Naval Architecture & Ocean Engineering

- Ship Stability -

Part.1-II Righting Force and Moment

2009 Fall

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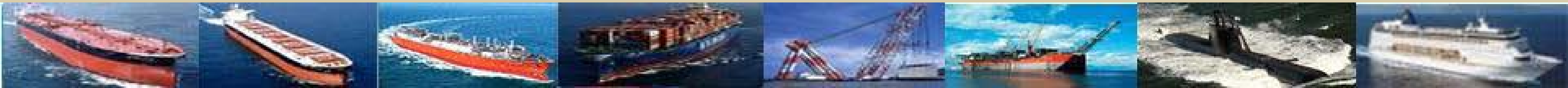


Overview of "Ship Stability"

F_B: Buoyancy force
ϕ : Angle of Heel, θ : Angle of Trim
(x_G, y_G, z_G) : Center of mass in waterplane fixed frame
(x_B, y_B, z_B) : Center of buoyancy in waterplane fixed frame

Fundamental of Ship Stability

Force & Moment on a Floating Body

Newton's 2nd Law
Euler Equation

Hydrostatic Values

• Properties which is related to hull form of the ship

y'_G , y'_B in body fixed frame

Rotational Transformation!

y_G , y_B in waterplane fixed frame

Righting Moment

Transverse Righting Moment : $F_B \times GZ$

Longitudinal Righting Moment : $F_B \times GZ_L$

GZ Calculation

<Method ①>

$$GZ = (-y_G + y_B)$$

$$GZ_L = (-x_G + x_B)$$

<Method ②>

$$GZ = GM \sin\phi \quad , GM = KB + BM - KG$$

$$GZ_L = GM_L \sin\theta \quad , GM_L = KB + BM_L - KG$$

Pressure Integration Technique

▪ **Calculation Method to find GZ with respect to IMO regulation**

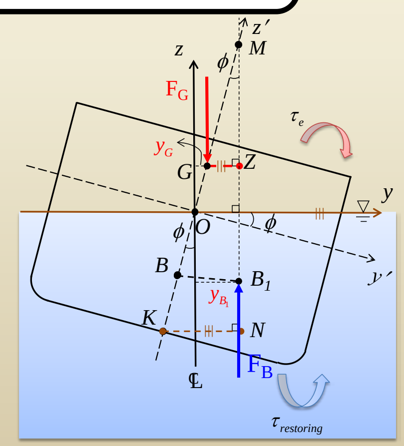
Stability Criteria

Intact Stability

- IMO Requirement (GZ)
- Grain Stability
- Floodable Length

Damage Stability

- MARPOL regulation



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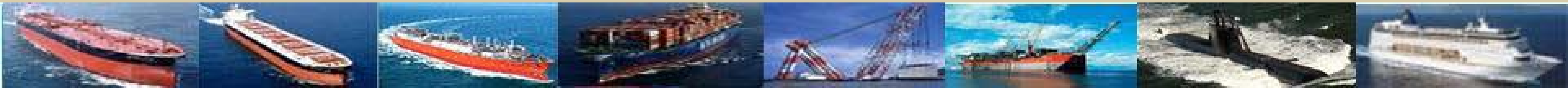

Sec.1 Calculation of Center of Buoyancy

Sec.2 Calculation of BM, GZ in Wall Sided Ship

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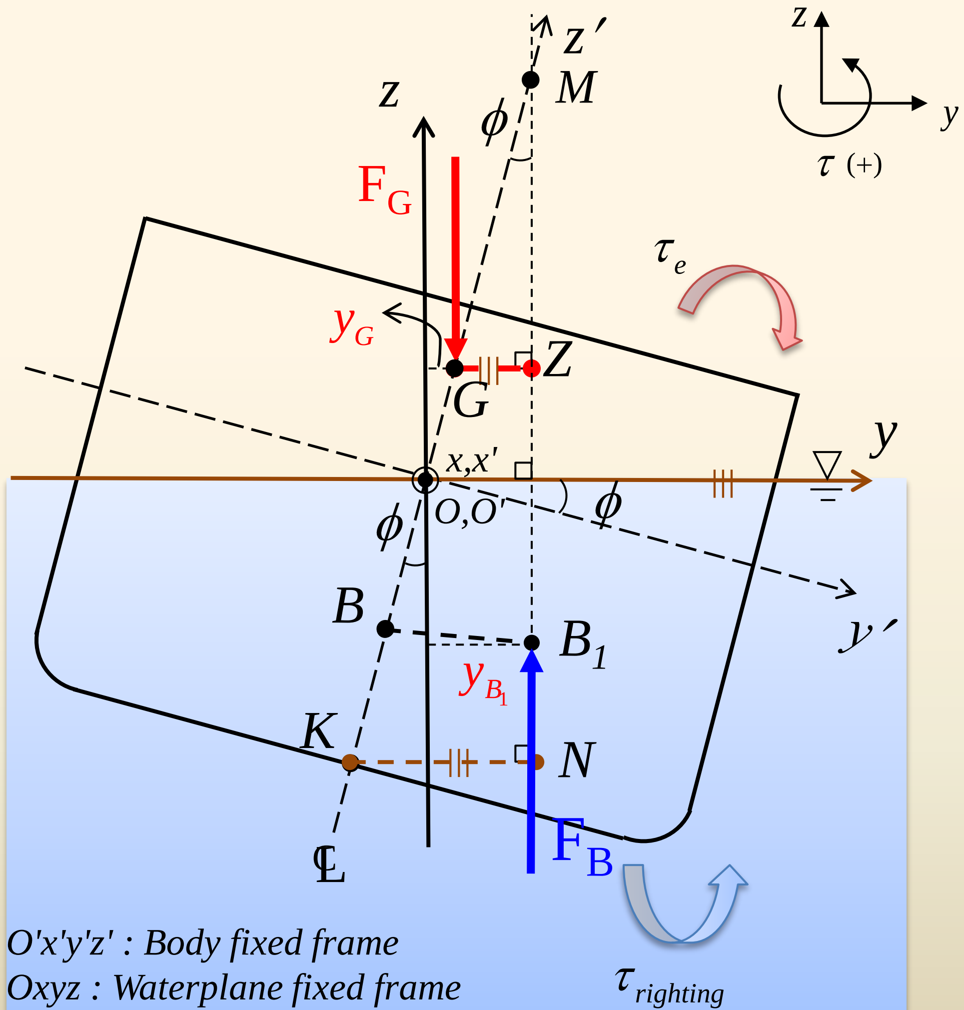
Sec.1 Calculation of Center of Buoyancy

- Rotational Transformation of Point and Frame
- Example) Calculation of Center of Buoyancy of Ship with Constant Section

  Method ① Direct calculating center of buoyancy in waterplane fixed frame

  Method ② Transformation center of buoyancy from body fixed frame to waterplane fixed frame
- Calculation of Center of Buoyancy of Ship with Various Station Shape

(Review)
Transverse Righting Moment



G: Center of mass
B: Center of buoyancy
B_1: Changed center of buoyancy
K : Keel
$\mathbf{F}_G$: Weight of ship
$\mathbf{F}_B$: Buoyant force acting on ship
Z : The intersection of the line of buoyant force through B_1 with the transverse line through G
M : The intersection of the line of buoyant force through B_1 with the centerline of the ship

- **Righting Moment :** Moment to return the ship to the upright floating position (Righting moment, Moment of statical stability))

- **Transverse Righting moment**

$$\tau_{righting} = (-y_G + y_{B_1}) \cdot F_B \, \mathbf{i} = \underbrace{GZ}_{\text{Righting arm}} \cdot F_B \, \mathbf{i}$$

- **Righting Arm** (GZ)

① From direct calculation

$$GZ = -y_G + y_{B_1}$$

We should know y_G, y_{B1} in waterplane fixed frame

② From geometrical figure with assumption that M does not change within small angle of heel (about 10°)

$$GZ = GM \cdot \sin\phi$$

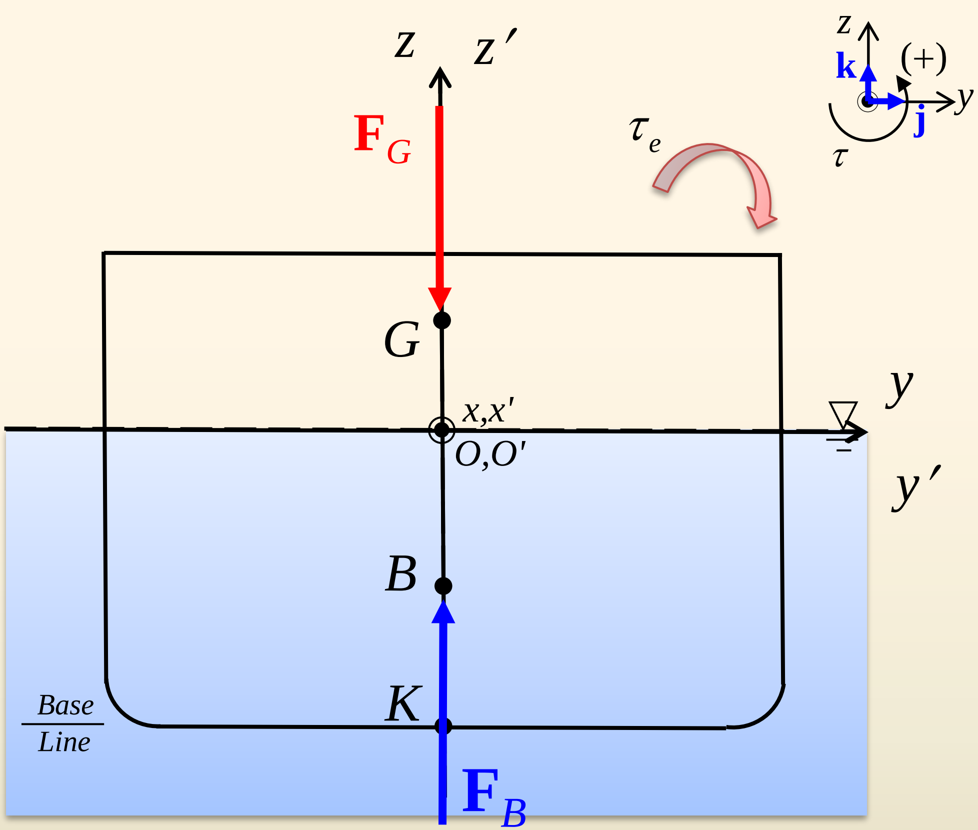
GM is related to below equation by geometrical figure

$$GM = KB + BM - KG$$

How to calculate y_{B_1}, z_{B_1} in waterplane fixed frame ?

Question : How to calculate center of the buoyancy(B_1) with respect to waterplane fixed frame?

✓Method 1. calculate center of the buoyancy(B_1) with respect to waterplane fixed frame



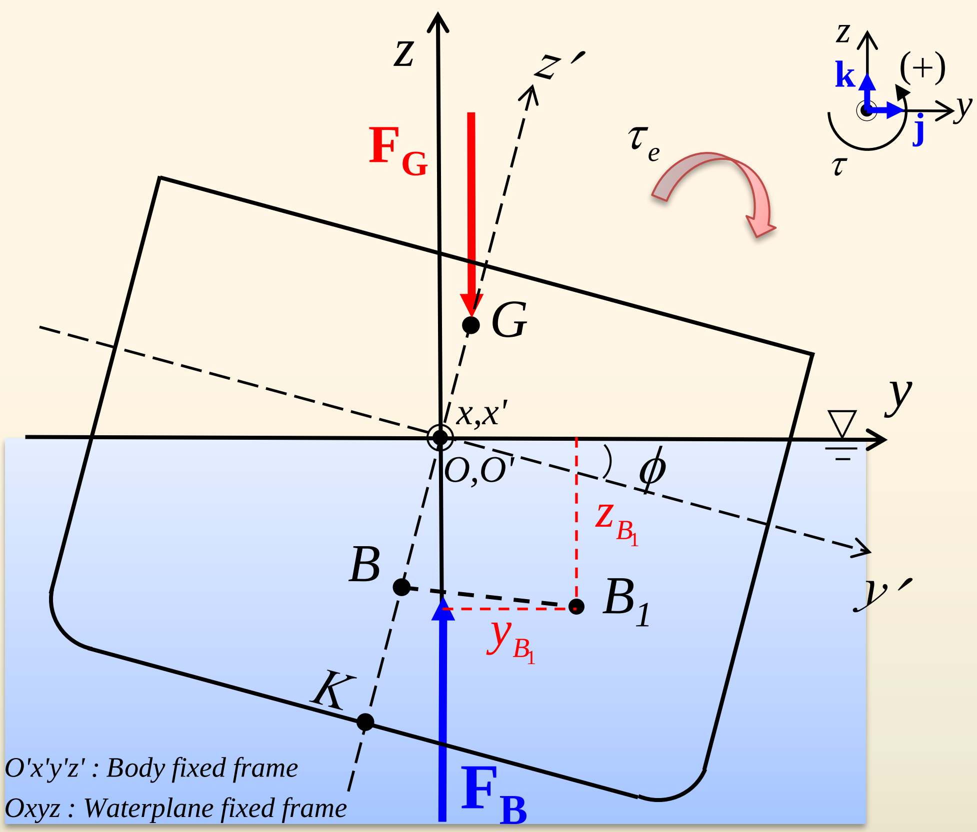
① External moment ($\boldsymbol{\tau}_e$) is applied on the ship in clockwise. A ship is heeled about origin O through an angle of ϕ

O'x'y'z' : Body fixed frame

Oxyz : Waterplane fixed frame

Question : How to calculate center of the buoyancy(B_1) with respect to waterplane fixed frame?

✓Method 1. calculate center of the buoyancy(B_1) with respect to waterplane fixed frame



① External moment ($\boldsymbol{\tau}_e$) is applied on the ship in clockwise. A ship is heeled about origin O through an angle of ϕ

② Center of buoyancy is changed from B to B_1.

How to calculate center of buoyancy B_1 with respect to waterplane fixed frame?

Method 1. Calculate Center of buoyancy B_1 with respect to waterplane fixed frame directly

✓ A, Mz, My (with respect to waterplane fixed frame)

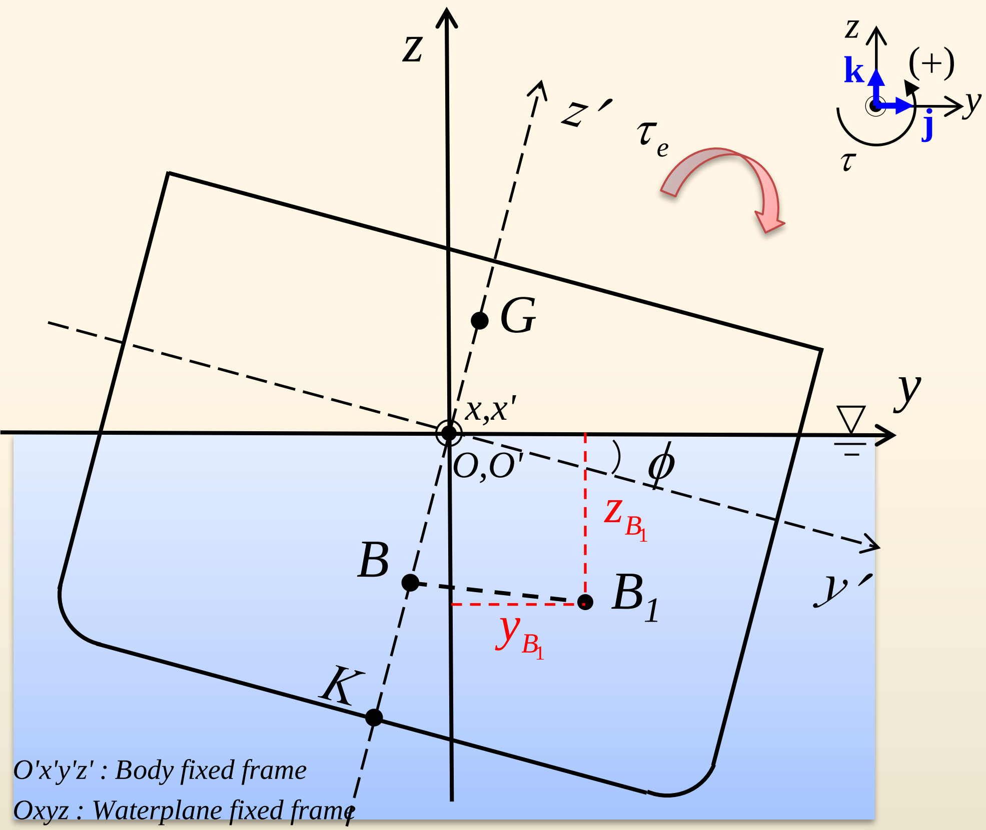
$$dA = dydz \qquad A = \int dA$$

$$M_{A,y} = \int y dA \qquad M_{A,z} = \int z dA$$

❖Integral value(area and 1st moment of area…) have to be calculated for every position when position of ship is changed.

Question : How to calculate center of the buoyancy(B_1) with respect to waterplane fixed frame?

✓Method 1. calculate center of the buoyancy(B_1) with respect to waterplane fixed frame



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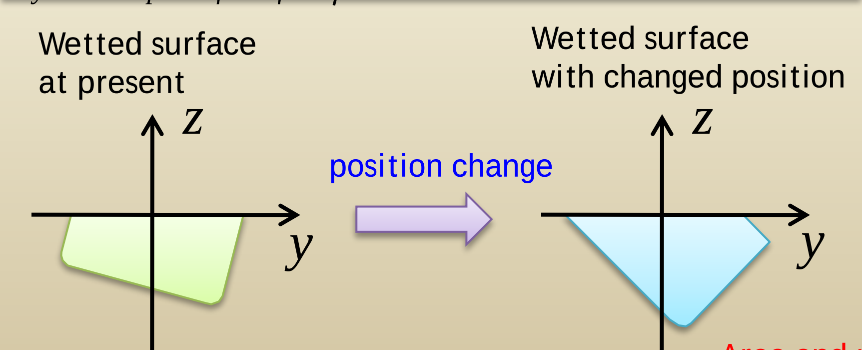


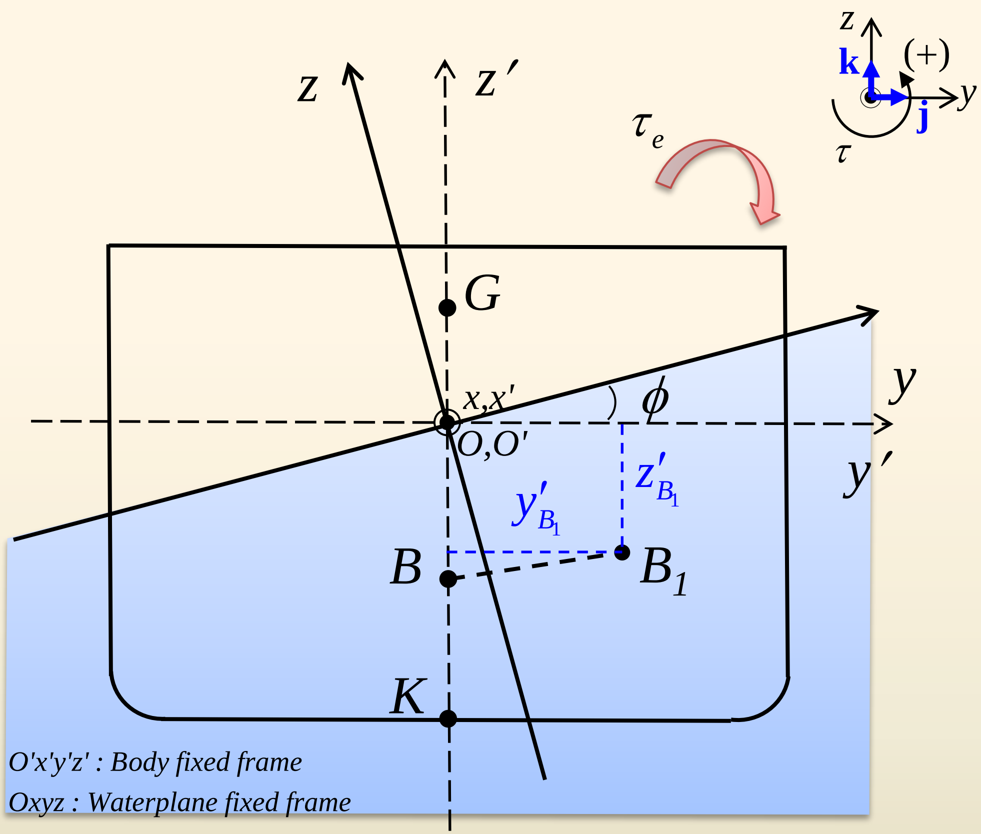
Area and moment of area have to be calculated again

✓Center of buoyancy with respect to waterplane fixed frame

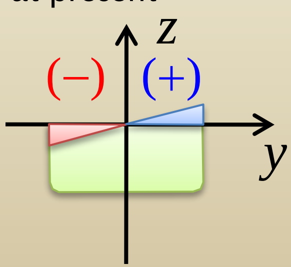
$$(y_{B_1}, z_{B_1}) = \left(\frac{M_{A,y}}{A}, \frac{M_{A,z}}{A} \right)$$

Question : How to calculate center of the buoyancy(B_1) with respect to waterplane fixed frame?

✓Method 2. calculate center of the buoyancy(B_1) with respect to <u>body fixed frame</u> then transform frame from body fixed frame to waterplane fixed frame



Wetted surface at present



How to calculate center of buoyancy B_1 with respect to waterplane fixed frame?

Method 2. Calculate center of buoyancy B_1 with respect to body fixed frame, then transform B_1 to waterplane fixed frame

✓ A, $M'_{A,y}$, $M'_{A,z}$ with respect to body fixed frame

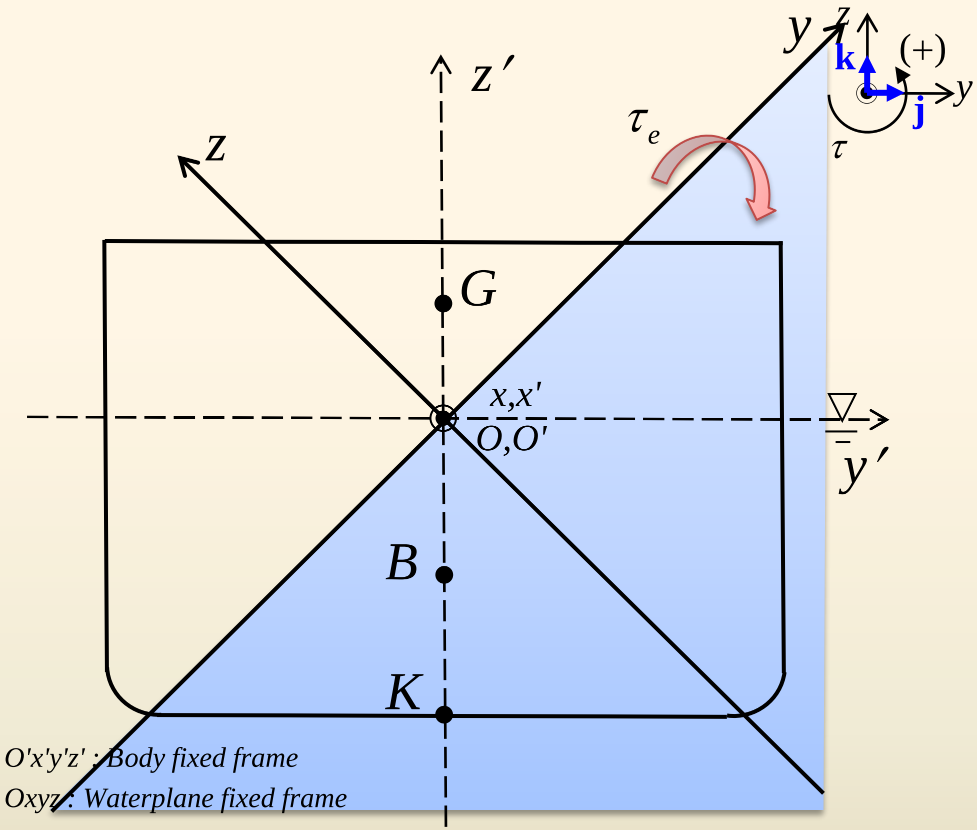
$$dA = dy'dz' \qquad A = \int dA$$

$$M_{A,y'} = \int y'dA \qquad M_{A,z'} = \int z'dA$$

❖Integral value <u>could be used as it is</u> except intersection region with waterplane area when position of ship is changed.

Question : How to calculate center of the buoyancy(B_1) with respect to waterplane fixed frame?

✓Method 2. calculate center of the buoyancy(B_1) with respect to body fixed frame then transform frame from body fixed frame to waterplane fixed frame



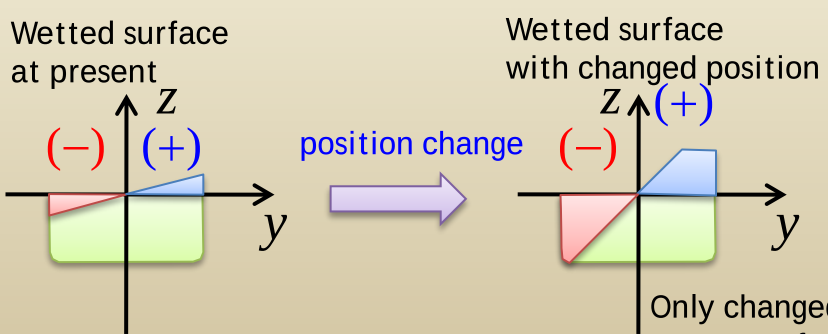
How to calculate center of buoyancy B_1 with respect to waterplane fixed frame?

Method 2. Calculate center of buoyancy B_1 with respect to body fixed frame, then transform B_1 to waterplane fixed frame

✓ $A,\ M'_{A,y},\ M'_{A,z}$ with respect to body fixed frame

$$dA = dy'dz' \qquad A = \int dA$$
$$M_{A,y'} = \int y'dA \qquad M_{A,z'} = \int z'dA$$

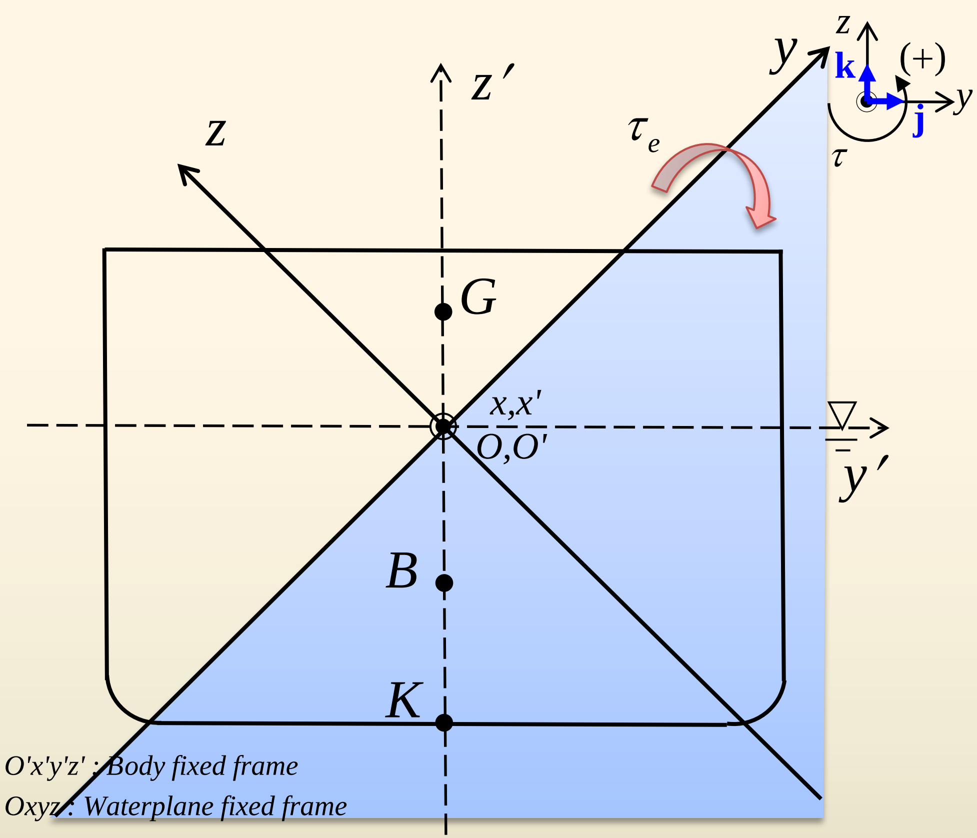
❖Integral value could be used as it is except intersection region with waterplane area when position of ship is changed.



Only changed area and moment of area have to calculated.

Question : How to calculate center of the buoyancy(B_1) with respect to waterplane fixed frame?

✓Method 2. calculate center of the buoyancy(B_1) with respect to body fixed frame then transform frame from body fixed frame to waterplane fixed frame



How to calculate center of buoyancy B_1 with respect to waterplane fixed frame?

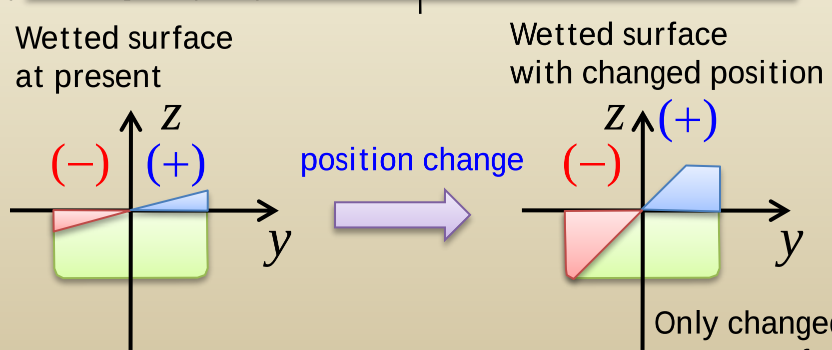
Method 2. Calculate center of buoyancy B_1 with respect to body fixed frame, then transform B_1 to waterplane fixed frame

✓ $A,\ M'_{A,y},\ M'_{A,z}$ with respect to body fixed frame

$$dA = dy'dz' \qquad A = \int dA$$

$$M_{A,y'} = \int y'dA \qquad M_{A,z'} = \int z'dA$$

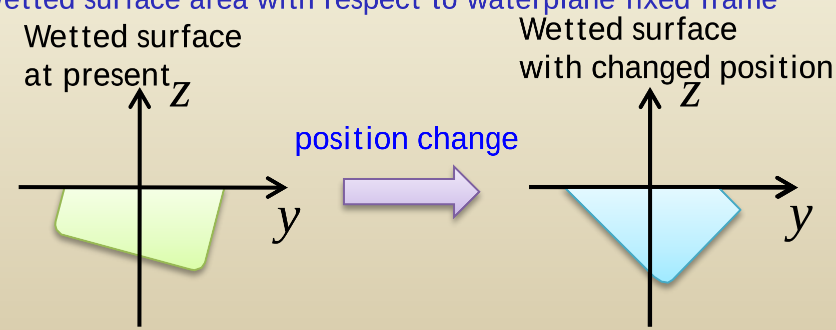
❖Integral value could be used as it is except intersection region with waterplane area when position of ship is changed.



Only changed area and moment of area have to calculated.

Is area invariant with respect to reference frame?

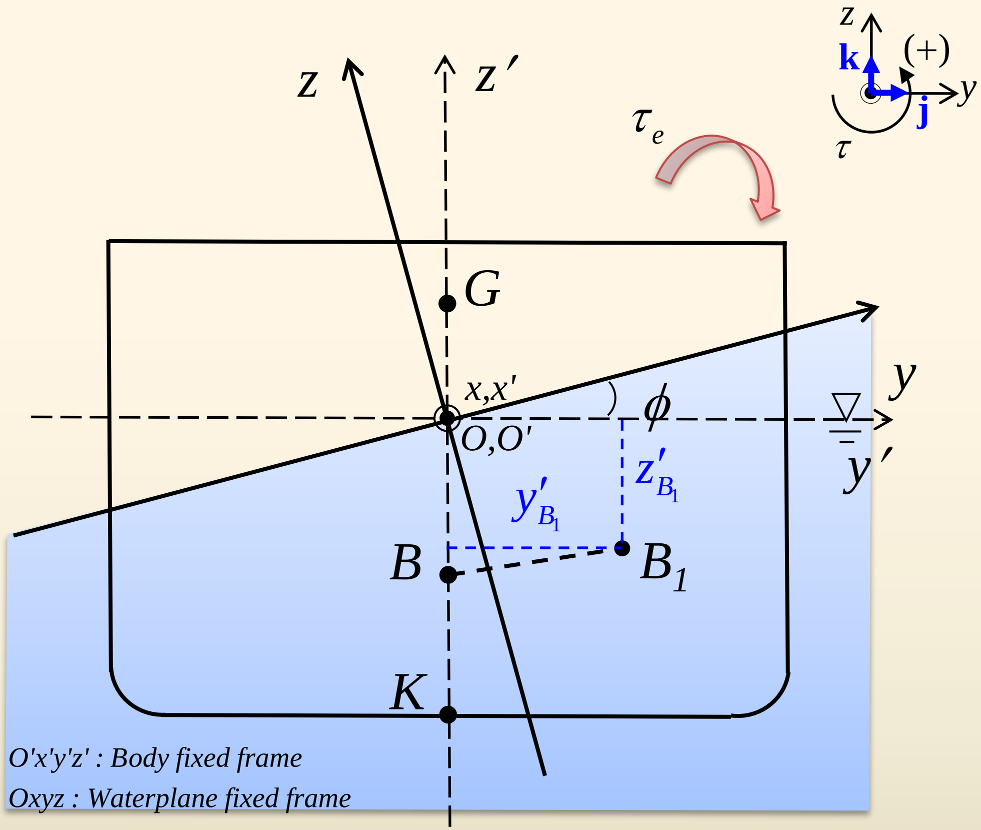
Wetted surface area with respect to waterplane fixed frame

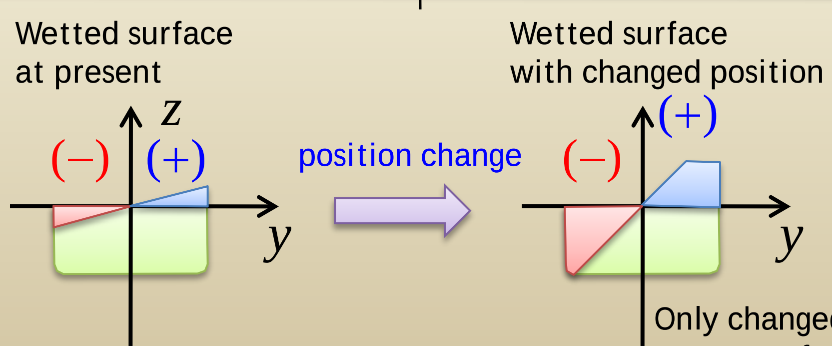


Area is invariant with respect to reference frame.

Question : How to calculate center of the buoyancy(B_1) with respect to waterplane fixed frame?

✓Method 2. calculate center of the buoyancy(B_1) with respect to body fixed frame then transform frame from body fixed frame to waterplane fixed frame





Only changed area and moment of area have to calculated.

How to calculate center of buoyancy B_1 with respect to waterplane fixed frame?

Method 2. Calculate center of buoyancy B_1 with respect to body fixed frame, then transform B_1 to waterplane fixed frame

✓ $A,\ M'_{A,y},\ M'_{A,z}$ with respect to body fixed frame

$$dA = dy'dz' \qquad A = \int dA$$

$$M_{A,y'} = \int y'dA \qquad M_{A,z'} = \int z'dA$$

❖Integral value could be used as it is except intersection region with waterplane area when position of ship is changed.

✓Center of buoyancy in body fixed frame

$$(y'_{B_1}, z'_{B_1}) = \left(\frac{M_{A,y'}}{A}, \frac{M_{A,z'}}{A} \right)$$

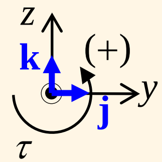
✓Center of buoyancy in waterplane fixed frame : Rotational Transformation

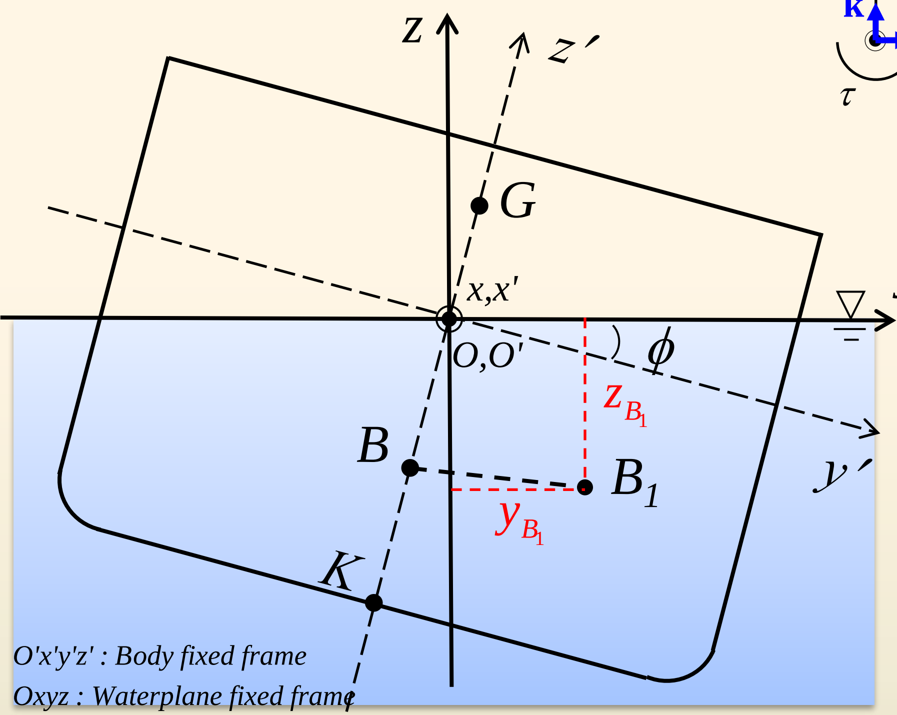
$$\begin{bmatrix} y_{B_1} \\ z_{B_1} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y'_{B_1} \\ z'_{B_1} \end{bmatrix}$$

Question : How to calculate center of the buoyancy(B_1) with respect to waterplane fixed frame?

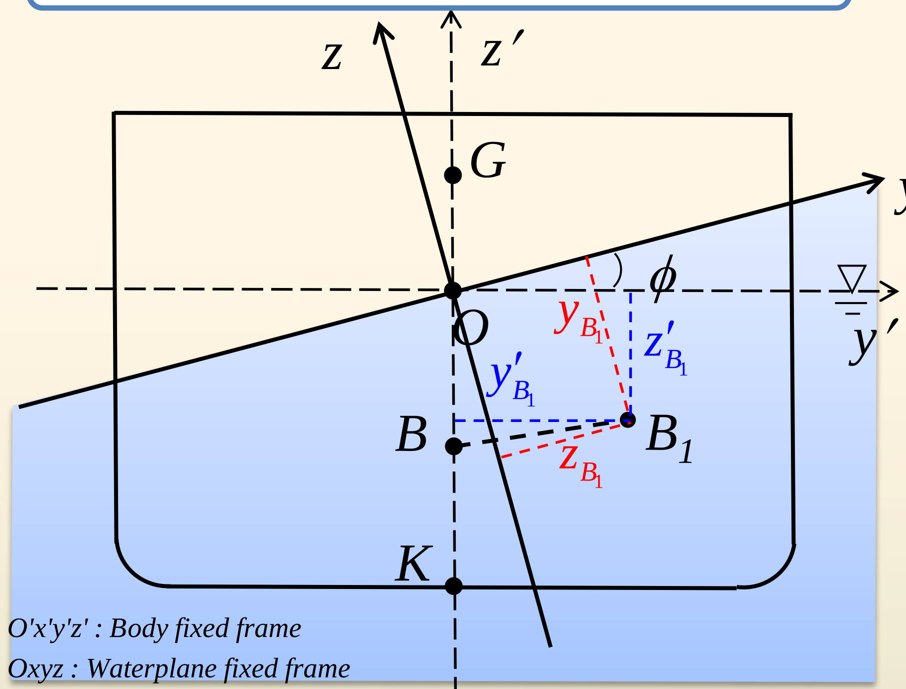
✓Comparison between Method 1 and Method 2

Method 1. Calculate Center of buoyancy B_1 with respect to waterplane fixed frame directly





Method 2. Calculate center of buoyancy B_1 with respect to body fixed frame, then transform B_1 to waterplane fixed frame



✓ A, M_z, M_y with respect to waterplane fixed frame

$$dA = dydz \qquad A = \int dA$$

$$M_{A,y} = \int ydA \qquad M_{A,z} = \int zdA$$

✓ Center of buoyancy with respect to waterplane fixed frame

$$(y_{B_1}, z_{B_1}) = \left(\frac{M_{A,y}}{A}, \frac{M_{A,z}}{A} \right)$$

Same

Convenient

✓ A, $M_{A,y'}$, $M_{A,z'}$ with respect to body fixed frame

$$dA' = dy'dz' \quad M_{A,y'} = \int y'dA \quad M_{A,z'} = \int z'dA$$

✓ Center of buoyancy with respect to waterplane fixed frame

$$(y'_{B_1}, z'_{B_1}) = \left(\frac{M_{A,y'}}{A}, \frac{M_{A,z'}}{A} \right)$$

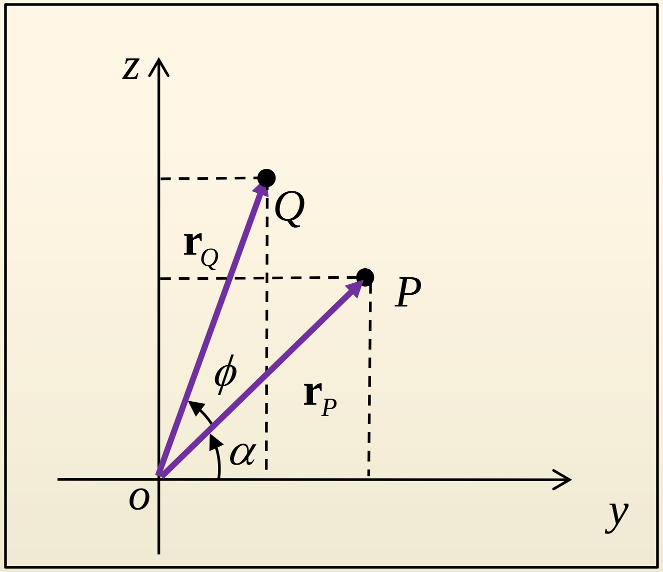
✓ Rotational transformation

$$\begin{bmatrix} y_{B_1} \\ z_{B_1} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y'_{B_1} \\ z'_{B_1} \end{bmatrix}$$

Rotational Transformation of Point and Frame

(A) Rotation of the point

Given: Coordinate of P with respect to oyz frame
Find : Coordinate of Q which is rotated coordinate of P about origin O in the yz plane through an angle of ϕ.

