 부분으로 나누는 것

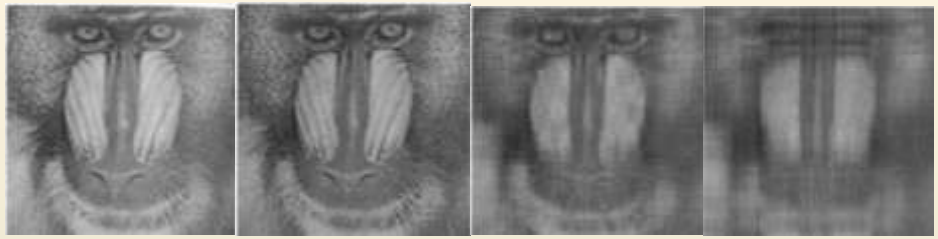
- rank가 *r*인 *m* by *n* 행렬 A를 *m* by *n* 행렬들 ($\mathbf{u}_i\mathbf{v}_i^{\mathrm{T}}$, i = 1,2,.., *r*)의 일차결합으로 표현할 수 있다.

$$\underset{[m\times n]}{\mathbf{A}} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}} = \underset{[m\times n]}{\sigma_1\mathbf{u}_1\mathbf{v}_1^{\mathrm{T}}} + \underset{[m\times n]}{\sigma_2\mathbf{u}_2\mathbf{v}_2^{\mathrm{T}}} + \cdots + \underset{[m\times n]}{\sigma_r\mathbf{u}_r\mathbf{v}_r^{\mathrm{T}}}$$

$$(\operatorname{rank}\mathbf{A} = r,\ \operatorname{rank}\mathbf{u}\mathbf{v}^{\mathrm{T}} = 1)$$
$$(\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0)$$

A를 구성하는 행렬 중에 σ가 클수록 주요한 성분이다. *σ*가 작은 부분을 소거(0으로 보냄)해도 A가 담고 있는 정보는 비슷하다. (압축)

예) A가 이미지를 담고 있는 행렬일 때,



$r = 128$ $r = 50$ $r = 10$ $r = 4$

Singular Value Decomposition

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
$$= (\text{orthonormal})(\text{diagonal})(\text{orthonormal})$$

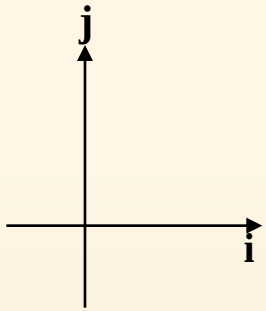
$$\sigma\mathbf{u}\mathbf{v}^{\mathrm{T}} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix}$$

2) 변환 측면에서의 SVD

$\mathbf{V}^{\mathrm{T}}\mathbf{x}$ $\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$ $\mathbf{U}\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$

벡터 x를 행렬 A로 변환하는 것은 x를 $\mathbf{V}^{\mathrm{T}}$로 변환하고, 다시 Σ로 변환하고, U로 변환하는 것이다.



$$\mathbf{x} = \begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

예) 2 by 2 matrix A

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$
$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Singular Value Decomposition

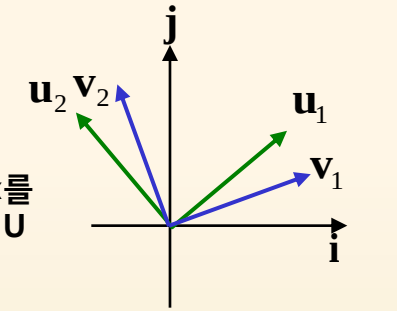
$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
$$= (\text{orthonormal})(\text{diagonal})(\text{orthonormal})$$

$$\sigma\mathbf{u}\mathbf{v}^{\mathrm{T}} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix}$$

2) 변환 측면에서의 SVD

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$

벡터 x를 행렬 A로 변환하는 것은 x를 $\mathbf{V}^{\mathrm{T}}$로 변환하고, 다시 Σ로 변환하고, U로 변환하는 것이다.



$\mathbf{V}^{\mathrm{T}}\mathbf{x}$ $\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$ $\mathbf{U}\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$

$$\mathbf{x} = \begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

예) 2 by 2 matrix A

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$
$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Singular Value Decomposition

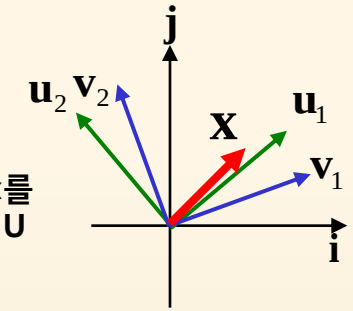
$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
$$= (\text{orthonormal})(\text{diagonal})(\text{orthonormal})$$

$$\sigma\mathbf{u}\mathbf{v}^{\mathrm{T}} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix}$$

2) 변환 측면에서의 SVD

$$\mathbf{V}^{\mathrm{T}}\mathbf{x} \qquad \Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x}) \qquad \mathbf{U}\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$$

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$



$$\mathbf{x} = \begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

벡터 x를 행렬 A로 변환하는 것은 x를 $\mathbf{V}^{\mathrm{T}}$로 변환하고, 다시 Σ로 변환하고, U로 변환하는 것이다.

예) 2 by 2 matrix A

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$
$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Singular Value Decomposition

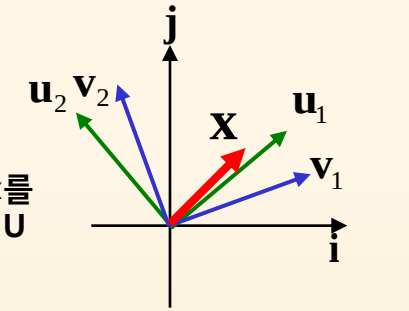
$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
$$= (\text{orthonormal})(\text{diagonal})(\text{orthonormal})$$

$$\sigma\mathbf{u}\mathbf{v}^{\mathrm{T}} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix}$$

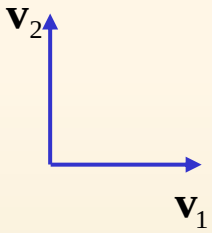
2) 변환 측면에서의 SVD

$$\mathbf{Ax} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$

벡터 x를 행렬 A로 변환하는 것은 x를 $\mathbf{V}^{\mathrm{T}}$로 변환하고, 다시 Σ로 변환하고, U로 변환하는 것이다.



$\mathbf{V}^{\mathrm{T}}\mathbf{x}$



$\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$

$\mathbf{U}\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$

예) 2 by 2 matrix A

$$\mathbf{x} = \begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\mathbf{Ax} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$
$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Singular Value Decomposition

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
$$= (\text{orthonormal})(\text{diagonal})(\text{orthonormal})$$

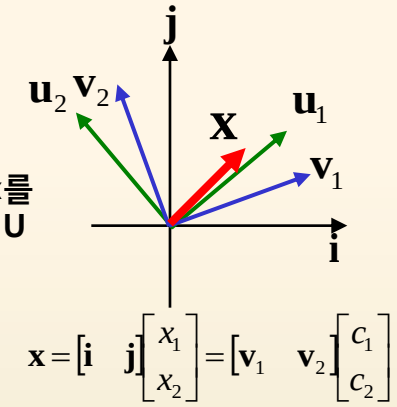
$$\sigma\mathbf{u}\mathbf{v}^{\mathrm{T}} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix}$$

2) 변환 측면에서의 SVD

$$\mathbf{Ax} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$

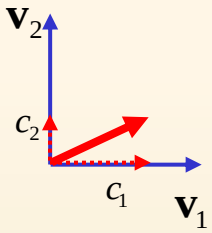
벡터 x를 행렬 A로 변환하는 것은 x를 $\mathbf{V}^{\mathrm{T}}$로 변환하고, 다시 Σ로 변환하고, U로 변환하는 것이다.

예) 2 by 2 matrix A



$$\mathbf{x} = \begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\mathbf{V}^{\mathrm{T}}\mathbf{x}$ $\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$ $\mathbf{U}\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$



$$\mathbf{Ax} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$
$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Singular Value Decomposition

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
$$= (\text{orthonormal})(\text{diagonal})(\text{orthonormal})$$

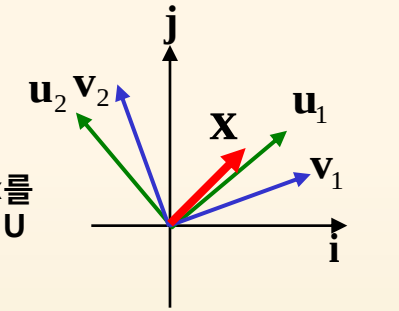
$$\sigma\mathbf{u}\mathbf{v}^{\mathrm{T}} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix}$$

2) 변환 측면에서의 SVD

$$\mathbf{Ax} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$

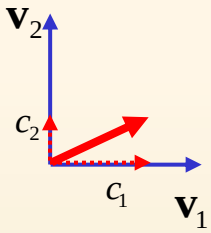
벡터 x를 행렬 A로 변환하는 것은 x를 $\mathbf{V}^{\mathrm{T}}$로 변환하고, 다시 Σ로 변환하고, U로 변환하는 것이다.

예) 2 by 2 matrix A



$$\mathbf{x} = \begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\mathbf{V}^{\mathrm{T}}\mathbf{x}$



$$\mathbf{V}^{\mathrm{T}}\mathbf{x} = \mathbf{V}^{-1}\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

rotation

$\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$

$\mathbf{U}\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$

$$\mathbf{Ax} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$
$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Singular Value Decomposition

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
$$= (\text{orthonormal})(\text{diagonal})(\text{orthonormal})$$

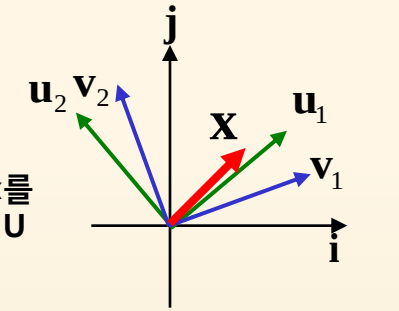
$$\sigma\mathbf{u}\mathbf{v}^{\mathrm{T}} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix}$$

2) 변환 측면에서의 SVD

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$

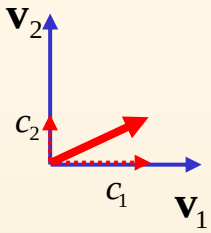
벡터 x를 행렬 A로 변환하는 것은 x를 $\mathbf{V}^{\mathrm{T}}$로 변환하고, 다시 Σ로 변환하고, U로 변환하는 것이다.

예) 2 by 2 matrix A



$$\mathbf{x} = \begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

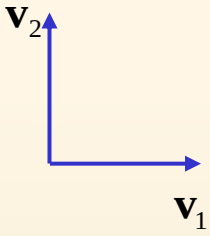
$\mathbf{V}^{\mathrm{T}}\mathbf{x}$



$$\mathbf{V}^{\mathrm{T}}\mathbf{x} = \mathbf{V}^{-1}\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

rotation

$\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$



$\mathbf{U}\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$
$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Singular Value Decomposition

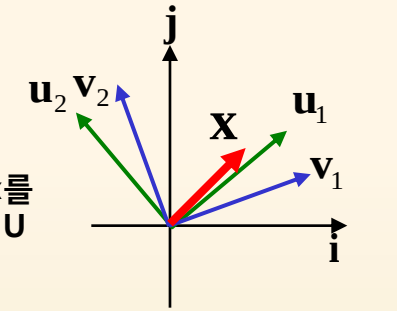
$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
$$= (\text{orthonormal})(\text{diagonal})(\text{orthonormal})$$

$$\sigma\mathbf{u}\mathbf{v}^{\mathrm{T}} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix}$$

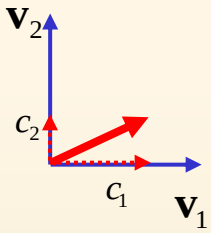
2) 변환 측면에서의 SVD

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$

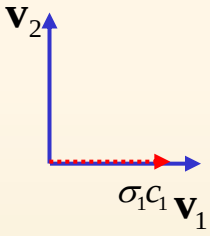
벡터 x를 행렬 A로 변환하는 것은 x를 $\mathbf{V}^{\mathrm{T}}$로 변환하고, 다시 Σ로 변환하고, U로 변환하는 것이다.



$\mathbf{V}^{\mathrm{T}}\mathbf{x}$



$\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$



$\mathbf{U}\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$

예) 2 by 2 matrix A

$$\mathbf{x} = \begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\mathbf{V}^{\mathrm{T}}\mathbf{x} = \mathbf{V}^{-1}\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

rotation

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$
$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Singular Value Decomposition

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
$$= (\text{orthonormal})(\text{diagonal})(\text{orthonormal})$$

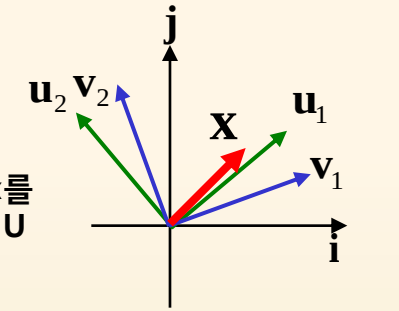
$$\sigma\mathbf{u}\mathbf{v}^{\mathrm{T}} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix}$$

2) 변환 측면에서의 SVD

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$

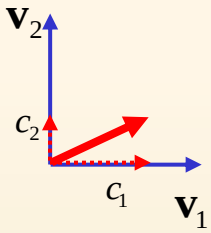
벡터 x를 행렬 A로 변환하는 것은 x를 $\mathbf{V}^{\mathrm{T}}$로 변환하고, 다시 Σ로 변환하고, U로 변환하는 것이다.

예) 2 by 2 matrix A



$$\mathbf{x} = \begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

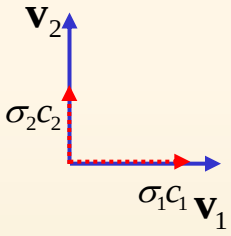
$\mathbf{V}^{\mathrm{T}}\mathbf{x}$



$$\mathbf{V}^{\mathrm{T}}\mathbf{x} = \mathbf{V}^{-1}\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

rotation

$\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$



$\mathbf{U}\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$
$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Singular Value Decomposition

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
$$= (\text{orthonormal})(\text{diagonal})(\text{orthonormal})$$

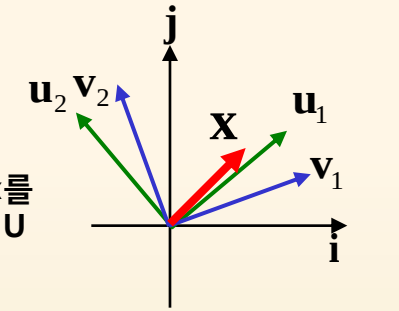
$$\sigma\mathbf{u}\mathbf{v}^{\mathrm{T}} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix}$$

2) 변환 측면에서의 SVD

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$

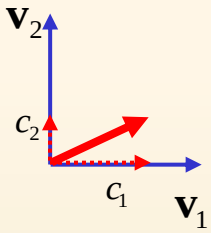
벡터 x를 행렬 A로 변환하는 것은 x를 $\mathbf{V}^{\mathrm{T}}$로 변환하고, 다시 Σ로 변환하고, U로 변환하는 것이다.

예) 2 by 2 matrix A



$$\mathbf{x} = \begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

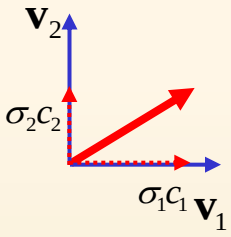
$\mathbf{V}^{\mathrm{T}}\mathbf{x}$



$$\mathbf{V}^{\mathrm{T}}\mathbf{x} = \mathbf{V}^{-1}\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

rotation

$\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$



$\mathbf{U}\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$
$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Singular Value Decomposition

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
$$= (\text{orthonormal})(\text{diagonal})(\text{orthonormal})$$

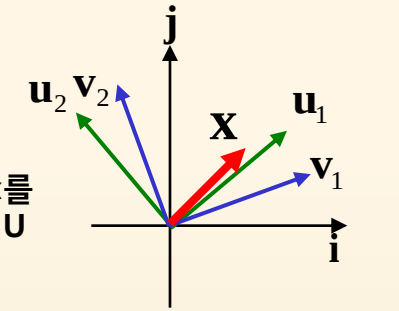
$$\sigma\mathbf{u}\mathbf{v}^{\mathrm{T}} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix}$$

2) 변환 측면에서의 SVD

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$

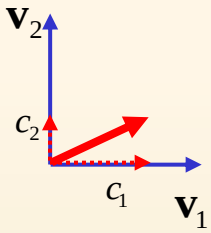
벡터 x를 행렬 A로 변환하는 것은 x를 $\mathbf{V}^{\mathrm{T}}$로 변환하고, 다시 Σ로 변환하고, U로 변환하는 것이다.

예) 2 by 2 matrix A



$$\mathbf{x} = \begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

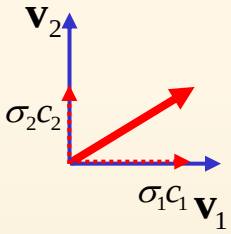
$\mathbf{V}^{\mathrm{T}}\mathbf{x}$



$$\mathbf{V}^{\mathrm{T}}\mathbf{x} = \mathbf{V}^{-1}\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

rotation

$\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$



$$\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x}) = \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 c_1 \\ \sigma_2 c_2 \end{bmatrix}$$

stretching

$\mathbf{U}\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$
$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Singular Value Decomposition

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
$$= (\text{orthonormal})(\text{diagonal})(\text{orthonormal})$$

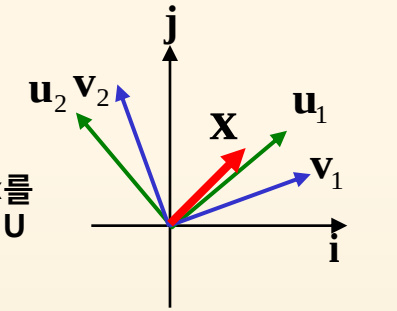
$$\sigma\mathbf{u}\mathbf{v}^{\mathrm{T}} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix}$$

2) 변환 측면에서의 SVD

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$

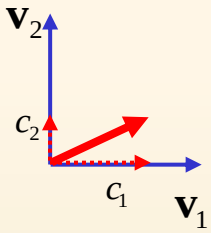
벡터 x를 행렬 A로 변환하는 것은 x를 $\mathbf{V}^{\mathrm{T}}$로 변환하고, 다시 Σ로 변환하고, U로 변환하는 것이다.

예) 2 by 2 matrix A



$$\mathbf{x} = \begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

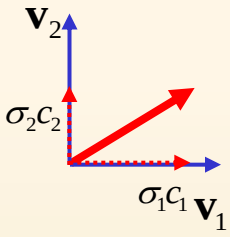
$\mathbf{V}^{\mathrm{T}}\mathbf{x}$



$$\mathbf{V}^{\mathrm{T}}\mathbf{x} = \mathbf{V}^{-1}\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

rotation

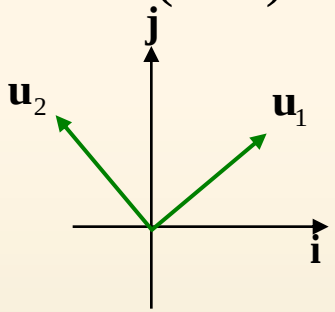
$\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$



$$\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x}) = \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 c_1 \\ \sigma_2 c_2 \end{bmatrix}$$

stretching

$\mathbf{U}\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$



$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$
$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Singular Value Decomposition

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
$$= (\text{orthonormal})(\text{diagonal})(\text{orthonormal})$$

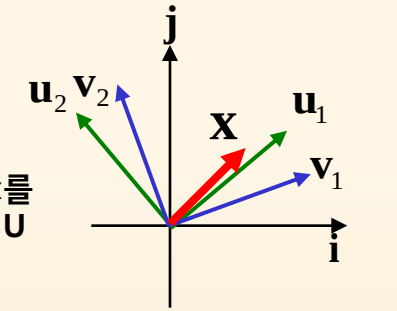
$$\sigma\mathbf{u}\mathbf{v}^{\mathrm{T}} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix}$$

2) 변환 측면에서의 SVD

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$

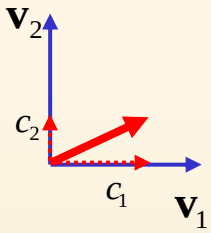
벡터 x를 행렬 A로 변환하는 것은 x를 $\mathbf{V}^{\mathrm{T}}$로 변환하고, 다시 Σ로 변환하고, U로 변환하는 것이다.

예) 2 by 2 matrix A



$$\mathbf{x} = \begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

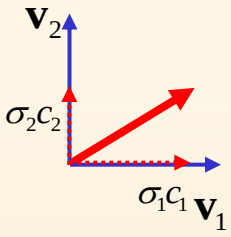
$\mathbf{V}^{\mathrm{T}}\mathbf{x}$



$$\mathbf{V}^{\mathrm{T}}\mathbf{x} = \mathbf{V}^{-1}\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

rotation

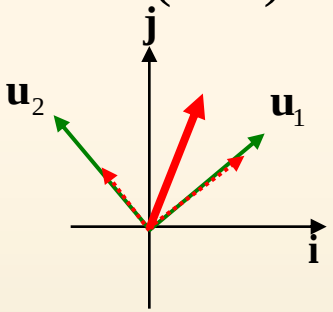
$\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$



$$\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x}) = \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 c_1 \\ \sigma_2 c_2 \end{bmatrix}$$

stretching

$\mathbf{U}\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$



$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$
$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Singular Value Decomposition

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
$$= (\text{orthonormal})(\text{diagonal})(\text{orthonormal})$$

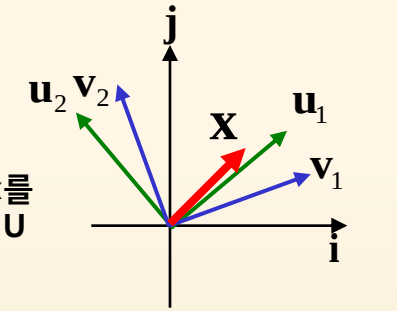
$$\sigma\mathbf{u}\mathbf{v}^{\mathrm{T}} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix}$$

2) 변환 측면에서의 SVD

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$

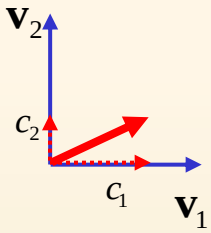
벡터 x를 행렬 A로 변환하는 것은 x를 $\mathbf{V}^{\mathrm{T}}$로 변환하고, 다시 Σ로 변환하고, U로 변환하는 것이다.

예) 2 by 2 matrix A



$$\mathbf{x} = \begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

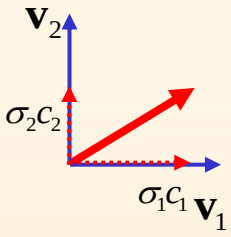
$\mathbf{V}^{\mathrm{T}}\mathbf{x}$



$$\mathbf{V}^{\mathrm{T}}\mathbf{x} = \mathbf{V}^{-1}\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

rotation

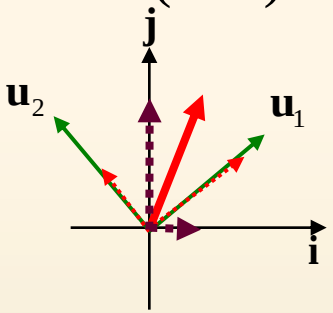
$\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$



$$\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x}) = \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 c_1 \\ \sigma_2 c_2 \end{bmatrix}$$

stretching

$\mathbf{U}\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$



$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$
$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Singular Value Decomposition

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
$$= (\text{orthonormal})(\text{diagonal})(\text{orthonormal})$$

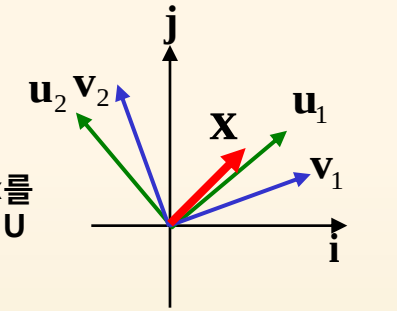
$$\sigma\mathbf{u}\mathbf{v}^{\mathrm{T}} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix}$$

2) 변환 측면에서의 SVD

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$

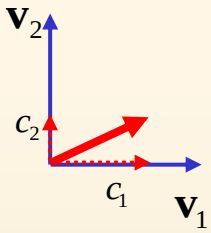
벡터 x를 행렬 A로 변환하는 것은 x를 $\mathbf{V}^{\mathrm{T}}$로 변환하고, 다시 Σ로 변환하고, U로 변환하는 것이다.

예) 2 by 2 matrix A



$$\mathbf{x} = \begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

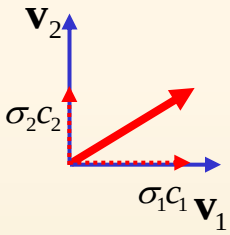
$\mathbf{V}^{\mathrm{T}}\mathbf{x}$



$$\mathbf{V}^{\mathrm{T}}\mathbf{x} = \mathbf{V}^{-1}\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

rotation

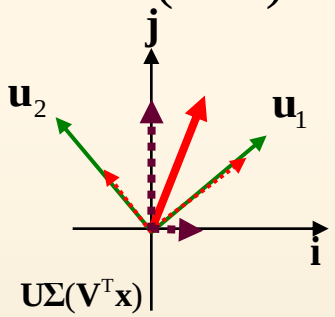
$\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$



$$\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x}) = \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 c_1 \\ \sigma_2 c_2 \end{bmatrix}$$

stretching

$\mathbf{U}\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x})$



$$\mathbf{U}\Sigma(\mathbf{V}^{\mathrm{T}}\mathbf{x}) = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 c_1 \\ \sigma_2 c_2 \end{bmatrix} = \sigma_1 c_1\mathbf{u}_1 + \sigma_2 c_2\mathbf{u}_2$$
$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix}\begin{bmatrix} \sigma_1 c_1 u_{11} + \sigma_2 c_2 u_{21} \\ \sigma_1 c_1 u_{12} + \sigma_2 c_2 u_{22} \end{bmatrix}$$

rotation

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x}$$
$$= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Singular Value Decomposition

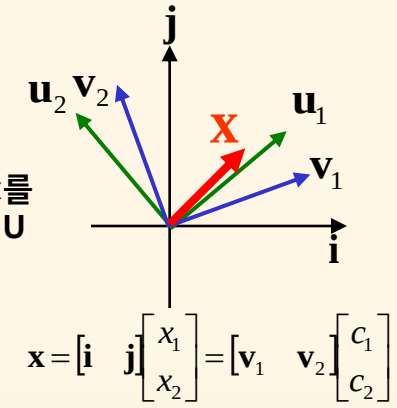
$$A = U\Sigma V^T$$
$$= (\text{orthonormal})(\text{diagonal})(\text{orthonormal})$$

2) 변환 측면에서의 SVD

$$Ax = U\Sigma V^T x$$

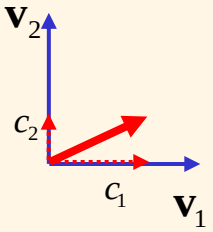
벡터 x를 행렬 A로 변환하는 것은 x를 V^T로 변환하고, 다시 Σ로 변환하고, U로 변환하는 것이다.