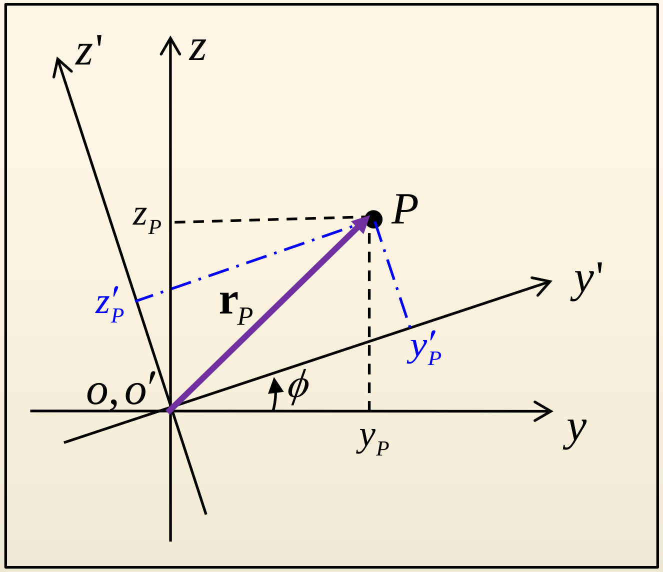


$$\begin{bmatrix} y_Q \\ z_Q \end{bmatrix} = \boxed{\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}} \begin{bmatrix} y_P \\ z_P \end{bmatrix} \quad ...(1)$$

Rotational transformation of point

(B) Rotation of the frame

Given: Coordinate of P with respect to oyz frame
Find: Coordinate of P with respect to oyz which is rotated frame about origin O′ from o′y′z′ through an angle of ϕ.



$$\begin{bmatrix} y'_P \\ z'_P \end{bmatrix} = \boxed{\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}} \begin{bmatrix} y_P \\ z_P \end{bmatrix} \quad ...(2-1)$$

Rotational transformation of frame

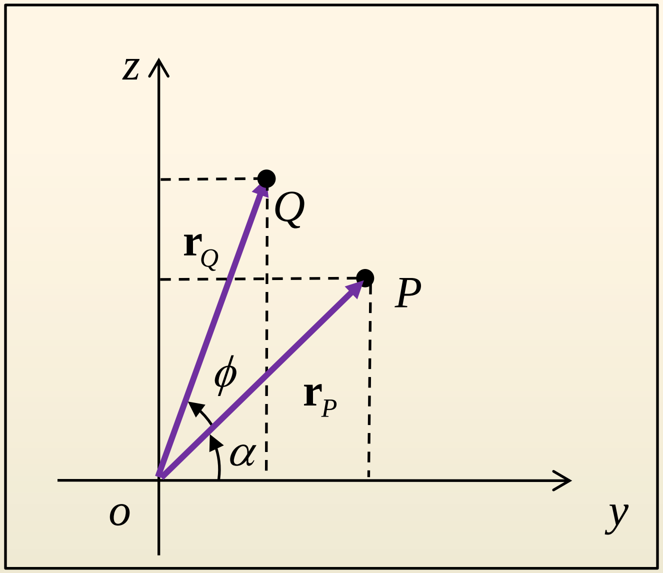
$$\begin{bmatrix} y_P \\ z_P \end{bmatrix} = \boxed{\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}} \begin{bmatrix} y'_P \\ z'_P \end{bmatrix} \quad ...(2-2)$$

Rotational transformation of frame

Rotational Transformation of Point and Frame

(A) Rotation of the point

Given: Coordinate of P with respect to oyz frame
Find : Coordinate of Q which is rotated coordinate of P about origin O in the yz plane through an angle of ϕ.

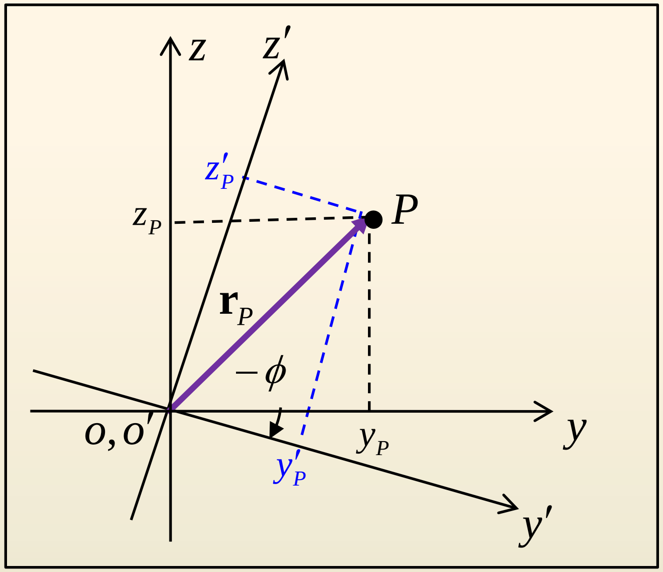


$$\begin{bmatrix} y_Q \\ z_Q \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y_P \\ z_P \end{bmatrix}$$

Rotational transformation of point

(B) Rotation of the frame

Given: Coordinate of P with respect to oyz frame
Find: Coordinate of P with respect to oyz which is rotated frame about origin O′ from o′y′z′ through an angle of $-\phi$.



$$\begin{bmatrix} y'_P \\ z'_P \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y_P \\ z_P \end{bmatrix} \quad \ldots(2-1)$$

If we substitude $-\phi$ into ϕ

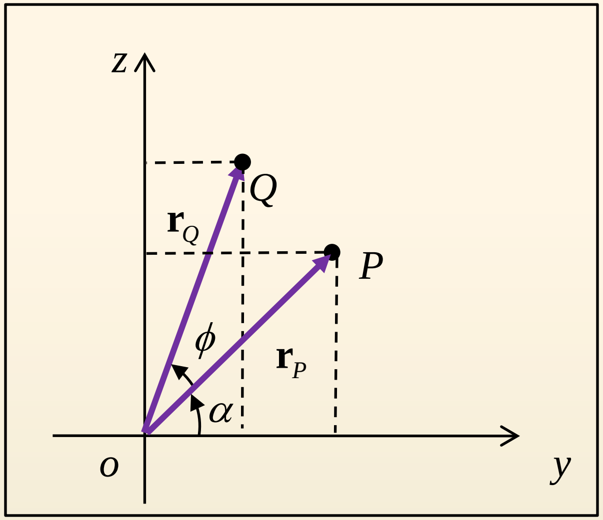
$$\begin{bmatrix} y'_P \\ z'_P \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y_P \\ z_P \end{bmatrix} \quad \ldots(2-1)'$$

Rotational transformation of point through an angle of ϕ **equals to** rotational transformation of frame through an angle of $-\phi$.

(Proof) Rotational Transformation of Point

Given: Coordinate of P with respect to oyz frame

Find : Coordinate of Q which is rotated coordinate of P about origin O in the yz plane through an angle of ϕ.



① Coordinate of point P, Q is expressed by an angle

$$y_P = |\mathbf{r}_P|\cos\alpha \qquad y_Q = |\mathbf{r}_Q|\cos(\alpha+\phi)$$

$$z_P = |\mathbf{r}_P|\sin\alpha \qquad z_Q = |\mathbf{r}_Q|\sin(\alpha+\phi)$$

② Summation formula of trigonometric function

$$\sin(\alpha+\phi) = \sin\alpha\cos\phi + \cos\alpha\sin\phi$$

$$\cos(\alpha+\phi) = \cos\alpha\cos\phi - \sin\alpha\sin\phi$$

③ Let coordinate of Q be expressed by difference formula of trigonometric function.

$$\begin{aligned} y_Q &= |\mathbf{r}_Q|\cos(\alpha+\phi) \\ &= |\mathbf{r}_Q|\cos\alpha\cos\phi - |\mathbf{r}_Q|\sin\alpha\sin\phi \\ &= (|\mathbf{r}_P|\cos\alpha)\cos\phi - (|\mathbf{r}_P|\sin\alpha)\sin\phi \quad ,(|\mathbf{r}_P| = |\mathbf{r}_Q|) \\ &= y_P\cos\phi - z_P\sin\phi \end{aligned}$$

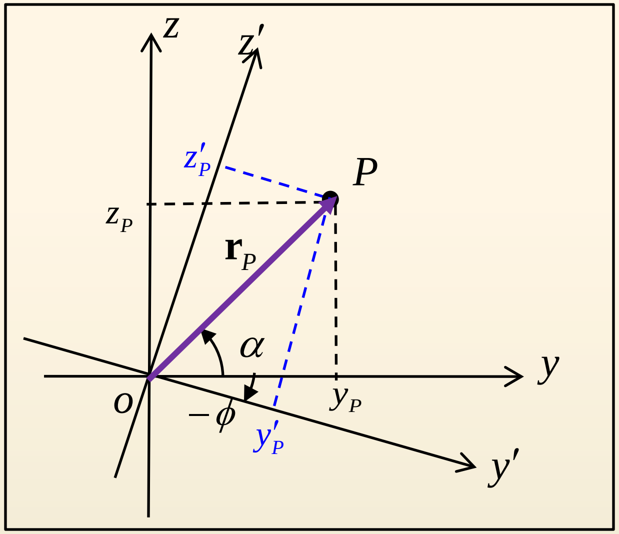
$$\begin{aligned} z_Q &= |\mathbf{r}_Q|\sin(\alpha+\phi) \\ &= |\mathbf{r}_Q|\sin\alpha\cos\phi + |\mathbf{r}_Q|\cos\alpha\sin\phi \\ &= (|\mathbf{r}_P|\sin\alpha)\cos\phi + (|\mathbf{r}_P|\cos\alpha)\sin\phi \quad ,(|\mathbf{r}_P| = |\mathbf{r}_Q|) \\ &= z_P\cos\phi + y_P\sin\phi \end{aligned}$$

④ In the matrix form

$$\boxed{\begin{bmatrix} y_Q \\ z_Q \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y_P \\ z_P \end{bmatrix}}$$

(Proof) Rotational Transformation of Frame

Given: Coordinate of P with respect to oyz frame
Find: Coordinate of P with respect to oyz which is rotated frame about origin O′ from o′y′z′ through an angle of -ϕ.



① Coordinate of point P is expressed by an angle

$$y_P = |\mathbf{r}_P|\cos\alpha \qquad y'_P = |\mathbf{r}_P|\cos(\alpha+\phi)$$

$$z_P = |\mathbf{r}_P|\sin\alpha \qquad z'_P = |\mathbf{r}_P|\sin(\alpha+\phi)$$

② Summation formula of trigonometric function

$$\sin(\alpha+\phi) = \sin\alpha\cos\phi + \cos\alpha\sin\phi$$

$$\cos(\alpha+\phi) = \cos\alpha\cos\phi - \sin\alpha\sin\phi$$

③ Let coordinate of P be expressed by difference formula of trigonometric function.

$$\begin{aligned} y'_P &= |\mathbf{r}_P|\cos(\alpha+\phi) \\ &= |\mathbf{r}_P|\cos\alpha\cos\phi - |\mathbf{r}_P|\sin\alpha\sin\phi \\ &= \left(|\mathbf{r}_P|\cos\alpha\right)\cos\phi - \left(|\mathbf{r}_P|\sin\alpha\right)\sin\phi \\ &= y_P\cos\phi - z_P\sin\phi \end{aligned}$$

$$\begin{aligned} z'_P &= |\mathbf{r}_P|\sin(\alpha+\phi) \\ &= |\mathbf{r}_P|\sin\alpha\cos\phi + |\mathbf{r}_P|\cos\alpha\sin\phi \\ &= \left(|\mathbf{r}_P|\sin\alpha\right)\cos\phi + \left(|\mathbf{r}_P|\cos\alpha\right)\sin\phi \\ &= z_P\cos\phi + y_P\sin\phi \end{aligned}$$

④ In the matrix form

$$\boxed{\begin{bmatrix} y'_P \\ z'_P \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y_P \\ z_P \end{bmatrix}}$$

Sec.1 Calculation of Center of Buoyancy

- Rotational Transformation of Point and Frame
- Example) Calculation of Center of Buoyancy of Ship with Constant Section

  Method ① Direct calculating center of buoyancy in waterplane fixed frame

  Method ② Transformation center of buoyancy from body fixed frame to waterplane fixed frame

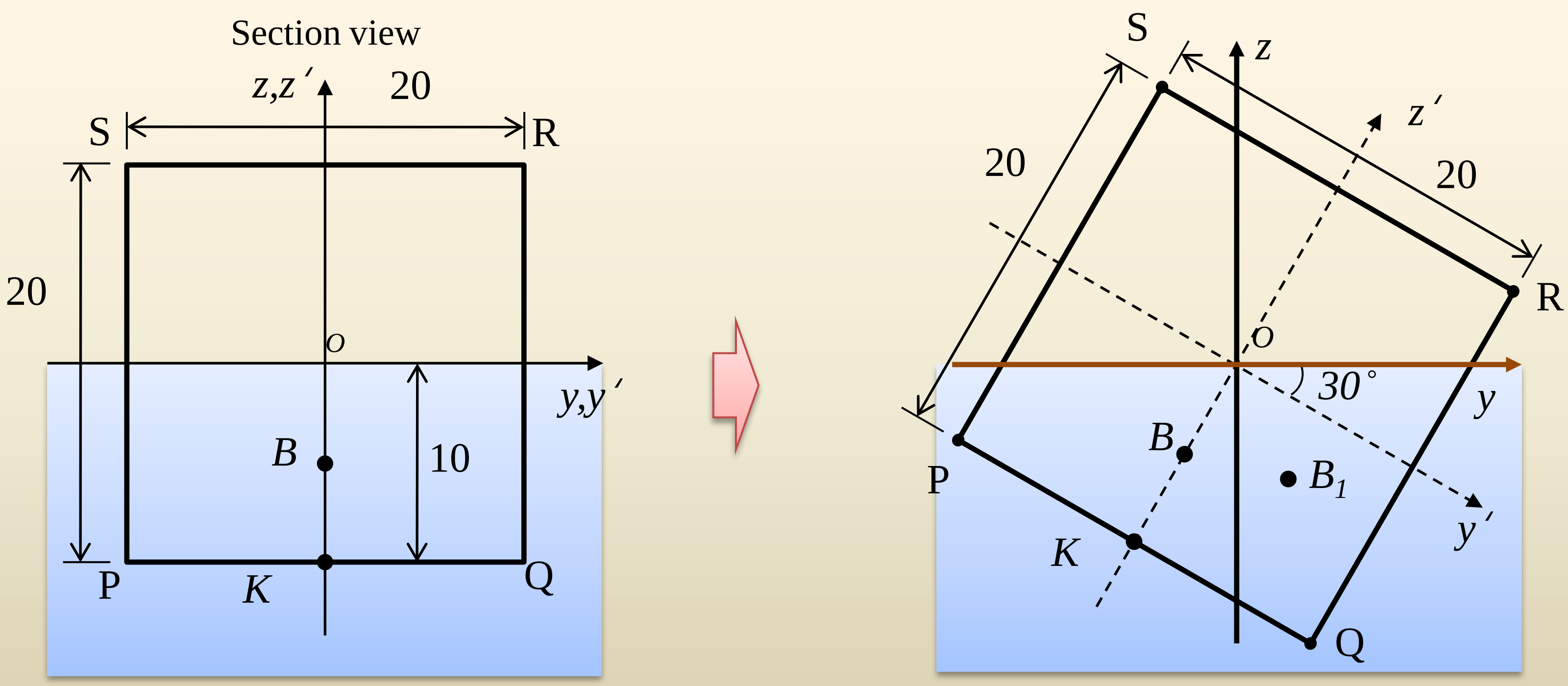
- Calculation of Center of Buoyancy of Ship with Various Station Shape

Example) Calculation of Center of Buoyancy of Ship with Constant Section : Method ① Direct calculating center of buoyancy in waterplane fixed frame

Question) A ship with a breadth 20m, depth 20m, draft 10m is heel about origin O through an angle of -30°. Calculate center of buoyancy with respect to waterplane fixed frame

- Given : Breadth (B):20, Depth (D):20, Draft(T) 10, Angle of Heel(ϕ) : -30˚
- Find : Center of buoyancy (y_B, z_B) in Waterplane fixed frame

G: Center of mass K:Keel
B: Center of buoyancy B_1 : Changed center of buoyancy



Example) Calculation of Center of Buoyancy of Ship with Constant Section : Method ① Direct calculating center of buoyancy in waterplane fixed frame

Question) A ship with a breadth 20m, depth 20m, draft 10m is heel about origin O through an angle of -30°. Calculate center of buoyancy with respect to waterplane fixed frame

- Given : Breadth (B):20, Depth (D):20, Draft(T) 10, Angle of Heel(ϕ) : -30˚
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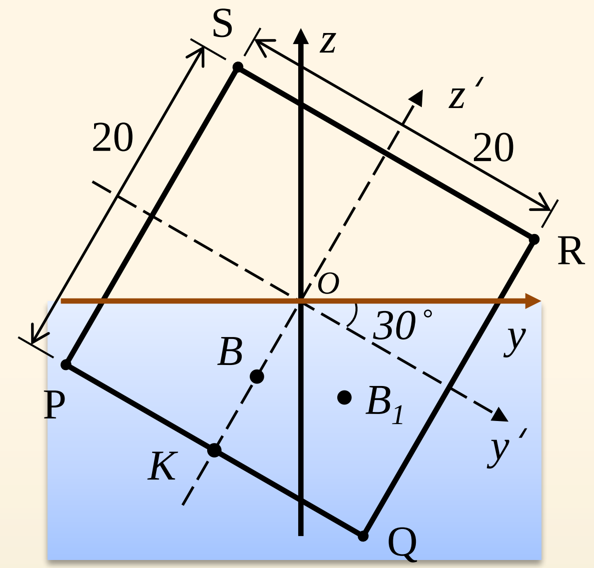

Sol.) P, Q, R,S with respect to waterplane fixed frame

$$(y_{B_1}, z_{B_1}) = \left(\frac{M_{A,y}}{A}, \frac{M_{A,z}}{A}\right)$$

$P(x_P, y_P)$ is calculated by rotational transformation from $P'(x'_P, y'_P)$

$$\begin{pmatrix} x_P \\ y_P \end{pmatrix} = \begin{pmatrix} \cos(30) & \sin(30) \\ -\sin(30) & \cos(30) \end{pmatrix} \begin{pmatrix} x'_P \\ y'_P \end{pmatrix}$$

$$= \begin{pmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{pmatrix} \begin{pmatrix} -10 \\ -10 \end{pmatrix} = \begin{pmatrix} -13.66 \\ -3.66 \end{pmatrix}$$

$Q(x_Q, y_Q)$, $R(x_R, y_R)$, $S(x_S, y_S)$ are calculated in the same way.

$$\begin{pmatrix} x_Q \\ y_Q \end{pmatrix} = \begin{pmatrix} \cos(30) & \sin(30) \\ -\sin(30) & \cos(30) \end{pmatrix} \begin{pmatrix} 10 \\ -10 \end{pmatrix} = \begin{pmatrix} 3.66 \\ -13.66 \end{pmatrix}$$

$$\begin{pmatrix} x_R \\ y_R \end{pmatrix} = \begin{pmatrix} \cos(30) & \sin(30) \\ -\sin(30) & \cos(30) \end{pmatrix} \begin{pmatrix} 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 13.66 \\ 3.66 \end{pmatrix}$$

$$\begin{pmatrix} x_S \\ y_S \end{pmatrix} = \begin{pmatrix} \cos(30) & \sin(30) \\ -\sin(30) & \cos(30) \end{pmatrix} \begin{pmatrix} -10 \\ 10 \end{pmatrix} = \begin{pmatrix} -3.66 \\ 13.66 \end{pmatrix}$$

Example) Calculation of Center of Buoyancy of Ship with Constant Section : Method ① Direct calculating center of buoyancy in waterplane fixed frame

Question) A ship with a breadth 20m, depth 20m, draft 10m is heel about origin O through an angle of -30°. Calculate center of buoyancy with respect to waterplane fixed frame

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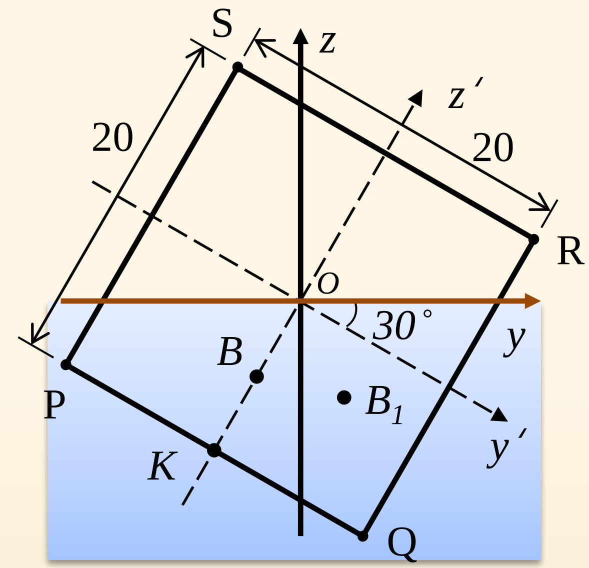
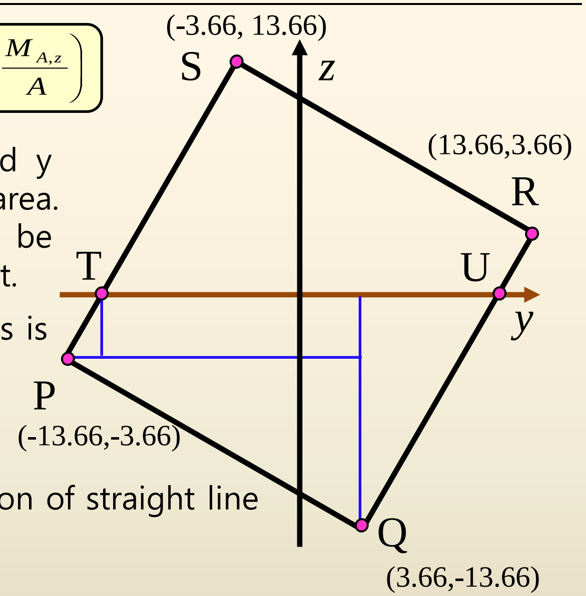


Sol.) Area $(y_{B_1}, z_{B_1}) = \left(\frac{M_{A,y}}{A}, \frac{M_{A,z}}{A} \right)$

Intersection point between straight line and y axis have to be calculated in order to know area. Equation of straight line P_1S_1 have to be calculated in order to know intersection point.

Equation of straight line through two points is as follows.

$$z - z_1 = \frac{z_2 - z_1}{y_2 - y_1}(y - y_1)$$



Substituting two points of P_1, S_1 into equation of straight line

$$z - (-3.66) = \frac{13.66 - (-3.66)}{(-3.66) - (-13.66)}(y - (-13.66))$$

$$\therefore z = 1.732y + 20 \xrightarrow[\text{point T}]{\text{Intersection}} T(-11.55, 0)$$

Equation of straight line of Q_1R_1 is obtained in the same way

$$\therefore z = 1.732y - 20 \xrightarrow[\text{point U}]{\text{Intersection}} U(11.55, 0)$$

Example) Calculation of Center of Buoyancy of Ship with Constant Section : Method ① Direct calculating center of buoyancy in waterplane fixed frame

Question) A ship with a breadth 20m, depth 20m, draft 10m is heel about origin O through an angle of -30°. Calculate center of buoyancy with respect to waterplane fixed frame

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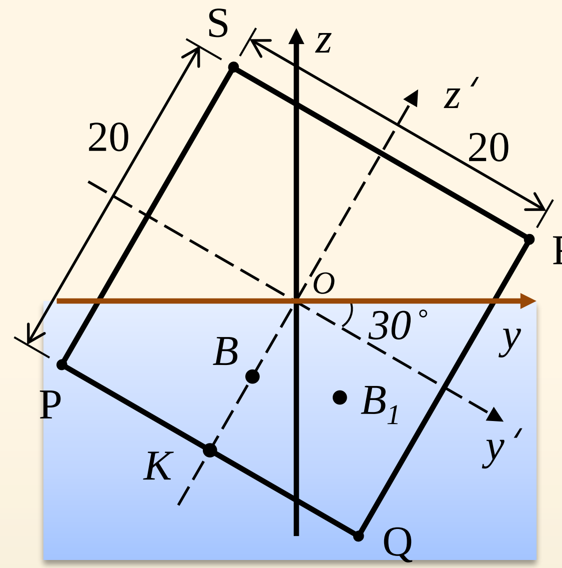


Sol.)

Area

$$(y_{B_1}, z_{B_1}) = \left(\frac{M_{A,y}}{\boxed{A}}, \frac{M_{A,z}}{\boxed{A}} \right)$$

Divide area into 4 part, A_1, A_2, A_3, A_4.
Then calculate area of A_1.

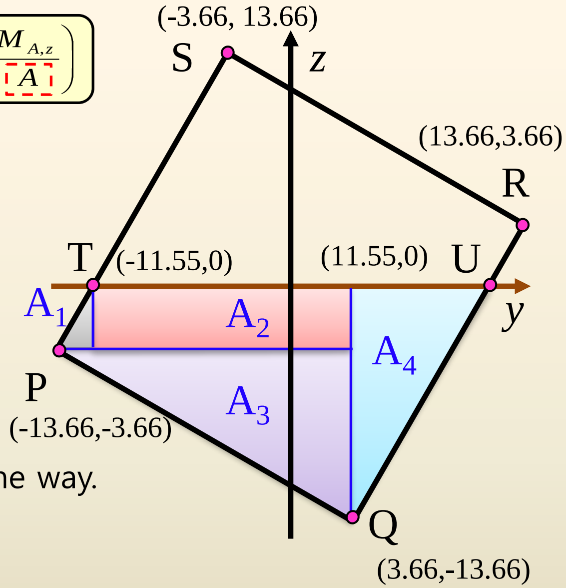
$$Area_{A_1} = \frac{1}{2}\left|-11.55-(-13.66)\right| \times \left|-3.66\right|$$
$$= 3.867$$

Area of A_2, A_3, A_4 can be calculated in the same way.

$$Area_{A_2} = 55.66,$$

$$Area_{A_3} = 86.6$$

$$Area_{A_4} = 53.87$$



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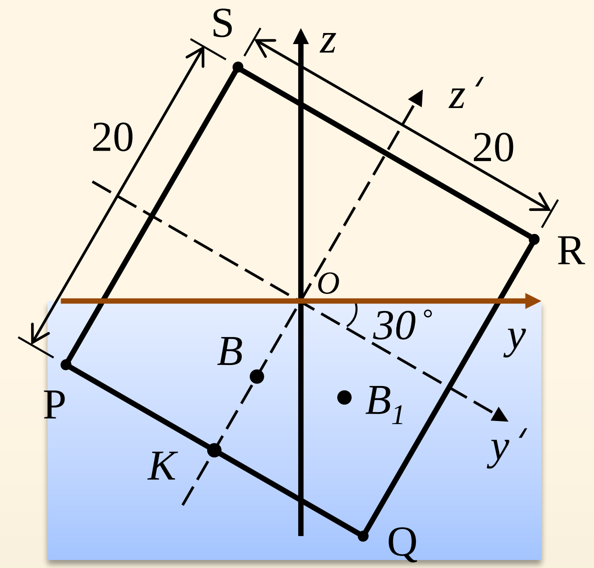
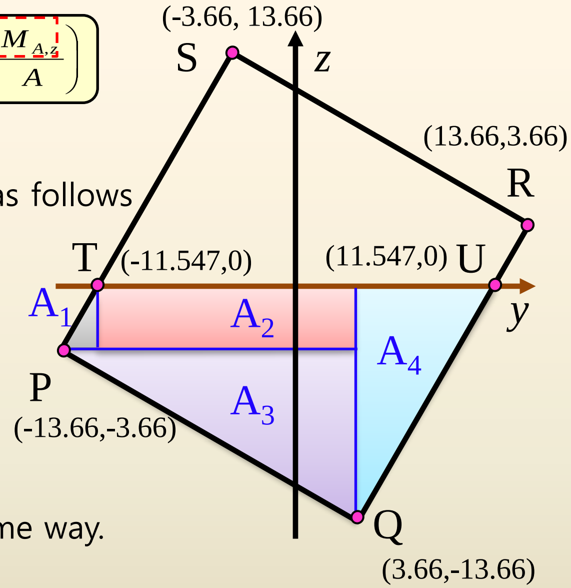


Sol.) Centroid

$$(y_{B_1}, z_{B_1}) = \left(\frac{M_{A,y}}{A}, \frac{M_{A,z}}{A}\right)$$



Centroid of $A_1(x_{c\_A_1}, y_{c\_A_1})$ can be calculated as follows

$$x_{c\_A_1} = -11.54 + \left(-\frac{2}{3}(-11.54 - (-13.66))\right)$$

$$= -12.25$$

$$y_{c\_A_1} = -3.66 \cdot \left(\frac{2}{3}\right) = -2.44$$

Centroids of A_2, A_3, A_4 are calculated in the same way.

$$(x_{c\_A_2}, y_{c\_A_2}) = (-3.96, -1.83)$$

$$(x_{c\_A_3}, y_{c\_A_3}) = (-2.11, -6.99)$$

$$(x_{c\_A_4}, y_{c\_A_4}) = (6.29, -4.55)$$

Example) Calculation of Center of Buoyancy of Ship with Constant Section : Method ① Direct calculating center of buoyancy in waterplane fixed frame

Question) A ship with a breadth 20m, depth 20m, draft 10m is heel about origin O through an angle of -30°. Calculate center of buoyancy with respect to waterplane fixed frame

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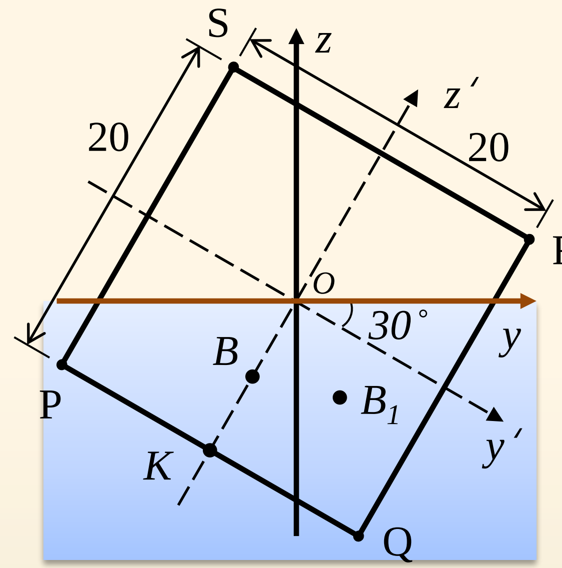
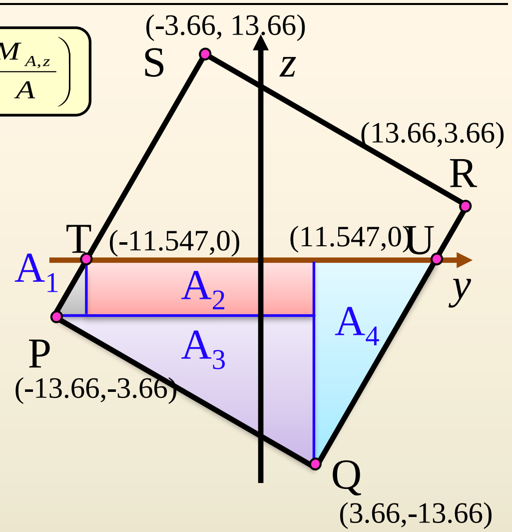


Sol.) 1st moment of area

$$\boxed{(y_{B_1}, z_{B_1})} = \left(\frac{M_{A,y}}{A}, \frac{M_{A,z}}{A} \right)$$



1st moments of area are calculated with areas and centroids which are calculated in previous.

	Area	y_c	z_c	$M_{A,y}$ Area x y_c	$M_{A,z}$ Area x z_c
A_1	3.87	-12.25	-2.44	-47.38	-9.44
A_2	55.66	-3.96	-1.83	-220.24	-101.85
A_3	86.60	-2.11	-6.99	-183.01	-605.62
A_4	53.87	6.29	-4.55	338.78	-245.28
Sum	200.00			-111.85	-962.19

Centoid of total area is calculated as follows.

$$(y_{B_1}, z_{B_1}) = \left(\frac{M_{A,y}}{A}, \frac{M_{A,z}}{A} \right) = \left(\frac{-111.85}{200}, \frac{-962.}{200} \right) = \boxed{(-0.56, -4.81)}$$

Example) Calculation of Center of Buoyancy of Ship with Constant Section
: Method ② Transformation of center of buoyancy from body fixed frame to waterplane fixed frame

Question) A ship with a breadth 20m, depth 20m, draft 10m is heel about origin O through an angle of -30°. Calculate center of buoyancy with respect to waterplane fixed frame

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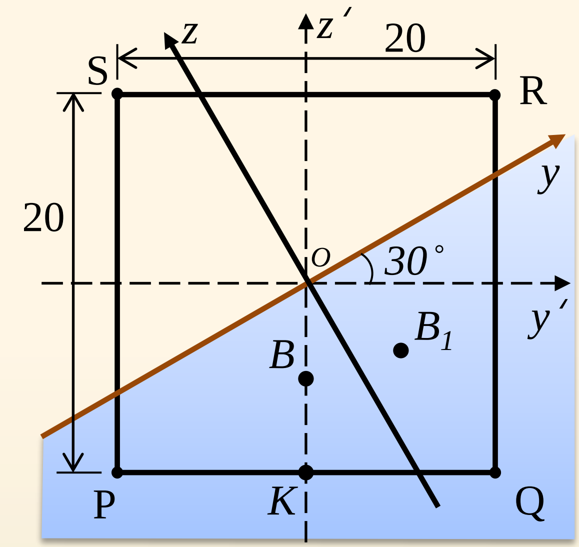
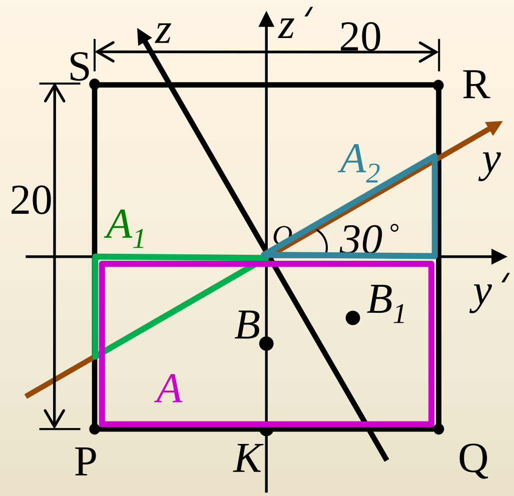


Sol.) Area $(y'_{B_1}, z'_{B_1}) = \left(\frac{M'_{A,y'}}{A}, \frac{M'_{A,z'}}{A} \right)$



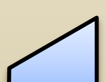
- Total area before heel

$$A = 20 \times 10 = 200$$

- Changed areas after heel

$$A_1,\ A_2 = \frac{1}{2} 10 \times 10 \times \tan 30 = 28.87$$

- Total area after heel



$$A' = A - A_1 + A_2$$
$$= 200 - 28.87 + 28.87 = 200$$

Area is invariant with respect to frame.

Example) Calculation of Center of Buoyancy of Ship with Constant Section
: Method ② Transformation of center of buoyancy from body fixed frame to waterplane fixed frame

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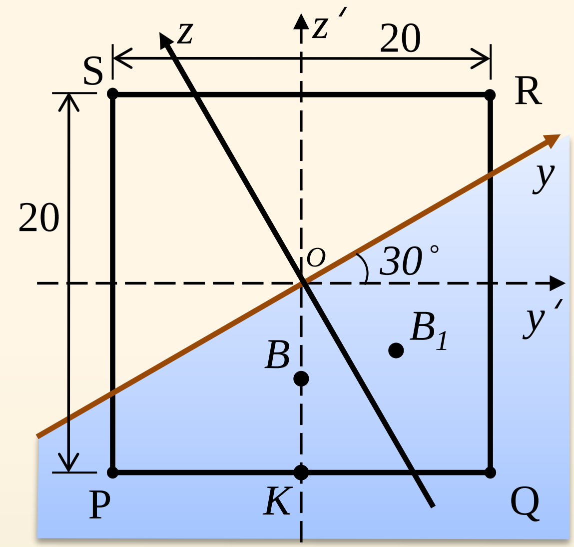


Sol.) Centroid

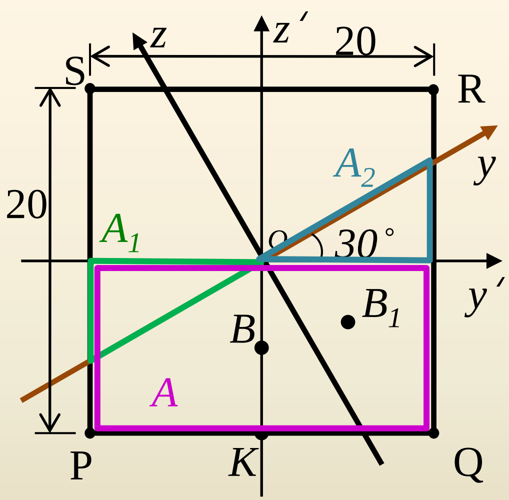
- Centroid of total area before heel

$$(y_{c\_A}, z_{c\_A}) = (0, -5)$$

- Centroids of changed area after heel

$$(y_{c\_A_1}, z_{c\_A_1}) = (-\frac{2}{3}\times 10,\ -\frac{1}{3}\times 10\times \tan 30^\circ)$$
$$= (-6.67, -1.92)$$

$$(y_{c\_A_2}, z_{c\_A_2}) = (\frac{2}{3}\times 10,\ \frac{1}{3}\times 10\times \tan 30^\circ)$$
$$= (6.67, 1.92)$$



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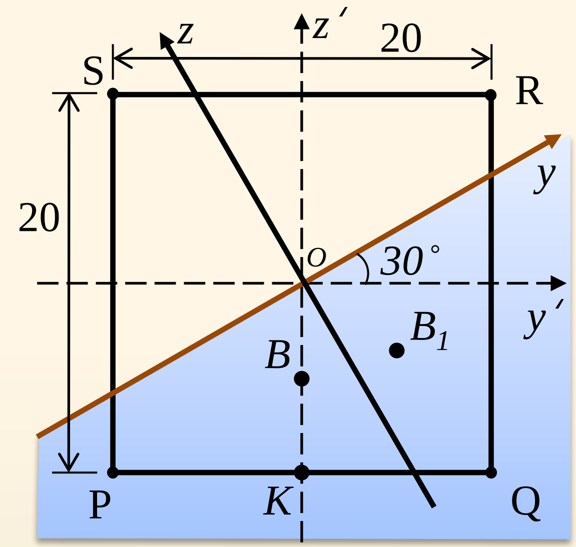
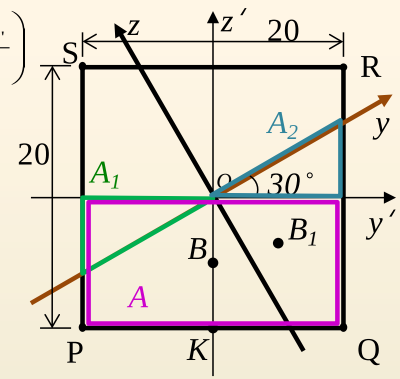


Sol.)

1st moment of area

$$(y'_{B_1}, z'_{B_1}) = \left(\frac{M'_{A,y'}}{A}, \frac{M'_{A,z'}}{A}\right)$$

1st moments of area are calculated with areas and centroids which are calculated in previous.