예) 2 by 2 matrix A



$$x = \begin{bmatrix} i & j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

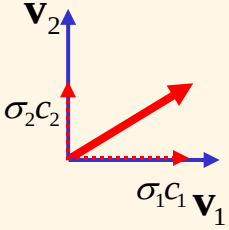
$V^T x$



$$V^T x = V^{-1} x = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

rotation

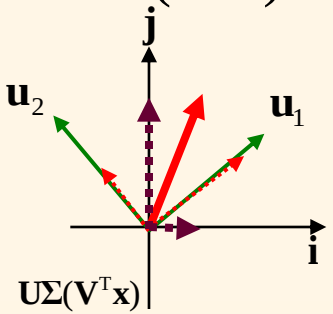
$\Sigma(V^T x)$



$$\Sigma(V^T x) = \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 c_1 \\ \sigma_2 c_2 \end{bmatrix}$$

stretching

$U\Sigma(V^T x)$

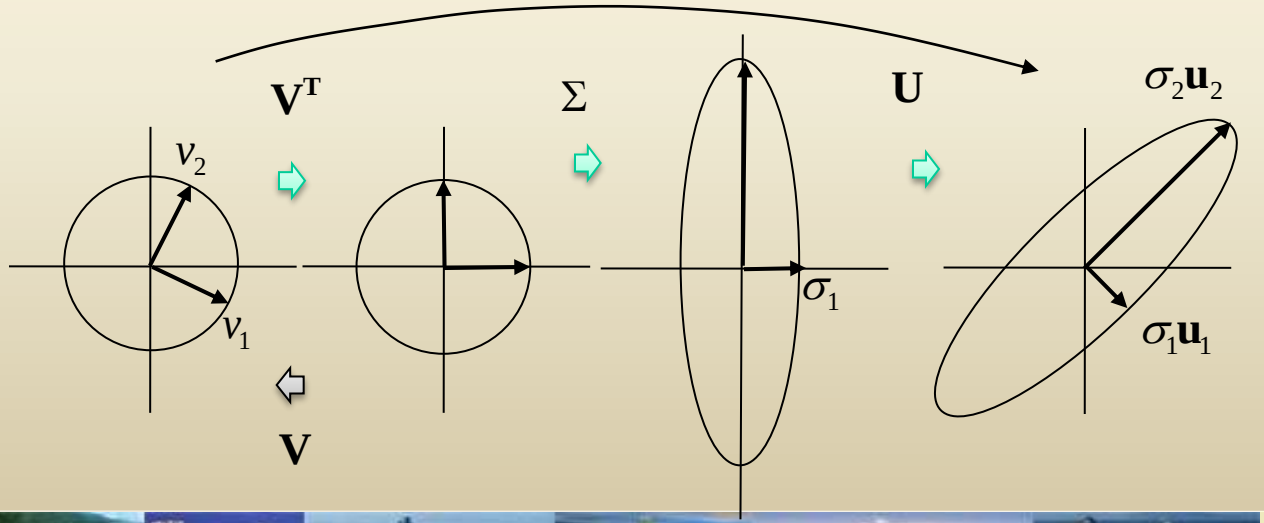


$$U\Sigma(V^T x) = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 c_1 \\ \sigma_2 c_2 \end{bmatrix} = \sigma_1 c_1 u_1 + \sigma_2 c_2 u_2$$
$$= \begin{bmatrix} i & j \end{bmatrix} \begin{bmatrix} \sigma_1 c_1 u_{11} + \sigma_2 c_2 u_{21} \\ \sigma_1 c_1 u_{12} + \sigma_2 c_2 u_{22} \end{bmatrix}$$

rotation

$$Ax = U\Sigma V^T x = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

A

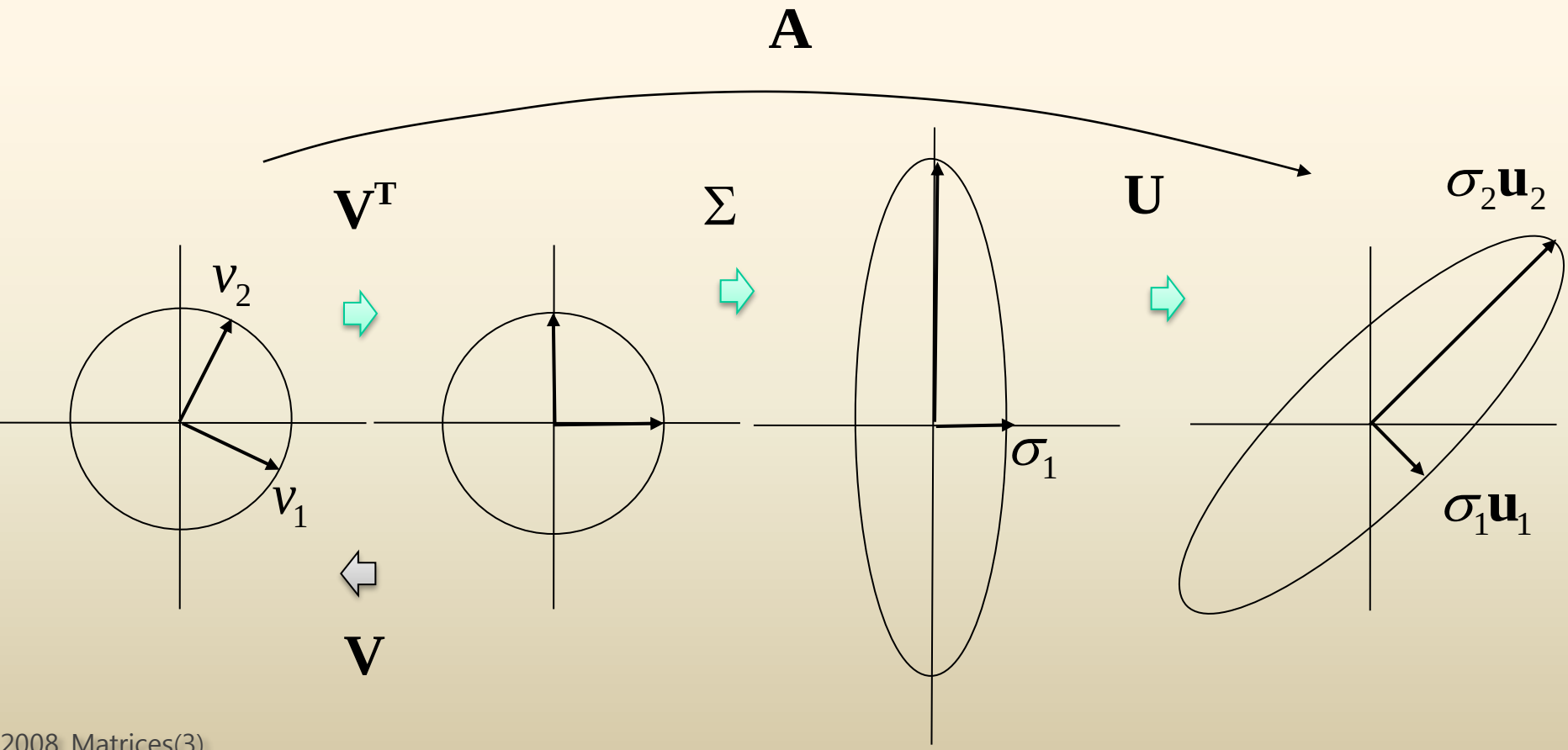


Singular Value Decomposition

$$\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i$$

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{\Sigma} = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

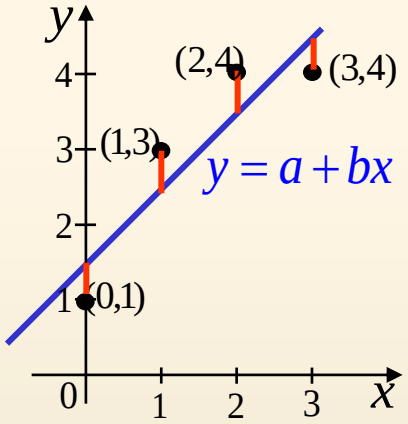
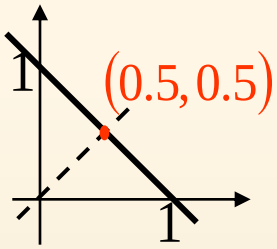
$$\mathbf{A}\mathbf{x} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}}\mathbf{x}$$



Singular Value Decomposition

$$\underset{[m\times n]}{\mathbf{A}}\ \underset{[n\times 1]}{\mathbf{x}}\ =\ \underset{[m\times 1]}{\mathbf{b}}$$

3) Ax = b의 유일한 해가 존재하지 않는 경우 (A^{-1}이 존재하지 않음)

	① 해가 없는 경우 (불능, $p > n$) → 최적의 근사해 찾기	② 해가 무수히 많은 경우 (부정, $p < n$) → norm이 가장 작은 해 찾기
예제	네 점을 지나는 직선 구하기 직선의 절편 : x 직선의 기울기 : y $a+0b=1,\ a+1b=3$ $a+2b=4,\ a+3b=4$ $\mathbf{Ax}=\begin{bmatrix}1&0\\1&1\\1&2\\1&3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\3\\4\\4\end{bmatrix}=\mathbf{b}$  네 점을 지나는 직선은 없으므로, 오차의 제곱의 합이 가장 작은 직선을 구함. x = (1.5, 1)	해가 직선일 때, 목적에 맞는 해 찾기 $\mathbf{Ax}=\begin{bmatrix}2&2\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}=\mathbf{b}$ $2x+2y=2,$ $x+y=1$  위 식을 만족하는 해는 무수히 많음(직선 상의 모든 점) 해 중에 norm이 최소인 해를 구함 원점을 지나고, 직선 $x + y = 1$과 수직인 직선의 교점이 norm이 최소인 해이다. (0.5, 0.5)
Pseudo inverse A^+	①, ②와 같이 exact solution $x = A^{-1} b$를 구할 수 없을 경우, Pseudo inverse를 사용하여, 각각의 목적에 맞는 해 (①의 경우 오차의 제곱이 가장 작게 하는 해, ②의 경우 norm이 가장 작은 해)를 구할 수 있다. $\mathbf{x}^+=\mathbf{A}^+\mathbf{b}=\begin{bmatrix}0.7&0.4&0.1&-0.2\\-0.3&-0.1&0.1&0.3\end{bmatrix}\begin{bmatrix}1\\3\\4\\4\end{bmatrix}=\begin{bmatrix}1.5\\1\end{bmatrix}$	$\mathbf{x}^+=\mathbf{A}^+\mathbf{b}=\begin{bmatrix}0.2&0.1\\0.2&0.1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}0.5\\0.5\end{bmatrix}$
SVD	$\underset{[m\times n]}{\mathbf{A}}=\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}=\underset{[m\times m]}{(\text{orthogonal})}\underset{[m\times n]}{(\text{diagonal})}\underset{[n\times n]}{(\text{orthogonal})}$	$\mathbf{A}^+=\mathbf{V}\boldsymbol{\Sigma}^+\mathbf{U}^{\mathrm{T}}=(\text{orthogonal})(\text{diagonal})(\text{orthogonal})$ $\mathbf{u}_1,\cdots,\mathbf{u}_m$ in $\mathbf{U}$ are eigenvectors of $\mathbf{AA}^{\mathrm{T}}$, $\mathbf{v}_1,\cdots,\mathbf{v}_n$ in $\mathbf{V}$ are eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ $\boldsymbol{\Sigma}=\begin{bmatrix}\sigma_1&&0\\&\ddots&\\0&&\sigma_r\end{bmatrix},\ \boldsymbol{\Sigma}^+=\begin{bmatrix}1/\sigma_1&&0\\&\ddots&\\0&&1/\sigma_r\end{bmatrix}$

Singular Value Decomposition

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \\ & & 0 \end{bmatrix}, \Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & 0 \\ & \ddots & \\ 0 & & 1/\sigma_r \\ & & 0 \end{bmatrix}$$

4) Ax = b의 유일 해 x = $A^{-1}b$ 를 구할 때, 오류가 생기는 경우

행렬 A의 역행렬 A^{-1}가 존재하는 경우, Pseudo inverse $A^+ = A^{-1}$ $\mathbf{A}^{-1} = \mathbf{A}^{+} = \mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}}$

역행렬 A^{-1}가 존재하나 컴퓨터 계산 상의 오류에 의해 역행렬을 구하지 못하는 경우가 발생한다.

$A^{-1} = A^+ = V\Sigma U^T$에서 σ가 매우 작을 때 1/ σ가 무한대가 되어 문제가 발생함을 알 수 있다. 따라서 충분히 작은 σ를 0으로 보냄으로써 문제를 해결한다.

예) $$\mathbf{Ax} = \begin{bmatrix} 1 & 1 \\ 1 & 1.0000000000000001 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.0000000000000001 \end{bmatrix} = \mathbf{b} \qquad \therefore \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Singular Value Decomposition

$$\Sigma=\begin{bmatrix}\sigma_1 & & 0\\ & \ddots & \\ 0 & & \sigma_r\\ & 0 & \end{bmatrix},\ \Sigma^{+}=\begin{bmatrix}1/\sigma_1 & & 0\\ & \ddots & \\ 0 & & 1/\sigma_r\\ & 0 & \end{bmatrix}$$

4) Ax = b의 유일 해 x = $A^{-1}b$ 를 구할 때, 오류가 생기는 경우

행렬 A의 역행렬 A^{-1}가 존재하는 경우, Pseudo inverse $A^{+} = A^{-1}$

$$\mathbf{A}^{-1}=\mathbf{A}^{+}=\mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}}$$

역행렬 A^{-1}가 존재하나 컴퓨터 계산 상의 오류에 의해 역행렬을 구하지 못하는 경우가 발생한다.

$A^{-1} = A^{+} = V\Sigma U^{T}$에서 σ가 매우 작을 때 1/ σ가 무한대가 되어 문제가 발생함을 알 수 있다. 따라서 충분히 작은 σ를 0으로 보냄으로써 문제를 해결한다.

예) $$\mathbf{Ax}=\begin{bmatrix}1 & 1\\ 1 & 1.0000000000000001\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}2\\ 2.0000000000000001\end{bmatrix}=\mathbf{b}\qquad \therefore \mathbf{x}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\mathbf{A}=\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}=\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & \varepsilon\end{bmatrix}\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\ (0<\varepsilon<<1)$$

Singular Value Decomposition

$$\Sigma=\begin{bmatrix}\sigma_1 & & 0\\ & \ddots & \\ 0 & & \sigma_r\\ & 0 & \end{bmatrix},\ \Sigma^{+}=\begin{bmatrix}1/\sigma_1 & & 0\\ & \ddots & \\ 0 & & 1/\sigma_r\\ & 0 & \end{bmatrix}$$

4) Ax = b의 유일 해 x = $A^{-1}b$ 를 구할 때, 오류가 생기는 경우

행렬 A의 역행렬 A^{-1}가 존재하는 경우, Pseudo inverse $A^{+} = A^{-1}$

$$\mathbf{A}^{-1}=\mathbf{A}^{+}=\mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}}$$

역행렬 A^{-1}가 존재하나 컴퓨터 계산 상의 오류에 의해 역행렬을 구하지 못하는 경우가 발생한다.

$A^{-1} = A^{+} = V\Sigma U^{T}$에서 σ가 매우 작을 때 1/ σ가 무한대가 되어 문제가 발생함을 알 수 있다. 따라서 충분히 작은 σ를 0으로 보냄으로써 문제를 해결한다.

예) $$\mathbf{Ax}=\begin{bmatrix}1 & 1\\ 1 & 1.0000000000000001\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}2\\ 2.0000000000000001\end{bmatrix}=\mathbf{b}\qquad \therefore \mathbf{x}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\mathbf{A}=\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}=\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & \varepsilon\end{bmatrix}\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\ (0<\varepsilon<<1)$$

$$\mathbf{A}^{-1}=\mathbf{A}^{+}=\mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}}=\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\begin{bmatrix}1/2 & 0\\ 0 & 1/\varepsilon\end{bmatrix}\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}$$

Singular Value Decomposition

$$\Sigma=\begin{bmatrix}\sigma_1 & & 0\\ & \ddots & \\ & \sigma_r & \\ 0 & & 0\end{bmatrix},\ \Sigma^{+}=\begin{bmatrix}1/\sigma_1 & & 0\\ & \ddots & \\ & 1/\sigma_r & \\ 0 & & 0\end{bmatrix}$$

4) Ax = b의 유일 해 x = $A^{-1}b$ 를 구할 때, 오류가 생기는 경우

행렬 A의 역행렬 A^{-1}가 존재하는 경우, Pseudo inverse $A^{+} = A^{-1}$

$$\mathbf{A}^{-1}=\mathbf{A}^{+}=\mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}}$$

역행렬 A^{-1}가 존재하나 컴퓨터 계산 상의 오류에 의해 역행렬을 구하지 못하는 경우가 발생한다.

$A^{-1} = A^{+} = V\Sigma U^{T}$에서 σ가 매우 작을 때 1/ σ가 무한대가 되어 문제가 발생함을 알 수 있다. 따라서 충분히 작은 σ를 0으로 보냄으로써 문제를 해결한다.

예) $$\mathbf{Ax}=\begin{bmatrix}1 & 1\\ 1 & 1.0000000000000001\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}2\\ 2.0000000000000001\end{bmatrix}=\mathbf{b}\qquad \therefore \mathbf{x}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\mathbf{A}=\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}=\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\ (0<\varepsilon<<1)$$

$$\mathbf{A}^{-1}=\mathbf{A}^{+}=\mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}}=\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\begin{bmatrix}1/2 & 0\\ 0 & 1/\varepsilon\end{bmatrix}\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}$$

Singular Value Decomposition

$$\Sigma=\begin{bmatrix}\sigma_1 & & 0\\ & \ddots & \\ & \sigma_r & \\ 0 & & 0\end{bmatrix},\ \Sigma^{+}=\begin{bmatrix}1/\sigma_1 & & 0\\ & \ddots & \\ & 1/\sigma_r & \\ 0 & & 0\end{bmatrix}$$

4) Ax = b의 유일 해 x = $A^{-1}b$ 를 구할 때, 오류가 생기는 경우

행렬 A의 역행렬 A^{-1}가 존재하는 경우, Pseudo inverse $A^{+} = A^{-1}$

$$\mathbf{A}^{-1}=\mathbf{A}^{+}=\mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}}$$

역행렬 A^{-1}가 존재하나 컴퓨터 계산 상의 오류에 의해 역행렬을 구하지 못하는 경우가 발생한다.

$A^{-1} = A^{+} = V\Sigma U^{T}$에서 σ가 매우 작을 때 1/ σ가 무한대가 되어 문제가 발생함을 알 수 있다. 따라서 충분히 작은 σ를 0으로 보냄으로써 문제를 해결한다.

예) $$\mathbf{Ax}=\begin{bmatrix}1 & 1\\ 1 & 1.0000000000000001\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}2\\ 2.0000000000000001\end{bmatrix}=\mathbf{b}\qquad \therefore \mathbf{x}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\mathbf{A}=\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}=\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\ (0<\varepsilon<<1)$$

$$\mathbf{A}^{-1}=\mathbf{A}^{+}=\mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}}=\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\begin{bmatrix}1/2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}$$

Singular Value Decomposition

$$\Sigma=\begin{bmatrix}\sigma_1 & & 0\\ & \ddots & \\ & \sigma_r & \\ 0 & & 0\end{bmatrix},\ \Sigma^{+}=\begin{bmatrix}1/\sigma_1 & & 0\\ & \ddots & \\ & 1/\sigma_r & \\ 0 & & 0\end{bmatrix}$$

4) Ax = b의 유일 해 x = $A^{-1}b$ 를 구할 때, 오류가 생기는 경우

행렬 A의 역행렬 A^{-1}가 존재하는 경우, Pseudo inverse $A^{+} = A^{-1}$

$$\mathbf{A}^{-1}=\mathbf{A}^{+}=\mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}}$$

역행렬 A^{-1}가 존재하나 컴퓨터 계산 상의 오류에 의해 역행렬을 구하지 못하는 경우가 발생한다.

$A^{-1} = A^{+} = V\Sigma U^{T}$에서 σ가 매우 작을 때 1/ σ가 무한대가 되어 문제가 발생함을 알 수 있다. 따라서 충분히 작은 σ를 0으로 보냄으로써 문제를 해결한다.

예) $$\mathbf{Ax}=\begin{bmatrix}1 & 1\\ 1 & 1.0000000000000001\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}2\\ 2.0000000000000001\end{bmatrix}=\mathbf{b}\qquad \therefore \mathbf{x}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\mathbf{A}=\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}=\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\ (0<\varepsilon<<1)$$

$$\mathbf{A}^{-1}=\mathbf{A}^{+}=\mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}}=\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\begin{bmatrix}1/2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}$$

ε을 0으로 바꿔서 A^{-1}를 계산하면 $$\tilde{\mathbf{A}}^{-1}=\tilde{\mathbf{A}}^{+}=\begin{bmatrix}0.25 & 0.25\\ 0.25 & 0.25\end{bmatrix}$$

Singular Value Decomposition

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \\ & & 0 \end{bmatrix}, \Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & 0 \\ & \ddots & \\ 0 & & 1/\sigma_r \\ & & 0 \end{bmatrix}$$

4) Ax = b의 유일 해 x = $A^{-1}b$ 를 구할 때, 오류가 생기는 경우

행렬 A의 역행렬 A^{-1}가 존재하는 경우, Pseudo inverse $A^+ = A^{-1}$ $\mathbf{A}^{-1} = \mathbf{A}^+ = \mathbf{V}\boldsymbol{\Sigma}^+ \mathbf{U}^T$

역행렬 A^{-1}가 존재하나 컴퓨터 계산 상의 오류에 의해 역행렬을 구하지 못하는 경우가 발생한다.

$A^{-1} = A^+ = V\Sigma U^T$에서 σ가 매우 작을 때 1/ σ가 무한대가 되어 문제가 발생함을 알 수 있다. 따라서 충분히 작은 σ를 0으로 보냄으로써 문제를 해결한다.

예) $$\mathbf{Ax} = \begin{bmatrix} 1 & 1 \\ 1 & 1.0000000000000001 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.0000000000000001 \end{bmatrix} = \mathbf{b} \qquad \therefore \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T = \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \quad (0 < \varepsilon << 1)$$

$$\mathbf{A}^{-1} = \mathbf{A}^+ = \mathbf{V}\boldsymbol{\Sigma}^+ \mathbf{U}^T = \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$$

ε을 0으로 바꿔서 A^{-1}를 계산하면 $$\tilde{\mathbf{A}}^{-1} = \tilde{\mathbf{A}}^+ = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}$$

Matlab으로 해 x = $A^{-1}b$ 를 구하면, $$\mathbf{x} = \tilde{\mathbf{A}}^{-1}\mathbf{b} = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix} \begin{bmatrix} 2 \\ 2.0000000000000001 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Singular Value Decomposition

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \\ & & 0 \end{bmatrix}, \Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & 0 \\ & \ddots & \\ 0 & & 1/\sigma_r \\ & & 0 \end{bmatrix}$$

4) Ax = b의 유일 해 x = $A^{-1}b$ 를 구할 때, 오류가 생기는 경우

행렬 A의 역행렬 A^{-1}가 존재하는 경우, Pseudo inverse $A^+ = A^{-1}$

$$\mathbf{A}^{-1} = \mathbf{A}^{+} = \mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}}$$

역행렬 A^{-1}가 존재하나 컴퓨터 계산 상의 오류에 의해 역행렬을 구하지 못하는 경우가 발생한다.

$A^{-1} = A^+ = V\Sigma U^T$에서 σ가 매우 작을 때 1/ σ가 무한대가 되어 문제가 발생함을 알 수 있다. 따라서 충분히 작은 σ를 0으로 보냄으로써 문제를 해결한다.

예) $$\mathbf{Ax} = \begin{bmatrix} 1 & 1 \\ 1 & 1.0000000000000001 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.0000000000000001 \end{bmatrix} = \mathbf{b} \qquad \therefore \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}} = \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \quad (0 < \varepsilon << 1)$$

$$\mathbf{A}^{-1} = \mathbf{A}^{+} = \mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}} = \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$$

해를 구했음을 알 수 있다.

ε을 0으로 바꿔서 A^{-1}를 계산하면

$$\tilde{\mathbf{A}}^{-1} = \tilde{\mathbf{A}}^{+} = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}$$

Matlab으로 해 x = $A^{-1}b$ 를 구하면,

$$\mathbf{x} = \tilde{\mathbf{A}}^{-1}\mathbf{b} = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix} \begin{bmatrix} 2 \\ 2.0000000000000001 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Quadratic Forms. Transformation to Principal Axes

Definition: a quadratic form Q in the components $x_1, \dots, x_n$ of a vector $\mathbf{x}$ is a sum of n^2 terms, namely,

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n$$
$$+ a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots +$$
$$+ a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2$$

$\mathbf{A} = [a_{jk}]$ is called of the *coefficient matrix* of the form. We may assume that $\mathbf{A}$ is *symmetric*, because we can take off-diagonal terms together in pairs and write the result as a sum of two equal terms.

Quadratic Forms. Transformation to Principal Axes

Let

$$\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Quadratic Forms. Transformation to Principal Axes

Let

$$\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2$$

Quadratic Forms. Transformation to Principal Axes

Let

$$\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2$$

$$= 3x_1^2 + 10x_1x_2 + 2x_2^2$$

Quadratic Forms. Transformation to Principal Axes

Let

$$\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2$$

$$= 3x_1^2 + 10x_1x_2 + 2x_2^2$$

Quadratic Forms. Transformation to Principal Axes

Let

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2$$

$$= 3x_1^2 + 10x_1x_2 + 2x_2^2$$

$$= 3x_1^2 + 5x_1x_2 + 5x_2x_1 + 2x_2^2$$

Quadratic Forms. Transformation to Principal Axes

Let

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2$$

$$= 3x_1^2 + 10x_1x_2 + 2x_2^2$$

$$= 3x_1^2 + 5x_1x_2 + 5x_2x_1 + 2x_2^2$$

Quadratic Forms. Transformation to Principal Axes

Let

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2$$

$$= 3x_1^2 + 10x_1x_2 + 2x_2^2$$

$$= 3x_1^2 + 5x_1x_2 + 5x_2x_1 + 2x_2^2$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Quadratic Forms. Transformation to Principal Axes

Let

$$\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2$$

$$= 3x_1^2 + 10x_1x_2 + 2x_2^2$$

$$= 3x_1^2 + 5x_1x_2 + 5x_2x_1 + 2x_2^2$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^{\mathbf{T}}\mathbf{C}\mathbf{x}$$

Quadratic Forms. Transformation to Principal Axes

Let

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2$$

$$= 3x_1^2 + 10x_1x_2 + 2x_2^2$$

$$= 3x_1^2 + 5x_1x_2 + 5x_2x_1 + 2x_2^2$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{x} \qquad \mathbf{C} = [c_{jk}],\ c_{jk} = \frac{1}{2}\left(a_{jk} + a_{kj}\right)$$

Principal Axes Theorem

Theorem 8.4.2 Symmetric Matrices

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

(Quadratic Forms) $$Q = \mathbf{x}^T \mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} x_j x_k$$

By the Theorem 8.4.2 the *symmetric* coefficient matrix $\mathbf{A}$ has an *orthonormal basis of eigenvectors* $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Let $\mathbf{X}$ be

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}$$

Symmetric: $\mathbf{A}^T=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^T=-\mathbf{A}$
Orthogonal: $\mathbf{A}^T=\mathbf{A}^{-1}$

Principal Axes Theorem

Theorem 8.4.2 Symmetric Matrices

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

(Quadratic Forms) $$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k$$

By the Theorem 8.4.2 the *symmetric* coefficient matrix $\mathbf{A}$ has an *orthonormal basis of eigenvectors* $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Let $\mathbf{X}$ be

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}$$

$\mathbf{X}$ is *orthogonal*, so that $\mathbf{X}^{-1}=\mathbf{X}^T$, we obtain

Symmetric: $\mathbf{A}^T=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^T=-\mathbf{A}$
Orthogonal: $\mathbf{A}^T=\mathbf{A}^{-1}$

Principal Axes Theorem

Theorem 8.4.2 Symmetric Matrices

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

(Quadratic Forms) $$Q = \mathbf{x}^T \mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} x_j x_k$$

By the Theorem 8.4.2 the *symmetric* coefficient matrix $\mathbf{A}$ has an *orthonormal basis of eigenvectors* $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Let $\mathbf{X}$ be

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}$$

$\mathbf{X}$ is *orthogonal*, so that $\mathbf{X}^{-1}=\mathbf{X}^T$, we obtain

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

Symmetric: $\mathbf{A}^T=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^T=-\mathbf{A}$
Orthogonal: $\mathbf{A}^T=\mathbf{A}^{-1}$

Principal Axes Theorem

Theorem 8.4.2 Symmetric Matrices

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

(Quadratic Forms) $$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k$$

By the Theorem 8.4.2 the *symmetric* coefficient matrix $\mathbf{A}$ has an *orthonormal basis of eigenvectors* $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Let $\mathbf{X}$ be

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}$$

$\mathbf{X}$ is *orthogonal*, so that $\mathbf{X}^{-1}=\mathbf{X}^T$, we obtain

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

$$\therefore \mathbf{A} = \mathbf{X}\mathbf{D}\mathbf{X}^{-1} = \mathbf{X}\mathbf{D}\mathbf{X}^T$$

Symmetric: $\mathbf{A}^T=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^T=-\mathbf{A}$
Orthogonal: $\mathbf{A}^T=\mathbf{A}^{-1}$

Principal Axes Theorem

Theorem 8.4.2 Symmetric Matrices

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

(Quadratic Forms) $$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k$$

By the Theorem 8.4.2 the *symmetric* coefficient matrix $\mathbf{A}$ has an *orthonormal basis of eigenvectors* $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Let $\mathbf{X}$ be

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}$$

$\mathbf{X}$ is *orthogonal*, so that $\mathbf{X}^{-1}=\mathbf{X}^T$, we obtain

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

$$\therefore \mathbf{A} = \mathbf{X}\mathbf{D}\mathbf{X}^{-1} = \mathbf{X}\mathbf{D}\mathbf{X}^T$$