	Area	y_c	z_c	$M_{A,y}'$ Area x y	$M_{A,z}'$ Area x z
A	200.00	0.00	-5.00	0.00	-1000.00
A1	-28.87	-6.67	-1.92	192.45	55.56
A2	28.87	6.67	1.92	192.45	55.56
Sum	200.00			384.90	-888.89

Centroid of total area after heel with respect to body fixed frame is as follows

$$(y'_{B_1}, z'_{B_1}) = \left(\frac{M'_{A,y}}{A}, \frac{M'_{A,z}}{A}\right) = \left(\frac{384.90}{200}, \frac{-888.89}{200}\right) = \boxed{(1.92, -4.44)}$$

Rotational transformation

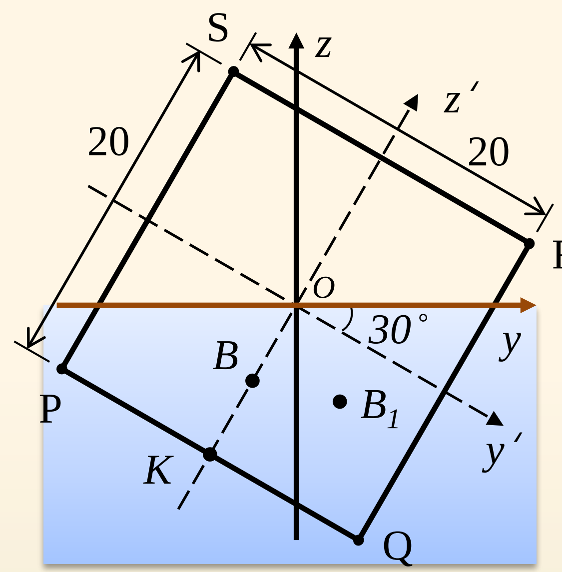
Transform coordinate of centroid of total area with respect to body fixed frame to centroid with respect to waterplane fixed frame by rotational transformation

$$\begin{pmatrix} y_{B_1} \\ z_{B_1} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} y'_{B_1} \\ z'_{B_1} \end{pmatrix} = \begin{pmatrix} \cos(30^\circ) & \sin(30^\circ) \\ -\sin(30^\circ) & \cos(30^\circ) \end{pmatrix} \begin{pmatrix} 1.92 \\ -4.44 \end{pmatrix} = \boxed{\begin{pmatrix} -0.56 \\ -4.81 \end{pmatrix}}$$

Same result

Example) Calculation of Center of Buoyancy of Ship with Constant Section

: Method ② Transformation of center of buoyancy from body fixed frame to waterplane fixed frame

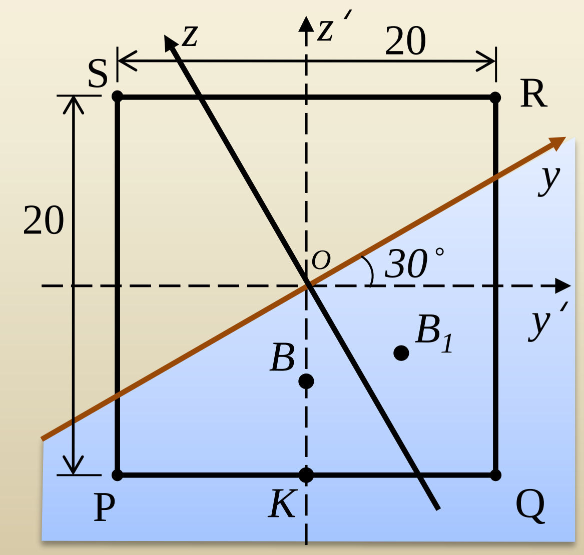


Method ① Direct calculating center of buoyancy in waterplane fixed frame

Calculation Result

$$\begin{pmatrix} y_{B_1} \\ z_{B_1} \end{pmatrix} = \begin{pmatrix} -0.56 \\ -4.81 \end{pmatrix}$$

Same



Method ② Transformation of center of buoyancy from body fixed frame to waterplane fixed frame

Calculation Result

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Sec.1 Calculation of Center of Buoyancy

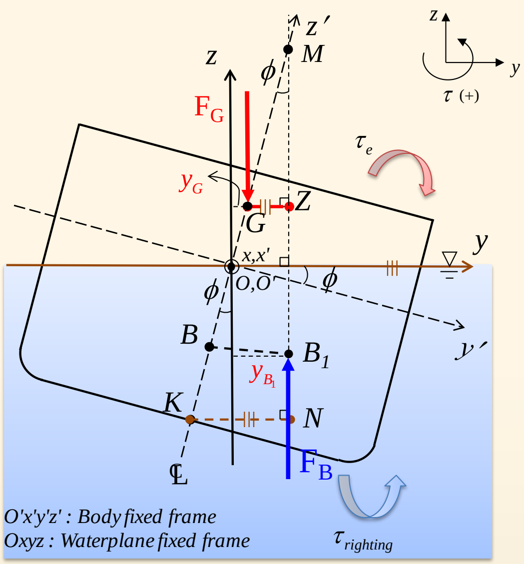
- Rotational Transformation of Point and Frame
- Example) Calculation of Center of Buoyancy of Ship with Constant Section

  Method ① Direct calculating center of buoyancy in waterplane fixed frame

  Method ② Transformation center of buoyancy from body fixed frame to waterplane fixed frame

- **Calculation of Center of Buoyancy of Ship with Various Station Shape**

Calculation of Center of Buoyancy of Ship with Various Station Shape - Introduction



- **Righting Moment :** Moment to return the ship to the upright floating position (Restoring moment, Moment of statical stability))

- **Transverse Restoring moment**

$$\tau_{righting} = (-y_G + y_{B_1}) \cdot F_B \mathbf{i} = \underbrace{GZ}_{\text{Righting arm}} \cdot F_B \mathbf{i}$$

- **Righting Arm** (GZ)

① From direct calculation

$$GZ = -y_G + y_{B_1}$$

We should know y_G, y_{B1} in waterplane fixed frame

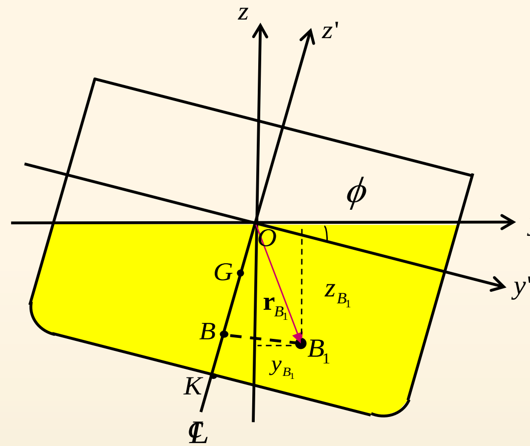
② From geometrical figure with assumption that M does not change within small angle of heel (about 10°)

$$GZ = GM \cdot \sin\phi$$

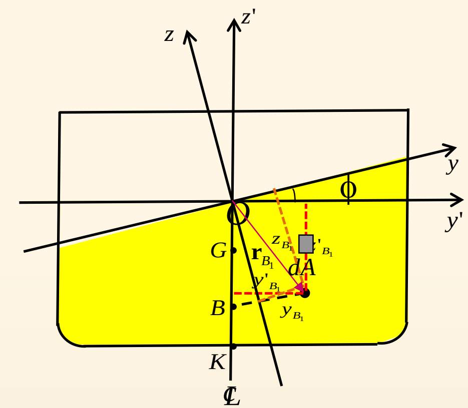
GM is related to below equation by geometrical figure

$$GM = KB + BM - KG$$

How to calculate y_{B_1}, z_{B_1} in waterplane fixed frame ?

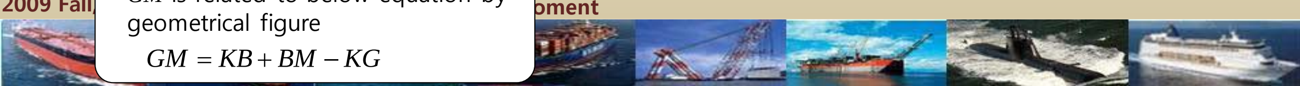


Method ① Direct calculating center of buoyancy in waterplane fixed frame



Method ② Transformation center of buoyancy from body fixed frame to waterplane fixed frame

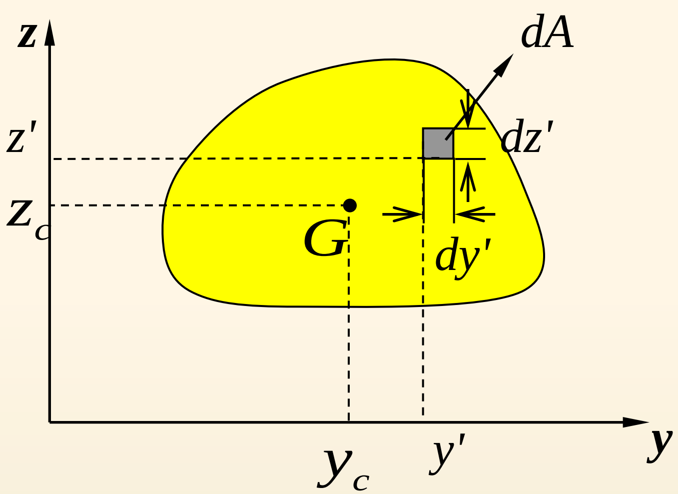
How to calculate center of buoyancy of ship with various sections ?



VCB(Vertical Center of Buoyancy)

Step① Area, 1st Moment of Area

1) James M. Gere, "Mechanics of materials", THOMSON, pp.828-850, 6th Edition



(y_c', z_c') : center of A

✓ **Differential element of area,** dA

$$dA = dy\,dz$$

✓ **Area**, A

$$A = \int dA = \iint dy'dz'$$

$$= \sum_{i=1}^{n} \Delta A_i$$

(ΔAi : Area of i-th element)

✓ **1st moment of area with respect to** z **axis,** $M_{A,x}$

$$M_{A,y} = \int y\,dA = \iint y\,dy\,dz$$

$$= y_c \cdot A$$

✓ **1st moment of area with respect to y axis,** $M_{A,y}$

$$M_{A,z} = \int z\,dA = \iint z\,dy\,dz$$

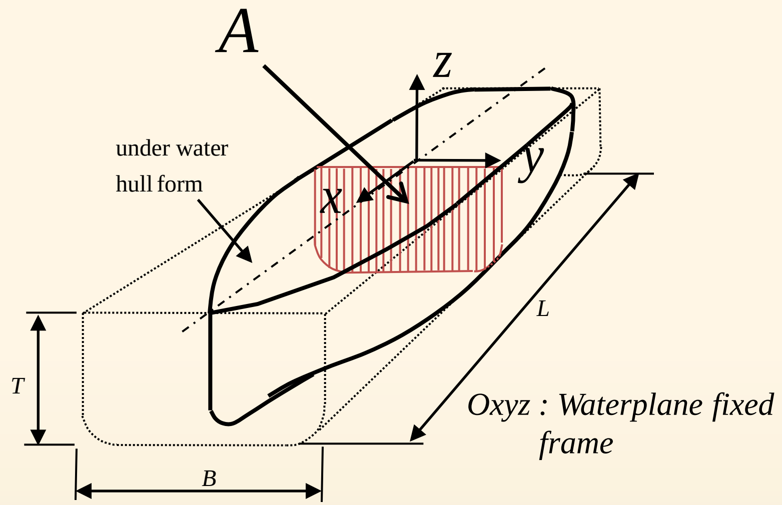
$$= z_c \cdot A$$

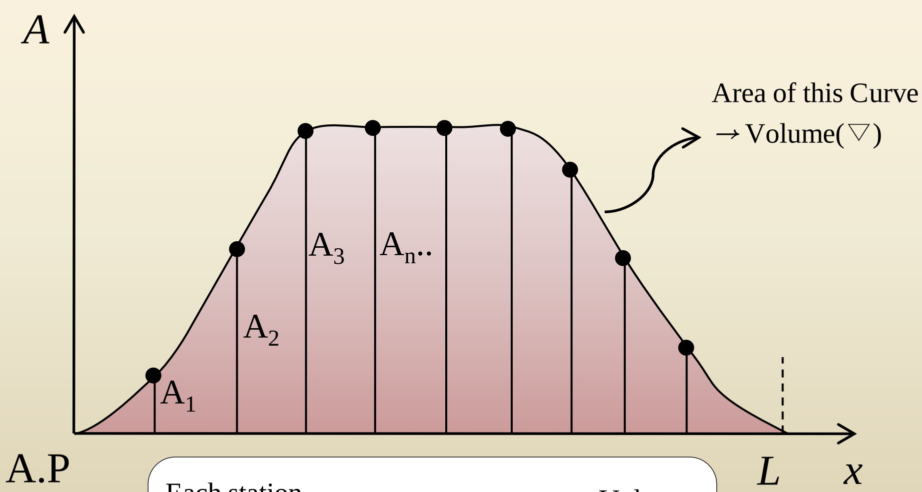
✓ **Centroid** G

$$\mathbf{G} = \left(\frac{M_{A,y}}{A}, \frac{M_{A,z}}{A} \right) = (y_c, z_c)$$

VCB(Vertical Center of Buoyancy)

Step② Sectional Area (A_M), Displacement Volume





Each station area A → Integral in longitudinal direction → Volume ∇

✓ Sectional Area

$$A = \int dA = \iint dy'\,dz'$$

✓ Displacement volume

$$\nabla = \int dV = \iiint dx'\,dy'\,dz'$$

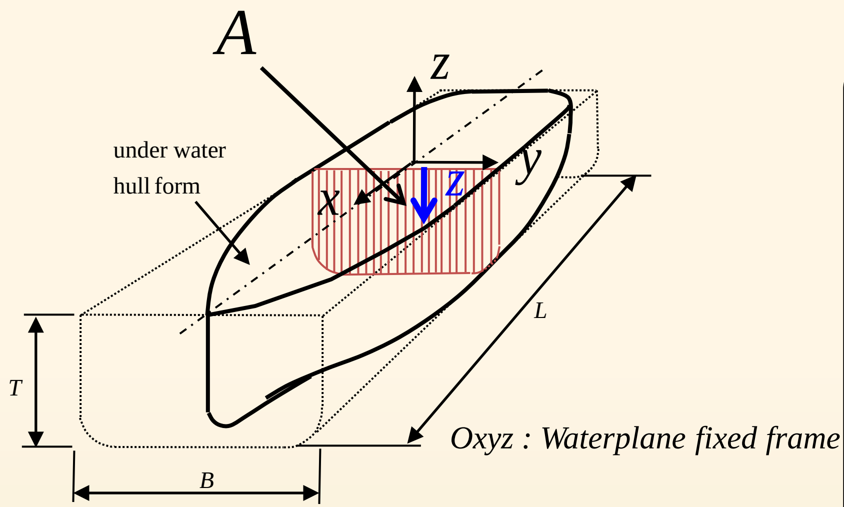
$$= \int \Big(\underbrace{\iint dy'\,dz'}_{\Rightarrow A(x)} \Big) dx'$$

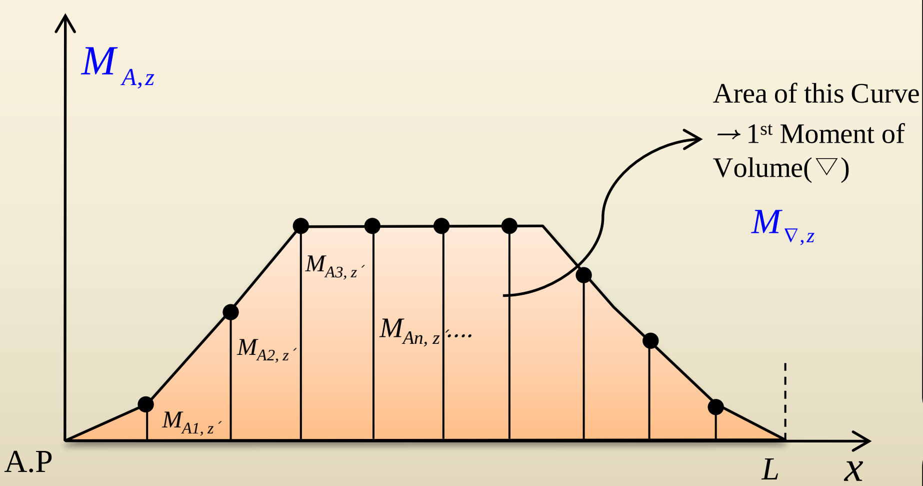
$$\therefore \nabla = \int A(x)dx'$$

After calculation of each station area, displacement volume can be calculated by integral of section area over the length of ship

VCB(Vertical Center of Buoyancy)

Step③ Vertical Moment of Volume, Vertical Center of Buoyancy(VCB)





vertical moment of station area about the y axis $\xrightarrow[\text{in longitudinal direction}]{\text{Integral}}$ Vertical moment of volume

$M_{A,z'}$ $\qquad$ $M_{\nabla,z'}$

✓Vertical Moment of Volume

$$M_{\nabla,z} = \int z\,dV$$
$$= \iiint z\,dx\,dy\,dz$$
$$= \int \Big(\underline{\iint z\,dy\,dz} \Big) dx$$
$$\Rightarrow M_{A,z}(x)$$

$M_{A,z}$: Vertical moment of area about y axis

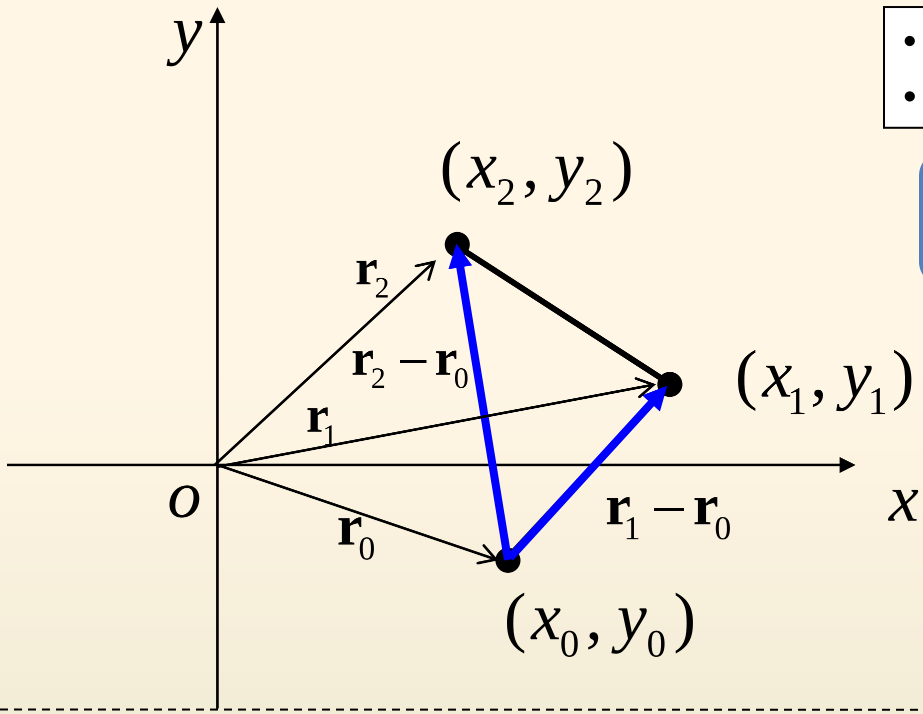
$$\therefore M_{\nabla,z} = \int M_{A,z}(x)dx$$

After calculation of each vertical moment of station area about the y axis($M_{A,z}$), <u>vertical moment of displaced volume</u> can be calculated by integral of vertical moment of section area over the length of ship

✓ **Vertical Center of Buoyancy**

$$VCB = \frac{M_{\nabla,z}}{\nabla}$$

Area of Triangle by Vector



- **Given :** Position vector $\mathbf{r}_0$, $\mathbf{r}_1$, $\mathbf{r}_2$, of vertex of triangle
- **Find :** Area of triangle

$$Area(\mathbf{r}) = \frac{1}{2}\left|(\mathbf{r}_1 - \mathbf{r}_0) \times (\mathbf{r}_2 - \mathbf{r}_0)\right|$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 - x_0 & y_1 - y_0 & 0 \\ x_2 - x_0 & y_2 - y_0 & 0 \end{vmatrix}$$

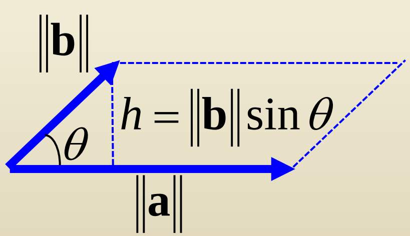
$$= \frac{1}{2}\left|(x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)\right|$$

Cross Product[1)]

$$\mathbf{a} \times \mathbf{b} = \left(\|\mathbf{a}\|\,\|\mathbf{b}\|\,\sin\theta\right)\mathbf{n}$$

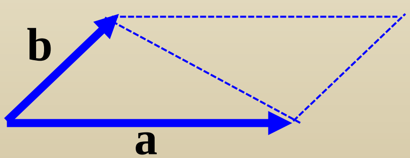
: Area of parellelogram

$$A = \|\mathbf{a} \times \mathbf{b}\|$$



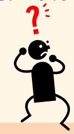
: Area of triangle with side a and b

$$A = \frac{1}{2}\|\mathbf{a} \times \mathbf{b}\|$$



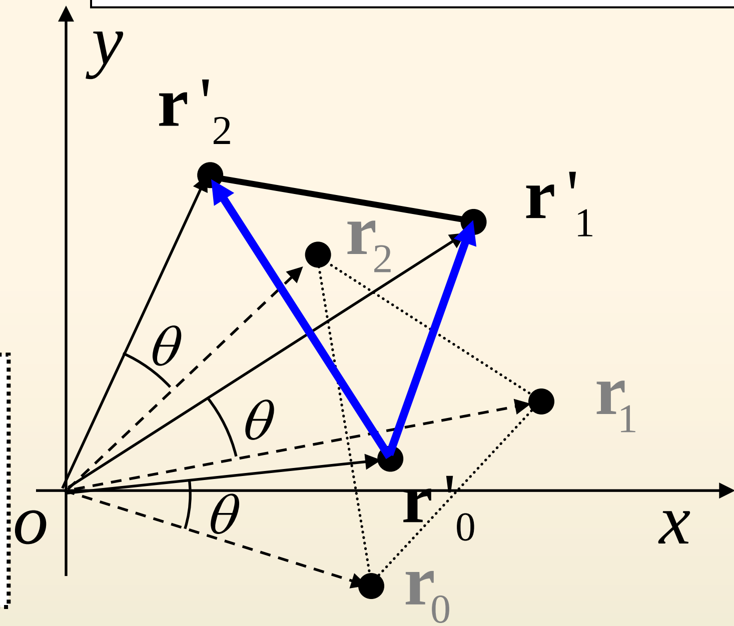
Area of Triangle by Rotational Transformation

1) Zill,C. Advanced Engineering Mathematics, p319, Jones and Bartlett, 3rd



Is a area changed after rotation about origin O?

- **Given : Area before rotation**
- **Find : Area after rotation**



① $$Area(\mathbf{r}) = \frac{1}{2}\left|(\mathbf{r}_1 - \mathbf{r}_0)\times(\mathbf{r}_2 - \mathbf{r}_0)\right|$$

$$= \frac{1}{2}\left|(x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)\right|$$

② $$Area(\mathbf{r}') = \frac{1}{2}\left|(\mathbf{r}'_1 - \mathbf{r}'_0)\times(\mathbf{r}'_2 - \mathbf{r}'_0)\right|$$

$$= \frac{1}{2}\left|(x'_1 - x'_0)(y'_2 - y'_0) - (x'_2 - x'_0)(y'_1 - y'_0)\right|$$

$\mathbf{r}' = \mathbf{R}\cdot\mathbf{r}$,(R:점의 회전변환 행렬)

$$\begin{bmatrix}\mathbf{r}'_0 & \mathbf{r}'_1 & \mathbf{r}'_2\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\mathbf{r}_0 & \mathbf{r}_1 & \mathbf{r}_2\end{bmatrix}$$

$$\begin{bmatrix}x'_0 & x'_1 & x'_2 \\ y'_0 & y'_1 & y'_2\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2\end{bmatrix}$$

$$= \frac{1}{2}\Big|((x_1\cos\theta - y_1\sin\theta) - (x_0\cos\theta - y_0\sin\theta))((x_2\sin\theta + y_2\cos\theta) - (x_0\sin\theta + y_0\cos\theta))$$
$$-((x_2\cos\theta - y_2\sin\theta) - (x_0\cos\theta - y_0\sin\theta))((x_1\sin\theta + y_1\cos\theta) - (x_0\sin\theta + y_0\cos\theta))\Big|$$

$$= \frac{1}{2}\Big|((x_1 - x_0)\cos\theta - (y_1 - y_0)\sin\theta)((x_2 - x_0)\sin\theta + (y_2 - y_0)\cos\theta)$$
$$-((x_2 - x_0)\cos\theta - (y_2 - y_0)\sin\theta)((x_1 - x_0)\sin\theta + (y_1 - y_0)\cos\theta)\Big|$$

$$= \frac{1}{2}\Big|(\cancel{(x_1 - x_0)(x_2 - x_0)\cos\theta}\sin\theta + (x_1 - x_0)(y_2 - y_0)\cos^2\theta - (x_2 - x_0)(y_1 - y_0)\sin^2\theta - \cancel{(y_1 - y_0)(y_2 - y_0)\sin\theta\cos\theta})$$
$$-\cancel{(x_1 - x_0)(x_2 - x_0)\cos\theta}\sin\theta - (x_2 - x_0)(y_1 - y_0)\cos^2\theta + (x_1 - x_0)(y_2 - y_0)\sin^2\theta + \cancel{(y_1 - y_0)(y_2 - y_0)\sin\theta\cos\theta})\Big|$$

$$= \frac{1}{2}\left|\left\{\cos^2\theta + \sin^2\theta\right\}\left\{(x_1 - x_0)(y_2 - y_0) - (y_1 - y_0)(y_2 - y_0)\right\}\right|$$

Zill,C. Advanced Engineering Mathematics, p319, Jones and Bartlett, 3rd [1]

Area of Triangle by Rotational Transformation

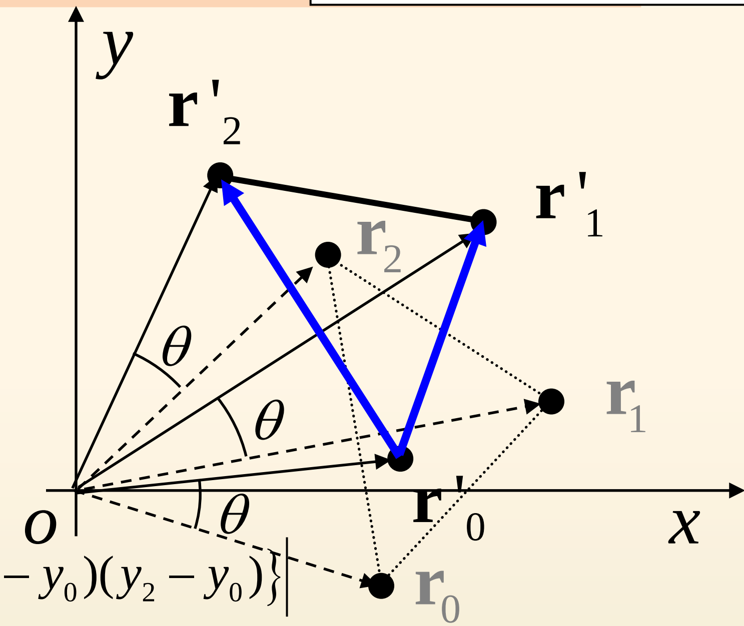
- **Given : 회전변환 전 면적**
- **Find : 회전변환 후 면적**

① $$Area(\mathbf{r}) = \frac{1}{2}\left|(\mathbf{r}_1 - \mathbf{r}_0)\times(\mathbf{r}_2 - \mathbf{r}_0)\right|$$

$$= \frac{1}{2}\left|(x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)\right|$$

② $$Area(\mathbf{r}') = \frac{1}{2}\left|(\mathbf{r}'_1 - \mathbf{r}'_0)\times(\mathbf{r}'_2 - \mathbf{r}'_0)\right|$$

$$= \frac{1}{2}\left|(x'_1 - x'_0)(y'_2 - y'_0) - (x'_2 - x'_0)(y'_1 - y'_0)\right|$$



$$= \frac{1}{2}\left|\left\{\cos^2\theta + \sin^2\theta\right\}\left\{(x_1 - x_0)(y_2 - y_0) - (y_1 - y_0)(y_2 - y_0)\right\}\right|$$

$$= \frac{1}{2}\left|(x_1 - x_0)(y_2 - y_0) - (y_1 - y_0)(y_2 - y_0)\right|$$

$$\therefore\ Area(\mathbf{r}) = Area(\mathbf{r})$$

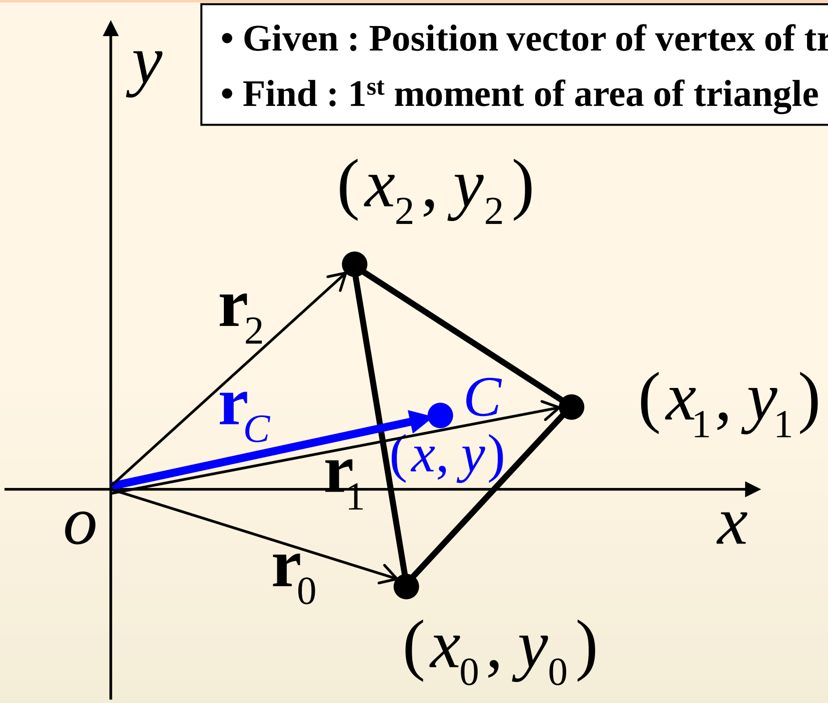
$$Area(\mathbf{Rr}) =$$
$$Area(\mathbf{r}') = Area(\mathbf{r})$$
($\mathbf{R}$: Rotational transformation matrix)

Area is invariant with respect to frame.

1st Moment of Area by Rotational Transformation

[1] Zill,C. Advanced Engineering Mathematics, p319, Jones and Bartlett, 3rd

- **Given : Position vector of vertex of triangle**
- **Find : 1st moment of area of triangle**



Area of triangle $\mathbf{r}_1\mathbf{r}_2\mathbf{r}_2$

$$Area(\mathbf{r}) = \frac{1}{2}\left|(\mathbf{r}_1 - \mathbf{r}_0)\times(\mathbf{r}_2 - \mathbf{r}_0)\right|$$

Position vector of centroid of triangle $\mathbf{r}_1\mathbf{r}_2\mathbf{r}_2$

$$\mathbf{r}_C = \frac{1}{3}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3)$$

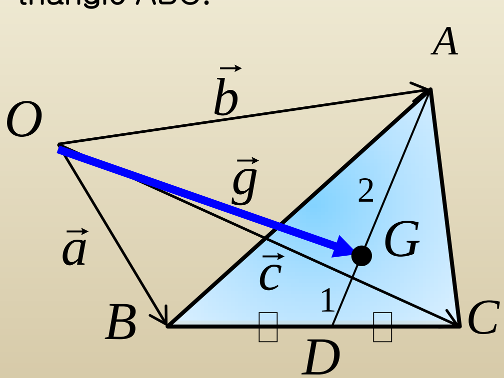
1st moment of area of triangle $\mathbf{r}_1\mathbf{r}_2\mathbf{r}_2$ with respect to x and y axis

$$\mathbf{M}(\mathbf{r}) = \mathbf{r}_C \cdot A$$

$$\mathbf{M}_x = \int x dA = \int x dx dy = \frac{x_1 + x_2 + x_3}{3} A$$

$$\mathbf{M}_y = \int y dA = \int y dx dy = \frac{y_1 + y_2 + y_3}{3} A$$

Position vector of centroid of triangle ABC.



Let cent of side BC as D

$$\overrightarrow{OD} = \frac{1}{2}\left(\vec{b} + \vec{c}\right)$$

Now, G is the point of internal division with ratio of 2 to 1

$$\overrightarrow{OG} = \frac{2\cdot\overrightarrow{OD} + 1\cdot\overrightarrow{OA}}{2+1} = \frac{2\overrightarrow{OD} + \overrightarrow{OA}}{3}$$

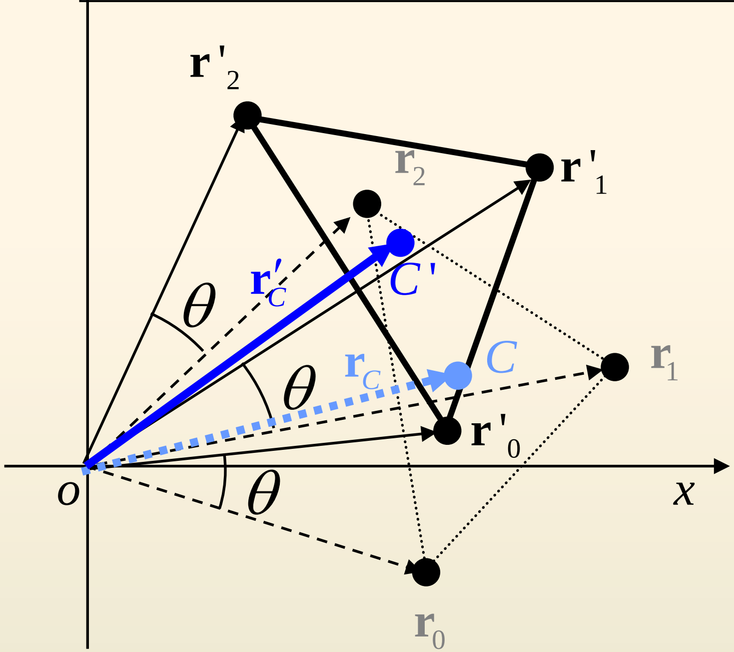
$$\vec{g} = \frac{1}{3}\left\{2\times\frac{1}{2}\left(\vec{b}+\vec{c}\right) + \vec{a}\right\} = \frac{1}{3}\left(\vec{a}+\vec{b}+\vec{c}\right)$$

(*) (기본)수학의 정석, 수학10-나,40th,2005,성지출판사,pp.15

1^{st} Moment of Area by Rotational Transformation

1) Zill,C. Advanced Engineering Mathematics, p319, Jones and Bartlett, 3rd

- **Given :** Position vector of triangle, angle of rotation θ
- **Find :** 1st moment of area of triangle after rotation about origin O



① 1^{st} moment of area of triangle before rotation

$$Area(\mathbf{r}) = \frac{1}{2}\left|(\mathbf{r}_1 - \mathbf{r}_0)\times(\mathbf{r}_2 - \mathbf{r}_0)\right|$$

$$\mathbf{r}_C = \frac{1}{3}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3)$$

$$Moment(\mathbf{r}) = \mathbf{r}_C\ A$$

② 1^{st} moment of area of triangle after rotation

$$Area(\mathbf{r}') = Area(\mathbf{r}) \quad ,(\rightarrow A' = A)$$

$$\mathbf{r}'_C = \frac{1}{3}(\mathbf{r}'_1 + \mathbf{r}'_2 + \mathbf{r}'_3)$$

$$= \frac{1}{3}(\mathbf{R}\,\mathbf{r}_1 + \mathbf{R}\,\mathbf{r}_2 + \mathbf{R}\,\mathbf{r}_3)$$

$$= \frac{1}{3}\mathbf{R}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3)$$

$$= \mathbf{R}\,\mathbf{r}_C$$

$$\mathbf{r}'_C = \mathbf{R}\ \mathbf{r}_C$$

$$Moment(\mathbf{r}') = \mathbf{r}'_C A'$$

$$= \mathbf{R}\ \mathbf{r}_C\ A$$

$$= \mathbf{R}\ Moment(\mathbf{r})$$

$$Moment(\mathbf{r}) = \mathbf{r}_C A$$

$$\mathbf{r}' = \mathbf{R}\cdot\mathbf{r}$$

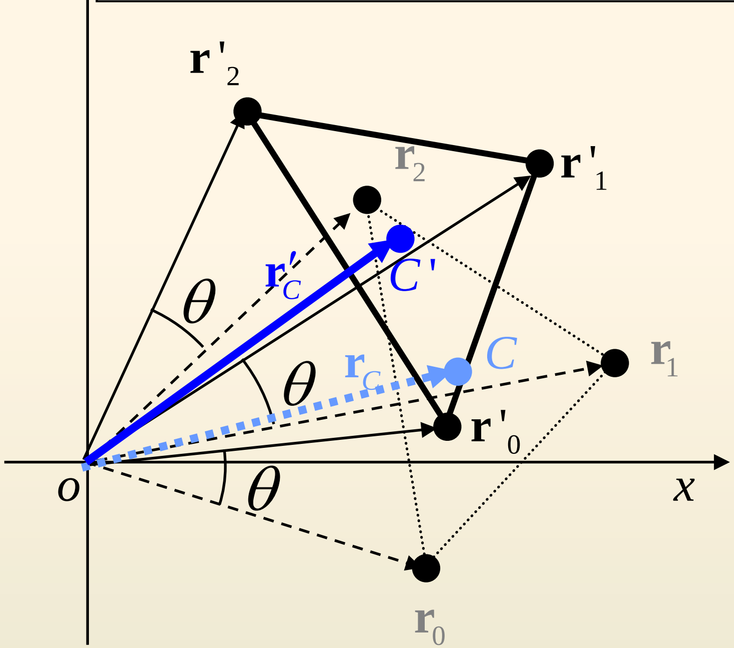
$$\begin{bmatrix}\mathbf{r}'_0 & \mathbf{r}'_1 & \mathbf{r}'_2\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\mathbf{r}_0 & \mathbf{r}_1 & \mathbf{r}_2\end{bmatrix}$$

$$\begin{bmatrix}x'_0 & x'_1 & x'_2 \\ y'_0 & y'_1 & y'_2\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2\end{bmatrix}$$

1) Zill,C. Advanced Engineering Mathematics, p319, Jones and Bartlett, 3rd

1st Moment of Area by Rotational Transformation

- **Given :** Position vector of triangle, angle of rotation θ
- **Find :** 1st moment of area of triangle after rotation about origin O



① 1st moment of area of triangle before rotation

$$Area(\mathbf{r}) = \frac{1}{2}\left|(\mathbf{r}_1 - \mathbf{r}_0)\times(\mathbf{r}_2 - \mathbf{r}_0)\right|$$

$$\mathbf{r}_C = \frac{1}{3}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3)$$

$$Moment(\mathbf{r}) = \mathbf{r}_C \; A$$

② 1st moment of area of triangle after rotation

$$Area(\mathbf{r}') = Area(\mathbf{r}) \quad ,(\rightarrow A' = A)$$

$$\mathbf{r}'_C = \mathbf{R}\,\mathbf{r}_C$$

$$\begin{aligned} Moment(\mathbf{r}') &= \mathbf{r}'_C A' \\ &= \mathbf{R}\,\mathbf{r}_C \; A \\ &= \mathbf{R} \; Moment(\mathbf{r}) \end{aligned}$$

$$\begin{aligned} \mathbf{R}\cdot Moment(\mathbf{r}) &= Moment(\mathbf{R}\,\mathbf{r}) \\ &= \mathbf{R}\,\mathbf{r}_C \; A \end{aligned}$$

($\mathbf{R}$: Rotational transformation matrix)

$$\mathbf{r}' = \mathbf{R}\cdot\mathbf{r}$$

$$\begin{bmatrix}\mathbf{r}'_0 & \mathbf{r}'_1 & \mathbf{r}'_2\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\mathbf{r}_0 & \mathbf{r}_1 & \mathbf{r}_2\end{bmatrix}$$

$$\begin{bmatrix}x'_0 & x'_1 & x'_2 \\ y'_0 & y'_1 & y'_2\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2\end{bmatrix}$$

1st moment of area is invariant with respect to frame.

Sec.1 Calculation of Center of Buoyancy

- Rotational Transformation of Point and Frame
- Example) Calculation of Center of Buoyancy of Ship with Constant Section

  Method ① Direct calculating center of buoyancy in waterplane fixed frame

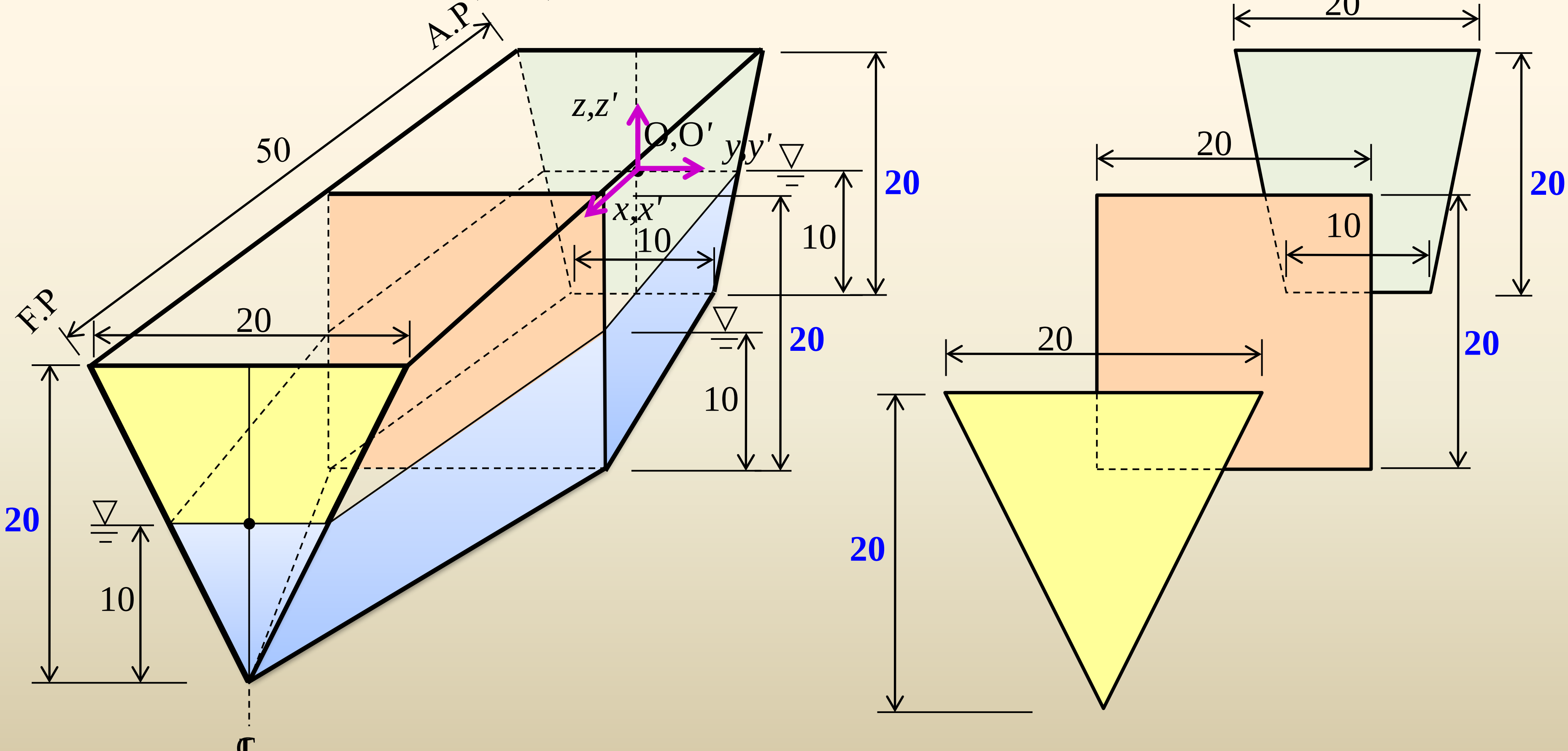
  Method ② Transformation center of buoyancy from body fixed frame to waterplane fixed frame

- **Calculation of Center of Buoyancy of Ship with Various Station Shape**

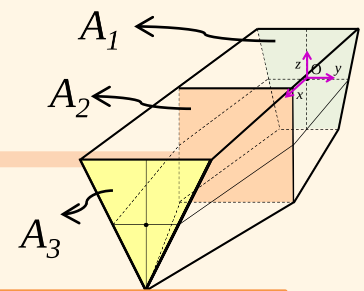
Example) Calculation of Center of Buoyancy of Ship with Various Station Shape

Problem) There is ship (L x B x D : 50 x 20 x 20) with various station shape. When this ship is heeled about x axis in counter-clock wise through an angle of 30° compulsorily, calculate y and z coordinates of center of buoyancy with respect to waterplane fixed frame.

- Given : Length(L) : 50, Breadth(B) : 20, Depth(D) : 20, Draft(T):10, Angle of Heel(ϕ) : -30
- Find : Center of buoyancy ($y_{\nabla,c}$, $z_{\nabla,c}$) after heel in waterplane fixed frame



Example) Calculation of Center of Buoyancy of Ship with Various Station Shape



① Sectional area of each section
(Body fixed frame)

$$A(x') = \int dA = \iint dy'\,dz'$$

② 1st moment of area of each section
(Body fixed frame)

$$M_{A,y'} = \iint y'\,dy'\,dz' = y'_c \cdot A(x')$$

$$M_{A,z'} = \iint z'\,dy'\,dz' = z'_c \cdot A(x')$$

③ Displacement volume
(Body fixed frame)

$$\nabla = \int dV = \iiint dz'\,dy'\,dx' = \int_{A.P}^{F.P} A(x')\,dx'$$

④ 1st moment of displacement volume
(Body fixed frame)

$$M_{\nabla,y'} = \iiint y'\,dz'\,dy'\,dx' = \int_{A.P}^{F.P} (y'_c \cdot A(x'))\,dx'$$

- 단면에 대한 모멘트를 길이방향으로 적분함

$$M_{\nabla,z'} = \iiint z'\,dz'\,dy'\,dx' = \int_{A.P}^{F.P} (z'_c \cdot A(x'))dx'$$

⑤ Center of buoyancy
(Body fixed frame)

$$y'_{\nabla,c} = \frac{M_{\nabla,y'}}{\nabla} = \frac{\iiint y'\,dz'\,dy'\,dx'}{\iiint dz'\,dy'\,dx'} = \frac{\int_{A.P}^{F.P} A(x')\cdot y'_c\,dx'}{\int_{A.P}^{F.P} A(x')\,dx'}$$

$$z'_{\nabla,c} = \frac{M_{\nabla,z'}}{\nabla} = \frac{\iiint z'\,dz'\,dy'\,dx'}{\iiint dz'\,dy'\,dx'} = \frac{\int_{A.P}^{F.P} A(x')\cdot z'_c\,dx'}{\int_{A.P}^{F.P} A(x')\,dx'}$$

⑥ Center of Buoyancy
(Waterplane fixed frame)

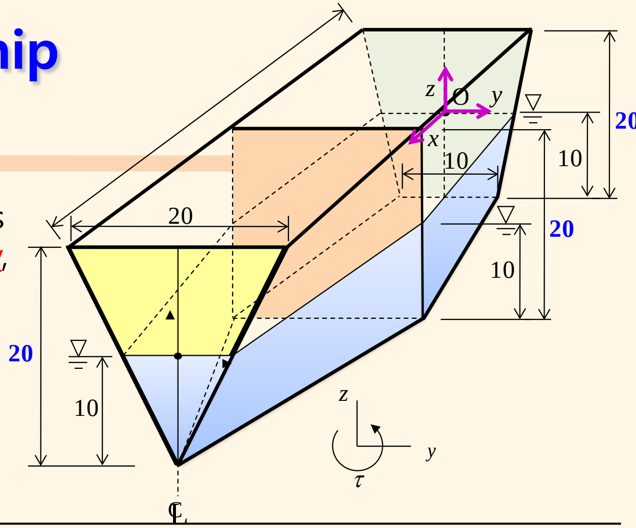
$$\begin{pmatrix} y_{\nabla,c} \\ z_{\nabla,c} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} y'_{\nabla,c} \\ z'_{\nabla,c} \end{pmatrix}$$

$A(x')$: Sectional area at x'

(y'_c, z'_c) : Centroid of section at x'

$(y'_{\nabla,c}, z'_{\nabla,c})$: Center of displaced volume

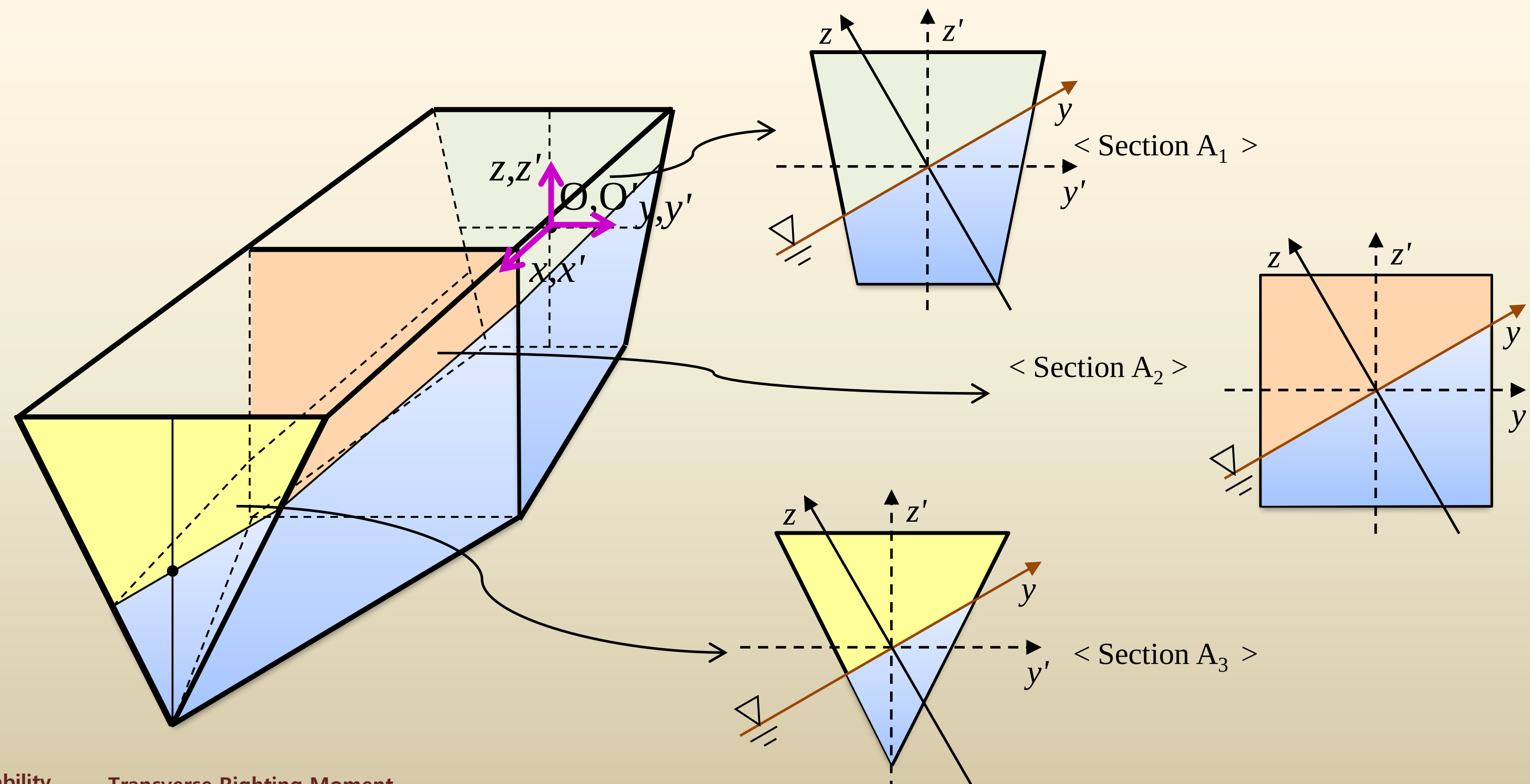
Example) Calculation of Center of Buoyancy of Ship with Various Station Shape



Problem) There is ship (L x B x D : 50 x 20 x 20) with various station shape. When this ship is heeled about x axis in counter-clock wise through an angle of 30° compulsorily, calculate y and z coordinates of center of buoyancy with respect to waterplane fixed frame.

- Given : Length(L) : 50, Breadth(B) : 20, Depth(D) : 20, Draft(T):10, Angle of Heel(ϕ) : -30
- Find : Center of buoyancy ($y_{\nabla,c}$, $z_{\nabla,\,c}$) after heel in waterplane fixed frame

풀이)



Example) Calculation of Center of Buoyancy of Ship with Various Station Shape

Problem) There is ship (L x B x D : 50 x 20 x 20) with various station shape. When this ship is heeled about x axis in counter-clock wise through an angle of 30° compulsorily, calculate y and z coordinates of center of buoyancy with respect to waterplane fixed frame.

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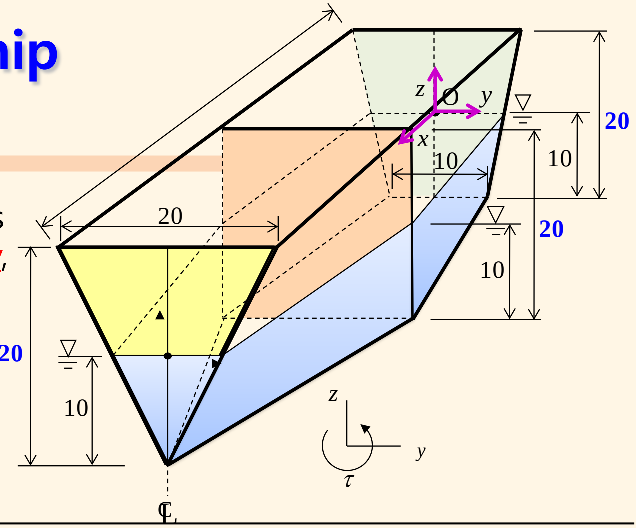

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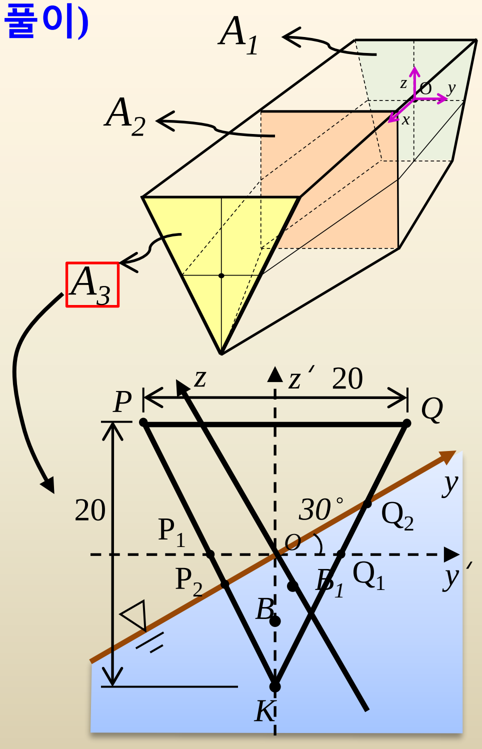


< Section A_3 >

In the same way of previous example, calculate center of buoyancy with respect to body fixed frame at first, then calculate center of buoyancy with respect to waterplane fixed frame by rotational transformation.

Coordinate of P_1, P_2, Q_1, Q_2 of section A3

- Coordinate of P_1, Q_1 is known as (-5,0), (5,0) by geometric shape.

Calculate equations of straight line PK, KQ in order to know P_2,Q_2

The equation of straight line PK $z' = -2y' - 10$

The equation of straight line KQ $z' = 2y' - 10$

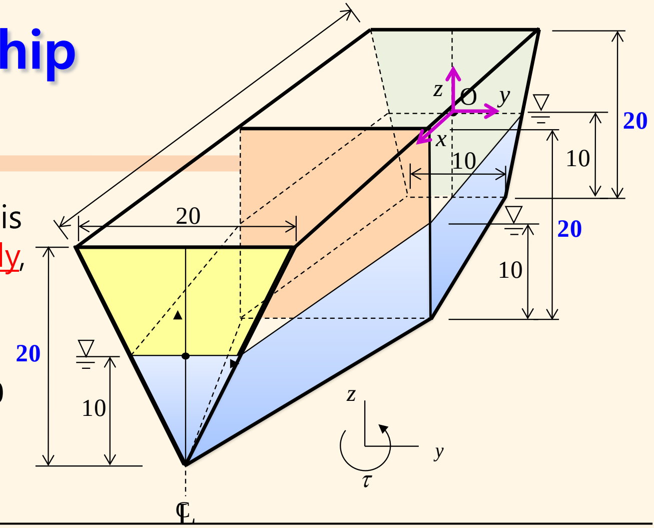
- The equation of line of waterplane with respect to body fixed frame is as follows, because waterplane is inclined through an angle of 30°.

$$z' = \tan 30\, y' = 0.5774 y'$$

- Intersection point P_2,Q_2 betwwen waterplane and straight line PK, KQ can be calculated as follows P_2(-3.88, -2.24), Q_2(7.03, 4.06)

※Area below waterplane can be calculated also by Gaussian Quadrature.

Example) Calculation of Center of Buoyancy of Ship with Various Station Shape



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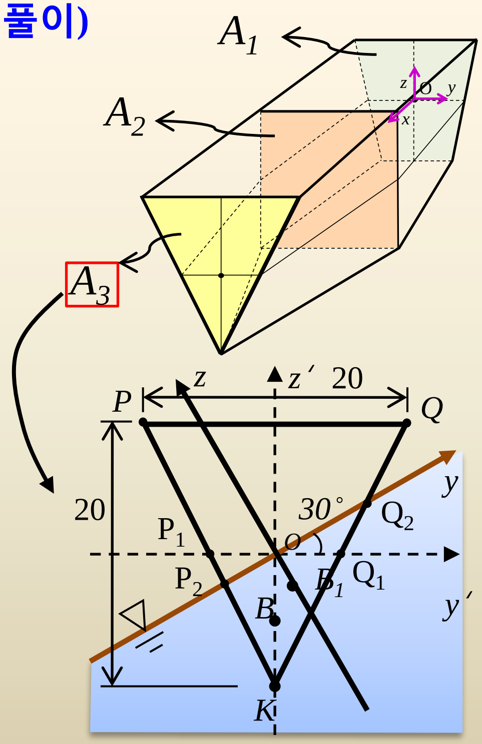
풀이)



< Section A_3 >

① - A_3 : Sectional area of A_3

$$A(x') = \int dA' = \iint dy'\,dz' = \sum A_i(x')$$

$P_1(-5,0)$, $Q_1(5,0)$
$P_2(-3.88,\ -2.24)$, $Q_2(7.03,\ 4.06)$

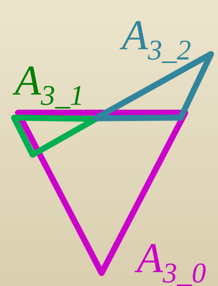
- Total sectional area before heel

$$A_{3\_0} = 0.5 \times 10 \times 10 = 50$$

- Changed area after heel

$$A_{3\_1} = \frac{1}{2}\left|(\mathbf{P}_1 - \mathbf{O}) \times (\mathbf{P}_2 - \mathbf{O})\right| = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -5 & 0 \\ 0 & -3.88 & -2.24 \end{vmatrix} = 5.60$$

$$A_{3\_2} = \frac{1}{2}\left|(\mathbf{Q}_1 - \mathbf{O}) \times (\mathbf{Q}_2 - \mathbf{O})\right| = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 5 & 0 \\ 0 & 7.03 & 4.06 \end{vmatrix} = 10.15$$



- Total sectional area after heel

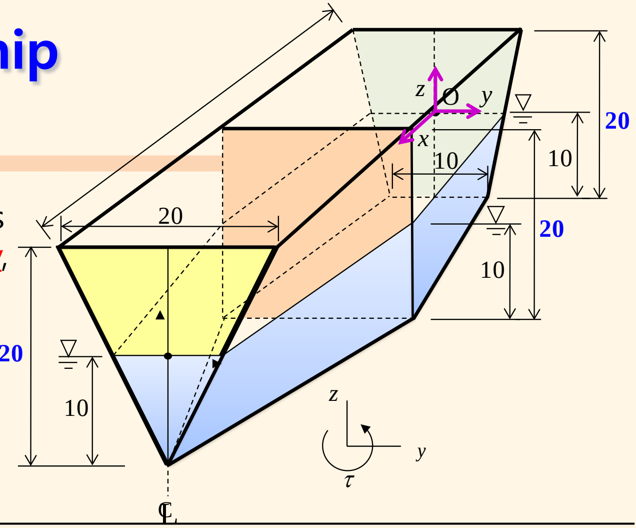
$$A_3' = 50 - 5.60 + 10.15 = 54.55$$

If a ship is not a wall sided ship, area below waterplane is different.

$A_{k\_i}(x')$: Partial area of section at x'

$A_k(x')$: Sectional area at x'

Example) Calculation of Center of Buoyancy of Ship with Various Station Shape



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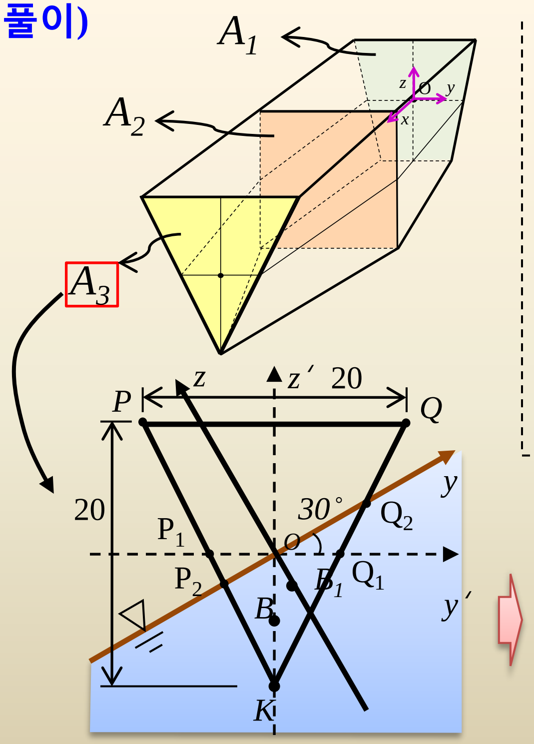
- Find : Center of buoyancy ($y_{\nabla,c}$, $z_{\nabla,c}$) after heel in waterplane fixed frame

풀이)

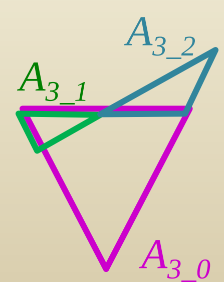


< Section A_3 >

$P_1(-5,0)$, $Q_1(5,0)$
$P_2(-3.88, -2.24)$, $Q_2(7.03, 4.06)$

Centroid of A_3

$$\text{Cetroid of } A_{3\_0} = (0, \frac{1}{3} \times (-10))$$
$$= (0, -\frac{10}{3})$$



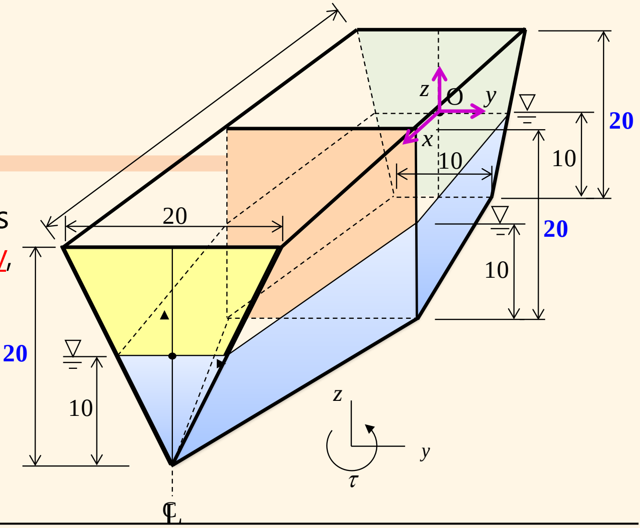
$$\text{Cetroid of } A_{3\_1} = (\frac{0-5-3.88}{3}, \frac{0+0-2.24}{3})$$
$$= (-2.96, -0.75)$$

$$\text{Cetroid of } A_{3\_2} = (\frac{0+7.03+5}{3}, \frac{0+0+4.06}{3})$$
$$= (4.01, 1.35)$$

Example) Calculation of Center of Buoyancy of Ship with Various Station Shape

Problem) There is ship (L x B x D : 50 x 20 x 20) with various station shape. When this ship is heeled about x axis in counter-clock wise through an angle of 30° compulsorily, calculate y and z coordinates of center of buoyancy with respect to waterplane fixed frame.

- Given : Length(L) : 50, Breadth(B) : 20, Depth(D) : 20, Draft(T):10, Angle of Heel(ϕ) : -30
- Find : Center of buoyancy ($y_{\nabla,c}$, $z_{\nabla,c}$) after heel in waterplane fixed frame



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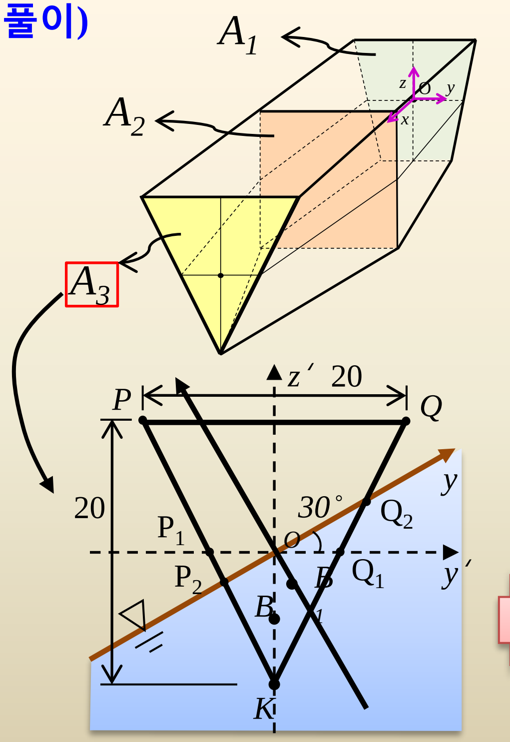
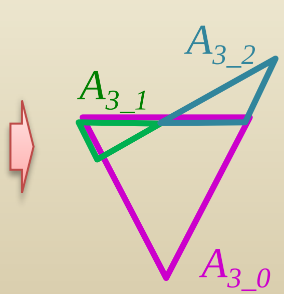


< Section A_3 >



②- A_3 : 1st moment of area of section A_3

- Calculate 1st moment of area in order to know centroid of section A_3 with respect to body fixed frame

$$M_{A_k,y'} = \iint y'\,dy'\,dz' = y'_{c\_k} \cdot A_k(x') = \sum y'_{c\_k\_i} \cdot A_{k\_i}(x')$$

	Area	$y'_{c\_3\_i}$	$z'_{c\_3\_i}$	$M_{A_3,y'}$ Area*y'c_3_i	$M_{A_3,z'}$ Area*z'c_3_i
① A_{3_0}	50.00	0.00	-3.33	0.00	-166.67
② A_{3_1}	5.60	-2.96	-0.75	-16.57	-4.18
③ A_{3_2}	10.15	4.01	1.35	40.69	13.73
①−②+③	54.55			57.26	-148.76

$M_{A_3,y'}$: 1st moment of area of section A_3 about z′ axis

$M_{A_3,z'}$: 1st moment of area of section A_3 about y′ axis

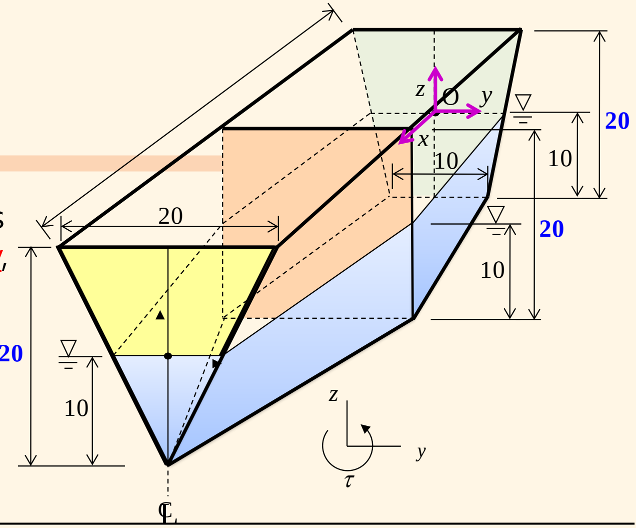
- Centroid of section A_3 with respect to body fixed frame is calculated as follows

$$(y'_{c\_3}, z'_{c\_3}) = \left(\frac{M_{A_3,y'}}{Area_{A_3}}, \frac{M_{A_3,z'}}{Area_{A_3}}\right) = \left(\frac{57.26}{54.55}, \frac{-148.76}{54.55}\right) = \boxed{(1.05, -2.73)}$$

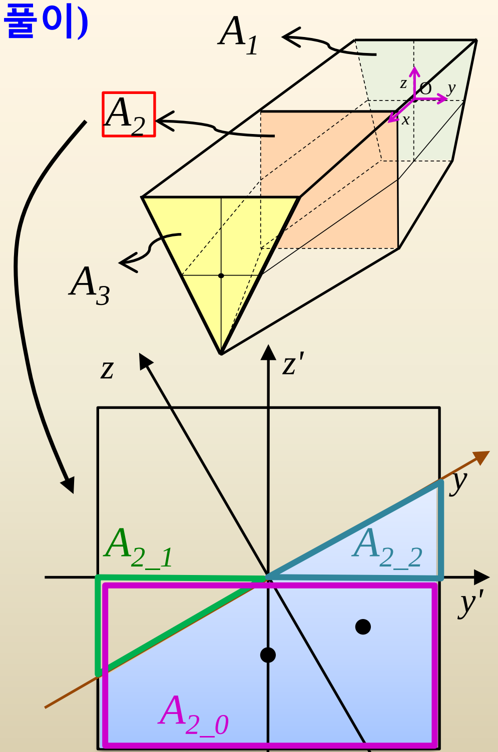
Example) Calculation of Center of Buoyancy of Ship with Various Station Shape

Problem) There is ship (L x B x D : 50 x 20 x 20) with various station shape. When this ship is heeled about x axis in counter-clock wise through an angle of 30° compulsorily, calculate y and z coordinates of center of buoyancy with respect to waterplane fixed frame.

- Given : Length(L) : 50, Breadth(B) : 20, Depth(D) : 20, Draft(T):10, Angle of Heel(ϕ) : -30
- Find : Center of buoyancy ($y_{\nabla,c}$, $z_{\nabla,c}$) after heel in waterplane fixed frame



풀이)



< Section A_2 >

①-A_2,②- A_2 : A_2 단면의 면적 및 면적1차 모멘트 계산

- Calculate 1st moment of area in order to know centroid of section A_2 with respect to body fixed frame

$$M_{A_k,y'} = \boxed{\iint y'\,dy'\,dz'} = y'_{c\_k} \cdot A_k(x') = \sum y'_{c\_k\_i} \cdot A_{k\_i}(x')$$

	Area	$y'_{c\_2\_i}$	$z'_{c\_2\_i}$	$M_{A_2,y'}$ Area*y'c_2_i	$M_{A_2,z'}$ Area*z'c_2_i
A_{3_0}	200.00	0.00	-5.00	0.00	-1000.00
A_{3_1}	-28.87	-6.67	-1.92	192.45	55.56
A_{3_2}	28.87	6.67	1.92	192.45	55.56
Sum	200.00			384.90	-888.89

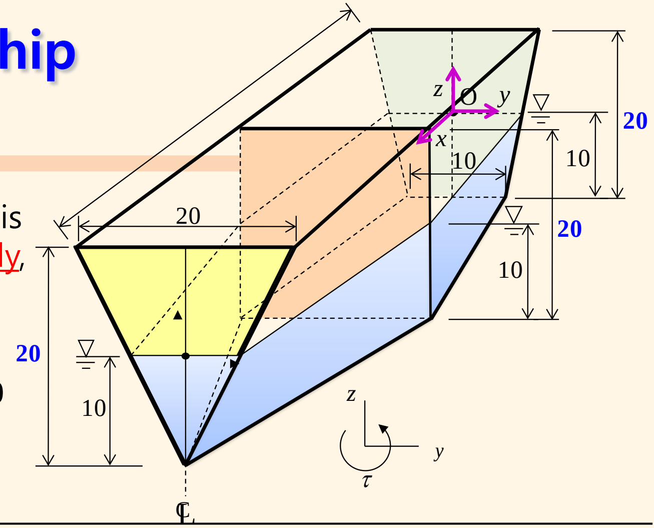
$M_{A_2,y'}$: 1st moment of area of section A_2 about z' axis

$M_{A_2,z'}$: 1st moment of area of section A_2 about y' axis

- Centroid of section A_2 with respect to body fixed frame is calculated as follows

$$(y'_{c\_2}, z'_{c\_2}) = \left(\frac{M_{A_3,y'}}{Area_{A_2}}, \frac{M_{A_3,z'}}{Area_{A_2}}\right) = \left(\frac{384.90}{200}, \frac{-888.89}{200}\right) \boxed{= (1.92, -4.44)}$$

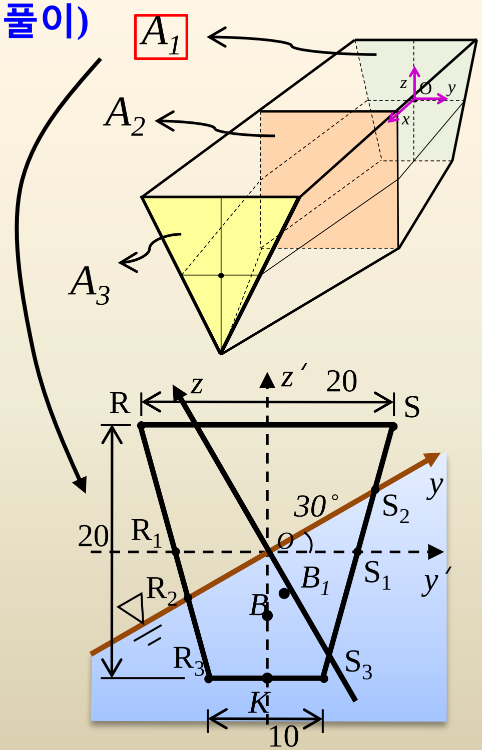
Example) Calculation of Center of Buoyancy of Ship with Various Station Shape



Problem) There is ship (L x B x D : 50 x 20 x 20) with various station shape. When this ship is heeled about x axis in counter-clock wise through an angle of 30° compulsorily, calculate y and z coordinates of center of buoyancy with respect to waterplane fixed frame.

- Given : Length(L) : 50, Breadth(B) : 20, Depth(D) : 20, Draft(T):10, Angle of Heel(ϕ) : -30
- Find : Center of buoyancy ($y_{\nabla,c}$, $z_{\nabla,c}$) after heel in waterplane fixed frame

풀이)



< Section A_1 >

Coordinate of R_1, R_2, S_1, S_2 of section A_1

- Coordinate of R_1, S_1 is known as (-7.5,0), (7.5,0) by geometric shape.
- Calculate equations of straight line RR_3, SS_3 in order to know R_2,S_2

The equation of straight line RR_3 $\quad z' = -4y' - 30$

The equation of straight line SS_3 $\quad z' = 4y' - 30$

- The equation of line of waterplane with respect to body fixed frame is as follows, because waterplane is inclined through an angle of 30°.

$$z' = \tan 30\, y' = 0.5774 y'$$

- Intersection point R_3,S_3 between waterplane and straight line RR_3, SS_3 can be calculated as follows

$P_2(-6.55,-3.78)$, $Q_2(8.77,5.06)$

Example) Calculation of Center of Buoyancy of Ship with Various Station Shape

Problem) There is ship (L x B x D : 50 x 20 x 20) with various station shape. When this ship is heeled about x axis in counter-clock wise through an angle of 30° compulsorily, calculate y and z coordinates of center of buoyancy with respect to waterplane fixed frame.

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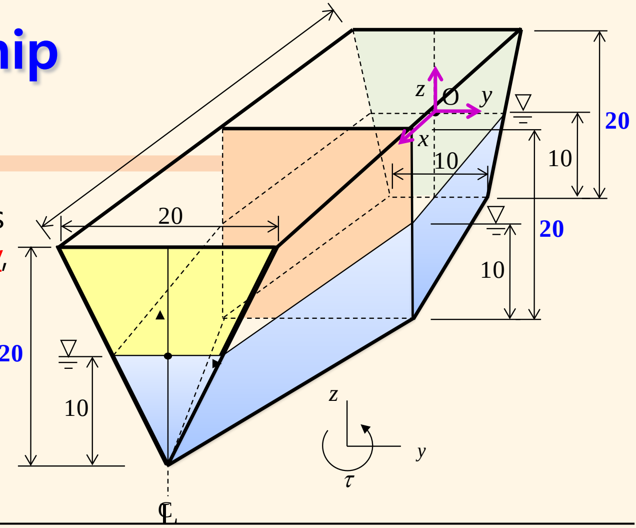

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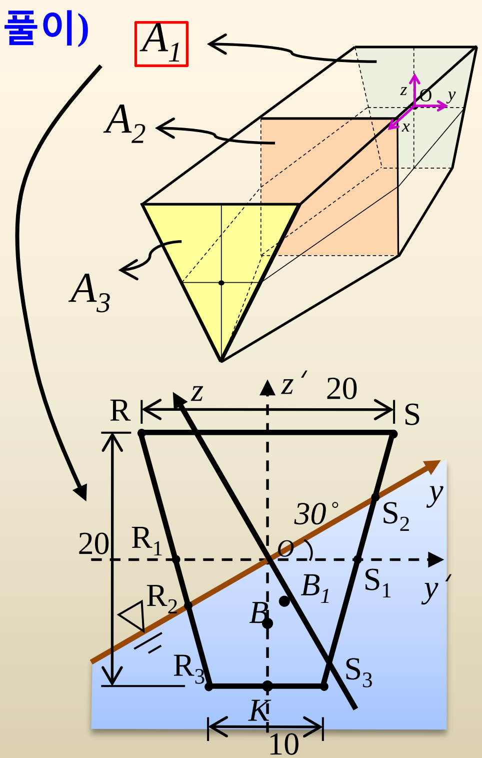


< Section A_1 >

① - A_1 : Sectional area of section A_1

$$A_k(x') = \int dA' = \iint dy'\,dz' = \sum_i A_{k\_i}(x')$$

$P_1(-7.5,0)$, $Q_1(7.5,0)$

$P_2(-6.55,-3.78)$, $Q_2(8.77,5.06)$

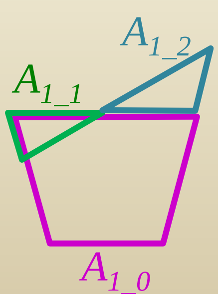
- Total sectional area before heel

$$A_{1\_0} = 0.5\times(15+10)\times10 = 125$$

- Changed area after heel

$$A_{1\_1} = \frac{1}{2}\left|(\mathbf{P}_1-\mathbf{O})\times(\mathbf{P}_2-\mathbf{O})\right| = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -7.5 & 0 \\ 0 & -6.55 & -3.78 \end{vmatrix} = 14.19$$

$$A_{1\_2} = \frac{1}{2}\left|(\mathbf{Q}_1-\mathbf{O})\times(\mathbf{Q}_2-\mathbf{O})\right| = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 7.5 & 0 \\ 0 & 8.77 & 5.06 \end{vmatrix} = 18.98$$

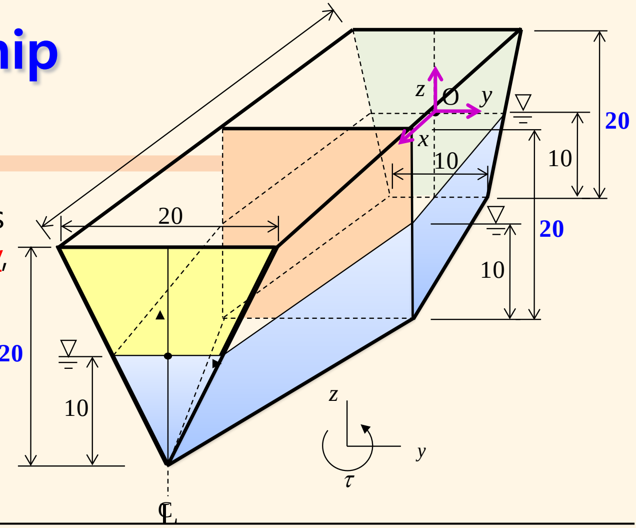


- Total sectional area after heel

$$A_1' = 125 - 14.19 + 18.98 = 129.79$$

If a ship is not a wall sided ship, area below waterplane is different.

Example) Calculation of Center of Buoyancy of Ship with Various Station Shape



Problem) There is ship (L x B x D : 50 x 20 x 20) with various station shape. When this ship is heeled about x axis in counter-clock wise through an angle of 30° compulsorily, calculate y and z coordinates of center of buoyancy with respect to waterplane fixed frame.

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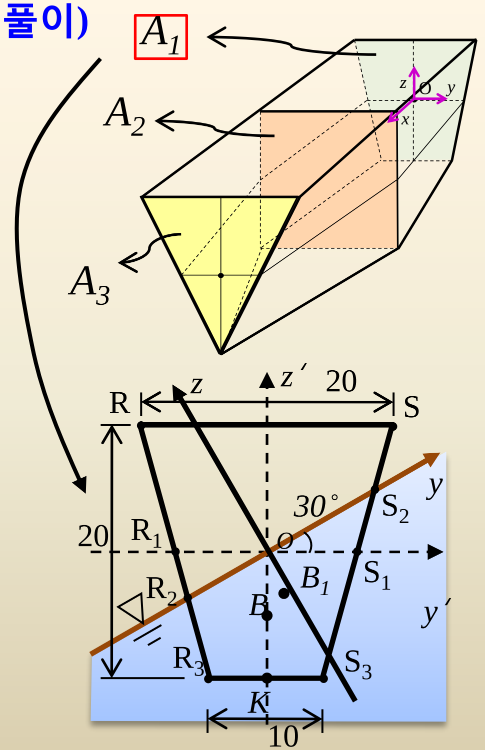
풀이)

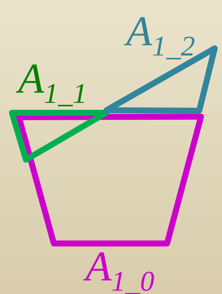


< Section A_1 >

Centroid of section A_1

$P_1(-7.5,0)$, $Q_1(7.5,0)$
$P_2(-6.55,-3.78)$, $Q_2(8.77,5.06)$

$$\text{Cetroid of } A_{3\_0} = \left(0, \frac{(15\times10)\times(-5)+(0.5\times2.5\times10)\times10\times(2/3)}{125}\right)$$
$$= (0,-4.67)$$



$$\text{Cetroid of } A_{3\_1} = \left(\frac{0-7.5-6.55}{3}, \frac{0+0-3.78}{3}\right)$$
$$= (-4.68,\ -1.26)$$

$$\text{Cetroid of } A_{3\_2} = \left(\frac{0+7.5+8.77}{3}, \frac{0+0+5.06}{3}\right)$$
$$= (5.42, 1.69)$$

Example) Calculation of Center of Buoyancy of Ship with Various Station Shape

Problem) There is ship (L x B x D : 50 x 20 x 20) with various station shape. When this ship is heeled about x axis in counter-clock wise through an angle of 30° compulsorily, calculate y and z coordinates of center of buoyancy with respect to waterplane fixed frame.

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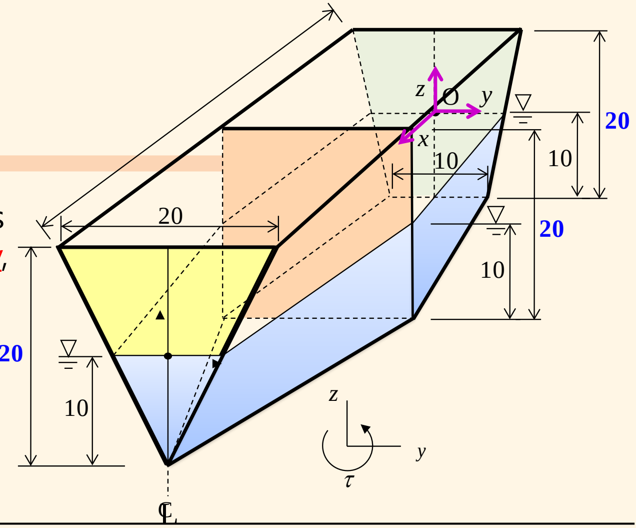

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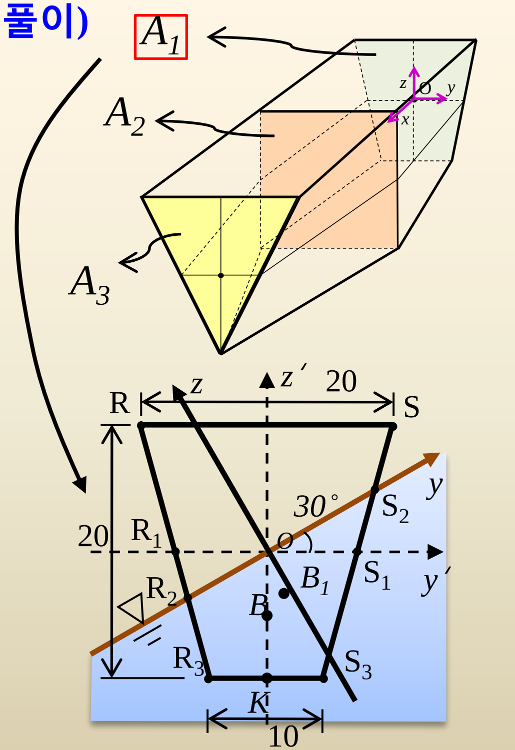
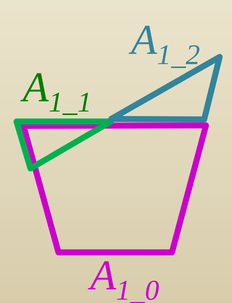


< Section A_1 >



②- A_1 : 1st moment of area of section A_1

- Calculate 1st moment of area in order to know centroid of section A_1 with respect to body fixed frame

$$M_{A,y'} = \iint y'\, dy'\, dz' = y'_c \cdot A(x') = \sum y'_{c\_i} \cdot A_i(x')$$

	Area	$y'_{c\_1\_i}$	$z'_{c\_1\_i}$	$M_{A_1,y'}$ Area*y'c_1_i	$M_{A_1,z'}$ Area*z'c_1_i
① A_{1_0}	125.00	0.00	-4.67	0.00	-583.34
② A_{1_1}	14.19	-4.68	-1.26	-66.48	-17.90
③ A_{1_2}	18.98	5.42	1.69	102.90	32.02
①−②+③	129.79			169.38	-533.42

$M_{A_1,y'}$: 1st moment of area of section A_1 about z′ axis

$M_{A_1,z'}$: 1st moment of area of section A_1 about y′ axis

- Centroid of section A_1 with respect to body fixed frame is calculated as follows

$$(y'_{c\_1}, z'_{c\_1}) = \left(\frac{M_{A_1,y'}}{Area_{A_1}}, \frac{M_{A_1,z'}}{Area_{A_1}}\right) = \left(\frac{169.38}{129.79}, \frac{-533.42}{129.79}\right) = \boxed{(1.31, -4.11)}$$

Example) Calculation of Center of Buoyancy of Ship with Various Station Shape

Problem) There is ship (L x B x D : 50 x 20 x 20) with various station shape. When this ship is heeled about x axis in counter-clock wise through an angle of 30° compulsorily, calculate y and z coordinates of center of buoyancy with respect to waterplane fixed frame.