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T(\mathbf{X}\mathbf{D}\mathbf{X}^T)\mathbf{x}$$

Symmetric: $\mathbf{A}^T=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^T=-\mathbf{A}$
Orthogonal: $\mathbf{A}^T=\mathbf{A}^{-1}$

Principal Axes Theorem

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{X}\mathbf{D}\mathbf{X}^T\mathbf{x}$$

If we set $\mathbf{X}^T\mathbf{x} = \mathbf{y}$, then, since $\mathbf{X}^T = \mathbf{X}^{-1}$, we get

Principal Axes Theorem

$$Q = \mathbf{x}^T \mathbf{A}\mathbf{x} = \mathbf{x}^T \mathbf{X}\mathbf{D}\mathbf{X}^T\mathbf{x}$$

If we set $\mathbf{X}^T\mathbf{x} = \mathbf{y}$, then, since $\mathbf{X}^T = \mathbf{X}^{-1}$, we get

$$\mathbf{x} = \left(\mathbf{X}^T\right)^{-1}\mathbf{y} = \mathbf{X}\mathbf{y}$$

Principal Axes Theorem

$$Q = \mathbf{x}^T \mathbf{A}\mathbf{x} = \mathbf{x}^T \mathbf{X}\mathbf{D}\mathbf{X}^T\mathbf{x}$$

If we set $\mathbf{X}^T\mathbf{x} = \mathbf{y}$, then, since $\mathbf{X}^T = \mathbf{X}^{-1}$, we get

$$\mathbf{x} = \left(\mathbf{X}^T\right)^{-1}\mathbf{y} = \mathbf{X}\mathbf{y}$$

Furthermore, we have

$$\mathbf{x}^T\mathbf{X} = \left(\mathbf{X}^T\mathbf{x}\right)^T = \mathbf{y}^T$$

Principal Axes Theorem

$$Q = \mathbf{x}^T \mathbf{A}\mathbf{x} = \mathbf{x}^T \mathbf{X}\mathbf{D}\mathbf{X}^T\mathbf{x}$$

If we set $\mathbf{X}^T\mathbf{x} = \mathbf{y}$, then, since $\mathbf{X}^T = \mathbf{X}^{-1}$, we get

$$\mathbf{x} = \left(\mathbf{X}^T\right)^{-1}\mathbf{y} = \mathbf{X}\mathbf{y}$$

Furthermore, we have

$$\mathbf{x}^T\mathbf{X} = \left(\mathbf{X}^T\mathbf{x}\right)^T = \mathbf{y}^T$$

So Q becomes simply

$$Q = \mathbf{x}^T\mathbf{X}\mathbf{D}\mathbf{X}^T\mathbf{x} = \mathbf{y}^T\mathbf{D}\mathbf{y}$$

Principal Axes Theorem

$$Q = \mathbf{x}^T \mathbf{A}\mathbf{x} = \mathbf{x}^T \mathbf{X}\mathbf{D}\mathbf{X}^T\mathbf{x}$$

If we set $\mathbf{X}^T\mathbf{x} = \mathbf{y}$, then, since $\mathbf{X}^T = \mathbf{X}^{-1}$, we get

$$\mathbf{x} = \left(\mathbf{X}^T\right)^{-1}\mathbf{y} = \mathbf{X}\mathbf{y}$$

Furthermore, we have

$$\mathbf{x}^T\mathbf{X} = \left(\mathbf{X}^T\mathbf{x}\right)^T = \mathbf{y}^T$$

So Q becomes simply

$$Q = \mathbf{x}^T\mathbf{X}\mathbf{D}\mathbf{X}^T\mathbf{x} = \mathbf{y}^T\mathbf{D}\mathbf{y}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

Principal Axes Theorem

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} \longleftarrow \mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$
$$= \mathbf{x}^T\mathbf{X}\mathbf{D}\mathbf{X}^T\mathbf{x} \longleftarrow \mathbf{y} = \mathbf{X}^{-1}\mathbf{x} = \mathbf{X}^T\mathbf{x}$$
$$= \mathbf{y}^T\mathbf{D}\mathbf{y} \qquad \mathbf{y}^T = (\mathbf{X}^T\mathbf{x})^T = \mathbf{x}\mathbf{X}^T$$

Theorem 8.4.5 Principal Axes Theorem

The substitution $\mathbf{x} = \mathbf{X}\mathbf{y}$ transforms a quadratic form

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k \qquad (a_{kj} = a_{jk})$$

to the *principal axes form* or *canonical form*

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where $\lambda_1, \ldots, \lambda_n$ are the (not necessarily distinct) eigenvalues of the *symmetric* matrix $\mathbf{A}$, and $\mathbf{X}$ is an *orthogonal* matrix with corresponding eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, respectively, as column vectors.

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

Find out what type of conic section the following quadratic form represents and transform it to principal axes. (Ex 8.2-1)

$$Q=17x_1^2-30x_1x_2+17x_2^2=128$$

$$Q=17x_1^2-30x_1x_2+17x_2^2=\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$$

$$=\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}17 & -15\\ -15 & 17\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix}17 & -15\\ -15 & 17\end{bmatrix}$$

Characteristic Equation:

$$\begin{vmatrix}17-\lambda & -15\\ -15 & 17-\lambda\end{vmatrix}=0$$

$$(17-\lambda)^2-15^2=0 \qquad \therefore \lambda_1=2,\ \lambda_2=32$$

Eigenvectors $\lambda_1=2, \mathbf{x}_1=\begin{bmatrix}1\\ 1\end{bmatrix}$

$$\lambda_2=32, \mathbf{x}_1=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$Q=\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}=\mathbf{x}^{\mathrm{T}}\mathbf{X}\mathbf{D}\mathbf{X}^{\mathrm{T}}\mathbf{x}=\mathbf{y}^{\mathrm{T}}\mathbf{D}\mathbf{y}$$

$$=[y_1,y_2]\begin{bmatrix}2 & 0\\ 0 & 32\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}$$

$$=2y_1^2+32y_2^2=128$$

$$\therefore \frac{y_1^2}{8^2}+\frac{y_2^2}{2^2}=1$$

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$$

$$\lambda_1 = 2,\ \lambda_2 = 32$$

Principal Axes

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

1) $\lambda = \lambda_1 = 2$

2) $\lambda = \lambda_2 = 32$

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$$

$$\lambda_1 = 2,\ \lambda_2 = 32$$

Principal Axes

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

1) $\lambda = \lambda_1 = 2$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

2) $\lambda = \lambda_2 = 32$

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$$

$$\lambda_1 = 2,\ \lambda_2 = 32$$

Principal Axes

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

1) $\lambda = \lambda_1 = 2$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$15x_1 - 15x_2 = 0$$

2) $\lambda = \lambda_2 = 32$

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$$

$$\lambda_1 = 2,\ \lambda_2 = 32$$

Principal Axes

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

1) $\lambda = \lambda_1 = 2$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$15x_1 - 15x_2 = 0$$

From this we get normalized eigenvector $\mathbf{x}_1$.

2) $\lambda = \lambda_2 = 32$

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$$

$$\lambda_1 = 2,\ \lambda_2 = 32$$

Principal Axes

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

1) $\lambda = \lambda_1 = 2$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$15x_1 - 15x_2 = 0$$

From this we get normalized eigenvector $\mathbf{x}_1$.

$$\mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

2) $\lambda = \lambda_2 = 32$

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$$

$$\lambda_1 = 2,\ \lambda_2 = 32$$

Principal Axes

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

1) $\lambda = \lambda_1 = 2$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$15x_1 - 15x_2 = 0$$

From this we get normalized eigenvector $\mathbf{x}_1$.

$$\mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

2) $\lambda = \lambda_2 = 32$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}$$

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$$

$$\lambda_1 = 2,\ \lambda_2 = 32$$

Principal Axes

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

1) $\lambda = \lambda_1 = 2$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$15x_1 - 15x_2 = 0$$

From this we get normalized eigenvector $\mathbf{x}_1$.

$$\mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

2) $\lambda = \lambda_2 = 32$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}$$

$$-15x_1 - 15x_2 = 0$$

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$$

$$\lambda_1 = 2,\ \lambda_2 = 32$$

Principal Axes

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

1) $\lambda = \lambda_1 = 2$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$15x_1 - 15x_2 = 0$$

From this we get normalized eigenvector $\mathbf{x}_1$.

$$\mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

2) $\lambda = \lambda_2 = 32$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}$$

$$-15x_1 - 15x_2 = 0$$

From this we get normalized eigenvector $\mathbf{x}_2$.

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$$

$$\lambda_1 = 2,\ \lambda_2 = 32$$

Principal Axes

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

1) $\lambda = \lambda_1 = 2$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$15x_1 - 15x_2 = 0$$

From this we get normalized eigenvector $\mathbf{x}_1$.

$$\mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

2) $\lambda = \lambda_2 = 32$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}$$

$$-15x_1 - 15x_2 = 0$$

From this we get normalized eigenvector $\mathbf{x}_2$.

$$\mathbf{x}_2 = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$$

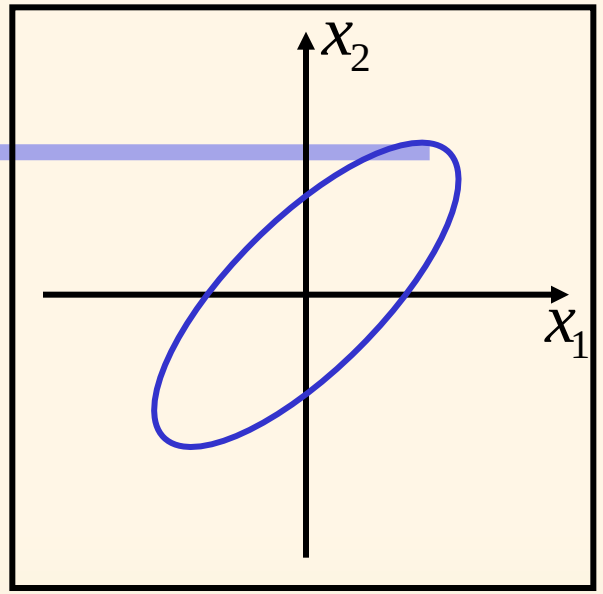
$$\mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathbf{T}}$$

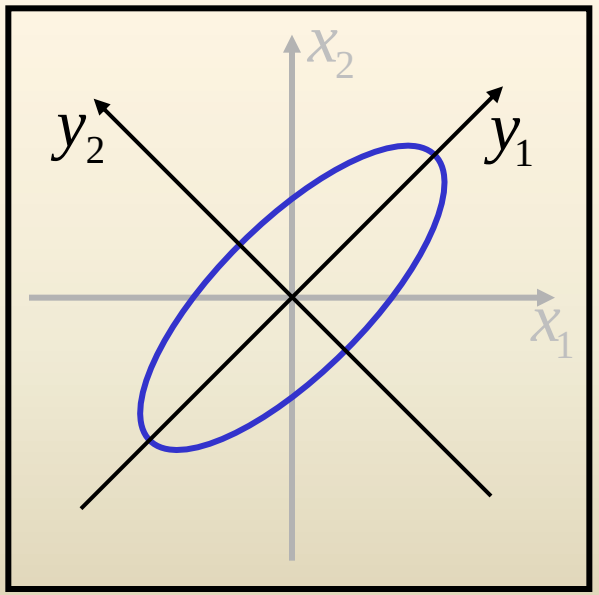
$$\mathbf{x}_2 = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathbf{T}}$$

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$





Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathbf{T}}$$

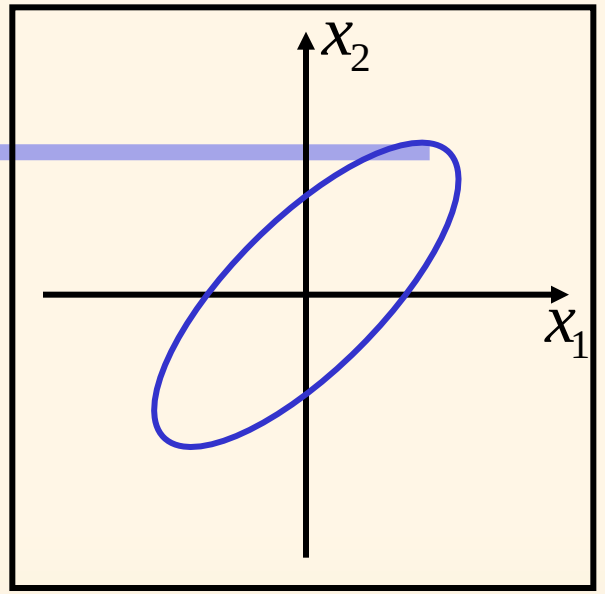
$$\mathbf{x}_2 = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathbf{T}}$$

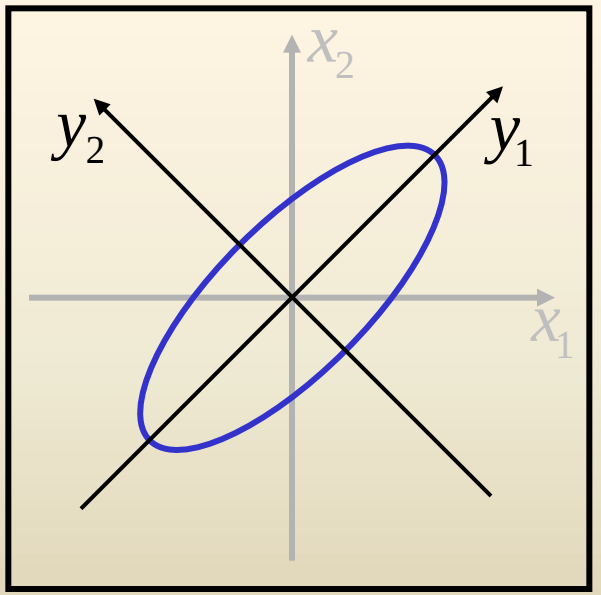
$$\therefore \mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$\mathbf{x} = \mathbf{X}\mathbf{y}$

$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$





Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathbf{T}}$$

$$\mathbf{x}_2 = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathbf{T}}$$

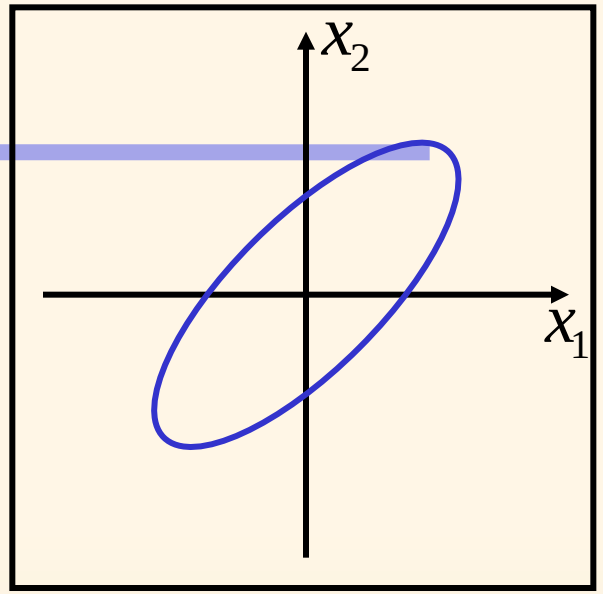
$$\therefore \mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

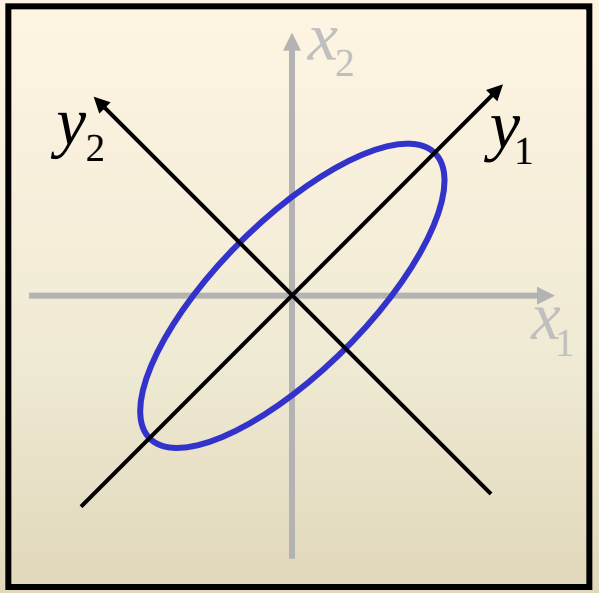
$$= \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$





Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathbf{T}}$$

$$\mathbf{x}_2 = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathbf{T}}$$

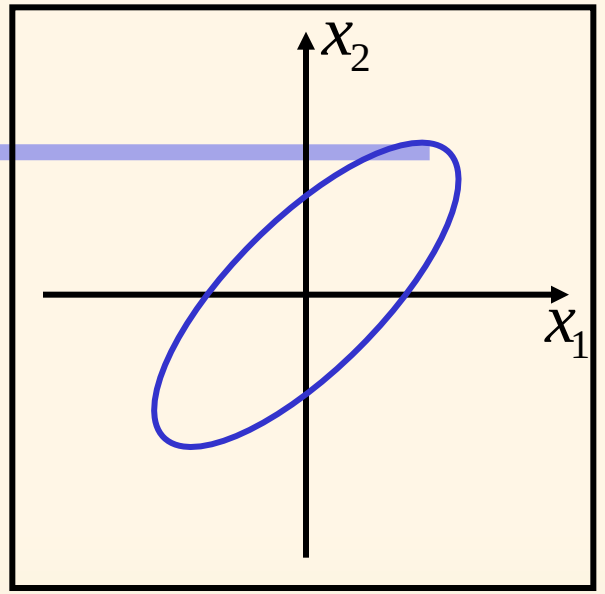
$$\therefore \mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

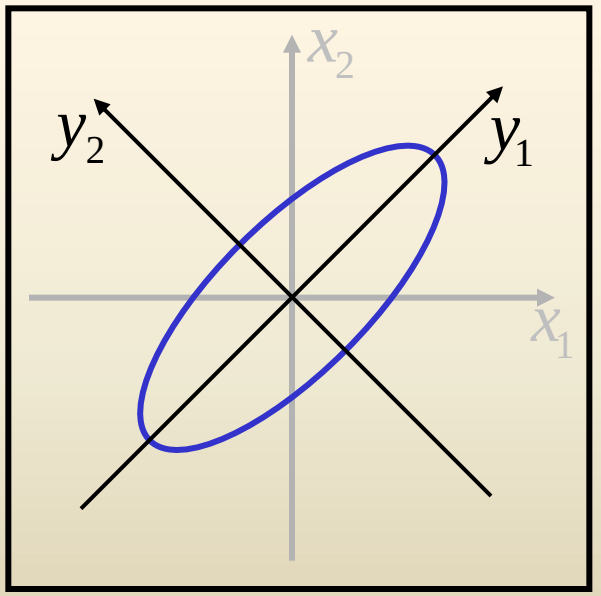
$$= \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$





➔This means a 45° rotation (of principal axes. (See example 8.2-1.)

Quadratic form(Definiteness)

A quadratic form $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ and its (symmetric!) matrix A are called

(a) positive definite if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$,

(b) negative definite if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$,

(c) indefinite if Q(x) takes both positive and negative values.

Quadratic form(Definiteness)

A quadratic form $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ and its (symmetric!) matrix A are called

(a) positive definite if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$,

(b) negative definite if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$,

(c) indefinite if Q(x) takes both positive and negative values.

A necessary and sufficient condition for positive definiteness is that all the "principal minors" are positive, that is,

Show that the form in Prob. 23 is positive definite, whereas that in Prob. 19 is indefinite.

Quadratic form(Definiteness)

A quadratic form $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ and its (symmetric!) matrix A are called

(a) positive definite if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$,

(b) negative definite if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$,

(c) indefinite if Q(x) takes both positive and negative values.

A necessary and sufficient condition for positive definiteness is that all the "principal minors" are positive, that is,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \cdots, \quad \det \mathbf{A} > 0$$

Show that the form in Prob. 23 is positive definite, whereas that in Prob. 19 is indefinite.

Quadratic form(Definiteness)

A necessary and sufficient condition for positive definiteness is that all the "principal minors" are positive, that is,

$$\mathbf{A}=\begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix}$$

Quadratic form(Definiteness)

A necessary and sufficient condition for positive definiteness is that all the "principal minors" are positive, that is,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \cdots, \quad \det \mathbf{A} > 0$$

$$\mathbf{A} = \begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix}$$

Quadratic form(Definiteness)

A necessary and sufficient condition for positive definiteness is that all the "principal minors" are positive, that is,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \cdots, \quad \det \mathbf{A} > 0$$

$$\mathbf{A} = \begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

$$a_{11} = 4 > 0$$

$$\mathbf{A} = \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix}$$

Quadratic form(Definiteness)

A necessary and sufficient condition for positive definiteness is that all the "principal minors" are positive, that is,

$$a_{11} > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \cdots, \det \mathbf{A} > 0$$

$$\mathbf{A} = \begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

$$a_{11} = 4 > 0$$

$$\begin{vmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{vmatrix} = 4 \cdot 2 - \left(\sqrt{3}\right)^2 = 5 > 0$$

$$\mathbf{A} = \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix}$$

Quadratic form(Definiteness)

A necessary and sufficient condition for positive definiteness is that all the "principal minors" are positive, that is,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \cdots, \quad \det \mathbf{A} > 0$$

$$\mathbf{A} = \begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

$$a_{11} = 4 > 0$$

$$\begin{vmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{vmatrix} = 4 \cdot 2 - \left(\sqrt{3}\right)^2 = 5 > 0$$

➔ **positive definite**

$$\mathbf{A} = \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix}$$

Quadratic form(Definiteness)

A necessary and sufficient condition for positive definiteness is that all the "principal minors" are positive, that is,

$$a_{11} > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \cdots, \det \mathbf{A} > 0$$

$$\mathbf{A} = \begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

$$a_{11} = 4 > 0$$

$$\begin{vmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{vmatrix} = 4 \cdot 2 - \left(\sqrt{3}\right)^2 = 5 > 0$$

➔ **positive definite**

$$\mathbf{A} = \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix}$$

$$a_{11} = 1 > 0$$

Quadratic form(Definiteness)

A necessary and sufficient condition for positive definiteness is that all the "principal minors" are positive, that is,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \cdots, \quad \det \mathbf{A} > 0$$

$$\mathbf{A} = \begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

$$a_{11} = 4 > 0$$

$$\begin{vmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{vmatrix} = 4 \cdot 2 - \left(\sqrt{3}\right)^2 = 5 > 0$$

➔ **positive definite**

$$\mathbf{A} = \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix}$$

$$a_{11} = 1 > 0$$

$$\begin{vmatrix} 1 & 12 \\ 12 & -6 \end{vmatrix} = -6 - 12^2 = -150 < 0$$

Quadratic form(Definiteness)

A necessary and sufficient condition for positive definiteness is that all the "principal minors" are positive, that is,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \cdots, \quad \det \mathbf{A} > 0$$

$$\mathbf{A} = \begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

$$a_{11} = 4 > 0$$

$$\begin{vmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{vmatrix} = 4 \cdot 2 - \left(\sqrt{3}\right)^2 = 5 > 0$$

➔ **positive definite**

$$\mathbf{A} = \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix}$$

$$a_{11} = 1 > 0$$

$$\begin{vmatrix} 1 & 12 \\ 12 & -6 \end{vmatrix} = -6 - 12^2 = -150 < 0$$

➔ **indefinite**

Quadratic form(Definiteness)

the eigenvalues of $\mathbf{A}$ are

(a) positive definite: all positive

(b) negative definite: all negative

(c) indefinite: both positive and negative

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}, \quad \mathbf{x} = \mathbf{X}\mathbf{y}$$

(1)

Quadratic form(Definiteness)

the eigenvalues of $\mathbf{A}$ are

(a) positive definite: all positive

(b) negative definite: all negative

(c) indefinite: both positive and negative

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}, \quad \mathbf{x} = \mathbf{X}\mathbf{y}$$
$$= \mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \quad \textbf{(1)}$$

Quadratic form(Definiteness)

the eigenvalues of $\mathbf{A}$ are

(a) positive definite: all positive

(b) negative definite: all negative

(c) indefinite: both positive and negative

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}, \quad \mathbf{x} = \mathbf{X}\mathbf{y}$$
$$= \mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \quad (1)$$

Because $y = X^{-1}x$, if $x \neq 0$, then $y \neq 0$.

Quadratic form(Definiteness)

the eigenvalues of $\mathbf{A}$ are

(a) positive definite: all positive

(b) negative definite: all negative

(c) indefinite: both positive and negative

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}, \quad \mathbf{x} = \mathbf{X}\mathbf{y}$$

$$= \mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \quad (1)$$

Because $y = X^{-1}x$, if $x \neq 0$, then $y \neq 0$.

From equation (1),

If all eigenvalues are positive, $Q(x)$ is positive.

Quadratic form(Definiteness)

the eigenvalues of $\mathbf{A}$ are

(a) positive definite: all positive

(b) negative definite: all negative

(c) indefinite: both positive and negative

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}, \quad \mathbf{x} = \mathbf{X}\mathbf{y}$$
$$= \mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \quad (1)$$

Because $\mathbf{y} = \mathbf{X}^{-1}\mathbf{x}$, if $\mathbf{x} \neq 0$, then $\mathbf{y} \neq 0$.

From equation (1),

If all eigenvalues are positive, $Q(x)$ is positive.
If all eigenvalues are negative, $Q(x)$ is negative.

Positive Definite Matrix & Optimization

최적성 조건을 이용한 최적해의 도출*

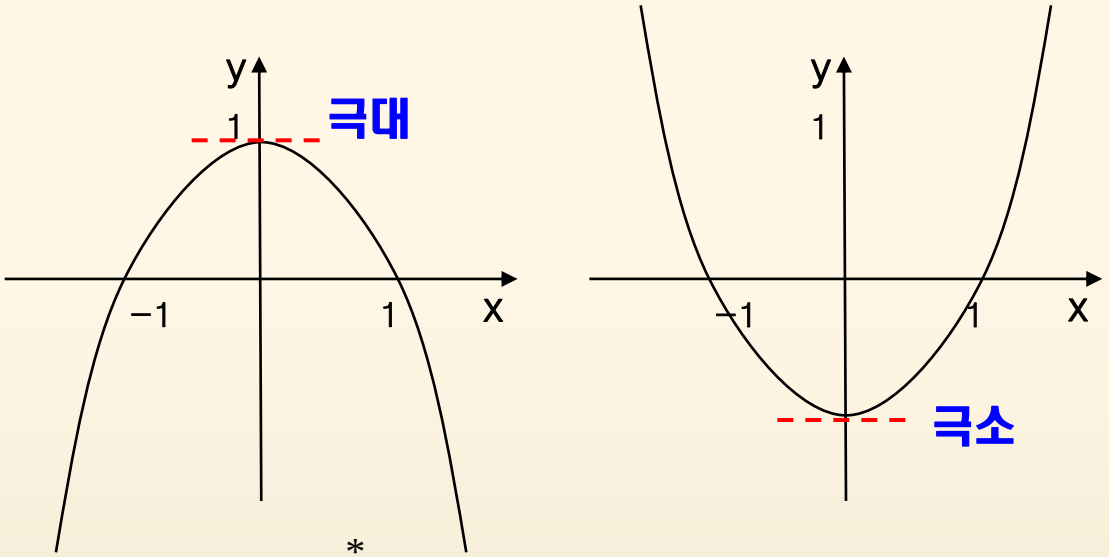
-Gradient 방법
- Steepest Descent 방법
- 공액 경사도 방법(Conjugate Gradient 방법)
- Newton의 방법
- Davidon-Fletcher-Powell(DFP) 방법
- Broyden-Fletcher-Goldfarb-Shanno(BFGS) 방법

-황금 분할에 의한 1차원 탐색 방법

-직접 탐사법(Direct Search Method)
- Hooke & Jeeves의 직접 탐사법
- Nelder & Mead의 Simplex 방법

*이규열, 전산 선박 설계 교재, Part1. 최적 설계 기법, 2007.9.