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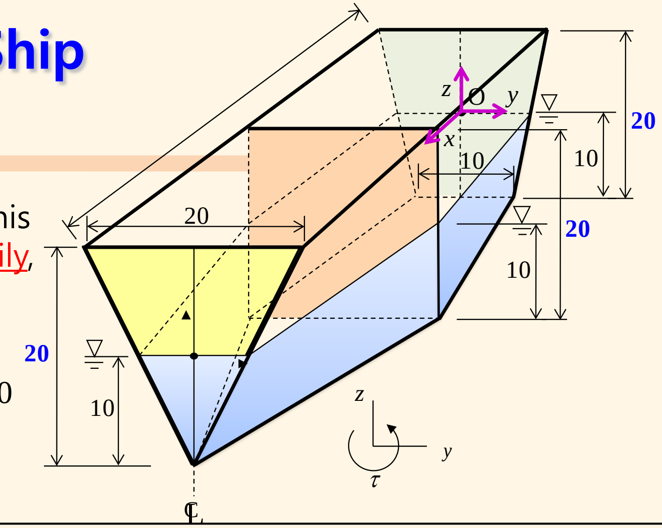
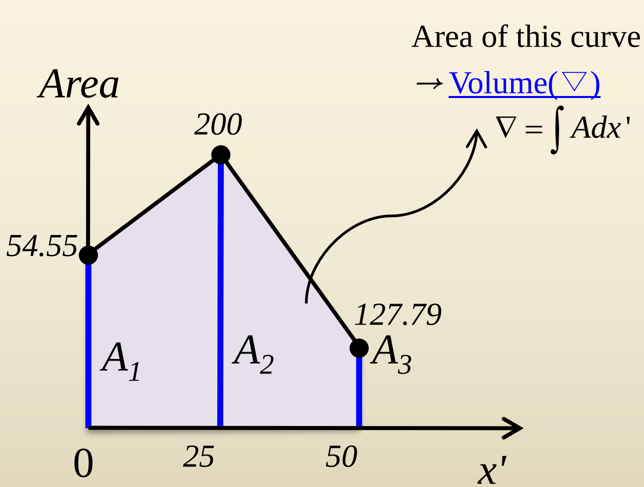
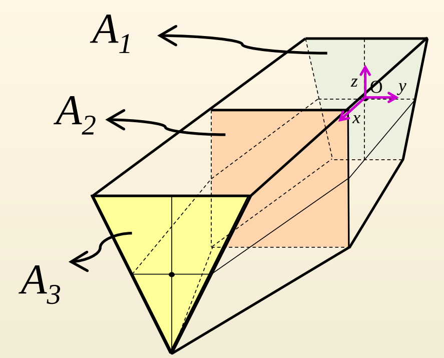


풀이) ③ Displacement Volume



$$\nabla = \int dV = \iiint dx'\,dy'\,dz'$$
$$= \int \boxed{\iint dz'\,dy'}\,dx'$$
$$= \int_{A.P}^{F.P} A(x')\,dx'$$



- Displacement volume can be calculated by integral of sectional area in longitudinal direction

$$\nabla = \int_0^{50} A(x')\,dx' = 7,304$$

Example) Calculation of Center of Buoyancy of Ship with Various Station Shape

Problem) There is ship (L x B x D : 50 x 20 x 20) with various station shape. When this ship is heeled about x axis in counter-clock wise through an angle of 30° compulsorily, calculate y and z coordinates of center of buoyancy with respect to waterplane fixed frame.

- Given : Length(L) : 50, Breadth(B) : 20, Depth(D) : 20, Draft(T):10, Angle of Heel(ϕ) : -30
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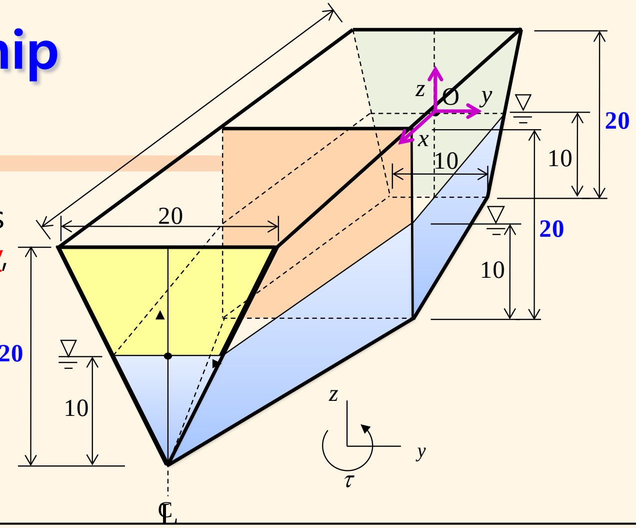
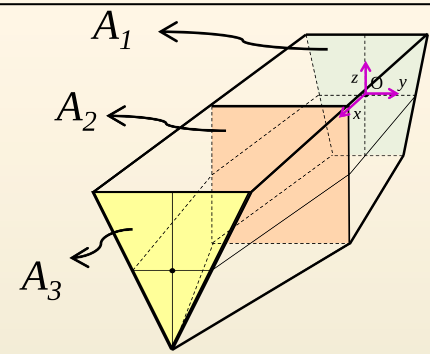
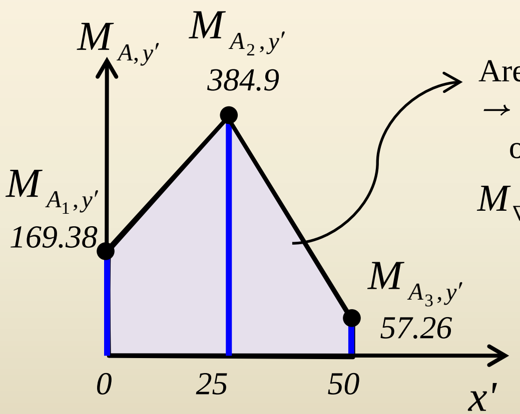


풀이)

④-1 : Transverse moment of displaced volume

(Body fixed frame)





Area of this curve
→ Transverse moment of volume

$$M_{\nabla,y'} = \int M_{A,y'} dx'$$

$$M_{\nabla,y'} = \int\!\!\iint y'\,dy'\,dz'\,dx'$$
$$= \int_{A.P}^{F.P} (y'_c \cdot A(x'))\,dx'$$
$$= y'_{\nabla,c} \cdot \nabla$$

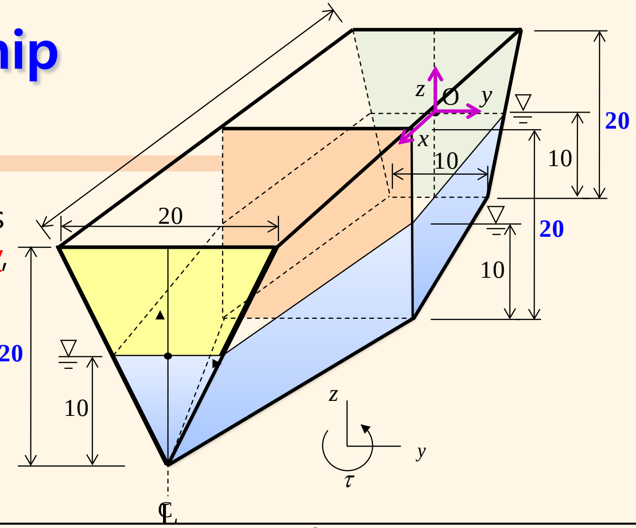
After calculation of each transverse moment of sectional area about the z' axis($M_{A,y'}$), transverse moment of displaced volume can be calculated by integral of transverse moment of section area over the length of ship

$$M_{\nabla,y'} = \int_0^{50} M_{A,y'}\,dx' = \boxed{12,455}$$

$M_{A,y'}$: Transverse moment of area about z' axis

$M_{A,z'}$: Vertical moment of area about y' axis

Example) Calculation of Center of Buoyancy of Ship with Various Station Shape



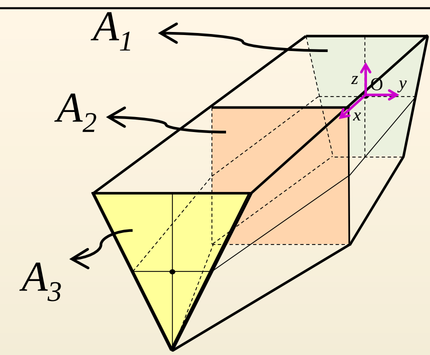
Problem) There is ship (L x B x D : 50 x 20 x 20) with various station shape. When this ship is heeled about x axis in counter-clock wise through an angle of 30° compulsorily, calculate y and z coordinates of center of buoyancy with respect to waterplane fixed frame.

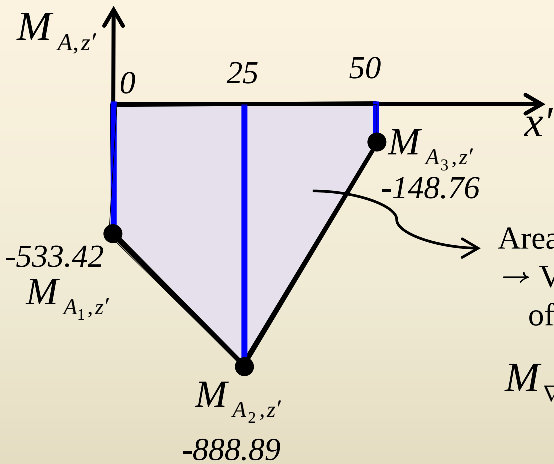
- Given : Length(L) : 50, Breadth(B) : 20, Depth(D) : 20, Draft(T):10, Angle of Heel(ϕ) : -30
- Find : Center of buoyancy ($y_{\nabla,c}$, $z_{\nabla,c}$) after heel in waterplane fixed frame

풀이)

④-2 : Vertical moment of displaced volume

(Body fixed frame)





Area of this curve → Vertical moment of volume

$$M_{\nabla,z'} = \int M_{A,z'} dx'$$

$$M_{\nabla,z'} = \int\!\!\int\!\!\int z'\,dy'\,dz'\,dx' = \int_{A.P}^{F.P} (z'_c \cdot A(x'))dx' = z'_{\nabla,c} \cdot \nabla$$

After calculation of each vertical moment of sectional area about the y axis($M_{A,z}$), vertical moment of displaced volume can be calculated by integral of vertical moment of section area over the length of ship

$$M_{\nabla,z'} = \int_0^{50} M_{A,z'}\,dx' = \boxed{-30,794}$$

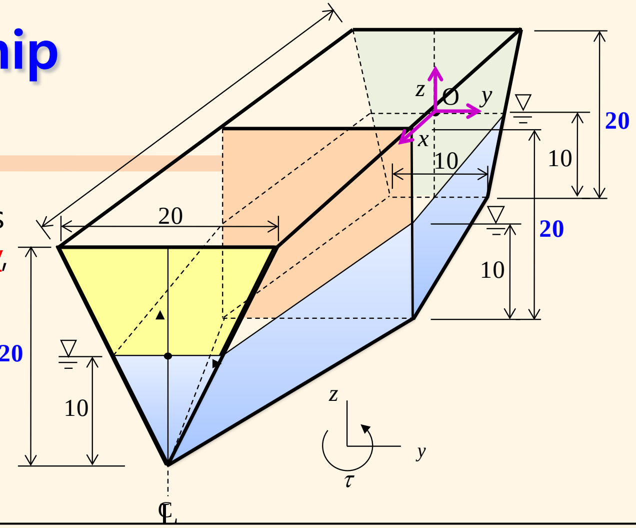
$M_{A,y'}$: Transverse moment of area about z′ axis

$M_{A,z'}$: Vertical moment of area about y′ axis

Example) Calculation of Center of Buoyancy of Ship with Various Station Shape

Problem) There is ship (L x B x D : 50 x 20 x 20) with various station shape. When this ship is heeled about x axis in counter-clock wise through an angle of 30° compulsorily, calculate y and z coordinates of center of buoyancy with respect to waterplane fixed frame.

- Given : Length(L) : 50, Breadth(B) : 20, Depth(D) : 20, Draft(T):10, Angle of Heel(ϕ) : -30
- Find : Center of buoyancy ($y_{\nabla,c}$, $z_{\nabla,c}$) after heel in waterplane fixed frame



풀이)

⑤ Center of buoyancy (Body fixed frame)

- Center of buoyancy can be calculated if we divide transverse, vertical moment of displaced volume by displaced volume.

$$y'_{\nabla,c} = \frac{M_{\nabla,y'}}{\nabla} = \frac{\iiint y'\, dy'\, dz'\, dx'}{\iiint dx'\, dy'\, dz'}$$

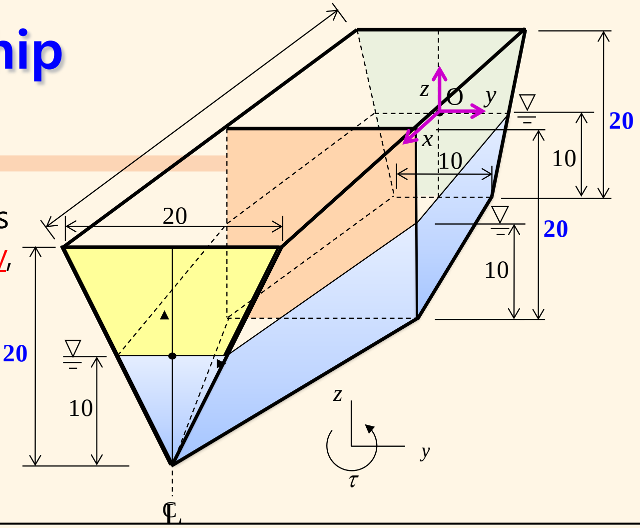
$$z'_{\nabla,c} = \frac{M_{\nabla,z'}}{\nabla} = \frac{\iiint z'\, dy'\, dz'\, dx'}{\iiint dx'\, dy'\, dz'}$$

$$(y'_{\nabla,c}\,,\ z'_{\nabla,c}) = \left(TCB = \frac{M_{\nabla,y'}}{\nabla}, VCB = \frac{M_{\nabla,z'}}{\nabla} \right)$$

$$= \left(\frac{12,455}{7,304}, \frac{-30,794}{7,304} \right) = \boxed{(1.71, -4.21)}$$

• Center of buoyancy with respect to waterplane fixed frame have to be calculated.
→ Rotational transformation

Example) Calculation of Center of Buoyancy of Ship with Various Station Shape



Problem) There is ship (L x B x D : 50 x 20 x 20) with various station shape. When this ship is heeled about x axis in counter-clock wise through an angle of 30° compulsorily, calculate y and z coordinates of center of buoyancy with respect to waterplane fixed frame.

- Given : Length(L) : 50, Breadth(B) : 20, Depth(D) : 20, Draft(T):10, Angle of Heel(ϕ) : -30
- Find : Center of buoyancy ($y_{\nabla,c}$, $z_{\nabla,c}$) after heel in waterplane fixed frame

풀이)

⑥ Center of buoyancy (Waterplane fixed frame)

- **Center of buoyancy with respect to waterplane fixed frame have to be calculated.**
  → Rotational transformation

$$\boxed{\begin{pmatrix} y_{\nabla,c} \\ z_{\nabla,c} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} y'_{\nabla,c} \\ z'_{\nabla,c} \end{pmatrix}}$$

$$\begin{pmatrix} y_{\nabla,c} \\ z_{\nabla,c} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1.71 \\ -4.21 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(30) & \sin(30) \\ -\sin(30) & \cos(30) \end{pmatrix} \begin{pmatrix} 1.71 \\ -4.21 \end{pmatrix} = \boxed{\begin{pmatrix} -0.63 \\ -4.50 \end{pmatrix}}$$

Naval Architecture & Ocean Engineering

- Ship Stability -

Ch.6 Transverse Righting Moment

- Sec.2 Calculating BM, GZ in Wall Sided Ship -

2009

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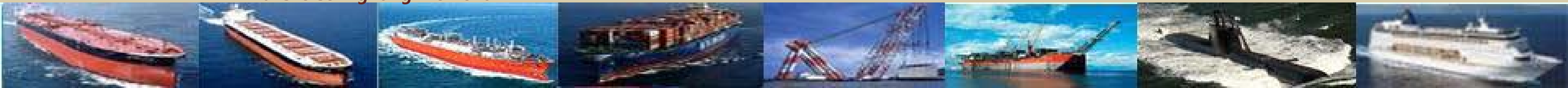


Sec.1 Calculation of Center of Buoyancy

Sec.2 Calculation of BM, GZ in Wall Sided Ship

Sec.3 Inclining Test

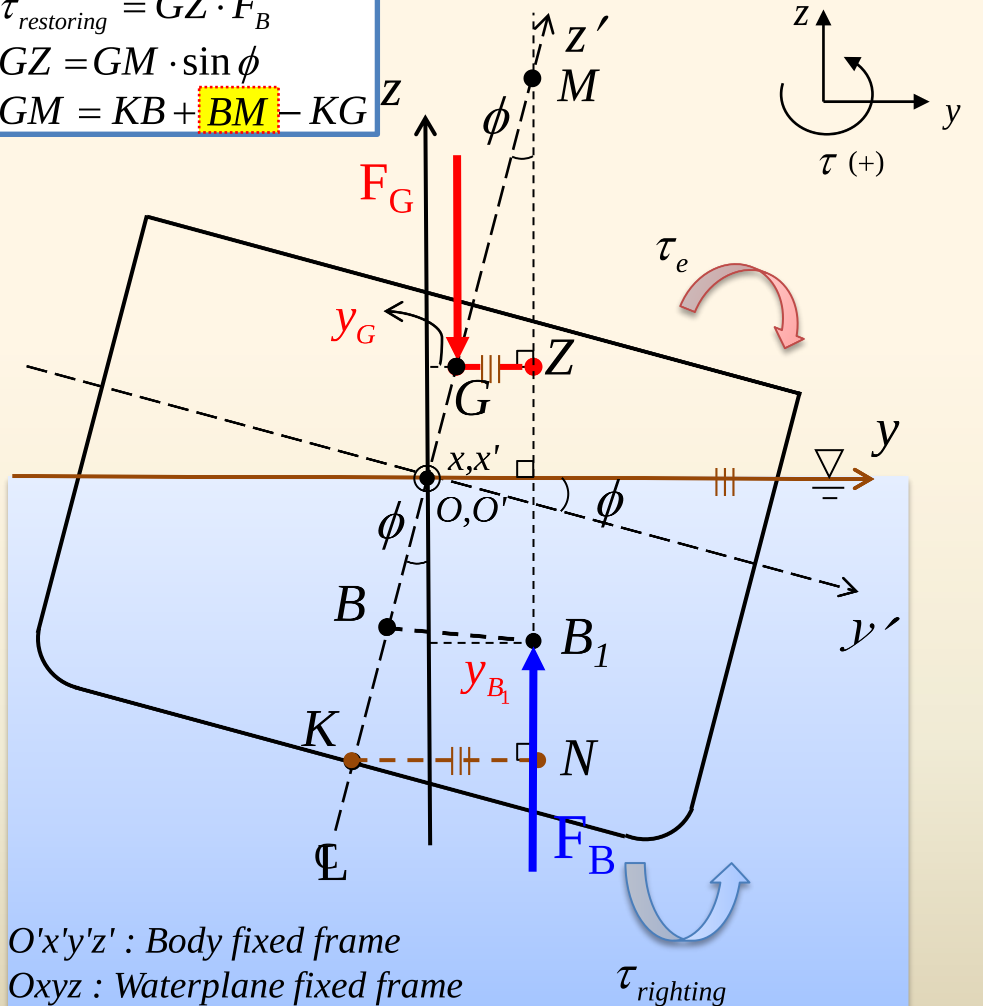
Sec.4 Transverse Stability of ship (Unstable condition)

Sec.5 Transverse Righting Moment due to Movement of Cargo

Sec.6 Calculation of Heeling Angle due to Shift of Center of Mass

Transverse Righting Moment

$$\tau_{restoring} = GZ \cdot F_B$$
$$GZ = GM \cdot \sin\phi$$
$$GM = KB + \boxed{BM} - KG$$



G: Center of mass B: Center of buoyancy B_1: Changed center of buoyancy

K: Keel $\mathbf{F}_G$: Weight of ship $\mathbf{F}_B$: Buoyant force acting on ship

Z: The intersection of the line of buoyant force through B_1 with the transverse line through G

M: The intersection of the line of buoyant force through B_1 with the centerline of the ship

- **Righting Moment :** Moment to return the ship to the upright floating position (Righting moment, Moment of statical stability))

- **Transverse Righting moment**

$$\tau_{righting} = (-y_G + y_{B_1}) \cdot F_B \mathbf{i} = \underbrace{GZ}_{\text{Righting arm}} \cdot F_B \mathbf{i}$$

- **Righting Arm** (GZ)

① From direct calculation

$$GZ = -y_G + y_{B_1}$$

We should know y_G, y_{B1} in waterplane fixed frame

② From geometrical figure with assumption that M does not change within small angle of heel (about 10°)

$$GZ = GM \cdot \sin\phi$$

GM is related to below equation by geometrical figure

$$GM = KB + \boxed{BM} - KG$$

KB : Vertical center of buoyancy

BM : Transverse Metacenter Radius

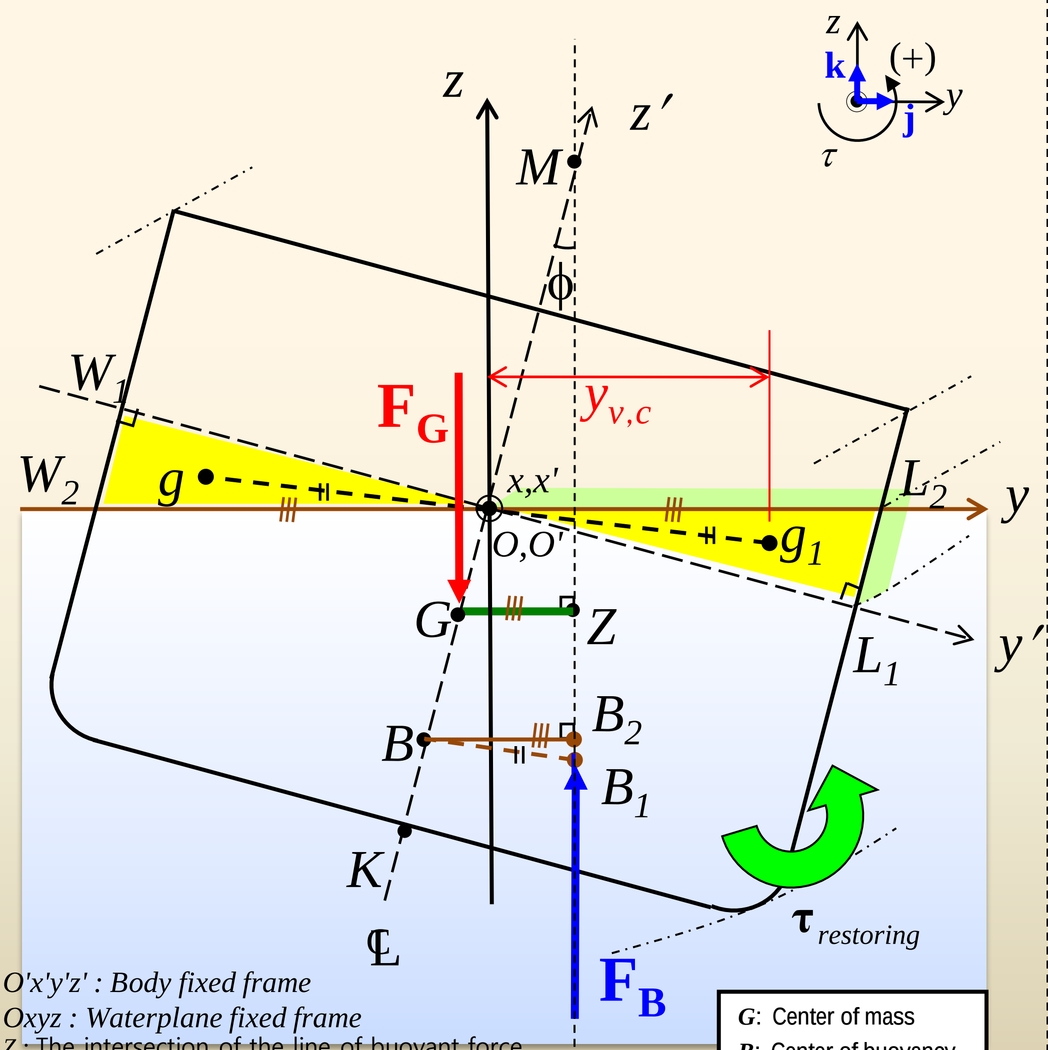
KG : Vertical center of mass of ship

KB, BM is determined by the shape of ship

KG is determined by loading condition of cargo

Calculation of BM (1)

(BM : Transverse Metacentric Radius)



O'x'y'z' : Body fixed frame

Oxyz : Waterplane fixed frame

Z : The intersection of the line of buoyant force through B1 with the transverse line through G

B_1: Changed center of buoyancy

B_2: The intersection of the line of buoyant force through B1 with the transverse line through B

M : The intersection of the line of buoyant force through B1 with the centerline of the ship

Let's derive BM in case of simple section like a wall sided ship.

- **Wall sided ship**

  : When a ship is in upright position, a ship which have perpendicular side shell to waterplane is called "wall sided ship".

Assumption

1. Wall sided ship

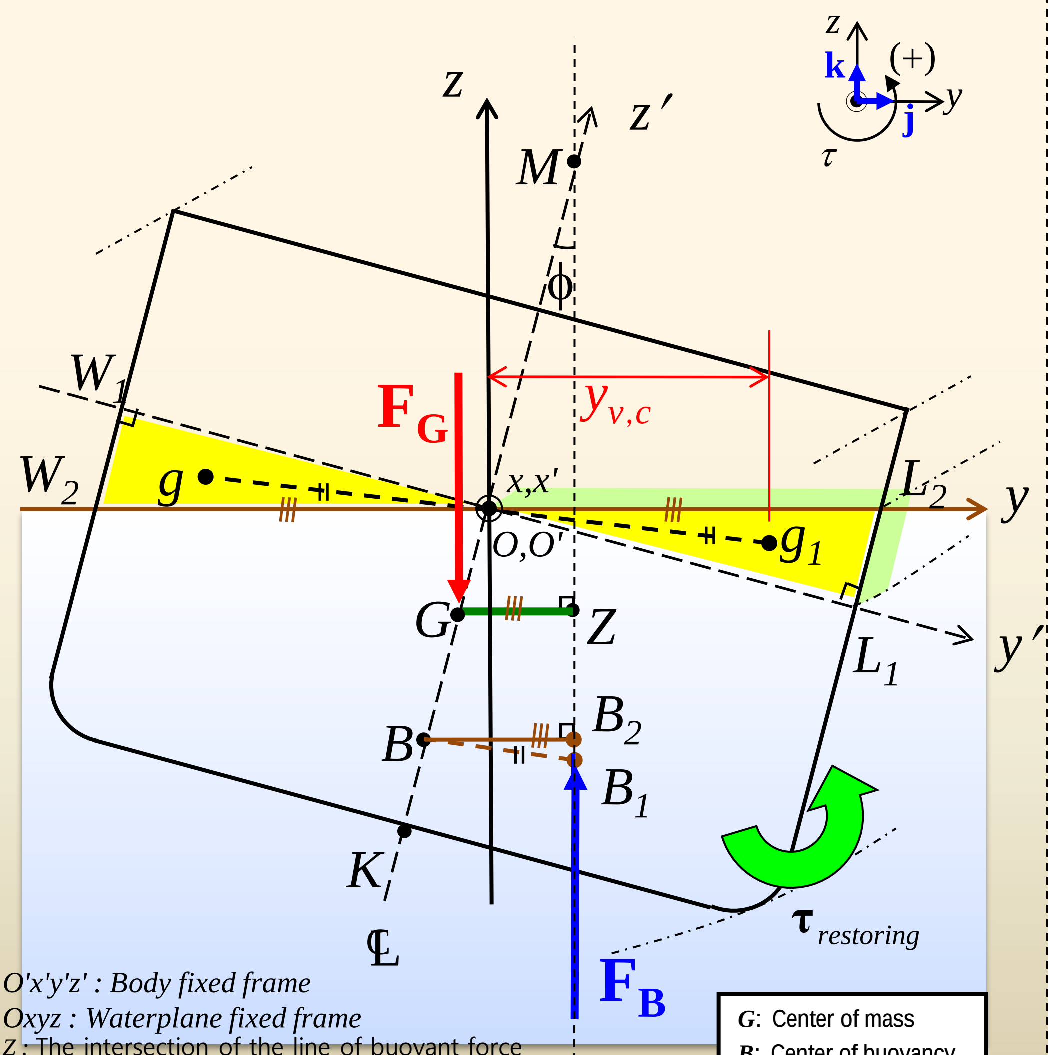
- Submerged volume is same with emerged volume when the ship is heeled. (A ship is heel without change of displacement volume)

2. A main deck is not flooded.

3. Center of rotation is not changed.

(*M* is not changed)

Calculation of BM (2)
(BM : Transverse Metacentric Radius)



The shape of displacement volume is changed as a ship is heeled.

Relation between moving distance of center of changed displacement volume and moving distance of center of center of buoyancy is as follows.

$$\rho g \nabla \cdot BB_1 = \rho g\, v \cdot gg_1$$

$$BB_1 = \frac{\rho g\, v}{\rho g \nabla} \cdot gg_1$$

$$(gg_1 = 2Og_1)$$

$$BB_1 = \frac{v}{\nabla} \cdot 2Og_1$$

O'x'y'z' : Body fixed frame

Oxyz : Waterplane fixed frame

Z : The intersection of the line of buoyant force through B1 with the transverse line through G

G: Center of mass
B: Center of buoyancy
$\mathbf{F}_G$: Weight of ship (=**W**)
$\mathbf{F}_B$: Buoyancy (=$\rho g \nabla$)

∇ : Displacement volume

v : Changed displacement volume

BB_1: Moving distance of center of buoyancy

gg_1 : Moving distance of center of changed displacement volume

B_1: Changed center of buoyancy

B_2: The intersection of the line of buoyant force through B1 with the transverse line through B

M : The intersection of the line of buoyant force through B1 with the centerline of the ship

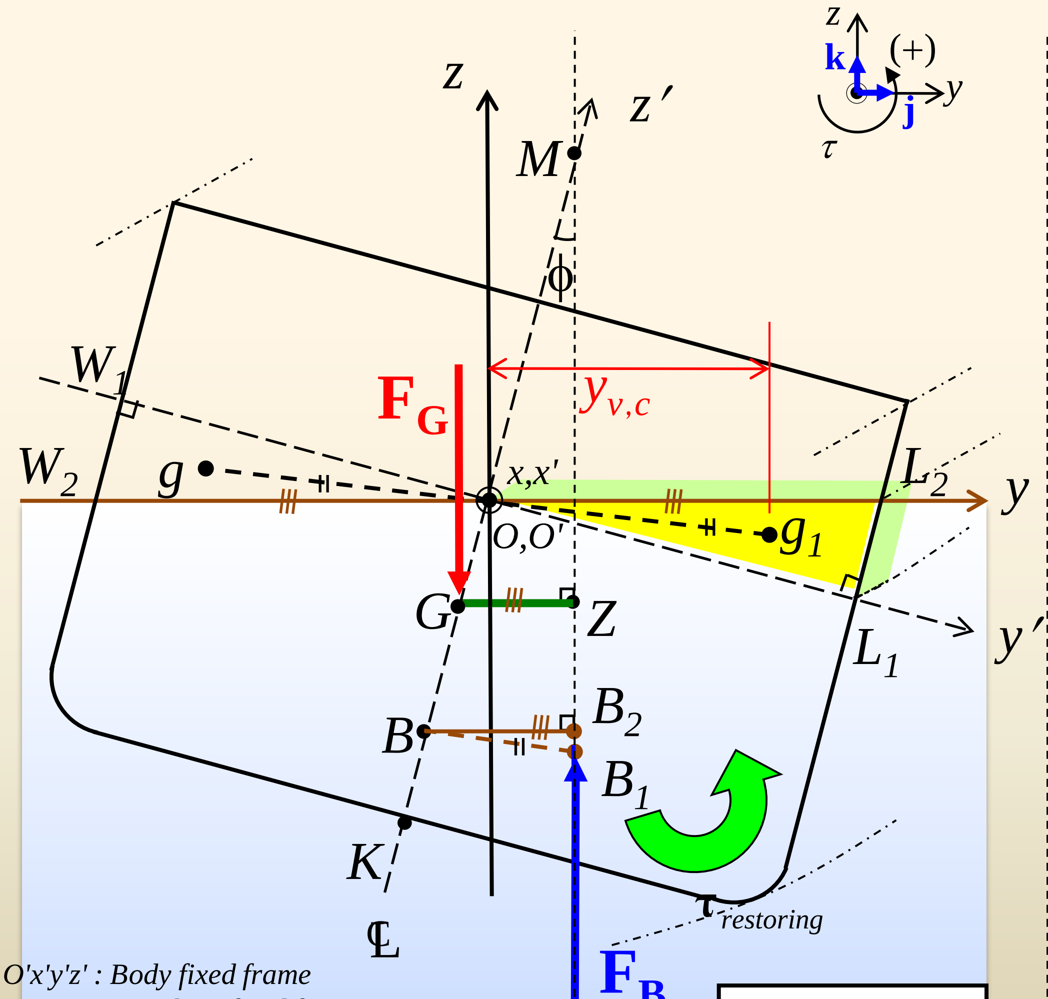
Calculation of BM (3)

(BM : Transverse Metacentric Radius)

$y_{v,c}$: y coordinate of changed displacement volume

Assumption

1. Wall sided ship.
2. A main deck is not flooded.
3. Center of rotation is not changed



O'x'y'z' : Body fixed frame

Oxyz : Waterplane fixed frame

Z : The intersection of the line of buoyant force through B1 with the transverse line through G

B_1: Changed center of buoyancy

B_2: The intersection of the line of buoyant force through B1 with the transverse line through B

M : The intersection of the line of buoyant force through B1 with the centerline of the ship

G: Center of mass

B: Center of buoyancy

$\mathbf{F}_G$: Weight of ship (=**W**)

$\mathbf{F}_B$: Buoyancy (=**ρg**∇)

$$BB_1 = \frac{v}{\nabla}\cdot 2Og_1$$

$$\angle B_1BB_2 = \angle g_1OL_2$$

$$\underbrace{BB_1\cos(\angle B_1BB_2)}_{BB_2} = \frac{v}{\nabla}\cdot 2\underbrace{Og_1\cos(\angle g_1OL_2)}_{y_{v,c}}$$

$$BB_2 = \frac{v}{\nabla}\cdot 2y_{v,c}$$

$$\boxed{BB_2 = \frac{2}{\nabla}\cdot v\cdot y_{v,c}}$$

L.H.S R.H.S

(L.H.S)

$$BB_2 = BM\cdot\sin|\phi|$$

In this case, an angle of heel is (-)

$$\boxed{BB_2 = -BM\cdot\sin\phi}$$

∇ : Displacement volume

v : Changed displacement volume

BB_1: Moving distance of center of buoyancy

gg_1 : Moving distance of changed displacement volume

Calculation of BM (4)
(BM : Transverse Metacentric Radius)

$$BB_2 = \frac{2}{\nabla} \cdot v \cdot y_{v,c}$$

$$A(x') = \int dA' = \iint dy'\,dz'$$
$$\nabla = \int dV = \iiint dz'\,dy'\,dx' = \int_{A.P}^{F.P} A(x')\,dx'$$

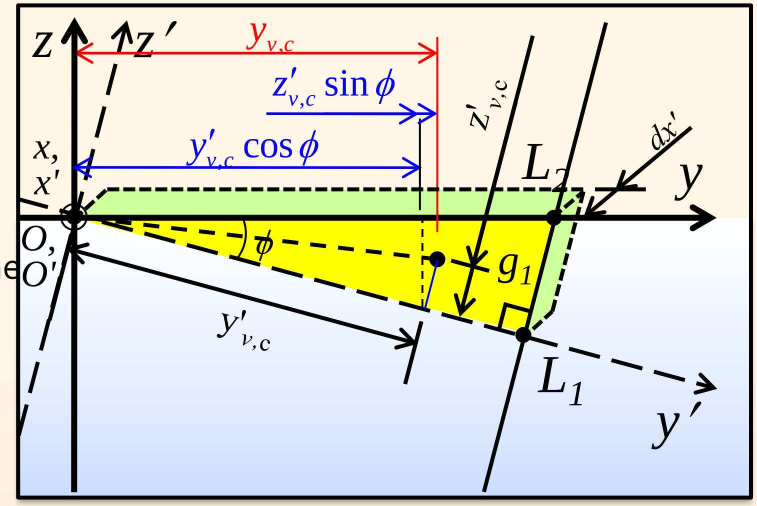
(R.H.S)

$$\frac{2}{\nabla} \cdot v \cdot y_{v,c}$$

Represent center of buoyancy with respect to waterplane fixed frame$(y_{v,c}, z_{v,c})$ as one with respect to body fixed frame

$$\begin{pmatrix} y_{v,c} \\ z_{v,c} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} y'_{v,c} \\ z'_{v,c} \end{pmatrix}$$

$$y_{v,c} = y'_{v,c} \cdot \cos\phi - z'_{v,c} \cdot \sin\phi$$



$$= \frac{2}{\nabla} \cdot v \cdot (y'_{v,c} \cdot \cos\phi - z'_{v,c} \cdot \sin\phi)$$

$$= \frac{2}{\nabla} \boxed{\cdot v \cdot y'_{v,c}} \cdot \cos\phi \quad - \quad \frac{2}{\nabla} \boxed{\cdot v \cdot z'_{v,c}} \cdot \sin\phi$$

Transverse moment of volume with respect to body fixed frame.

Vertical moment of volume with respect to body fixed frame.

$y_{v,c}$: y coordinate of center of changed displacement volume
v : Changed displacement volume
∇ : Displacement volume

Calculation of BM (5)

(BM : Transverse Metacentric Radius)

$$BB_2 = \frac{2}{\nabla}\cdot v\cdot y_{v,c}$$

$$A(x') = \int dA' = \iint dy'\,dz'$$
$$\nabla = \int dV = \iiint dz'\,dy'\,dx' = \int_{A.P}^{F.P} A(x')\,dx'$$

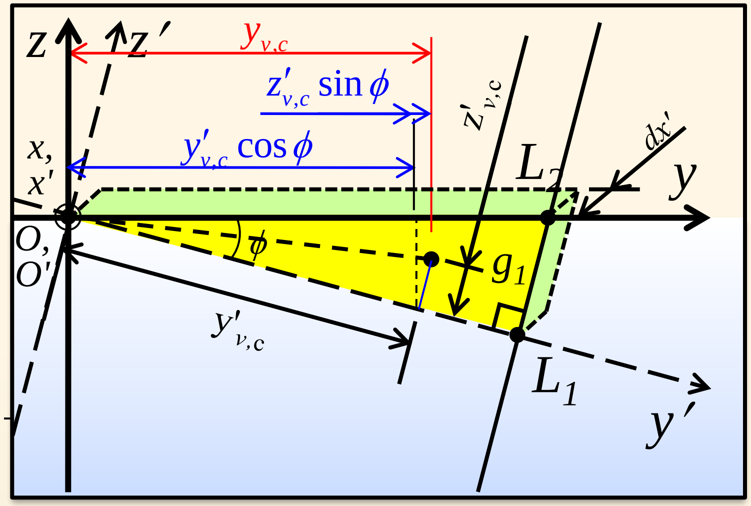
(R.H.S)

$$\frac{2}{\nabla}\cdot v\cdot\ y_{v,c}$$

$$= \frac{2}{\nabla}\boxed{\cdot v\cdot y'_{v,c}}\cdot\cos\phi \quad - \quad \frac{2}{\nabla}\boxed{\cdot v\cdot z'_{v,c}}\cdot\sin\phi$$

Transverse moment of volume with respect to body fixed frame.

Vertical moment of volume with respect to body fixed frame.



How to calculate 1st moment of volume?

: It can be calculated by integral of 1st moment of area over the the length of ship.

$$M_{v,y'} = v\cdot y'_{v,c} = \iiint y'\,dy'\,dz'\,dx' = \int_{x_A}^{x_F} A(x')\,y'_c\,dx'$$

$, y'_c$: Transverse center of section with respect to body fixed frame

$$M_{v,z'} = v\cdot z'_{v,c} = \iiint z'\,dy'\,dz'\,dx' = \int_{x_A}^{x_F} A(x')\,z'_c\,dx'$$

$, z'_c$: Vertical center of section with respect to body fixed frame

$$= \frac{2}{\nabla}\cos\phi\int_{x_A}^{x_F}\left(A(x')\cdot y'_c\right)dx' - \frac{2}{\nabla}\sin\phi\int_{x_A}^{x_F}\left(A(x')\cdot z'_c\right)dx'$$

$y_{v,c}$: y coordinate of center of changed displacement volume

v : Changed displacement volume

∇ : Displacement volume

Calculation of BM (6)

(BM : Transverse Metacentric Radius)

$$BB_2 = \frac{2}{\nabla}\cdot v \cdot y_{v,c}$$

$$A(x') = \int dA' = \boxed{\iint dy'\,dz'}$$

$$\nabla = \int dV = \iiint \boxed{dz'\,dy'}\,dx'$$

$$= \int_{A.P}^{F.P} A(x')\,dx'$$

(R.H.S)

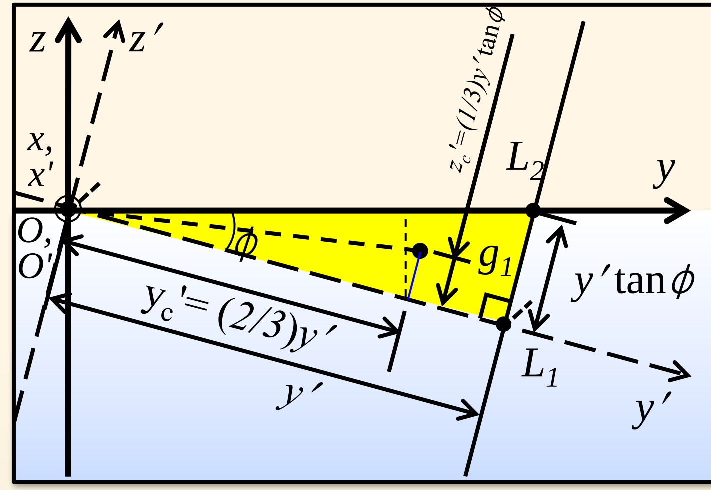
$$\frac{2}{\nabla}\cdot v \cdot \; y_{v,c}$$

$$= \frac{2}{\nabla}\cdot\boxed{v\cdot y'_{v,c}}\cdot\cos\phi \quad - \quad \frac{2}{\nabla}\cdot\boxed{v\cdot z'_{v,c}}\cdot\sin\phi$$

Transverse moment of volume with respect to body fixed frame.

Vertical moment of volume with respect to body fixed frame.

$$= \frac{2}{\nabla}\cos\phi\int_{x_A}^{x_F}\left(A(x')\cdot y'_c\right)dx' - \frac{2}{\nabla}\sin\phi\int_{x_A}^{x_F}\left(A(x')\cdot z'_c\right)dx'$$



$y_{v,c}$: y coordinate of center of changed displacement volume

v : Changed displacement volume

∇ : Displacement volume

How to calculate sectional area?