1변수 함수의 최적성 조건
- 함수의 극대, 극소(고교 과정 Review)*



1) 극대값 : $x = x^*$ 에서 연속 함수 *f*(*x*)가 증가에서 감소 상태로 변함

2) 극소값 : $x = x^*$ 에서 연속 함수 *f*(*x*)가 감소에서 증가 상태로 변함

$$f'(x^*) = 0$$

($x = x^*$에서 극대 또는 극소값을 가질 필요 조건)

홍성대, 기본 수학 I 의 정석, 1996, 성지사, p247

1변수 함수의 최적성 조건
- 1계 필요 조건(1)

- 변수가 하나일 때, 극값을 가질 필요 조건 : $f'(x^*) = 0$

pf) 주어진 점 x^* 에서 $f(x)$의 테일러 급수는 다음과 같다.

$$f(x) = f(x^*) + \frac{df(x^*)}{dx}(x - x^*) + \frac{1}{2}\frac{d^2 f(x^*)}{dx^2}(x - x^*)^2 + R$$

나머지항(Remainder)
: x가 x^* 에 충분히 가까우면 그 값이 매우 작음

$x - x^* = d$ 라고 놓으면

$$f(x) = f(x^*) + f'(x^*)d + \frac{1}{2}f''(x^*)d^2 + R$$

함수 값의 변화량 $f(x) - f(x^*) = \Delta f(x)$

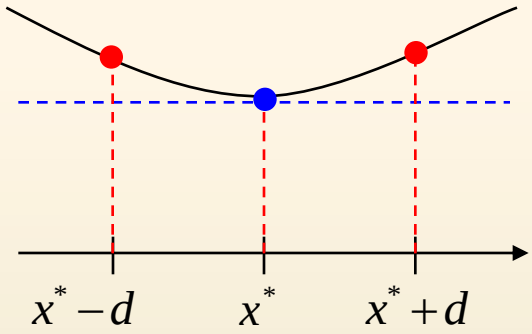
$$\Delta f(x) = f'(x^*)d + \frac{1}{2}f''(x^*)d^2 + R$$

1변수 함수의 최적성 조건
- 1계 필요 조건(2)

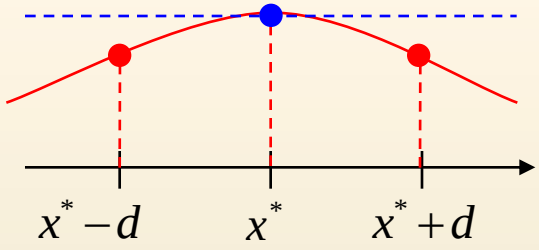
$$\Delta f(x) = f(x) - f(x^*) = f'(x^*)d + \frac{1}{2} f''(x^*)d^2 + R$$ ➡

$x = x^*$ 에서 국부적 후보 최소점이기 위해서는 $\Delta f \geq 0$

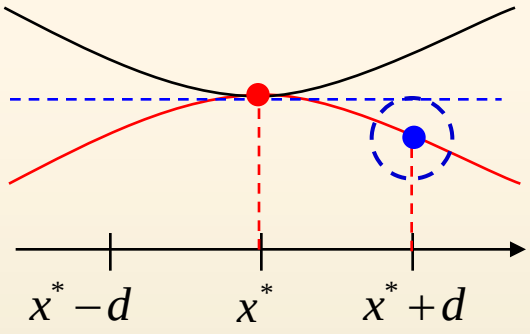
의미 : x^*에서의 함수 값이 $x^* \pm d$에서의 함수 값보다 작으면 x^*는 국부적 후보 최소점이 될 가능성이 있음



$f(x^*)$ 에서 최소값



$f(x^*)$ 에서 최대값



$f(x^*)$ 보다 함수 값이 작은 점(x^*+d)이 존재하므로 x^*는 최소점이 아님

따라서, $d(= x - x^*)$ 의 부호에 상관없이 $\Delta f \geq 0$ 를 만족하려면 $f'(x^*) = 0$ 이어야 함

이와 유사한 개념으로 $x = x^*$ 에서 국부적 후보 최대점이 되기 위해서는 $\Delta f \leq 0$ 이어야 하며,

$d(= x - x^*)$ 의 부호에 상관없이 $\Delta f \leq 0$ 를 만족하려면 $f'(x^*) = 0$ 이어야 함

cf) $f'(x^*) = 0$ 를 만족하는 점 : 국부적으로 최소, 최대, 또는 변곡점

총칭하여 상점(Stationary point)이라고 함

1변수 함수의 최적성 조건
- 충분 조건과 2계 필요 조건

$$\Delta f(x) = f(x) - f(x^*) = f'(x^*)d + \frac{1}{2} f''(x^*)d^2 + R$$

$x = x^*$ 에서 국부적 후보 최소점이기 위해서는 $\Delta f \geq 0$

따라서, $d(= x - x^*)$ 의 부호에 상관없이 $\Delta f \geq 0$ 를 만족하려면 $f'(x^*) = 0$ 이어야 함

- 상점($f'(x^*) = 0$ 를 만족하는 점) 중 어느 점이 최소점인지 결정하는 방법

상점이므로, $f'(x^*) = 0$. 따라서, $\Delta f(x) = \frac{1}{2} f''(x^*)d^2 + R$

2차항이 다른 모든 고차항에 비해 지배적인 항이므로,

d≠0에 대하여 다음 조건을 만족하면 항상 $\Delta f \geq 0$

$f''(x^*) > 0$ (충분 조건(Sufficient condition))

cf) $f''(x^*) = 0$ 인 경우 국부적으로 최소점인지 알 수 없다.

∵ 1계, 2계 미분이 모두 0이므로, 국부적으로 최소이기 위해서는
3계 미분계수는 0이 되어야 하고, 4계 미분계수가 양수(>0)여야 한다.
즉, R의 부호에 따라 달라지게 된다.

- 2계 필요 조건 :

국부적으로 최소값을 가지면, $f''(x^*) \geq 0$ (※ 단, 역은 성립하지 않는다.)

1변수 함수의 최적성 조건(요약)

▪ 충분 조건, 필요 조건, 필요 충분 조건

(고등학교 수학의 정석 '명제와 조건' 참고)

1) A는 B의 충분 조건이다.

"A이면 B이다." ($A \Rightarrow B$)

2) C는 D의 필요 조건이다.

"D이면 C이다." ($C \Leftarrow D$)

3) E는 F의 필요 충분 조건이다.

"E이면 F이고, F이면 E이다." ($E \Leftrightarrow F$)

▪ 1계 필요 조건(1st Order necessary condition)

x^*가 국부적 후보 최소점이면, $f'(x^*) = 0$

cf) $f'(x^*) = 0$ 이면, x^*가 상점(극소, 극대, 변곡점)이다.

▪ 충분 조건(Sufficient condition)

상점($f'(x^*) = 0$)일 경우

$f''(x^*) > 0$ 이면 x^*가 국부적 최소점이다.

▪ 2계 필요 조건(2nd Order necessary condition)

x^*가 국부적 최소점이면, $f''(x^*) \geq 0$

2변수 함수의 테일러 전개(Taylor Series Expansion)

- 2변수 함수 $f(x_1, x_2)$ 에 대한 점 (x_1^*, x_2^*) 에서의 테일러 전개식

$$f(x_1,x_2) = f(x_1^*,x_2^*) + \frac{\partial f}{\partial x_1}(x_1 - x_1^*) + \frac{\partial f}{\partial x_2}(x_2 - x_2^*)$$
$$+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2}(x_1 - x_1^*)^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1 - x_1^*)(x_2 - x_2^*) + \frac{\partial^2 f}{\partial x_2^2}(x_2 - x_2^*)^2 \right) + R$$

⬇ 이를 다시 표현하면,

$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \mathbf{H}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) + R$$

2x2 Matrix의 원소

$$\left(\mathbf{x} = (x_1, x_2)^T,\ \mathbf{x}^* = (x_1^*, x_2^*)^T,\ \mathbf{H} \in M_{2\times 2} \right)$$

$$\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}^T \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$$

$$\frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \mathbf{H}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) = \frac{1}{2}\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(x_1 - x_1^*) + \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_2 - x_2^*) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1 - x_1^*) + \frac{\partial^2 f}{\partial x_2^2}(x_2 - x_2^*) \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} x_1 - x_1^* & x_2 - x_2^* \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$$

2변수 함수의 테일러 전개

- 2변수 테일러 전개식의 행렬 표기법

$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \mathbf{H}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) + R$$

$$\left(\mathbf{x} = (x_1, x_2)^T,\ \mathbf{x}^* = (x_1^*, x_2^*)^T,\ \mathbf{H} \in M_{2\times 2} \right)$$

2x2 Matrix의 원소

- 다변수 함수의 테일러 전개식의 행렬 표기법 : 2변수 전개식과 동일

$\mathbf{x},\ \mathbf{x}^*, \nabla f$: *n* 차원 Vector

$$\mathbf{H} \in M_{n\times n}$$

- $\mathbf{x} - \mathbf{x}^* = \mathbf{d}$ 로 정의 하면, 다음과 같이 표현 가능

$$f(\mathbf{x}^* + \mathbf{d}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T \mathbf{d} + \frac{1}{2}\mathbf{d}^T \mathbf{H}(\mathbf{x}^*)\mathbf{d} + R$$

$\nabla f(\mathbf{x}^*)^T = 0,\ \frac{1}{2}\mathbf{d}^T \mathbf{H}(\mathbf{x}^*)\mathbf{d} > 0$ ➡ $\mathbf{x} = \mathbf{x}^*$에서 최소이기 위한 충분 조건

[참고] 헷세 행렬(Hessian Matrix)

- 헷세 행렬(Hessian Matrix) : Gradient Vector를 한번 더 미분하면(각각, x_i 에 대하여 미분), 함수 $f(x)$ 에 대한 2계 편도함수의 행렬을 얻게 되는데, 이를 헷세 행렬이라고 정의함.
  즉, Gradient Vector의 각 성분을 $x_1, x_2, \cdots, x_n$ 으로 미분하면 다음을 얻는다.

$$\frac{\partial^2 f}{\partial \mathbf{x} \partial \mathbf{x}} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T$$

: *n*개의 변수를 가지는 column Vector(열벡터)

- 헷세 행렬은 보통 H 또는 $\nabla^2 f$ 로 표시

$$\mathbf{H} = \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right] \quad (i = 1,2,\cdots,n \; ; \; j = 1,2,\cdots,n)$$

- 헷세 행렬의 대칭성

$$\frac{\partial f}{\partial x_i \partial x_j} = \frac{\partial f}{\partial x_j \partial x_i}$$

따라서, 헷세 행렬은 항상 대칭 행렬

Quadratic Form(2차 형식)

- 2차 형식 : 한 변수의 제곱항과 서로 다른 두 변수의 곱항(다변수 곱항)의 합으로 이루어진 함수

ex) $F(x_1,x_2,x_3)=\frac{1}{2}\left(2x_1^2+2x_1x_2+4x_1x_3-6x_2^2-4x_2x_3+5x_3^2\right)$

2차 형식은 다음과 같이 행렬로 표현이 가능하다.

A : 대칭 행렬 (Symmetric Matrix)

$$F(x_1,x_2,x_3)=\frac{1}{2}\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}2 & 1 & 2\\ 1 & -6 & -2\\ 2 & -2 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} \Leftrightarrow \frac{1}{2}\mathbf{d}^T\mathbf{H}\mathbf{d}$$

- 대칭행렬 A의 구성 방법 (A의 (i,j) 성분을 a_{ij}로 나타냄)
  1) 대각 성분은 제곱항의 계수 $a_{ii}=\left(x_i^2\text{의 계수}\right)$
  2) 대각 성분 이외의 성분은 다변수 곱항 계수의 ½ $a_{ij}=\left(x_ix_j\text{의 계수}\right)\times\frac{1}{2}$

Quadratic Form

- 2차 형식의 정의

1) 양정 형식(Positive Definite)

: $\mathbf{x} \neq 0$ 인 모든 x에 대하여 $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$

2) 양반정 형식(Positive Semidefinite)

: 모든 x에 대하여 $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$ 이고

$\mathbf{x}^T\mathbf{A}\mathbf{x} = 0$인 $\mathbf{x} \neq 0$가 존재

3) 음정 형식(Negative Definite)

: $\mathbf{x} \neq 0$ 인 모든 x에 대하여 $\mathbf{x}^T\mathbf{A}\mathbf{x} < 0$

4) 음반정 형식(Negative Semidefinite)

: 모든 x에 대하여 $\mathbf{x}^T\mathbf{A}\mathbf{x} \leq 0$

5) 부정 형식(Indefinite)

: 어떤 x에 대해서는 양수 또는 음수

- 2차 형식 정의의 이용

① 1변수 함수의 최소 조건

$f'(x^*) = 0$ 인 *x**(상점)에서

$f''(x^*) > 0$ 이면

*x**는 국부적 최소점이다.

② 다변수 함수의 최소 조건

$\nabla f(\mathbf{x}^*)^T = 0 \Rightarrow \nabla f(\mathbf{x}^*) = 0$ 인

x*(상점)에서 $\mathbf{d}^T\mathbf{H}(\mathbf{x}^*)\mathbf{d} > 0$이면

즉, $\mathbf{d}^T\mathbf{H}(\mathbf{x}^*)\mathbf{d} > 0$ 가 양정 형식이면

x*는 국부적 최소점이다.

여기서, $\mathbf{H}(\mathbf{x}^*)$ 가 양정 행렬이면

$\mathbf{d}^T\mathbf{H}(\mathbf{x}^*)\mathbf{d} > 0$

Quadratic Form 고유치 검사

2차 형식 $F(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x}$에 관련된 대칭 $n \times n$ 행렬 $\mathbf{A}$의 n개의 고유치를 $\lambda_i, i = 1,\ldots,n$ 이라고 가정

1) $F(\mathbf{x})$ 가 양정(Positive Definite)이기 위한 필요 충분 조건: 모든 고유치가 양수

$$\lambda_i > 0, i = 1,\ldots,n$$

2) $F(\mathbf{x})$ 가 양반정(Positive Semidefinite)이기 위한 필요 충분 조건: 모든 고유치가 음수가 아님

$$\lambda_i \geq 0, i = 1,\ldots,n$$

3) $F(\mathbf{x})$ 가 음정(Negative Definite)이기 위한 필요 충분 조건: 모든 고유치가 음수

$$\lambda_i < 0, i = 1,\ldots,n$$

4) $F(\mathbf{x})$ 가 음반정(Negative Semidefinite)이기 위한 필요 충분 조건: 모든 고유치가 양수가 아님

$$\lambda_i \leq 0, i = 1,\ldots,n$$

5) 적어도 하나의 λ_i 가 음수이고 또 적어도 하나의 λ_i 가 양수이면 $F(\mathbf{x})$ 는 부정(Indefinite)임

고유치 검사에 의한 Quadratic Form 행렬의 형태 결정 예제

고유치 구하는 방법

$$Av = \lambda v \quad \Rightarrow \quad (A - \lambda I)v = 0 \quad \Rightarrow \quad \det(A - \lambda I) = 0$$

다음 행렬의 고유치를 구하여 행렬의 형태를 결정하시오.