In case of triangular section

$$A(x') = \int dA = \int_0^{y'}\int_0^{z'} dz'\,dy' = \int_0^{y'}\int_0^{-y'\tan\phi} dz'\,dy'$$

$$= -\int_0^{y'}\left(y'\tan\phi\right)dy' = -\left[\frac{1}{2}y'^2\tan\phi\right]_0^{y'} = -\frac{1}{2}y'^2\tan\phi$$

How to calculate centroid of section?

$$y'_c = \frac{2}{3}y' \qquad , z'_c = -\frac{1}{3}y'\tan\phi$$,(In case of triangular section)

$$= -\frac{2}{\nabla}\cos\phi\int_{x_A}^{x_F}\left(\frac{1}{2}y'^2\tan\phi\cdot\frac{2}{3}y'\right)dx' \; - \; \frac{2}{\nabla}\sin\phi\int_{x_A}^{x_F}\left(\frac{1}{2}y'^2\tan\phi\cdot\frac{1}{3}y'\tan\phi\right)dx'$$

Calculation of BM (7)
(BM : Transverse Metacentric Radius)

$$BB_2 = \frac{2}{\nabla}\cdot v\cdot y_{v,c}$$

$$A(x') = \int dA' = \iint dy'\,dz'$$

$$\nabla = \int dV = \iiint dz'\,dy'\,dx' = \int_{A.P}^{F.P} A(x')\,dx'$$

(R.H.S)

$$\frac{2}{\nabla}\cdot v\cdot\ y_{v,c}$$

$, y_{v,c}$: y coordinate of changed displacement volume

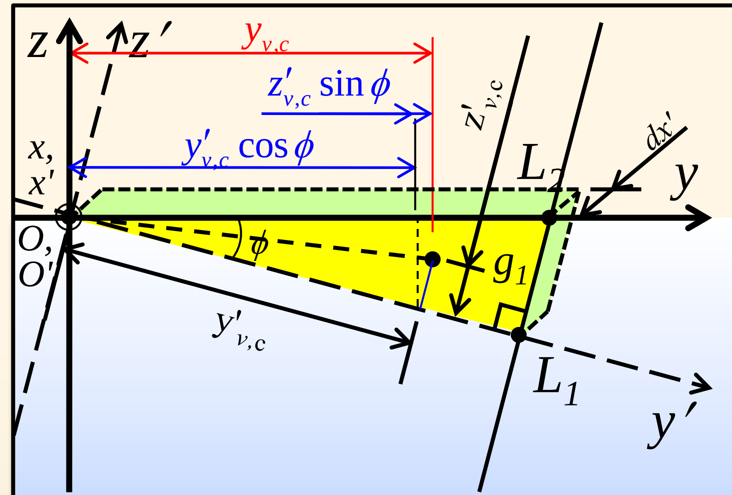
$, v$: Changed displacement volume

$, \nabla$: Displacement volume

$$= \frac{2}{\nabla}\boxed{\cdot v\cdot y'_{v,c}}\cdot\cos\phi + \frac{2}{\nabla}\boxed{\cdot v\cdot z'_{v,c}}\cdot\sin\phi$$

Transverse moment of volume with respect to body fixed frame.

Vertical moment of volume with respect to body fixed frame.



$$= \frac{2}{\nabla}\cos\phi\int_{x_A}^{x_F}\left(A(x')\cdot y'_c\right)dx' - \frac{2}{\nabla}\sin\phi\int_{x_A}^{x_F}\left(A(x')\cdot z'_c\right)dx'$$

$$= -\frac{2}{\nabla}\cos\phi\int_{x_A}^{x_F}\left(\frac{1}{2}y'^2\tan\phi\cdot\frac{2}{3}y'\right)dx' - \frac{2}{\nabla}\sin\phi\int_{x_A}^{x_F}\left(\frac{1}{2}y'^2\tan\phi\cdot\frac{1}{3}y'\tan\phi\right)dx'$$

$$= -\frac{1}{\nabla}\cos\phi\tan\phi\frac{2}{3}\int_{x_A}^{x_F}y'^3dx' - \frac{1}{2\nabla}\sin\phi\tan^2\phi\frac{2}{3}\int_{x_A}^{x_F}y'^3dx'$$

$$\because I_T = \frac{2}{3}\int_{x_A}^{x_F}y'^3dx'$$

$$= -\frac{1}{\nabla}\cos\phi\tan\phi\, I_T - \frac{1}{2\nabla}\sin\phi\tan^2\phi\, I_T$$

$$= \boxed{-\frac{1}{\nabla}\left(\sin\phi\ I_T + \frac{1}{2}\sin\phi\tan^2\phi\, I_T\right)}$$

$y_{v,c}$: y coordinate of center of changed displacement volume

v : Changed displacement volume

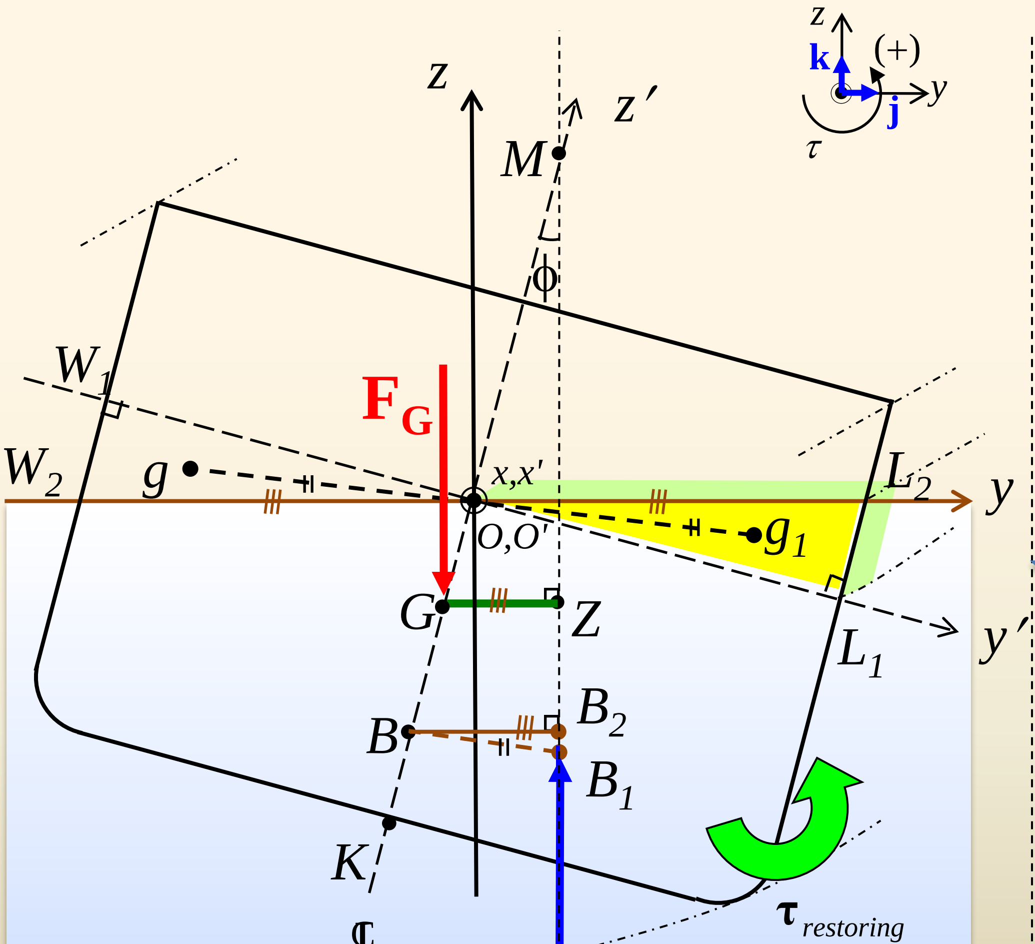
∇ : Displacement volume

Calculation of BM (8)

(BM : Transverse Metacentric Radius)

Assumption

1. Wall sided ship.
2. A main deck is not flooded.
3. Center of rotation is not changed



O'x'y'z' : Body fixed frame

Oxyz : Waterplane fixed frame

Z : The intersection of the line of buoyant force through B1 with the transverse line through G

B$_1$: Changed center of buoyancy

B$_2$: The intersection of the line of buoyant force through B1 with the transverse line through B

M : The intersection of the line of buoyant force through B1 with the centerline of the ship

G: Center of mass

B: Center of buoyancy

$\mathbf{F}_G$: Weight of ship (=**W**)

$\mathbf{F}_B$: Buoyancy (=**ρg**▽)

- Derivation of BM

L.H.S R.H.S

$$\boxed{BB_2 = \frac{2}{\nabla}\cdot v\cdot y_{v,c}}$$

(L.H.S) $BB_2 = -BM\cdot\sin\phi$

(R.H.S) $\frac{2}{\nabla}\cdot v\cdot\ y_{v,c} = -\frac{1}{\nabla}\left(\sin\phi\ I_T + \frac{1}{2}\sin\phi\tan^2\phi\, I_T\right)$

$$-BM\cdot\sin\phi = -\frac{1}{\nabla}(\sin\phi\, I_T + \frac{1}{2}\sin\phi\tan^2\phi\, I_T)$$

$$BM = \frac{1}{\nabla}(\ I_T + \frac{1}{2}\tan^2\phi\, I_T)$$

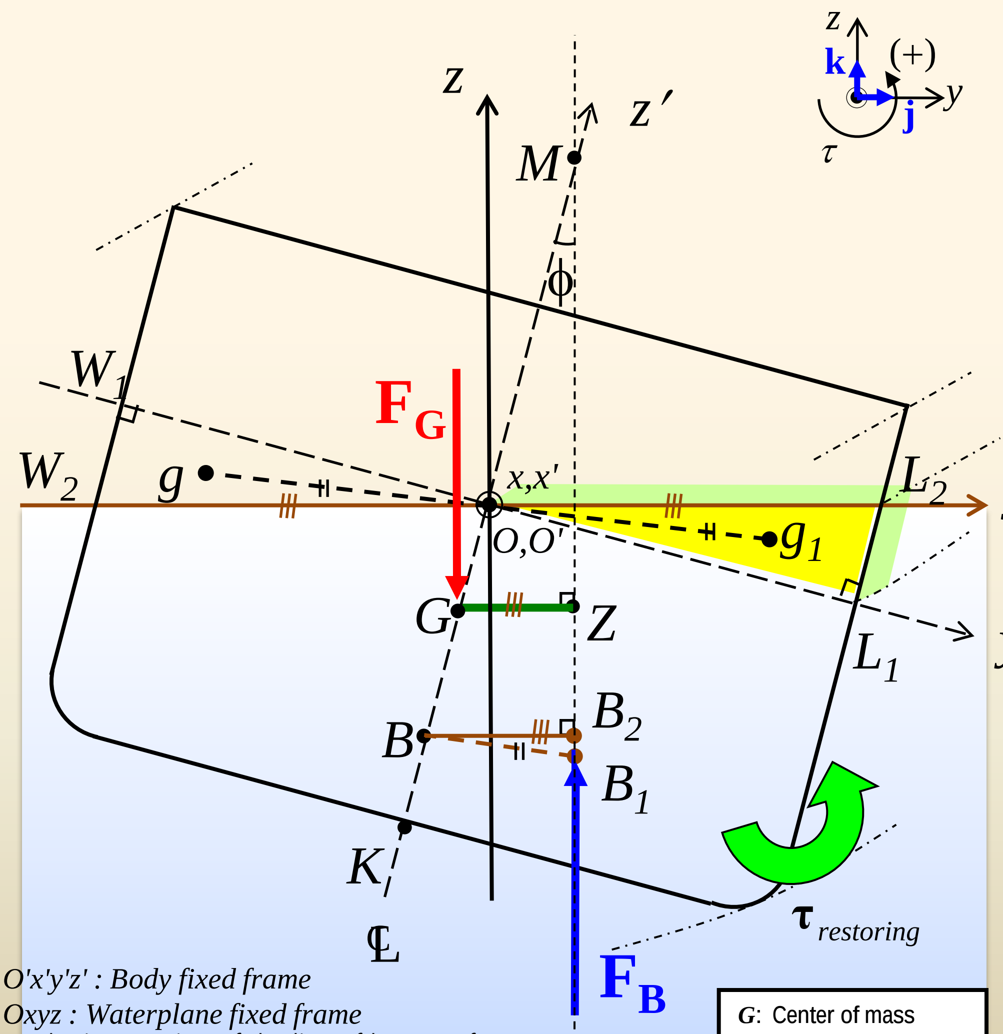
$$\boxed{BM = \frac{I_T}{\nabla}(1+\frac{1}{2}\tan^2\phi)}$$

Calculation of BM (9)

(BM : Transverse Metacentric Radius)

Assumption

1. Wall sided ship.
2. A main deck is not flooded.
3. Center of rotation is not changed
4. An angle of heel ϕ is small.



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M : The intersection of the line of buoyant force through B1 with the centerline of the ship

G: Center of mass

B: Center of buoyancy

F_G : Weight of ship (=W)

F_B : Buoyancy (=$\rho g \nabla$)

- Derivation of BM in case of small angle of heel

$$BM = \frac{I_T}{\nabla}\left(1+\frac{1}{2}\tan^2\phi\right)$$

if ϕ is small

$\tan^2\phi \approx \phi^2 = 0$

$$BM = \frac{I_T}{\nabla}$$

If we assume that ϕ is small,

$$BM = \frac{I_T}{\nabla}$$

which is generally known as BM.

That BM does not consider change of center of buoyancy in vertical direction.

In order to distinguish those, we will indicate two as follows

$$BM_0 = \frac{I_T}{\nabla}\left(1+\frac{1}{2}\tan^2\phi\right)$$

,(Considering change of center of buoyancy in vertical direction)

$$BM = \frac{I_T}{\nabla}$$

,(Not considering change of center of buoyancy in vertical direction)

Calculation of GZ

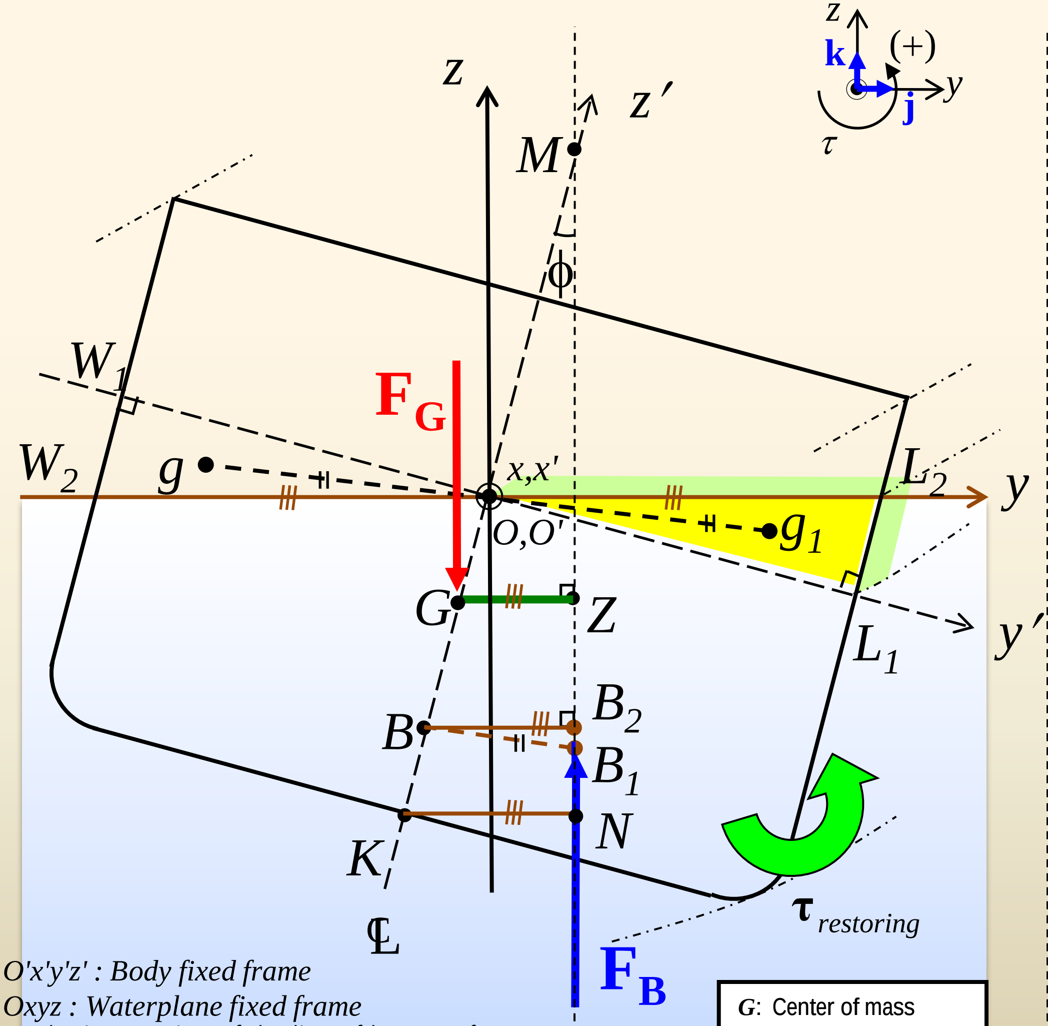
(GZ : Righting arm)

$$\overline{BM}_0 = \frac{I_T}{\nabla}\left(1+\frac{1}{2}\tan^2\phi\right)$$

$$\overline{BM} = \frac{I_T}{\nabla}$$

Assumption

1. Wall sided ship.
2. A main deck is not flooded.
3. Center of rotation is not changed



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B: Center of buoyancy
$\mathbf{F}_G$: Weight of ship (=**W**)
$\mathbf{F}_B$: Buoyancy (=$\rho g \nabla$)

- Derivation of GZ

$$GZ = KN - KG\sin\phi$$

$$= KM\sin\phi - KG\sin\phi$$

$$= (KB + BM_0)\sin\phi - KG\sin\phi$$

$$\downarrow \left(BM_0 = \frac{I_T}{\nabla}\left(1+\frac{1}{2}\tan^2\phi\right) \right)$$

$$= \left(KB + \frac{I_T}{\nabla}\left(1+\frac{1}{2}\tan^2\phi\right)\right)\sin\phi - KG\sin\phi$$

$$= \left(KB + \frac{I_T}{\nabla} - KG\right)\sin\phi + \frac{1}{2}\frac{I_T}{\nabla}\tan^2\phi\sin\phi$$

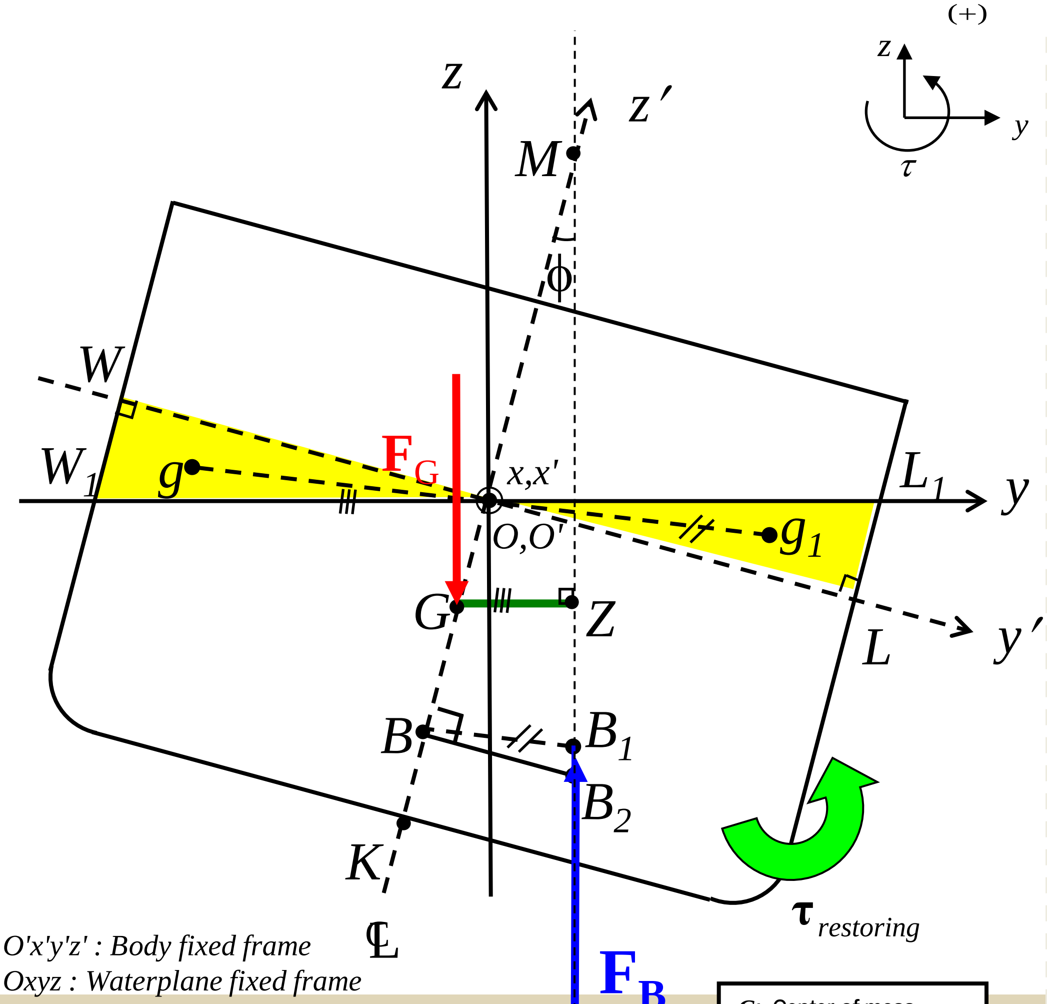
$$\downarrow \left(BM = \frac{I_T}{\nabla} \right)$$

$$\boxed{= (KB + BM - KG)\sin\phi + \frac{1}{2}BM\tan^2\phi\sin\phi}$$

$$\boxed{= GM\sin\phi + \frac{1}{2}BM\tan^2\phi\sin\phi}$$

Righting arm in wall sided ship!

(Ref.) Calculation of BM – Another method (1)



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B_1: Changed center of buoyancy

B_2: The point at which a vertical line through B1 crosses parallel line with line WL through B

M: The intersection of the line of buoyant force through B1 with the centerline of the ship

G: Center of mass
B: Center of buoyancy
F_G : Weight of ship (=W)
F_B : Buoyancy (=$\rho g \nabla$)

BM : Transverse Metacenter Radius

Assumption

1. **Wall sided ship.**
2. **A main deck is not flooded.**
3. **Center of rotation is not changed**

Displacement volume of WOW_1

is same with displacement volume LOL_1

$$BB_1 // gg_1 \qquad , BB_1 = \frac{v}{\nabla} gg_1$$

$$\tan\phi = \frac{BB_2}{BM} \quad \Rightarrow \quad BM = \frac{BB_2}{\tan\phi}$$

Assumption 4. An angle of heel ϕ is small

$$BM = \frac{BB_2}{\tan\phi} \approx \frac{BB_1}{\tan\phi} = \frac{v \cdot gg_1}{\nabla \cdot \tan\phi}$$

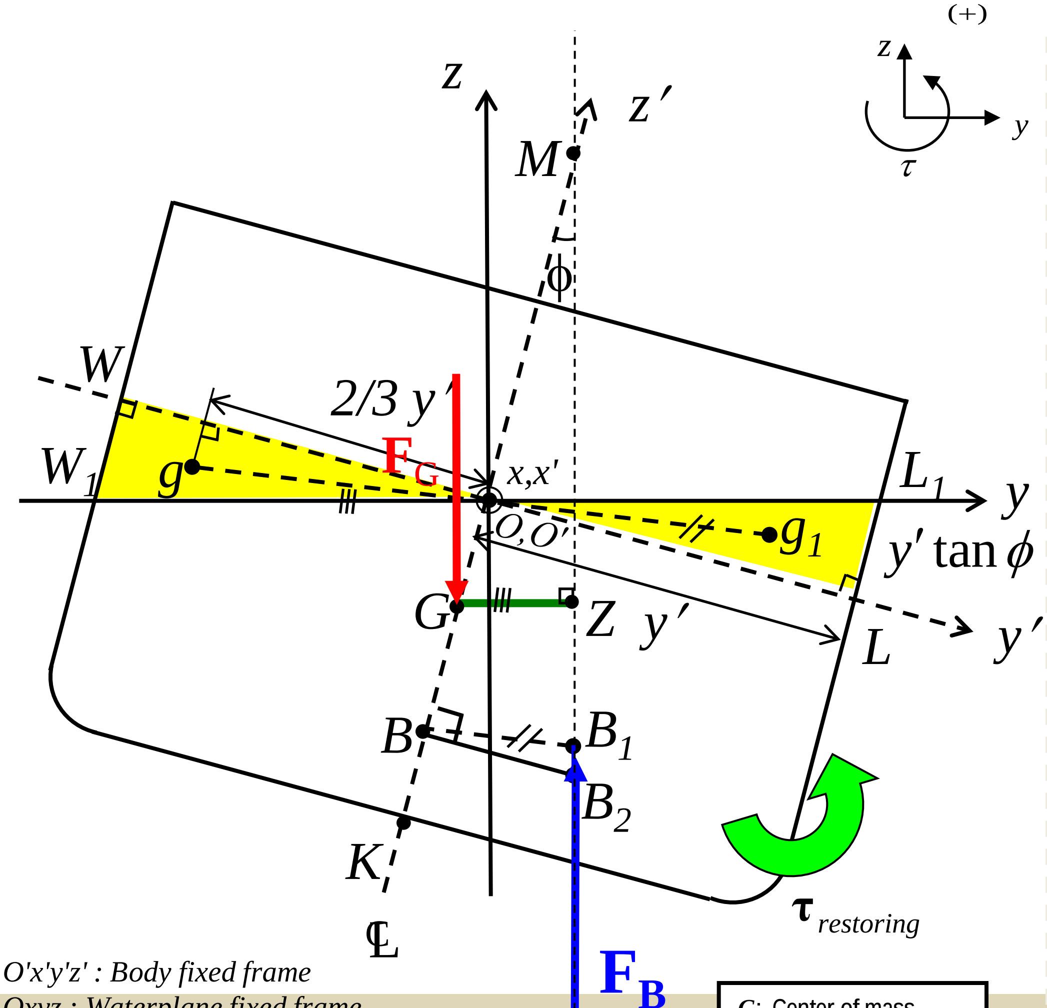
[Ref.] Calculation of BM – Another method (2)

$$\overline{BM} = \frac{\overline{BB_2}}{\tan\phi} \approx \frac{\overline{BB_1}}{\tan\phi} = \frac{v \cdot gg_1}{\nabla \cdot \tan\phi}$$

$$BM \approx \frac{v \cdot gg_1}{\nabla \cdot \tan\phi}$$

Assumption

1. Wall sided ship.
2. A main deck is not flooded.
3. Center of rotation is not changed
4. An angle of heel ϕ is small.



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$\mathbf{F_G}$: Weight of ship (=**W**)
$\mathbf{F_B}$: Buoyancy (=$\rho g \nabla$)

Sectional Area of WOW_1 and LOL_1 are as follows

$$\frac{1}{2} y' \cdot y' \tan\phi$$

Differential area of ship = dx'

Differential volume of WOW_1 and LOL_1

$$dv = \frac{1}{2} y' \cdot y' \tan\phi \cdot dx'$$

Because ϕ is small

$$gg_1 = 2 \cdot Og \approx 2 \cdot \frac{2}{3} y'$$

$$v \cdot gg_1 = \int dv \cdot gg_1 = \tan\phi \cdot \frac{2}{3} \int_0^L y'^3 dx'$$

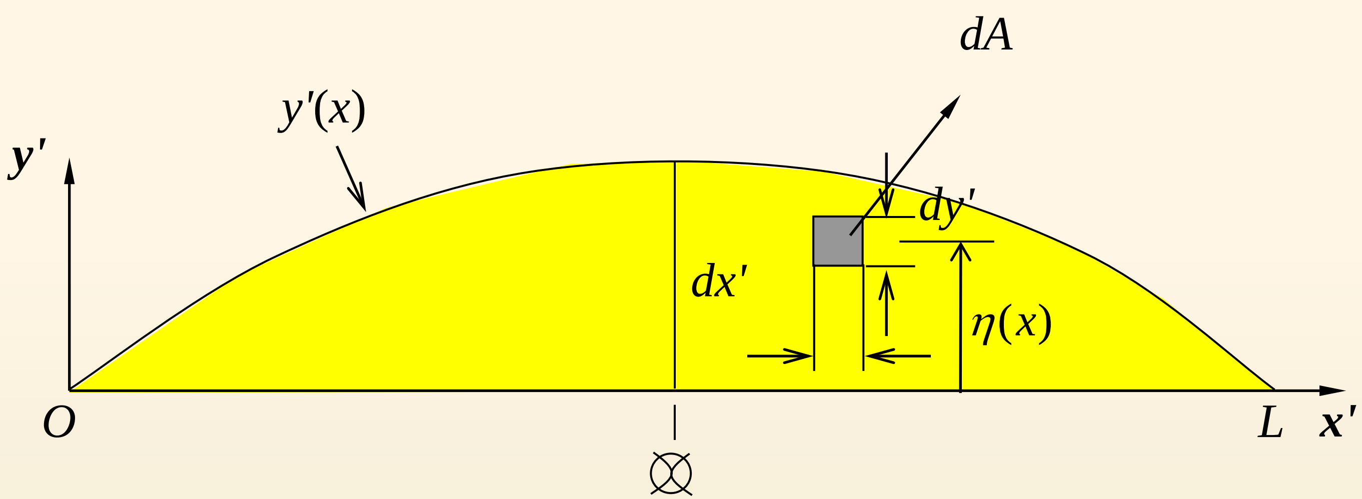
$$= \tan\phi \cdot I_T \quad , \quad \left(I_T = \frac{2}{3} \int_0^L y'^3 dx' \right)$$

I_T : 2nd moment of waterplane area about x′ axis with respect to body fixed frame.

$$BM = \frac{v \cdot gg_1}{\nabla \cdot \tan\phi} = \frac{I_T \cdot \tan\phi}{\nabla \cdot \tan\phi} = \frac{I_T}{\nabla}$$

$$\boxed{BM = \frac{I_T}{\nabla}}$$

(Ref.) Transverse Moment of Inertia (I_T)



$$I_{T,\text{half breadth}} = \int \eta(x)^2 dA = \int_0^L \int_0^{y'(x)} \eta(x)^2 dy' dx' = \int_0^L \frac{\{y'(x)\}^3}{3} dx'$$

2nd moment of waterplane area about x' axis is as follows

$$\boxed{I_T = \frac{2}{3}\int_0^L \{y'(x)\}^3 dx'}$$

Sec.1 Calculation of Center of Buoyancy

Sec.2 Calculation of BM, GZ in Wall Sided Ship

Sec.3 Inclining Test

Sec.4 Transverse Stability of ship (Unstable condition)

Sec.5 Transverse Righting Moment due to Movement of Cargo

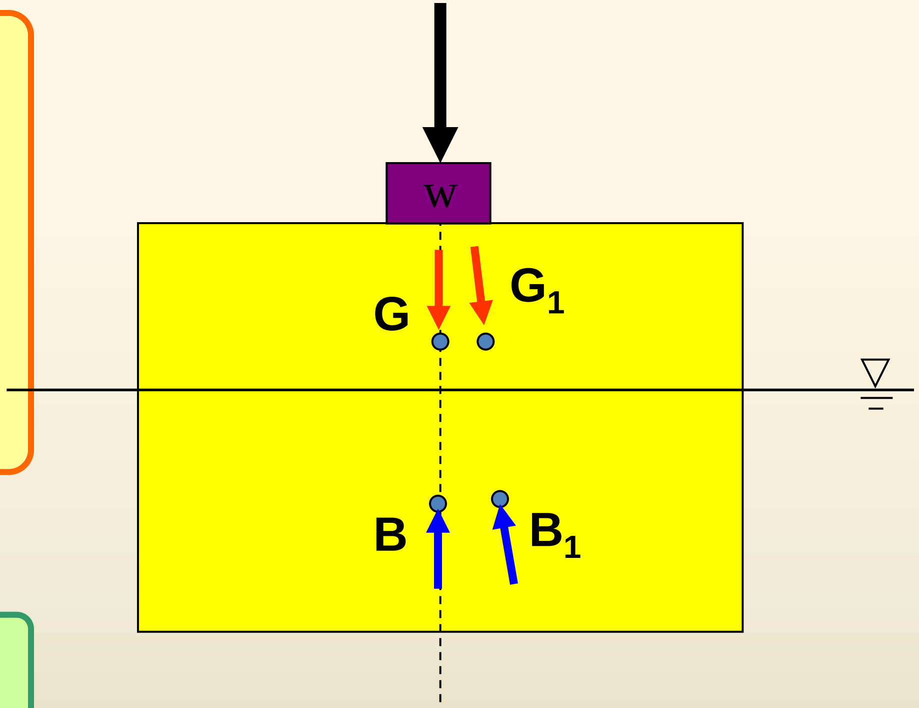
Sec.6 Calculation of Heeling Angle due to Shift of Center of Mass

Inclining Test (1)

$$\tau_{restoring} = GZ \cdot F_B$$
$$GZ = GM \cdot \sin\phi$$
$$GM = KB + BM - \boxed{KG}$$

GIVEN

- Draft
- Weight of ship(W)
- Hydrostatic values (KB,BM)
- Angle of heel ϕ when cargo is moved rightward through a distance ‘d’

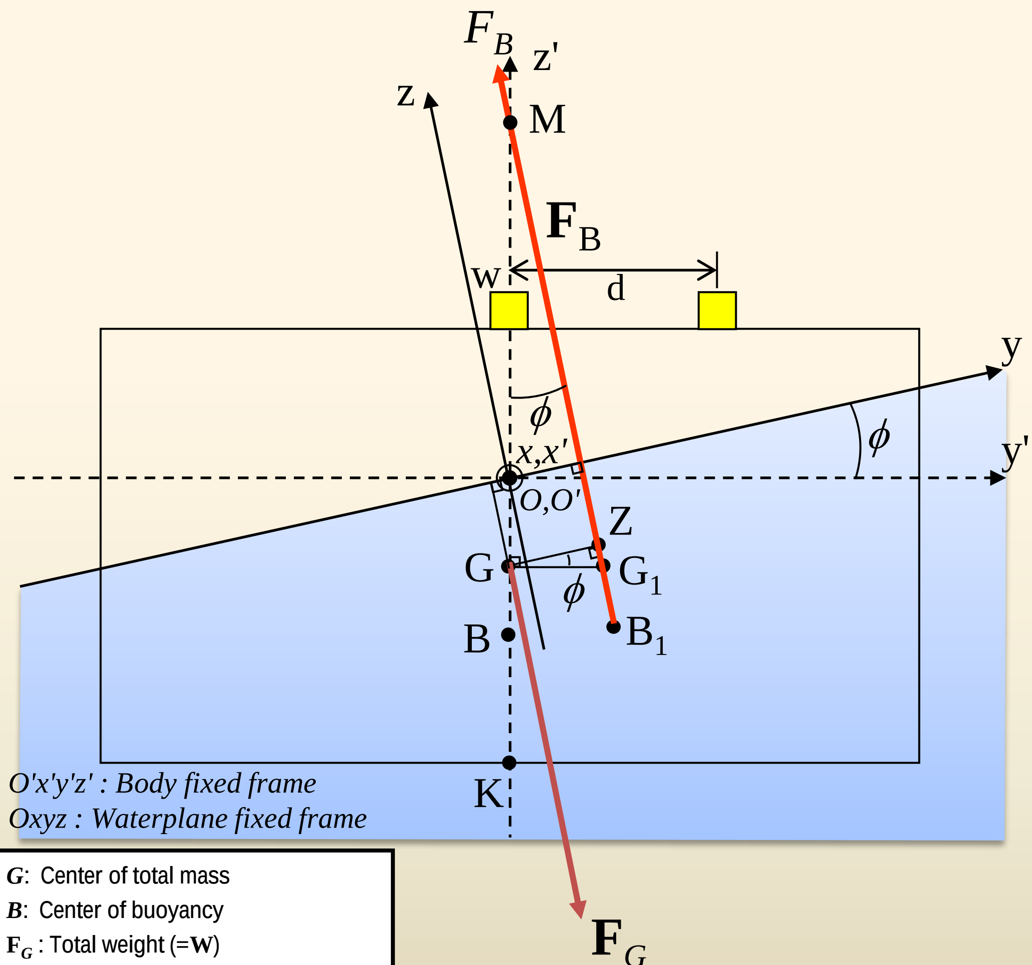


FIND

- KG (Transverse center of gravity)

Inclining Test (2)

$$\tau_{restoring} = GZ \cdot F_B$$
$$GZ = GM \cdot \sin\phi$$
$$GM = KB + BM - \boxed{KG}$$



G: Center of total mass
B: Center of buoyancy
$\mathbf{F}_G$: Total weight (=**W**)
$\mathbf{F}_B$: Buoyant force (=$\rho g \nabla = \Delta g$)
d : Moving distance of cargo
ϕ: An angle of heel
M : Metacenter

If we know an angle of heel ϕ, we can calculate KG.

Shift of center of total mass

$$GG_1 = \frac{w \cdot d}{W}$$

Transverse heeling moment

$$W \cdot GG_1 \cos\phi$$

Transverse righting moment

$$F_B \cdot GZ \approx F_B \cdot GM \sin\phi$$

Static equilibrium of moment

$$\cancel{W} \cdot GG_1 \cos\phi = \cancel{F_B} \cdot GM \sin\phi$$

$$\therefore GM = \frac{GG_1}{\tan\phi} = \frac{w \cdot d}{W \cdot \tan\phi}$$

KG

$$GM = KB + BM - KG$$

$$KG = \underset{\text{known}}{KB} + \underset{\text{known}}{BM} - \frac{w \cdot d}{W \cdot \tan\phi}$$

Sec.1 Calculation of Center of Buoyancy

Sec.2 Calculation of BM, GZ in Wall Sided Ship

Sec.3 Inclining Test

Sec.4 Transverse Stability of ship (Unstable condition)

Sec.5 Transverse Righting Moment due to Movement of Cargo

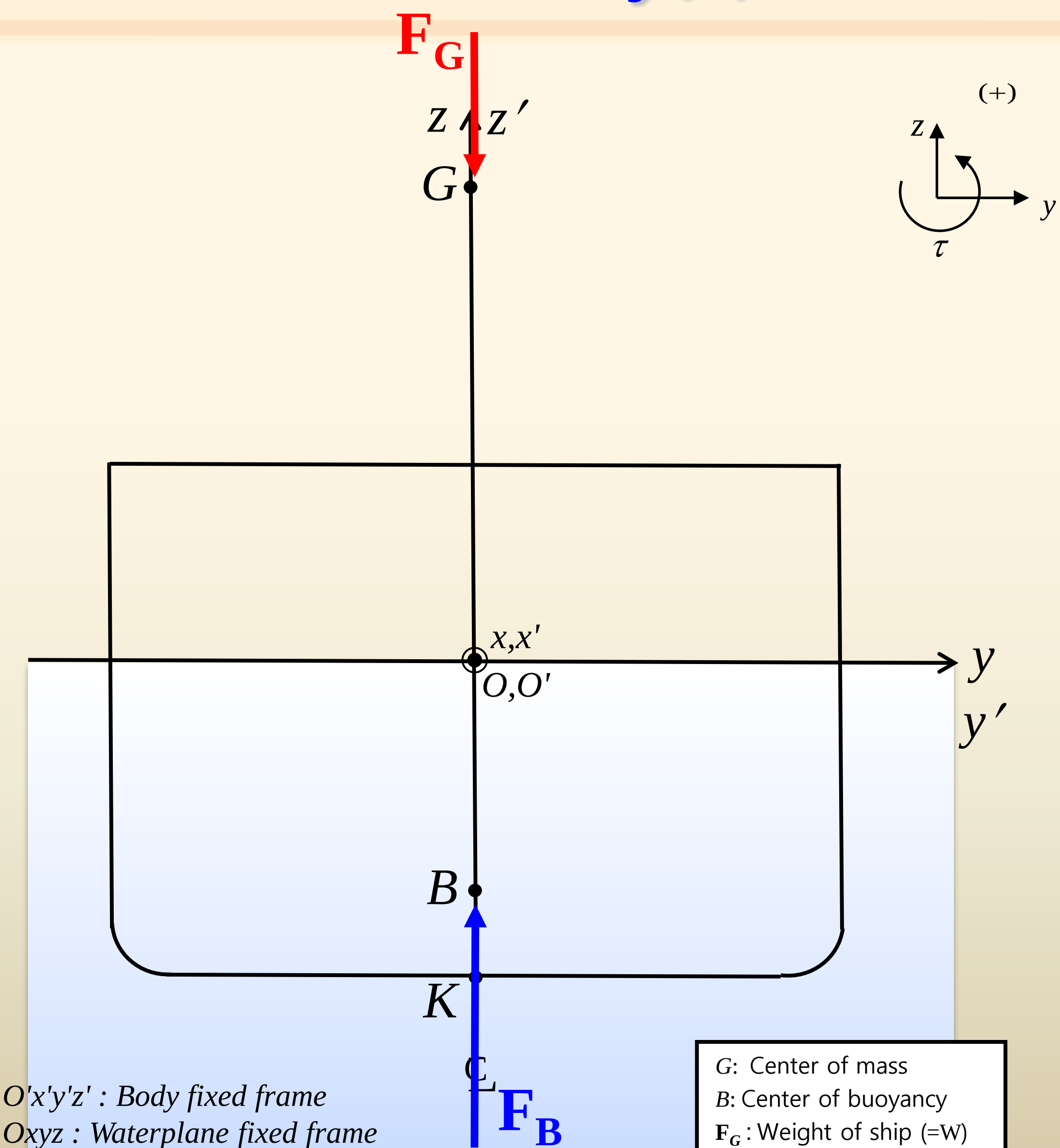
Sec.6 Calculation of Heeling Angle due to Shift of Center of Mass

Transverse Stability(1)

Heel case in ship hydrostaics (ϕ :angle of heel)

$$0 = \mathbf{M}_{Gravity}(\phi+\Delta\phi) + \mathbf{M}_{Buoyancy}(\phi+\Delta\phi) + \mathbf{M}_{external}$$

$$0 = \mathbf{r}_G \times \mathbf{F}_{Gravity}(\phi+\Delta\phi) + \mathbf{r}_B \times \mathbf{F}_{Buoyancy}(\phi+\Delta\phi) + \mathbf{M}_{external}$$



G: Center of mass
B: Center of buoyancy
$\mathbf{F}_G$: Weight of ship (=W)
$\mathbf{F}_B$: Buoyant force acting on ship (=ρg▽)

① $$\sum \mathbf{F} = \mathbf{F}_G + \mathbf{F}_B = 0$$

(static equilibrium of force)

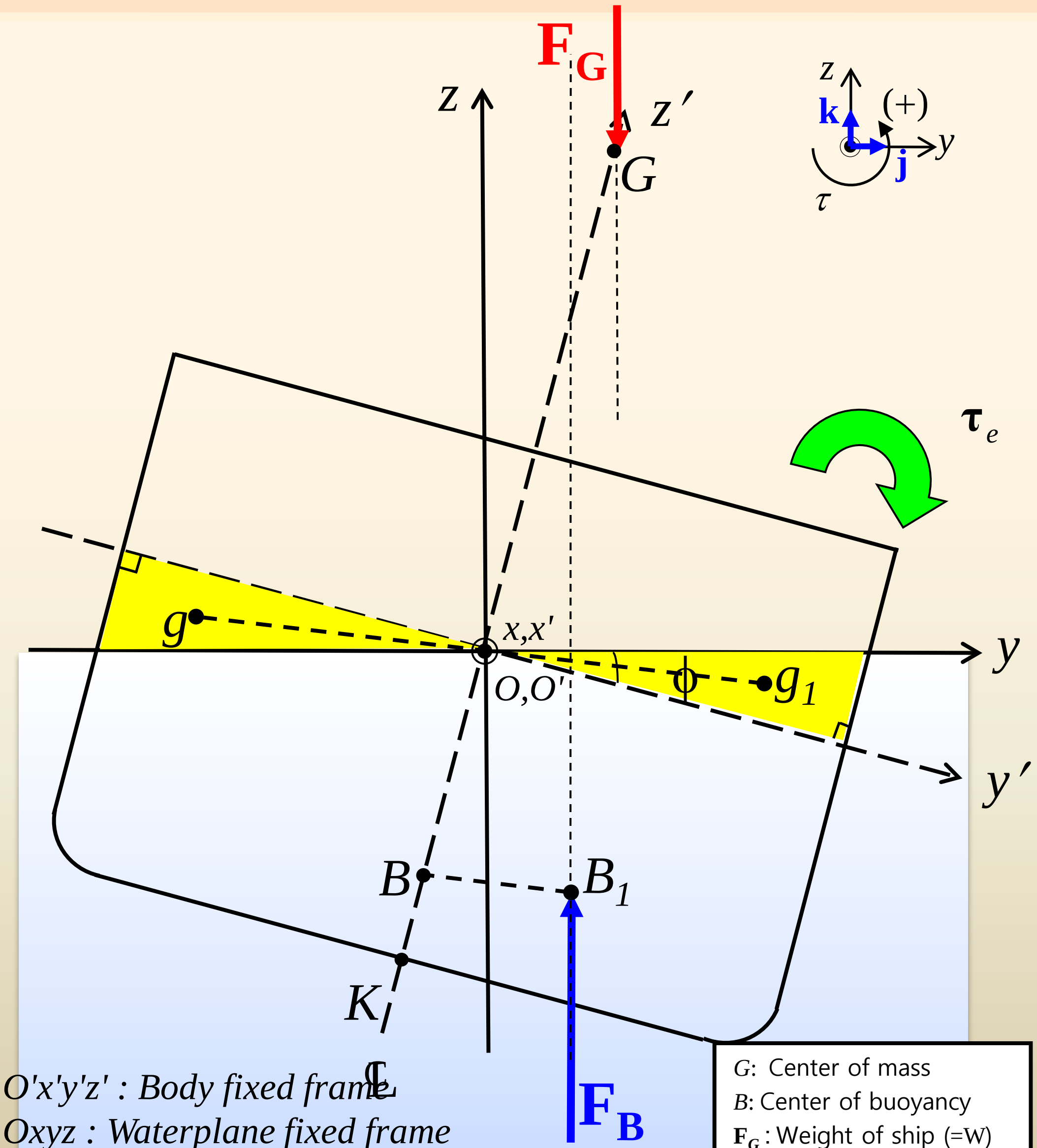
② Center of mass(G) and center of buoyancy (B) are in the same vertical line which is perpendicular to waterplane → Transverse moment arms about origin Origin O about z axis are same. (static equilibrium of moment)

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_B = \mathbf{r}_G \times \mathbf{F}_G + \mathbf{r}_B \times \mathbf{F}_B$$

$$\boldsymbol{\tau}_G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & z_G \\ 0 & 0 & F_{G,z} \end{vmatrix} = 0, \quad \boldsymbol{\tau}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & z_B \\ 0 & 0 & F_{B,z} \end{vmatrix} = 0$$

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_B = 0 + 0 = 0$$

Transverse Stability(2)



$O'x'y'z'$: Body fixed frame

$Oxyz$: Waterplane fixed frame

G: Center of mass

B: Center of buoyancy

$\mathbf{F}_G$: Weight of ship (=W)

$\mathbf{F}_B$: Buoyant force acting on ship (=ρg▽)

③ External moment ($\boldsymbol{\tau}_e$) is applied on the ship in clockwise.

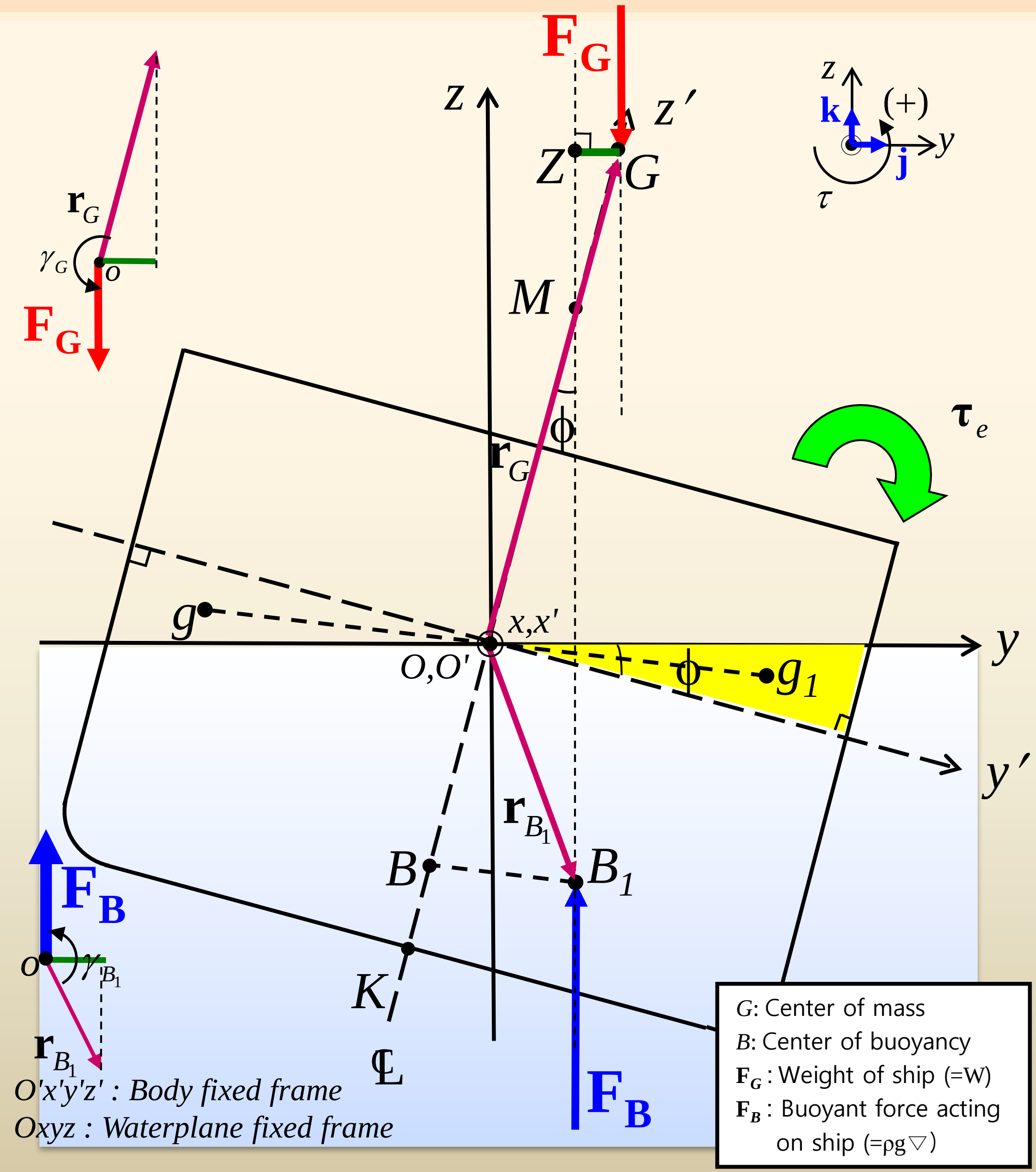
(Negative moment is applied)

④ A ship is heeled about origin O through an angle of ϕ.

⑤ Center of buoyancy is changed from B to B_1.

B_1: Changed center of buoyancy

Transverse Stability(3)



⑥ Moments due to weight of ship and buoyant force are calculated as follows

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_B$$

$$= \mathbf{r}_G \times \mathbf{F}_G + \mathbf{r}_{B_1} \times \mathbf{F}_B$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & y_G & z_G \\ 0 & 0 & F_G \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & y_{B_1} & z_{B_1} \\ 0 & 0 & F_B \end{vmatrix}$$

$$= y_G \cdot F_G\,\mathbf{i} + y_{B_1} \cdot F_B\,\mathbf{i}$$

$$\sum \mathbf{F} = \mathbf{F}_G + \mathbf{F}_B = 0 \quad ⇨ \quad F_G = -F_B$$

$$\therefore \boldsymbol{\tau}\mathbf{i} = (-y_G + y_{B_1}) \cdot F_B$$

$$, GZ = -y_G + y_{B_1}$$

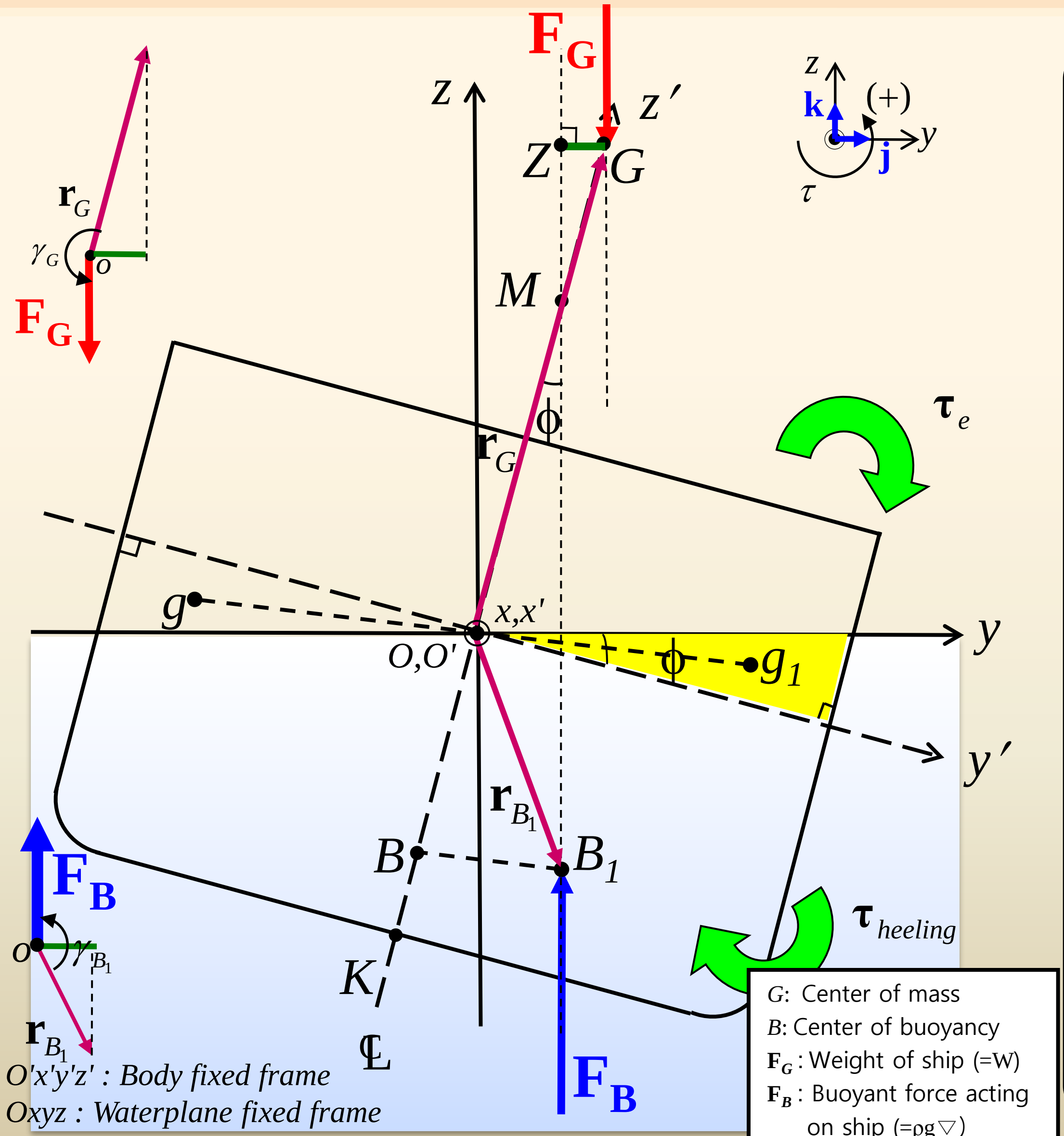
Z: B_1을 통한 부력작용선과 G를 지나고 y축과 평행한 선이 만나는 점

Z : The intersection of the line of buoyant force through B1 with the transverse line through G

M : The intersection of the line of buoyant force through B1 with the centerline of the ship

B_1: Changed center of buoyancy

Transverse Stability(4)



G: Center of mass
B: Center of buoyancy
$\mathbf{F}_G$: Weight of ship (=W)
$\mathbf{F}_B$: Buoyant force acting on ship (=ρg▽)

⑥ Moments due to weight of ship and buoyant force are calculated as follows

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_B$$

$$= \mathbf{r}_G \times \mathbf{F}_G + \mathbf{r}_{B_1} \times \mathbf{F}_B$$

$$\therefore \boldsymbol{\tau}\mathbf{i} = (-y_G + y_{B_1}) \cdot F_B$$

$$,(GZ = -y_G + y_{B_1})$$

If y_G is larger than y_B

(= y_G is located in right side more than y_G)

(= Center of mass is located in a high level)

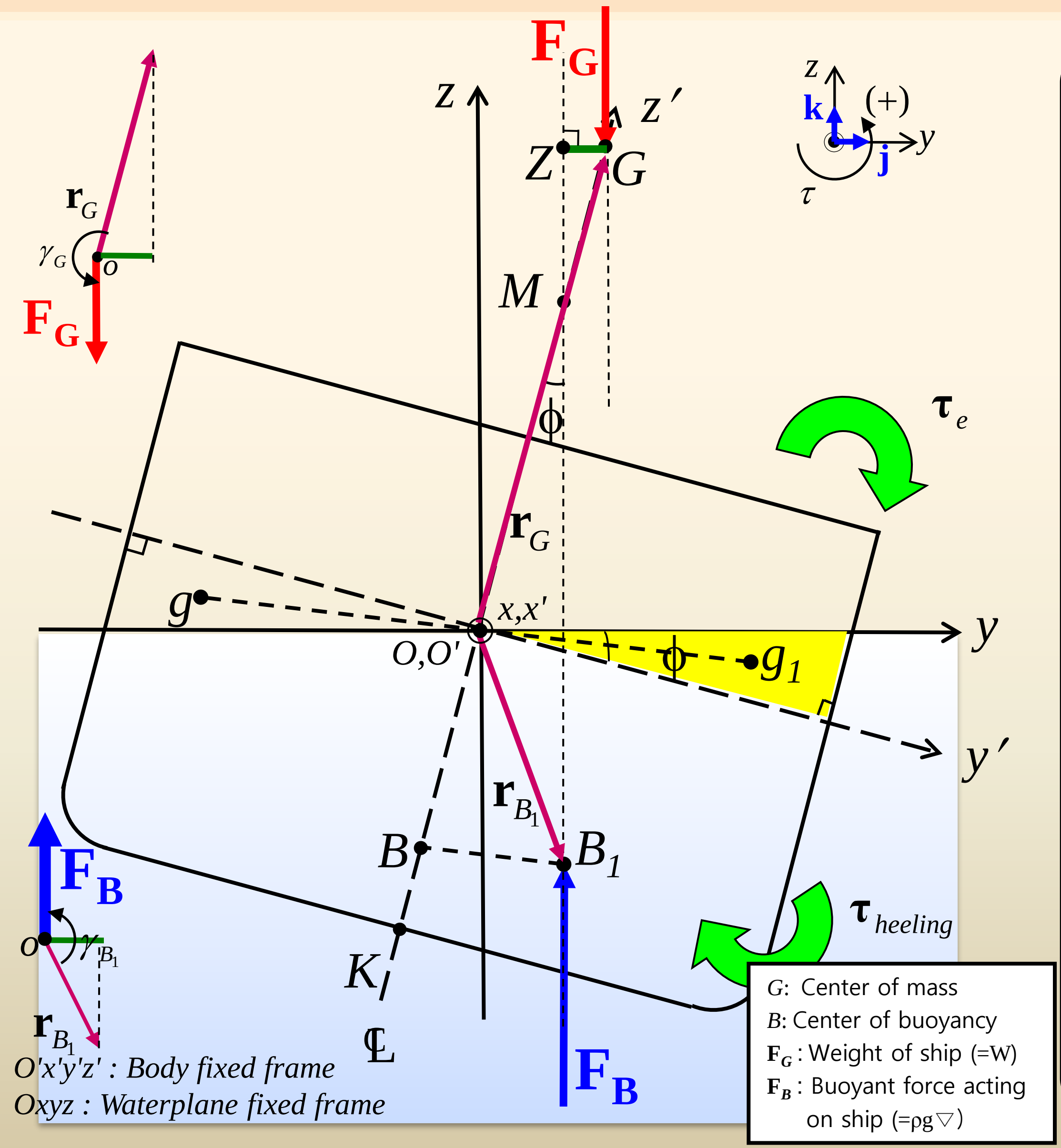
Because GZ=($-y_G+y_B$) is (−)

$$\boldsymbol{\tau}\mathbf{i} = \boxed{(-y_G + y_{B_1})} \cdot F_B$$

Heeling moment about origin O in (-i) direction is applied on a ship.
(A ship become heeled more
: **unstable condition**)

Z: The intersection of the line of buoyant force through B1 with the transverse line through G
M: The intersection of the line of buoyant force through B1 with the centerline of the ship B_1: Changed center of buoyancy

Transverse Stability(5)



⑥ Moments due to weight of ship and buoyant force are calculated as follows

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_B$$

$$= \mathbf{r}_G \times \mathbf{F}_G + \mathbf{r}_{B_1} \times \mathbf{F}_B$$

$$\therefore \boldsymbol{\tau}\mathbf{i} = (-y_G + y_{B_1}) \cdot F_B$$

$$,(GZ = -y_G + y_{B_1})$$

Or, Substituting -$\mathbf{F_G}$ into $\mathbf{F_B}$

$$\therefore \boldsymbol{\tau}\mathbf{i} = (y_G - y_{B_1}) \cdot F_G$$

Thought '$GZ=(y_G-y_B)$' is positive, but F_G is applied in $-\mathbf{k}$ direction

Heeling moment about origin O in (-i) direction is applied on a ship.
(A ship become heeled more
: **unstable condition**)

Z: The intersection of the line of buoyant force through B1 with the transverse line through G
M: The intersection of the line of buoyant force through B1 with the centerline of the ship B_1: Changed center of buoyancy

Sec.1 Calculation of Center of Buoyancy

Sec.2 Calculation of BM, GZ in Wall Sided Ship

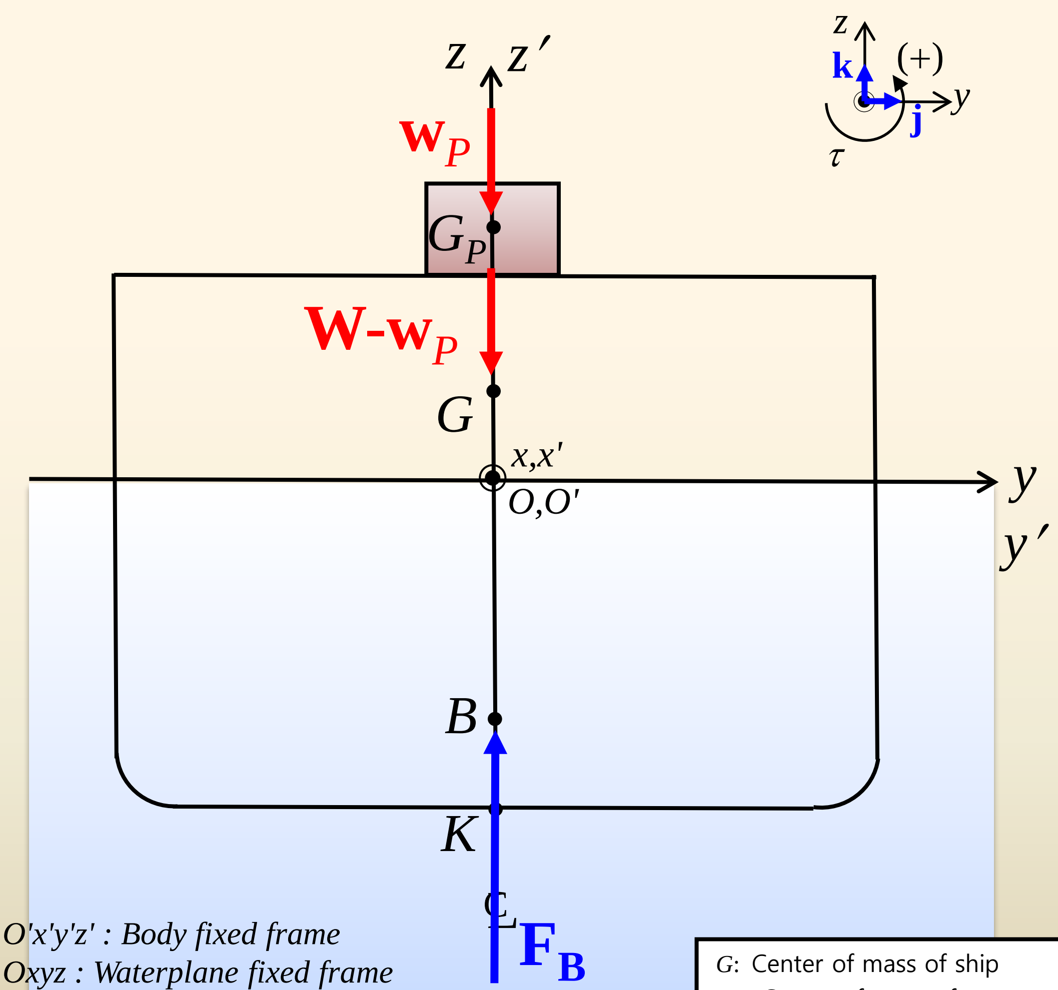
Sec.3 Inclining Test

Sec.4 Transverse Stability of ship (Unstable condition)

Sec.5 Transverse Righting Moment due to Movement of Cargo

Sec.6 Calculation of Heeling Angle due to Shift of Center of Mass

Transverse Righting Moment due to Movement of Cargo
Case1 : Considering weight of ship and weight of cargo separately (1)



$\boldsymbol{\tau}_G$: Moment due to weight of ship
$\boldsymbol{\tau}_P$: Moment due to weight of cargo
$\boldsymbol{\tau}_B$: Moment due to buoyancy

G: Center of mass of ship
G_P: Center of mass of cargo
B: Center of buoyancy, K : Keel
$\mathbf{W}$: Total Weight
$\mathbf{W}-\mathbf{w}_P$: Weight of ship
$\mathbf{w}_P$: Weight of cargo
$\mathbf{F}_B$: Buoyant force acting on ship (=ρg▽)

Heel case in ship hydrostatics (ϕ: Angle of Heel)

$$\mathbf{i}M_{T,gravity} + \mathbf{i}M_{T,Buoyancy} + \mathbf{i}M_{T,Ext,static} = 0$$
$$\mathbf{r}_G \times \mathbf{k}F_{gravity} + \mathbf{r}_B \times \mathbf{k}F_{Buoyancy} + \mathbf{i}M_{T,Ext,static} = 0$$

Case1) Considering weight of ship and weight of cargo separately

① $\sum \mathbf{F} = (\mathbf{W} - \mathbf{w}_P) + \mathbf{w}_P + \mathbf{F}_B = 0$ (static equilibrium of force)

② Center of mass of ship(G) and center of buoyancy (B) and center of mass of cargo are(G_P) in the same vertical line which is perpendicular to waterplane → y components of moment arms about origin O about z axis are same. (static equilibrium of moment)

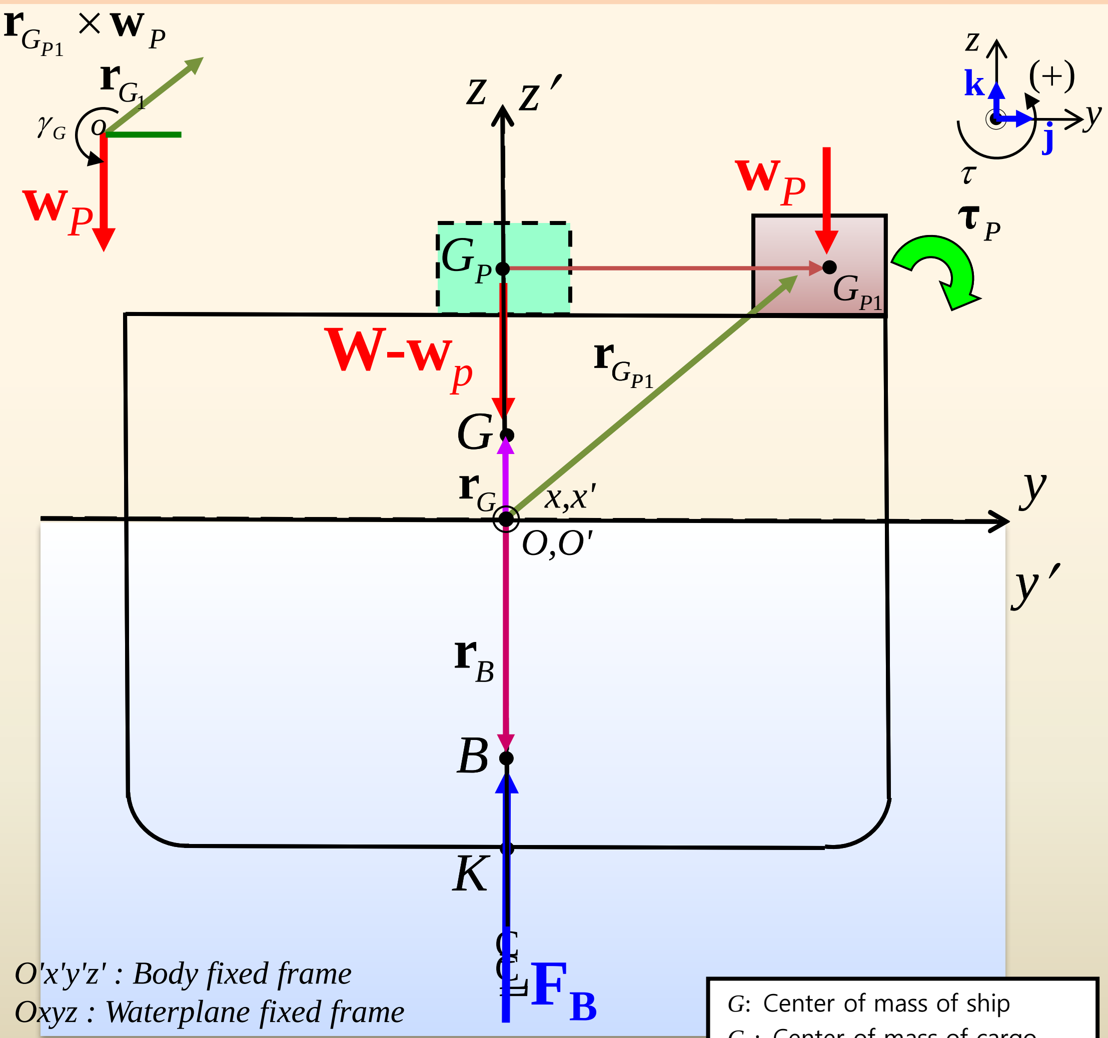
$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_B + \boldsymbol{\tau}_P$$
$$= \mathbf{r}_G \times (\mathbf{W} - \mathbf{w}_P) + \mathbf{r}_B \times \mathbf{F}_B + \mathbf{r}_{G_P} \times \mathbf{w}_P$$

$$\boldsymbol{\tau}_G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & z_G \\ 0 & 0 & (W_z - w_P) \end{vmatrix} = 0$$

$$\boldsymbol{\tau}_P = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & z_{G_P} \\ 0 & 0 & w_P \end{vmatrix} = 0, \quad \boldsymbol{\tau}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & z_B \\ 0 & 0 & F_B \end{vmatrix} = 0$$

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_P + \boldsymbol{\tau}_B$$
$$= 0 + 0 + 0 = 0$$

Transverse Righting Moment due to Movement of Cargo
Case1 : Considering weight of ship and weight of cargo separately (2)



Heel case in ship hydrostatics (ϕ: Angle of Heel)

$$\mathbf{i}M_{T,gravity} + \mathbf{i}M_{T,Buoyancy} + \mathbf{i}M_{T,Ext,static} = 0$$
$$\mathbf{r}_G \times \mathbf{k}F_{gravity} + \mathbf{r}_B \times \mathbf{k}F_{Buoyancy} + \mathbf{i}M_{T,Ext,static} = 0$$

③ A moment due to weight of cargo

The cargo is moved right side.
Moment due to weight of cargo about origin O is as follows

$$\boldsymbol{\tau}_P = \mathbf{r}_{G_{P1}} \times \mathbf{w}_P = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & y_{G_{P1}} & z_{G_{P1}} \\ 0 & 0 & w_P \end{vmatrix} = \mathbf{i}\, y_{P_1} \cdot w_P$$

(Heeling moment)

④ A moment due to weight of ship

Moment due to weight of ship about origin O is as follows

$$\boldsymbol{\tau}_G = \mathbf{r}_G \times (\mathbf{W} - \mathbf{w}_P) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & z_G \\ 0 & 0 & w_G \end{vmatrix} = 0$$

⑤ A ship is heeled in clockwise direction by a moment due to weight of cargo.

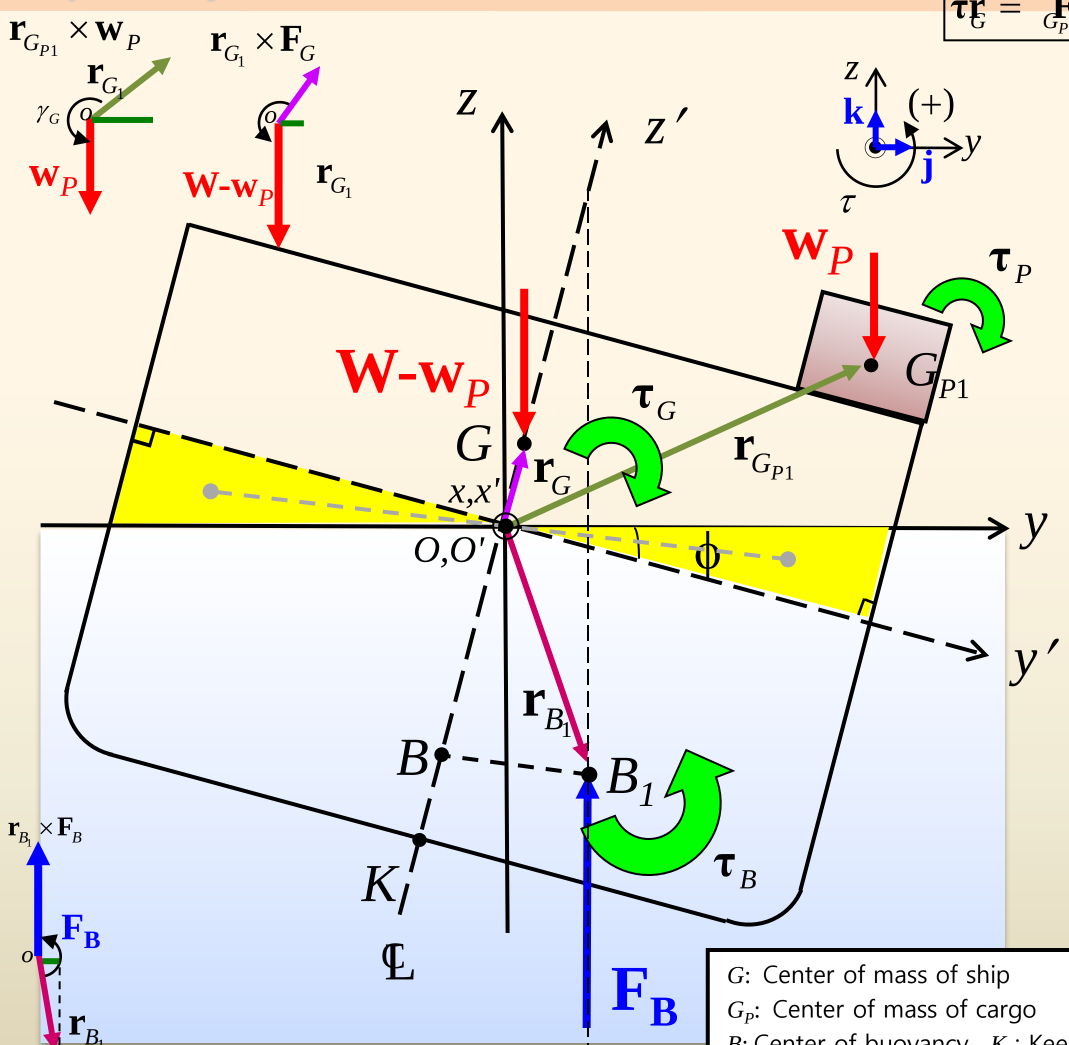
G: Center of mass of ship
G_P: Center of mass of cargo
B: Center of buoyancy, K : Keel
$\mathbf{W}$: Total Weight
$\mathbf{W} - \mathbf{w}_P$: Weight of ship
$\mathbf{w}_P$: Weight of cargo
$\mathbf{F}_B$: Buoyant force acting on ship (=ρg▽)

$\boldsymbol{\tau}_G$: Moment due to weight of ship
$\boldsymbol{\tau}_P$: Moment due to weight of cargo
$\boldsymbol{\tau}_B$: Moment due to buoyancy

Transverse Righting Moment due to Movement of Cargo
Case1 : Considering weight of ship and weight of cargo separately (3)

$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$

$\boldsymbol{\tau}_G = \mathbf{r}_{G_{P1}} \times \mathbf{F}_G$



O'x'y'z' : Body fixed frame

Oxyz : Waterplane fixed frame

$\boldsymbol{\tau}_G$: Moment due to weight of ship
$\boldsymbol{\tau}_P$: Moment due to weight of cargo
$\boldsymbol{\tau}_B$: Moment due to buoyancy

G: Center of mass of ship
G_P: Center of mass of cargo
B: Center of buoyancy, K : Keel
B_1 : Changed center of buoyancy
$\mathbf{W}$: Total Weight
$\mathbf{W} - \mathbf{w}_P$: Weight of ship
$\mathbf{w}_P$: Weight of cargo
$\mathbf{F_B}$: Buoyant force acting on ship (=ρg▽)

Heel case in ship hydrostatics (ϕ: Angle of Heel)

$$\mathbf{i}M_{T,gravity} + \mathbf{i}M_{T,Buoyancy} + \mathbf{i}M_{T,Ext,static} = 0$$
$$\mathbf{r}_G \times \mathbf{k}F_{gravity} + \mathbf{r}_B \times \mathbf{k}F_{Buoyancy} + \mathbf{i}M_{T,Ext,static} = 0$$

⑥ Center of buoyancy is changed from B to B_1.

⑦ If we assume that moment due to weight of ship, moment due to weight of cargo and moment due to buoyancy are in static equilibrium at an angle of heel ϕ

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_B + \boldsymbol{\tau}_P$$
$$= \mathbf{r}_G \times (\mathbf{W} - \mathbf{w}_P) + \mathbf{r}_{B_1} \times \mathbf{F}_B + \mathbf{r}_{G_{P1}} \times \mathbf{w}_P$$

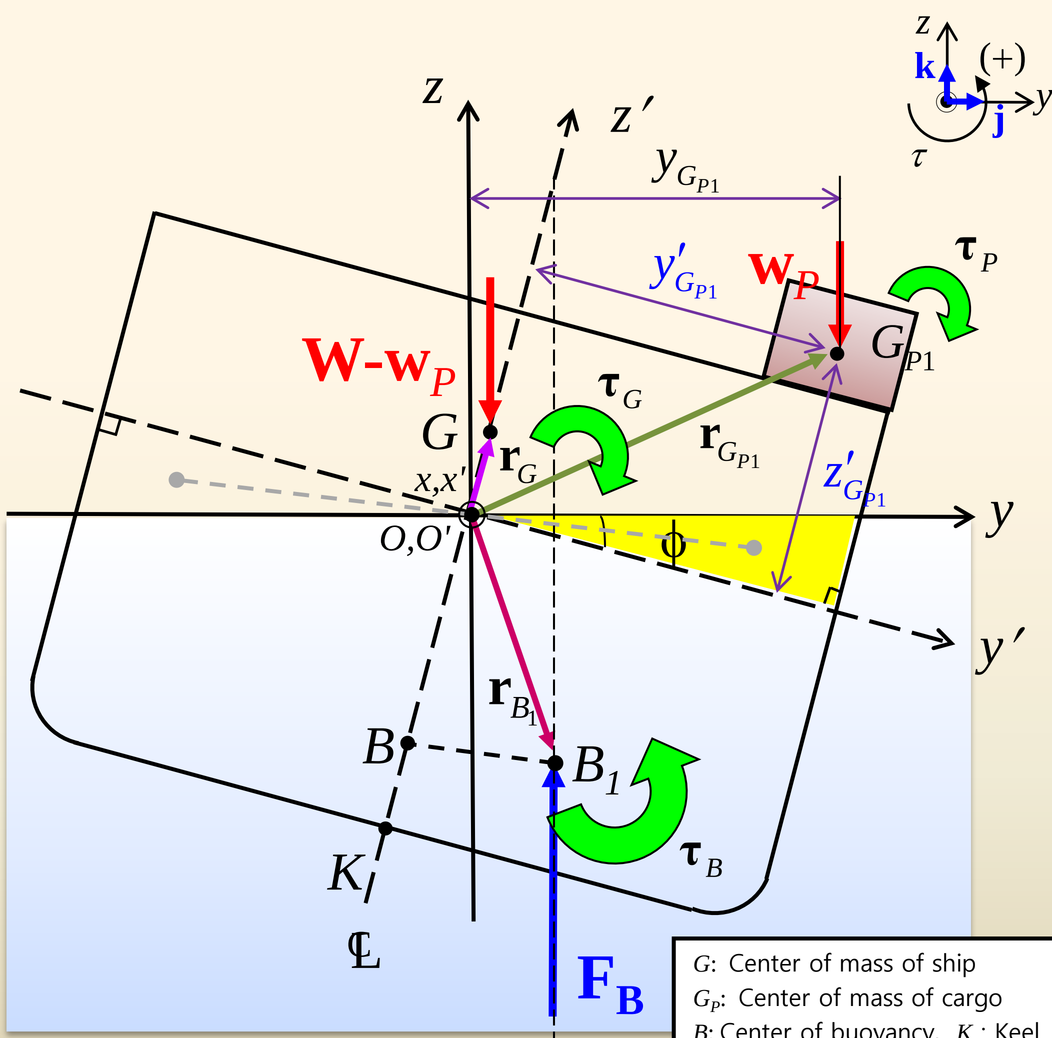
$$\boldsymbol{\tau}_G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & y_G & z_G \\ 0 & 0 & (W_z - w_P) \end{vmatrix} = y_G \cdot (W_z - w_P)\,\mathbf{i}$$

Position of G with respect to waterplane fixed frame is changed as a ship is heeled.
-> Heeling moment is caused.

$$\boldsymbol{\tau}_P = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & y_{G_{P1}} & z_{G_{P1}} \\ 0 & 0 & w_P \end{vmatrix} = \mathbf{i}(y_{G_{P1}} \cdot w_P), \quad \boldsymbol{\tau}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & y_{B_1} & z_{B_1} \\ 0 & 0 & F_B \end{vmatrix} = \mathbf{i}(y_{B_1} \cdot F_B)$$

$$\sum \boldsymbol{\tau} = \mathbf{i}\, y_G \cdot (W_z - w_P) + \mathbf{i}(y_{G_{P1}} \cdot w_P) + \mathbf{i}(y_{B_1} \cdot F_B)$$
$$= 0 \quad \text{,(In a static equilibrium of moment)}$$

Transverse Righting Moment due to Movement of Cargo
Case1 : Considering weight of ship and weight of cargo separately (4)



$O'x'y'z'$: *Body fixed frame*

$Oxyz$: *Waterplane fixed frame*

$\boldsymbol{\tau}_G$: Moment due to weight of ship
$\boldsymbol{\tau}_P$: Moment due to weight of cargo
$\boldsymbol{\tau}_B$: Moment due to buoyancy

G: Center of mass of ship
G_P: Center of mass of cargo
B: Center of buoyancy, K : Keel
B_1 : Changed center of buoyancy
$\mathbf{W}$: Total Weight
$\mathbf{W}-\mathbf{w}_P$: Weight of ship
$\mathbf{w}_P$: Weight of cargo
$\mathbf{F_B}$: Buoyant force acting on ship ($=\rho g \nabla$)

Heel case in ship hydrostatics (ϕ: Angle of Heel)

$$\mathbf{i}M_{T,gravity} + \mathbf{i}M_{T,Buoyancy} + \mathbf{i}M_{T,Ext,static} = 0$$
$$\mathbf{r}_G \times \mathbf{k}F_{gravity} + \mathbf{r}_B \times \mathbf{k}F_{Buoyancy} + \mathbf{i}M_{T,Ext,static} = 0$$

⑦ If we assume that moment due to weight of ship, moment due to weight of cargo and moment due to buoyancy are in static equilibrium at an angle of heel ϕ

$$\sum \boldsymbol{\tau}_G + \boldsymbol{\tau}_B + \boldsymbol{\tau}_P = 0$$
$$= \mathbf{r}_G \times (\mathbf{W} - \mathbf{w}_P) + \mathbf{r}_{B_1} \times \mathbf{F}_B + \mathbf{r}_{G_{P1}} \times \mathbf{w}_P = 0$$

If we consider components of moment due to weight of cargo

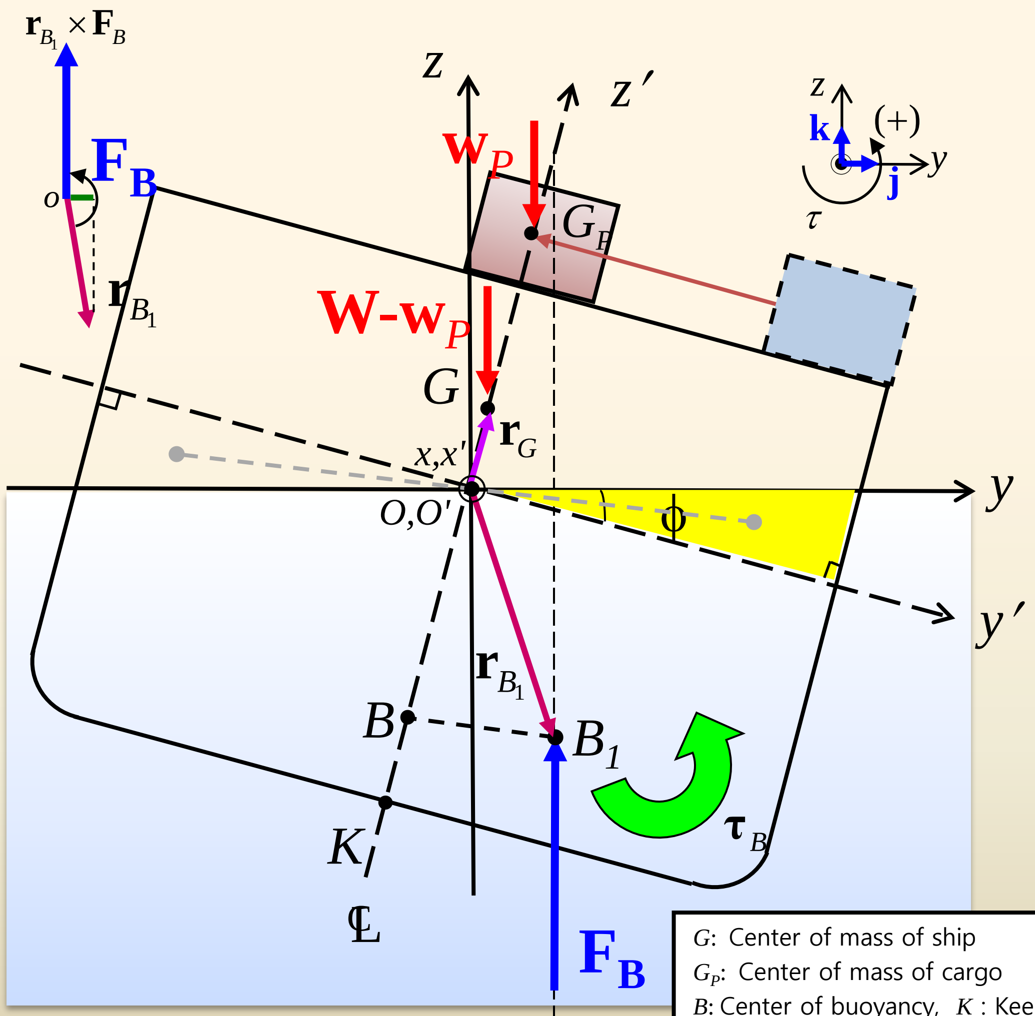
$$\boldsymbol{\tau}_P = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & y_{G_{P1}} & z_{G_{P1}} \\ 0 & 0 & w_P \end{vmatrix}$$
$$= \mathbf{i}(y_{G_{P1}} \cdot w_P)$$

Remind! : Rotational transformation!

$$= \mathbf{i}(y'_{G_{P1}} \cos\phi + z'_{G_{P1}} \sin\phi) w_P$$

Moment arm due to weight of cargo with respect to waterplane fixed frame also can be represented by moments arm in body fixed frame by rotational transformation

Transverse Righting Moment due to Movement of Cargo
Case1 : Considering weight of ship and weight of cargo separately (5)



O'x'y'z' : Body fixed frame

Oxyz : Waterplane fixed frame

⑧ The cargo is moved to centerline of ship again.