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$$\det \begin{bmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{bmatrix} = (2-\lambda)^2(8-\lambda) = 0$$

$$\therefore \lambda = 2(\text{중근}), 8$$

모든 고유치가 양수이므로 주어진 행렬은 양정(Positive Definite) 행렬임

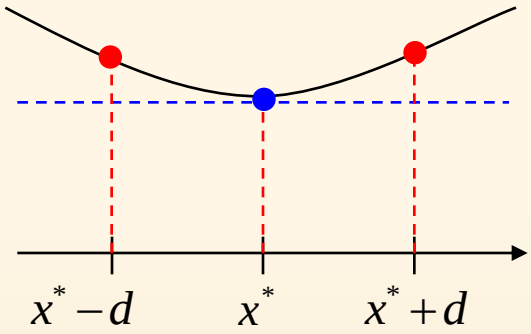
다변수 함수의 후보 최적성 조건(요약)

- n개의 변수로 이루어진 다변수 함수 $f(\mathbf{x})$ 에 대한 테일러 전개식

$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T \mathbf{d} + \frac{1}{2}\mathbf{d}^T \mathbf{H}(\mathbf{x}^*)\mathbf{d} + R$$

- 함수의 변화량으로 위 식을 다시 쓰면,

$$\Delta f = \nabla f(\mathbf{x}^*)^T \mathbf{d} + \frac{1}{2}\mathbf{d}^T \mathbf{H}(\mathbf{x}^*)\mathbf{d} + R$$



$f(x^*)$ 에서 최소값

- $\mathbf{x}^*$ 가 국부적으로 후보 최소점이면, $\Delta f \geq 0$ 이어야 함

1) x*가 국부적 후보 최소점일 필요 조건 :

➡ $\nabla f(\mathbf{x}^*)^T = 0$ 즉, $\frac{\partial f(\mathbf{x}^*)}{\partial x_i} = 0,\ (i = 1,2,\cdots n)$

2) $\nabla f(\mathbf{x}^*)^T = 0 \Rightarrow \nabla f(\mathbf{x}^*) = 0$ 인 상점에서 x*가 국부적 최소점일 조건 :

➡ $\mathbf{d}^T \mathbf{H}(\mathbf{x}^*)\mathbf{d} > 0$ ➡ 위 조건은 $\mathbf{H}(\mathbf{x}^*)$ 가 양정 행렬일 경우 성립한다.

Gradient 방법- Newton의 방법

☑ 헷세(Hassian) 행렬을 사용하여 탐색 방법을 개선한 방법

- 단계 1 : 시작점 $\mathbf{x}^{(0)}$ 을 추정한다. 반복 회수 번호를 $k=0$ 으로 둔다. 그리고 종료 기준으로 허용치 ε 을 선정한다.
- 단계 2 : $i=1$ 에서 n 까지 $c_i^{(k)} = \partial f(\mathbf{x}^{(k)})/\partial x_i$ 를 계산한다.
- 단계 3 : 헷세 행렬을 계산한다.

$$H(\mathbf{x}^{(k)}) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right], \qquad i=1,\cdots,n;\ j=1,\cdots,n$$

- 단계 4 : 탐색 방향을 계산한다.

$$\Delta\mathbf{x}^{(k)} = \mathbf{d}^{(k)} = -\mathbf{H}^{-1}\mathbf{c}^{(k)}$$

- 단계 5 : 설계점을 $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha_k \mathbf{d}^{(k)}$ 로 수정한다. 이때 α_k 는 $f(\mathbf{x}^{(k)} + \alpha\mathbf{d}^{(k)})$ 를 최소화하도록 계산된다. 를 계산하기 위해서는 어떠한 1차원 탐색법도 사용할 수 있다. 초기 이동 거리 추정으로 $\alpha=1$ 로 가정한다.
- 단계 6 : $k=k+1$ 로 두고 단계 2를 수행한다.

Gradient 방법- Newton의 방법

☑ 헷세(Hassian) 행렬을 사용하여 탐색 방법을 개선한 방법

- 단계 1 : 시작점 $\mathbf{x}^{(0)}$ 을 추정한다. 반복 회수 번호를 $k=0$ 으로 둔다. 그리고 종료 기준으로 허용치 ε 을 선정한다.
- 단계 2 : $i=1$ 에서 n 까지 $c_i^{(k)} = \partial f(\mathbf{x}^{(k)})/\partial x_i$ 를 계산한다.
- 단계 3 : 헷세 행렬을 계산한다.

$$H(\mathbf{x}^{(k)}) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right], \qquad i=1,\cdots,n;\ j=1,\cdots,n$$

- 단계 4 : 탐색 방향을 계산한다.

$$\Delta\mathbf{x}^{(k)} = \mathbf{d}^{(k)} = -\mathbf{H}^{-1}\mathbf{c}^{(k)}$$

$f(\mathbf{x}) = f(\mathbf{x}^*) + \mathbf{c}^T\mathbf{d} + \frac{1}{2}\mathbf{d}^T\mathbf{H}(\mathbf{x}^*)\mathbf{d}$ 일 때

최소값을 가질 조건

$f'(\mathbf{x}) = \mathbf{c} + \mathbf{H}(\mathbf{x}^*)\mathbf{d} = 0$

$\Rightarrow \mathbf{H}(\mathbf{x}^*)\mathbf{d} = -\mathbf{c} \Rightarrow \mathbf{d} = -\mathbf{H}(\mathbf{x}^*)^{-1}\mathbf{c}$

- 단계 5 : 설계점을 $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha_k \mathbf{d}^{(k)}$ 로 수정한다. 이때 α_k 는 $f(\mathbf{x}^{(k)} + \alpha\mathbf{d}^{(k)})$ 를 최소화하도록 계산된다. α 를 계산하기 위해서는 어떠한 1차원 탐색법도 사용할 수 있다. 초기 이동 거리 추정으로 $\alpha=1$ 로 가정한다.
- 단계 6 : $k=k+1$ 로 두고 단계 2를 수행한다.

비제약 최적화 문제- Newton 방법을 이용한 해법

$$\textit{Minimize}\quad f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$$

☑ 단계 1 - $\mathbf{x}^{(1)}$구하기

이 방법은 출발점 $\mathbf{x}^{(k)}$에서 $\Delta\mathbf{x}^{(k)}$만큼 증가한점$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \Delta\mathbf{x}^{(k)}$에서
함수 f가 최소값을갖는다고가정한다.

여기서, $\Delta\mathbf{x}^{(0)} = \mathbf{x}^{(1)} - \mathbf{x}^{(0)}$

즉, $\Delta x_1^{(0)} = x_1^{(1)} - x_1^{(0)}$, $\Delta x_2^{(1)} = x_2^{(2)} - x_2^{(1)}$이다.

Start point $\mathbf{x}^{(0)} = (0,0)$

오차 범위: $\varepsilon = 0.001$

$$\nabla f(\mathbf{x}) = \nabla f(x_1, x_2) = \begin{pmatrix} f_{x_1} \\ f_{x_2} \end{pmatrix} = \begin{pmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{pmatrix}$$

$$H(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_1x_2} & f_{x_2x_2} \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$$

비제약 최적화 문제- Newton 방법을 이용한 해법

$$\textit{Minimize}\quad f(x_1,x_2)=x_1-x_2+2x_1^2+2x_1x_2+x_2^2$$

$$\begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_1x_2} & f_{x_2x_2} \end{pmatrix}\begin{pmatrix} \Delta x_1^{(0)} \\ \Delta x_2^{(0)} \end{pmatrix}=-\begin{pmatrix} f_{x_1} \\ f_{x_2} \end{pmatrix}\text{이고}$$

$$f_{x_1}=1+4x_1+2x_2$$

$$f_{x_2}=-1+2x_1+2x_2$$

$$f_{x_1x_1}=4$$

$$f_{x_2x_2}=2$$

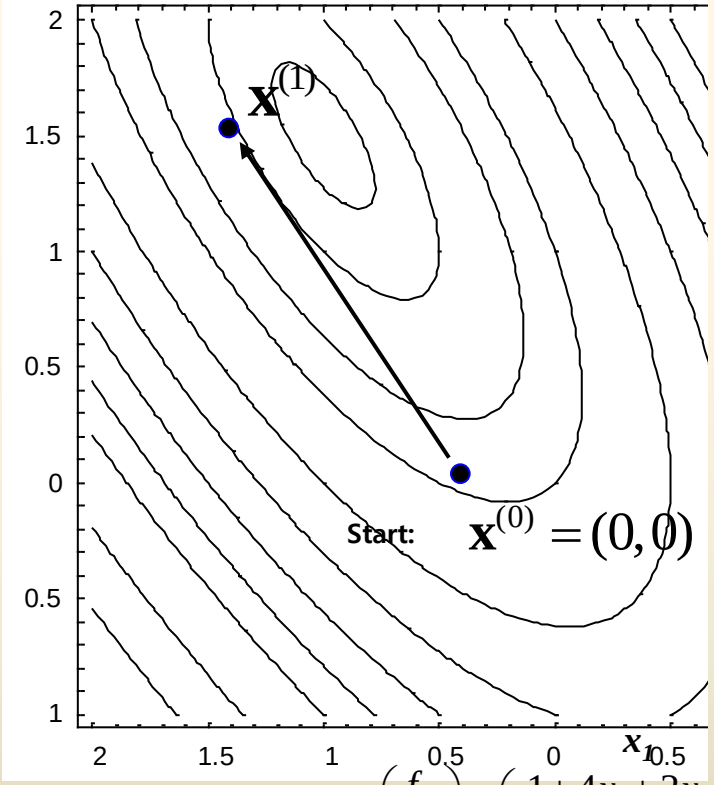
$$f_{x_1x_2}=f_{x_2x_1}=2\text{이므로}$$

$$\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} \Delta x_1^{(0)} \\ \Delta x_2^{(0)} \end{pmatrix}=-\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

에서 $\Delta x_1^{(0)}=-1,\ \Delta x_2^{(0)}=1.5$이다.

$$\therefore \mathbf{x}^{(1)}=\mathbf{x}^{(0)}+\alpha\Delta\mathbf{x}^{(0)}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}+\begin{pmatrix} -1 \\ 1.5 \end{pmatrix}=\begin{pmatrix} -1 \\ 1.5 \end{pmatrix}$$

$$\text{assum.}:\alpha=1$$



$$\nabla f(\mathbf{x})=\nabla f(x_1,x_2)=\begin{pmatrix} f_{x_1} \\ f_{x_2} \end{pmatrix}=\begin{pmatrix} 1+4x_1+2x_2 \\ -1+2x_1+2x_2 \end{pmatrix}$$

$$H(\mathbf{x})=\begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\partial x_2} \\ \dfrac{\partial^2 f}{\partial x_1\partial x_2} & \dfrac{\partial^2 f}{\partial x_2^2} \end{pmatrix}=\begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_1x_2} & f_{x_2x_2} \end{pmatrix}=\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$$

비제약 최적화 문제- Newton 방법을 이용한 해법

$$\textit{Minimize}\quad f(x_1,x_2)=x_1-x_2+2x_1^2+2x_1x_2+x_2^2$$

☑ **단계 2** $\mathbf{x}^{(2)}$구하기

단계1과 같은 과정을 거치면

$$\begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_1x_2} & f_{x_2x_2} \end{pmatrix}\begin{pmatrix} \Delta x_1^{(1)} \\ \Delta x_2^{(1)} \end{pmatrix}=-\begin{pmatrix} f_{x_1} \\ f_{x_2} \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} \Delta x_1^{(1)} \\ \Delta x_2^{(1)} \end{pmatrix}=-\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

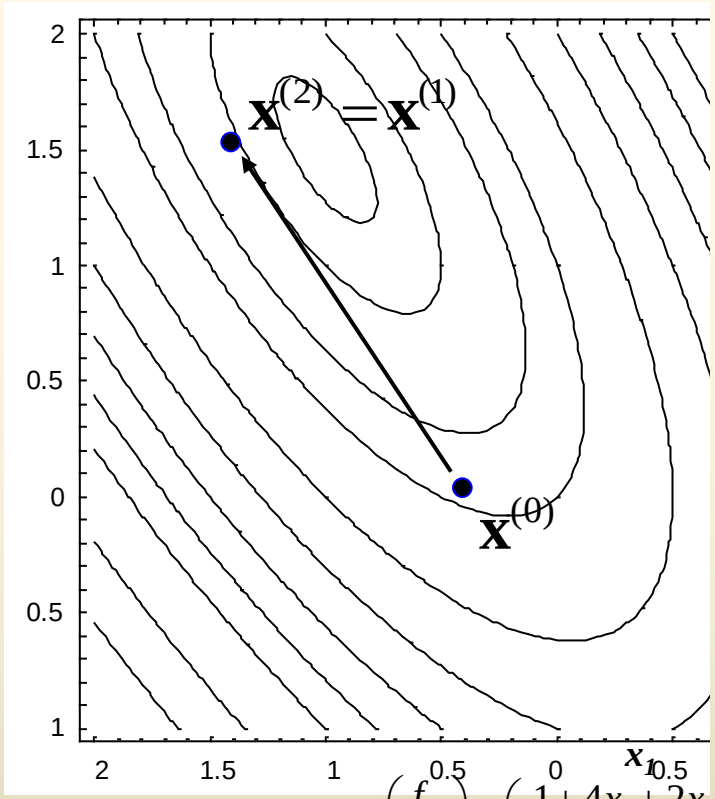
에서 $\Delta x_1^{(1)}=0,\ \Delta x_2^{(1)}=0$이다.

$$\therefore \mathbf{x}^{(2)}=\mathbf{x}^{(1)}+\alpha\Delta\mathbf{x}^{(1)}=\begin{pmatrix} -1 \\ 1.5 \end{pmatrix}+\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\begin{pmatrix} -1 \\ 1.5 \end{pmatrix}$$

$$\text{assum.}:\alpha=1$$

따라서 $\left|\mathbf{x}^{(2)}-\mathbf{x}^{(1)}\right|\le\varepsilon=0.001$이므로

$\mathbf{x}^{(2)}$이 최적해가 된다.



$$\nabla f(\mathbf{x})=\nabla f(x_1,x_2)=\begin{pmatrix} f_{x_1} \\ f_{x_2} \end{pmatrix}=\begin{pmatrix} 1+4x_1+2x_2 \\ -1+2x_1+2x_2 \end{pmatrix}$$

$$H(\mathbf{x})=\begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\partial x_2} \\ \dfrac{\partial^2 f}{\partial x_1\partial x_2} & \dfrac{\partial^2 f}{\partial x_2^2} \end{pmatrix}=\begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_1x_2} & f_{x_2x_2} \end{pmatrix}=\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$$