⑨ $\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_B + \boldsymbol{\tau}_P$

Because moment due to weight of cargo is decreased from static equilibrium of moment,

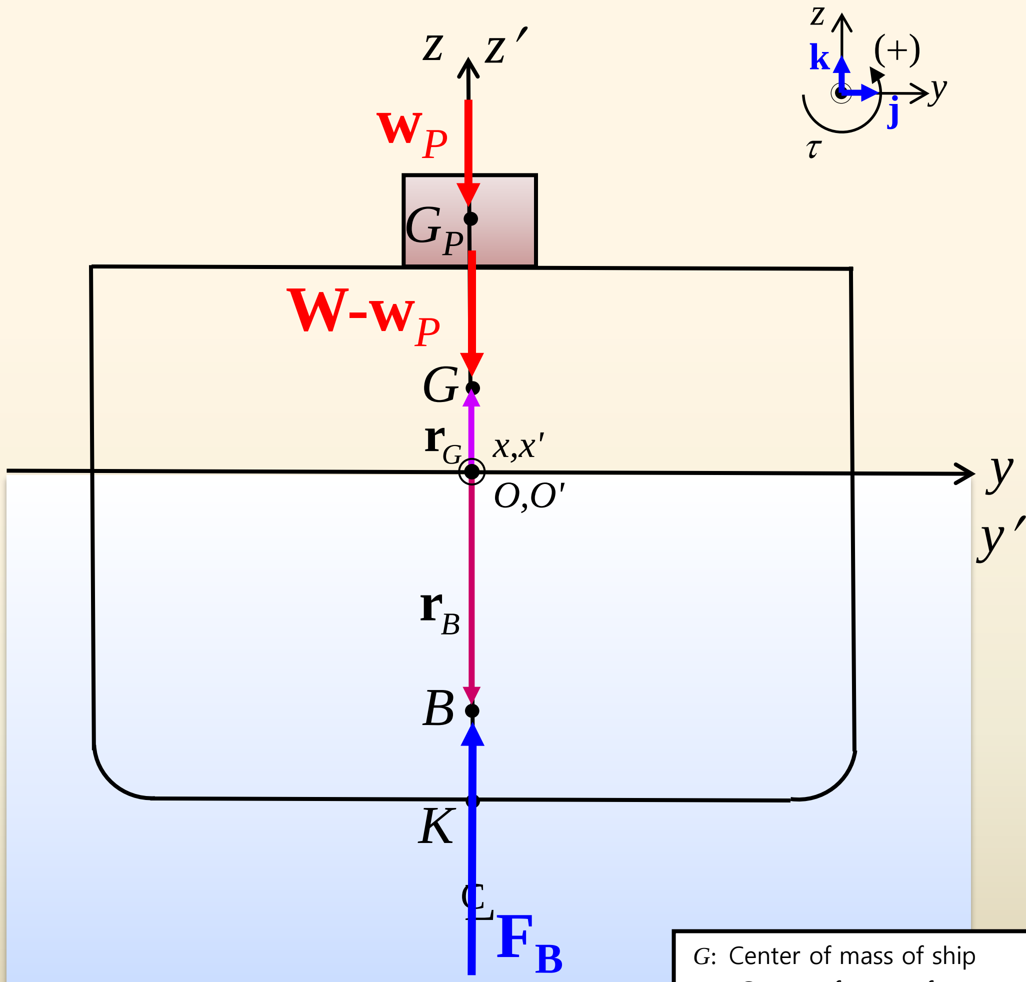
moment due to buoyancy is larger than heeling moment.

A ship returns to upright floating position due to transverse righting moment.

G: Center of mass of ship
G_P: Center of mass of cargo
B: Center of buoyancy, K : Keel
B_1 : Changed center of buoyancy
$\mathbf{W}$: Total Weight
$\mathbf{W} - \mathbf{w}_P$: Weight of ship
$\mathbf{w}_P$: Weight of cargo
$\mathbf{F_B}$: Buoyant force acting on ship ($=\rho g \nabla$)

$\boldsymbol{\tau}_G$: Moment due to weight of ship
$\boldsymbol{\tau}_P$: Moment due to weight of cargo
$\boldsymbol{\tau}_B$: Moment due to buoyancy

Transverse Righting Moment due to Movement of Cargo
Case1 : Considering weight of ship and weight of cargo separately (6)



O'x'y'z' : Body fixed frame
Oxyz : Waterplane fixed frame

$\boldsymbol{\tau}_G$: Moment due to weight of ship
$\boldsymbol{\tau}_P$: Moment due to weight of cargo
$\boldsymbol{\tau}_B$: Moment due to buoyancy

G: Center of mass of ship
G_P: Center of mass of cargo
B: Center of buoyancy
$\mathbf{W}$: Total Weight
$\mathbf{W}-\mathbf{w}_P$: Weight of ship
$\mathbf{w}_P$: Weight of cargo
$\mathbf{F_B}$: Buoyant force acting on ship (=ρg▽)

⑩ A ship is rotated in counter-clock wise direction.

⑪ Center of mass of ship(G) and center of buoyancy (B) and center of mass of cargo are(G_P) in the same vertical line which is perpendicular to waterplane. It becomes <u>in static equilibrium of moment.</u>

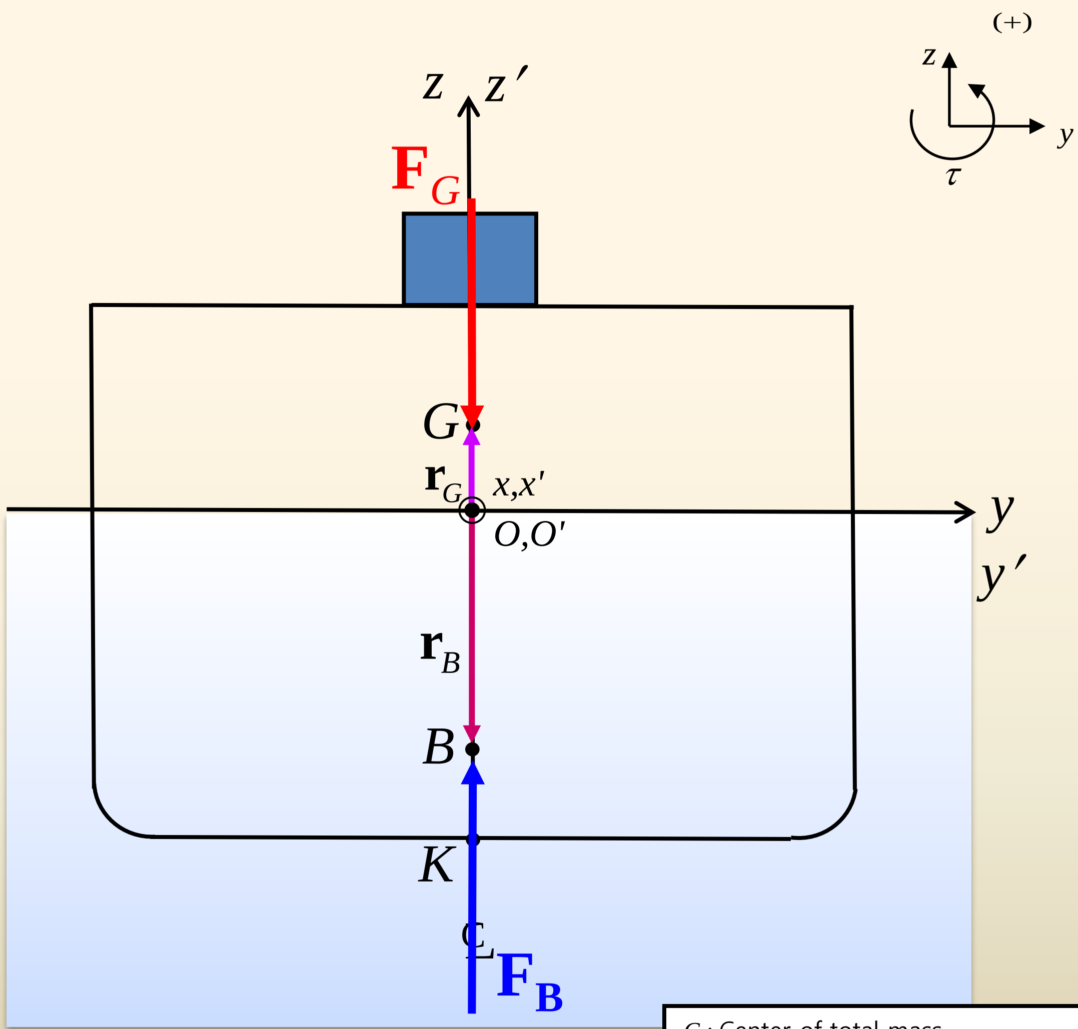
$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_B + \boldsymbol{\tau}_P$$
$$= \mathbf{r}_G \times (\mathbf{W} - \mathbf{w}_P) + \mathbf{r}_B \times \mathbf{F}_B + \mathbf{r}_{G_P} \times \mathbf{w}_P$$

$$\boldsymbol{\tau}_G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & z_G \\ 0 & 0 & (W_z - w_{P,z}) \end{vmatrix} = 0$$

$$\boldsymbol{\tau}_P = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & z_{G_P} \\ 0 & 0 & w_P \end{vmatrix} = 0 \ , \boldsymbol{\tau}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & z_B \\ 0 & 0 & F_B \end{vmatrix} = 0$$

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_P + \boldsymbol{\tau}_B$$
$$= 0 + 0 + 0 = 0$$

Transverse Righting Moment due to Movement of Cargo
Case2 : Considering Weight of Cargo is included in Weight of Ship (1)



O'x'y'z' : Body fixed frame
Oxyz : Waterplane fixed frame

G : Center of total mass
G_1 : Changed center of total mass
B: Center of buoyancy
B_1: Changed center of buoyancy
$\mathbf{F}_G$: Total weight ($=\mathbf{W}=\mathbf{W}_{Ship}+\mathbf{W}_{Weight}$)
$\mathbf{F}_B$: Buoyant force acting on ship ($=\rho g \nabla$)

$\boldsymbol{\tau}_G$: Moment due to total weight
$\boldsymbol{\tau}_B$: Moment due to buoyancy

Case2 : Considering Weight of Cargo **is included in** Weight of Ship

① $\sum \mathbf{F} = \mathbf{F}_G + \mathbf{F}_B = 0$

$,(\mathbf{F}_G = \mathbf{W}_{Ship} + \mathbf{w}_{Weight})$

(static equilibrium of force)

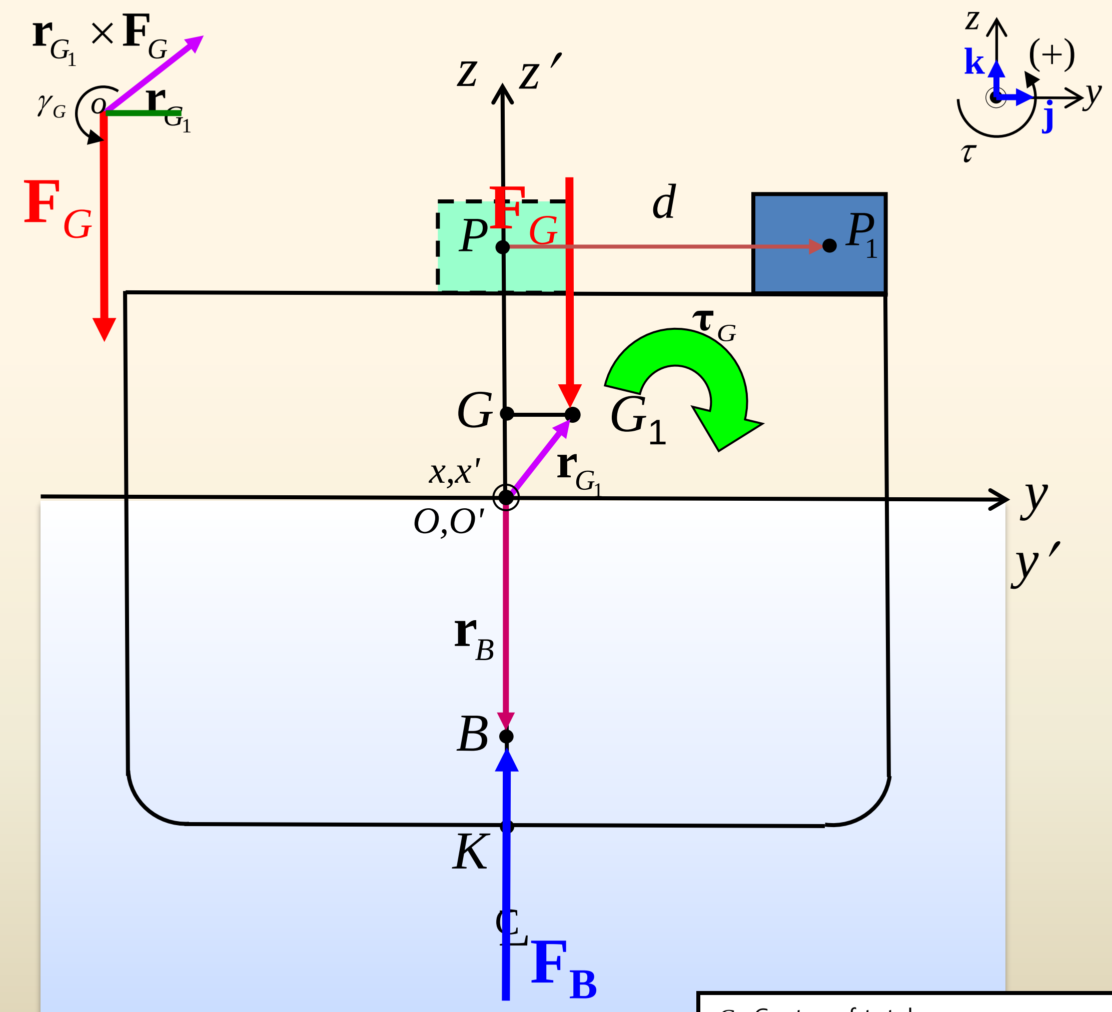
② Center of total mass(G) and center of buoyancy (B) are in the same vertical line which is perpendicular to waterplane → y components of moment arms about origin Origin O about z axis are same. (static equilibrium of moment)

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_B$$

$$\boldsymbol{\tau}_G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & z_G \\ 0 & 0 & F_G \end{vmatrix} = 0 \quad , \boldsymbol{\tau}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & z_B \\ 0 & 0 & F_B \end{vmatrix} = 0$$

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_B = 0 + 0 = 0$$

Transverse Righting Moment due to Movement of Cargo
Case2 : Considering Weight of Cargo is included in Weight of Ship (2)



O'x'y'z' : Body fixed frame
Oxyz : Waterplane fixed frame

③ The cargo is moved right side

④ Center of mass of total weight is moved from G to G_1.

$$GG_1 = \frac{w_p}{F_G} d \quad ,(w_p\text{: The weight of cargo})$$

⑤ Moment due to total weight about origin O about z axis

$$\boldsymbol{\tau}_G = \mathbf{r}_{G_1} \times \mathbf{F}_G$$

⑥ A ship is heeled about origin O through an angle of ϕ by a moment due to total weight

G : Center of total mass
G_1 : Changed center of total mass
B: Center of buoyancy
B_1: Changed center of buoyancy
$\mathbf{F}_G$: Total weight ($=\mathbf{W}=\mathbf{W}_{\mathbf{Ship}}+\mathbf{W}_{\mathbf{Weight}}$)
$\mathbf{F}_B$: Buoyant force acting on ship ($=\rho g \nabla$)

$\boldsymbol{\tau}_G$: Moment due to total weight
$\boldsymbol{\tau}_B$: Moment due to buoyancy

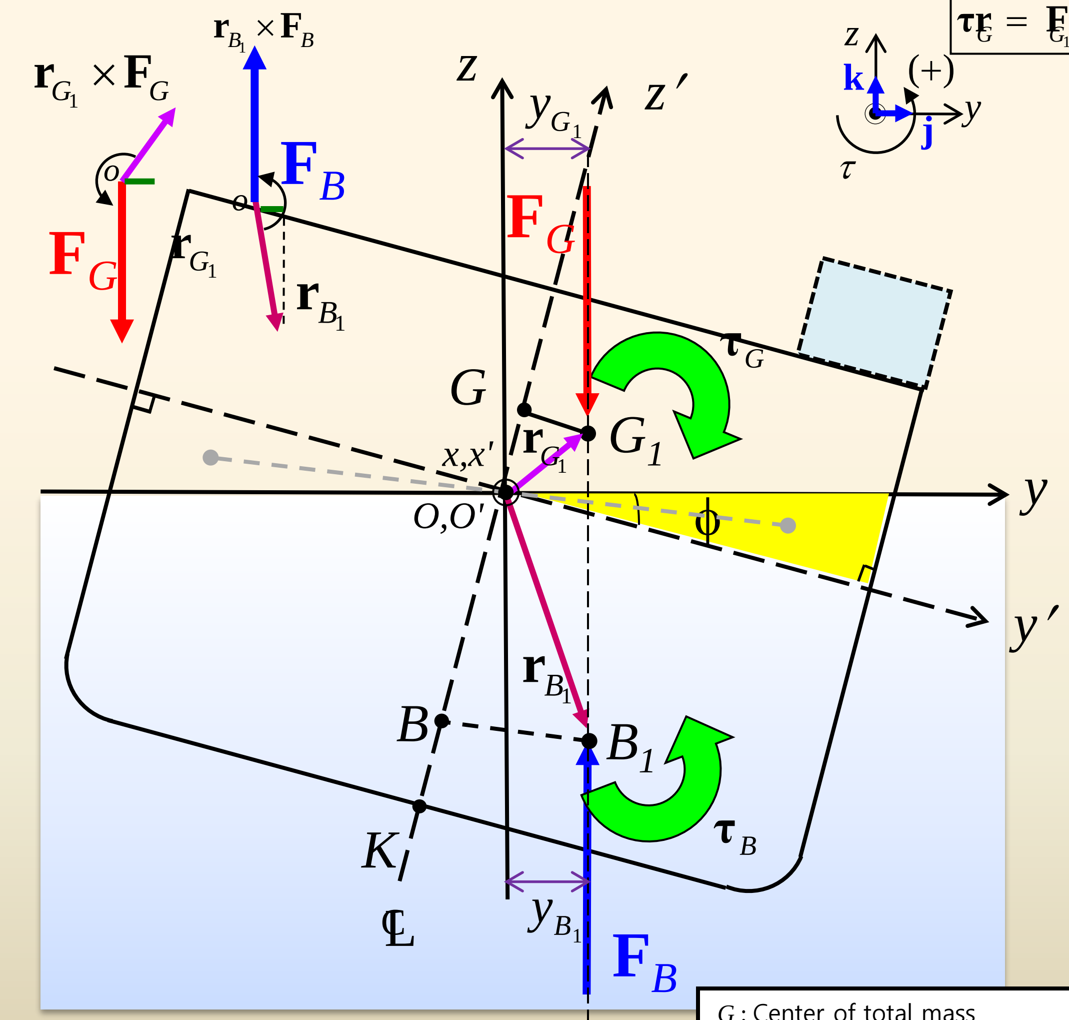
Transverse Righting Moment due to Movement of Cargo
Case2 : Considering Weight of Cargo is included in Weight of Ship (3)

선박 유체정역학 중 Heel case

$$\boxed{\mathbf{i}M_{T,gravity}} + \boxed{\mathbf{i}M_{T,Buoyancy}} + \boxed{\mathbf{i}M_{T,Ext,static}} = 0$$
$$\boxed{\mathbf{r}_G \times \mathbf{k}F_{gravity}} + \boxed{\mathbf{r}_B \times \mathbf{k}F_{Buoyancy}} + \boxed{\mathbf{i}M_{T,Ext,static}} = 0$$

$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$

$\boldsymbol{\tau}_G = \mathbf{r}_{G_1} \times \mathbf{F}_G$



⑦ Center of buoyancy is changed from B to B_1.

⑧ Changed center of mass(G_1) and changed center of buoyancy (B_1) are in the same vertical line which is perpendicular to waterplane. Then it become in static equilibrium at angle of heel ϕ.

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_B = 0$$

$$= \boxed{\mathbf{r}_{G_1} \times \mathbf{F}_G} + \boxed{\mathbf{r}_{B_1} \times \mathbf{F}_B}$$

$$\boldsymbol{\tau}_G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & y_{G_1} & z_{G_1} \\ 0 & 0 & F_G \end{vmatrix}, \quad \boldsymbol{\tau}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & y_{B_1} & z_{B_1} \\ 0 & 0 & F_B \end{vmatrix}$$

$$= \mathbf{i}\, y_{G_1} \cdot F_G \qquad\qquad = \mathbf{i}\, y_{B_1} \cdot F_B$$

$$= \mathbf{i}\left(y_{G_1} \cdot F_G + y_{B_1} \cdot F_B\right)$$

$$\sum \mathbf{F} = \mathbf{F}_G + \mathbf{F}_B = 0 \Rightarrow F_G = -F_B$$

because they lay in the same vertical line $y_{G_1} = y_{B_1}$

$$\sum \boldsymbol{\tau} = \mathbf{i}\left(-y_{B_1} \cdot F_B + y_{B_1} \cdot F_B\right)$$
$$= 0$$

O′x′y′z′ : Body fixed frame
Oxyz : Waterplane fixed frame

G : Center of total mass
G_1 : Changed center of total mass
B: Center of buoyancy
B_1: Changed center of buoyancy
$\mathbf{F}_G$: Total weight (=$\mathbf{W}$=$\mathbf{W}_{Ship}$+$\mathbf{W}_{Weight}$)
$\mathbf{F}_B$: Buoyant force acting on ship (=ρg▽)

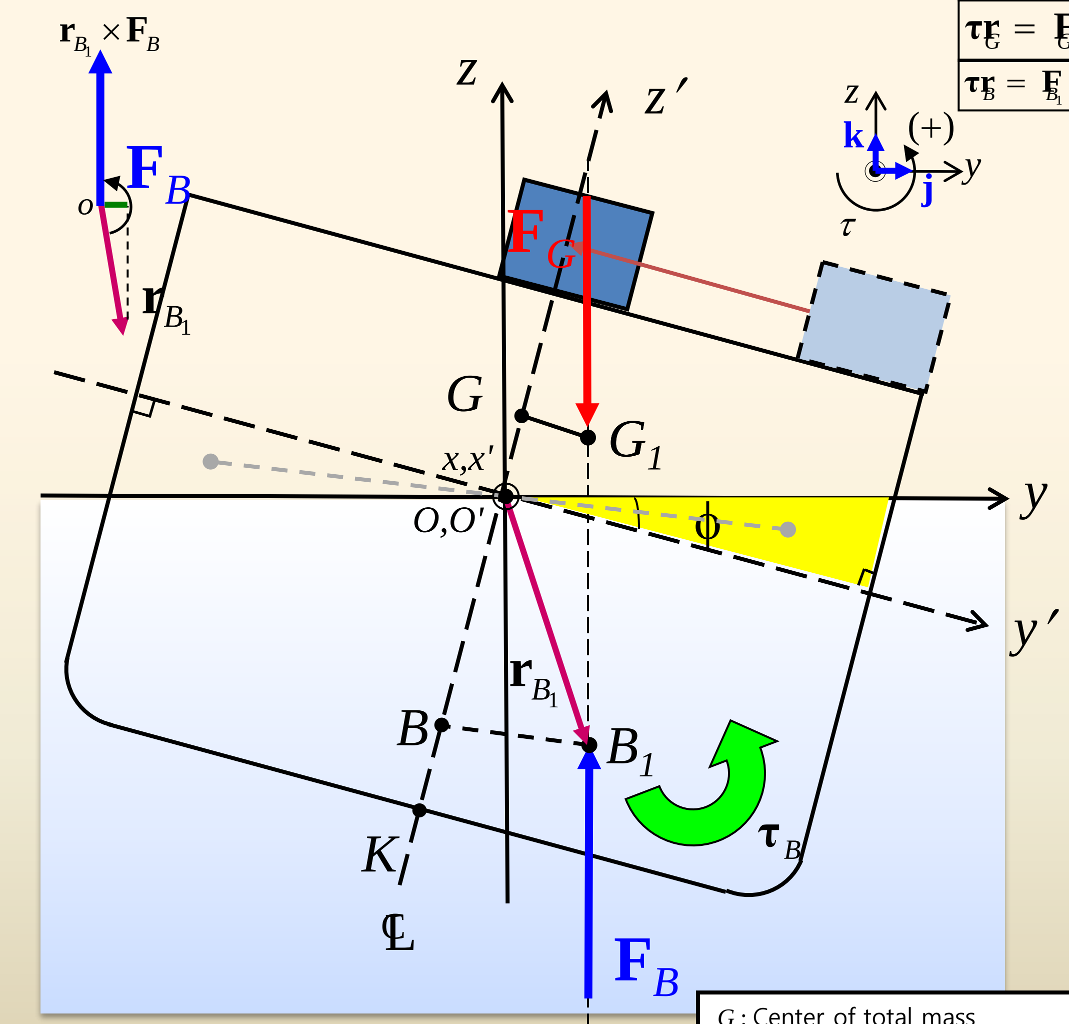
$\boldsymbol{\tau}_G$: Moment due to total weight
$\boldsymbol{\tau}_B$: Moment due to buoyancy

Transverse Righting Moment due to Movement of Cargo
Case2 : Considering Weight of Cargo is included in Weight of Ship (4)

$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$

$\boldsymbol{\tau}_G = \mathbf{r}_{G_1} \times \mathbf{F}_G$

$\boldsymbol{\tau}_B = \mathbf{r}_{B_1} \times \mathbf{F}_B$



⑨ The cargo is moved to centerline of ship again.

⑩ Center of total mass is moved from G_1 to G.

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_W$$

Because moment due to total weight is decreased from static equilibrium of moment, moment due to buoyancy is larger than heeling moment.

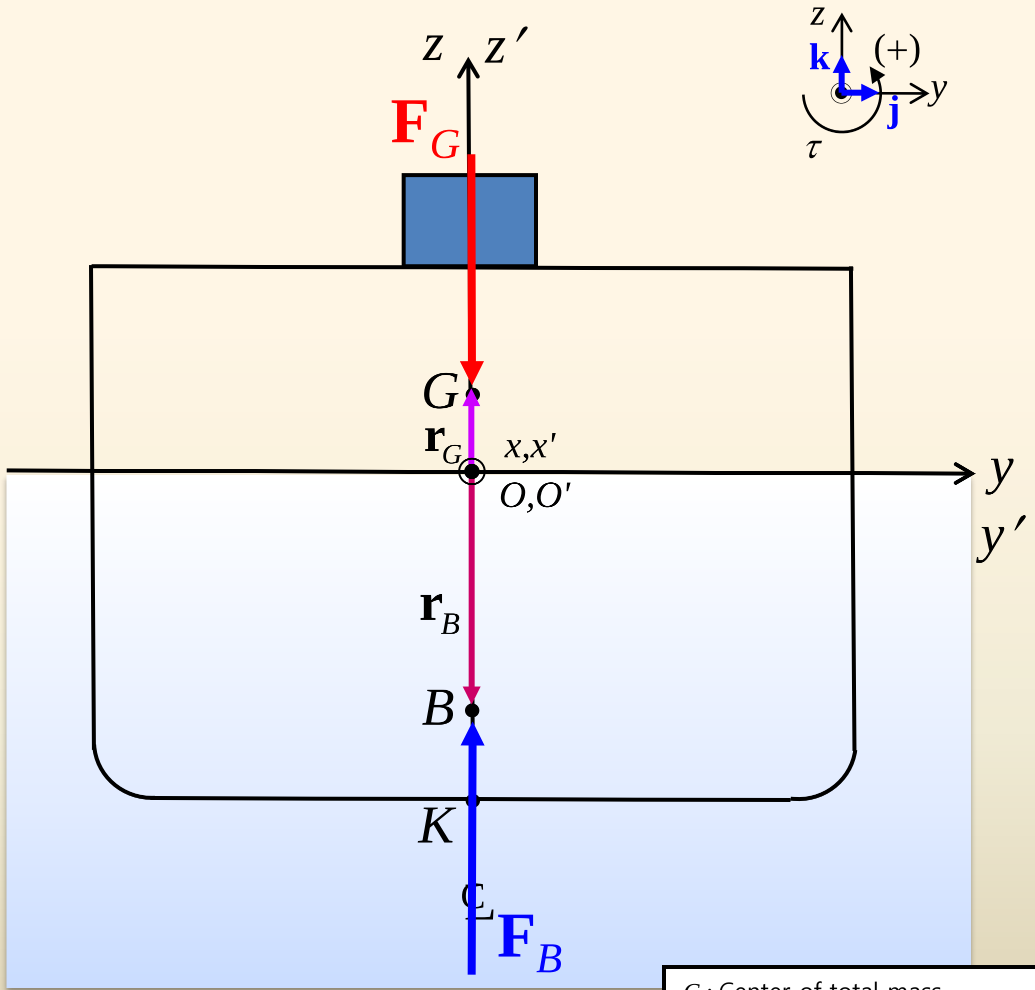
A ship returns to upright floating position due to transverse righting moment.

O'x'y'z' : Body fixed frame
Oxyz : Waterplane fixed frame

G : Center of total mass
G_1 : Changed center of total mass
B: Center of buoyancy
B_1: Changed center of buoyancy
$\mathbf{F}_G$: Total weight ($=\mathbf{W}=\mathbf{W}_{Ship}+\mathbf{W}_{Weight}$)
$\mathbf{F}_B$: Buoyant force acting on ship ($=\rho g \nabla$)

$\boldsymbol{\tau}_G$: Moment due to total weight
$\boldsymbol{\tau}_B$: Moment due to buoyancy

Transverse Righting Moment due to Movement of Cargo Case2 : Considering Weight of Cargo is included in Weight of Ship (5)



O'x'y'z' : Body fixed frame

Oxyz : Waterplane fixed frame

⑪ A ship is rotated in counter-clock wise direction.

② Center of total mass(G) and center of buoyancy (B) are in the same vertical line which is perpendicular to waterplane. It becomes in static equilibrium of moment.

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_B$$

$$\boldsymbol{\tau}_G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & z_G \\ 0 & 0 & F_G \end{vmatrix} = 0 \quad , \boldsymbol{\tau}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & z_B \\ 0 & 0 & F_B \end{vmatrix} = 0$$

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_B$$

$$= 0 + 0 = 0$$

G : Center of total mass

G_1 : Changed center of total mass

B: Center of buoyancy

B_1: Changed center of buoyancy

$\mathbf{F}_G$: Total weight (=$\mathbf{W}$=$\mathbf{W}_{Ship}$+$\mathbf{W}_{Weight}$)

$\mathbf{F}_B$: Buoyant force acting on ship (=ρg▽)

$\boldsymbol{\tau}_G$: Moment due to total weight

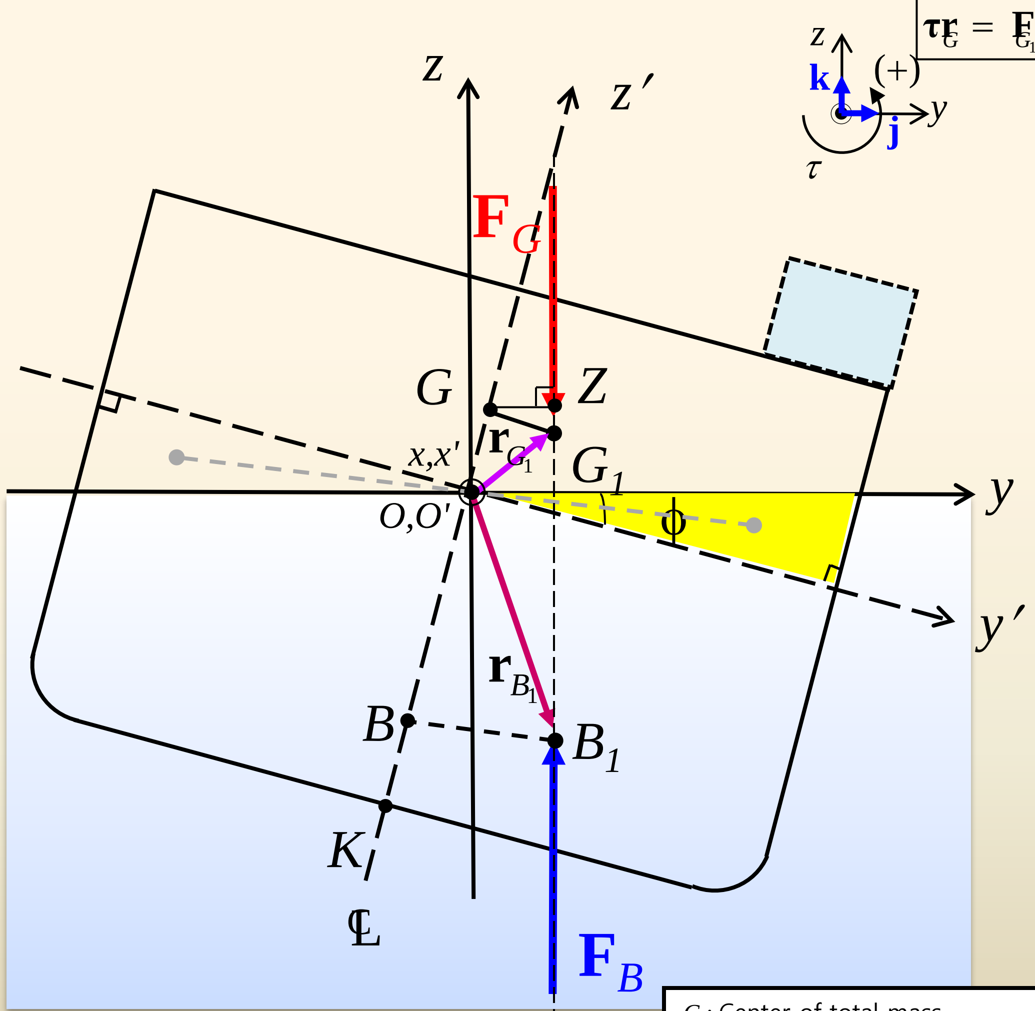
$\boldsymbol{\tau}_B$: Moment due to buoyancy

Transverse Righting Moment due to Movement of Cargo
Case2 : Considering Weight of Cargo is included in Weight of Ship (6)

$$\tau_{restoring} = GZ \cdot F_B$$
$$GZ = GM \cdot \sin\phi$$
$$GM = KB + BM - \boxed{KG}$$

$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$

$\boldsymbol{\tau}_G = \mathbf{r}_{G_1} \times \mathbf{F}_G$



O'x'y'z' : Body fixed frame
Oxyz : Waterplane fixed frame

$\boldsymbol{\tau}_G$: Moment due to total weight
$\boldsymbol{\tau}_B$: Moment due to buoyancy

G : Center of total mass
G_1 : Changed center of total mass
B: Center of buoyancy
B_1: Changed center of buoyancy
$\mathbf{F}_G$: Total weight ($=\mathbf{W}=\mathbf{W}_{Ship}+\mathbf{W}_{Weight}$)
$\mathbf{F}_B$: Buoyant force acting on ship ($=\rho g \nabla$)

Calculation of KG in static equilibrium of moment at angle of heel ϕ

$$GG_1 \cos\phi = GZ$$

$$\downarrow \quad GG_1 = \frac{w}{W} d$$

$$\frac{w}{W} d \cos\phi = GZ$$

By geometric shape

$$\downarrow \quad GZ = GM \sin\phi$$

$$\frac{w}{W} d \cos\phi = GM \sin\phi$$

$$\boxed{\therefore GM = \frac{w \cdot d}{W \cdot \tan\phi} = \frac{w \cdot d}{\Delta_g \cdot \tan\phi}}$$

$$GM = KB + BM - KG$$

$$KG = KB + BM - GM$$

$$\boxed{KG = \underset{\uparrow \text{ known}}{KB} + \underset{\uparrow \text{ known}}{BM} - \frac{w \cdot d}{\Delta_g \cdot \tan\phi}}$$

If we know an angle of heel ϕ, we can calculate KG.

That is a same way for 'Inclining Test'.

Sec.1 Calculation of Center of Buoyancy

Sec.2 Calculation of BM, GZ in Wall Sided Ship

Sec.3 Inclining Test

Sec.4 Transverse Stability of ship (Unstable condition)

Sec.5 Transverse Righting Moment due to Movement of Cargo

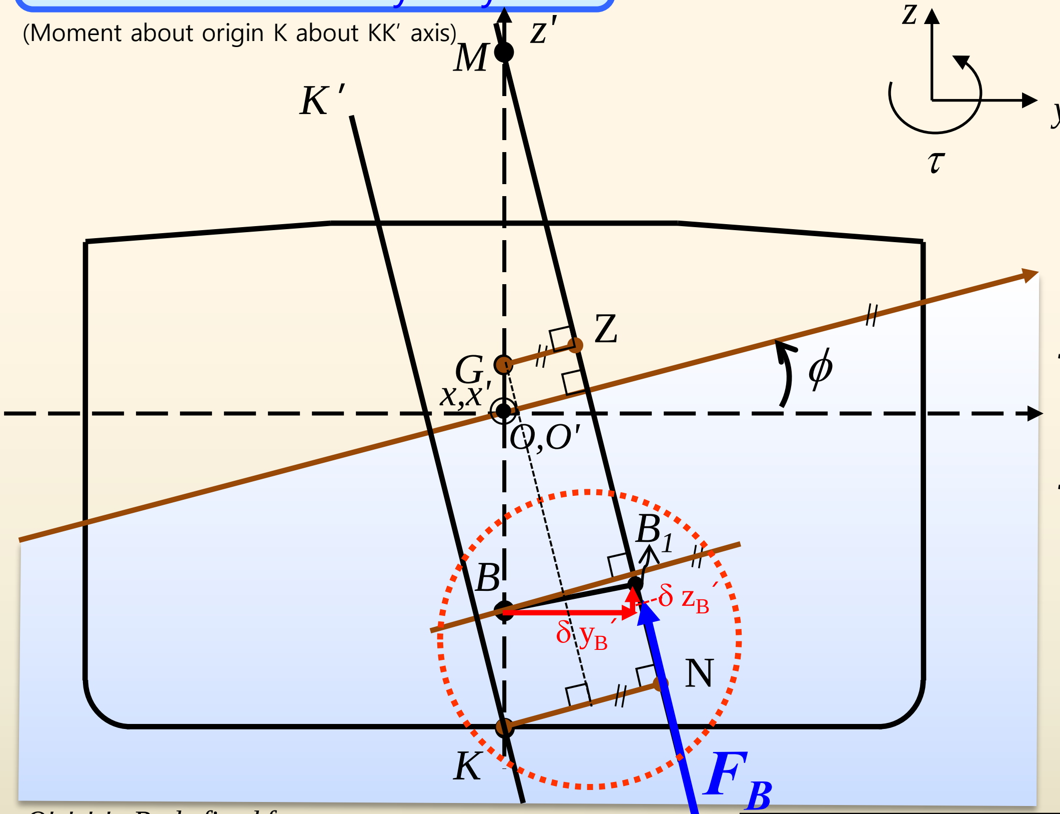
Sec.6 Calculation of Heeling Angle due to Shift of Center of Mass

Calculation of Heel Angle due to the shift of center of mass
- Righting Arm due to Shift of Center of Buoyancy(KN)

• Given : Shift distance of center of buoyancy ($\delta y'_B, \delta z'_B$)
• Find : Righting arm KN → An angle of heel ϕ

Righting arm due to shift of center of buoyancy KN

(Moment about origin K about KK' axis)

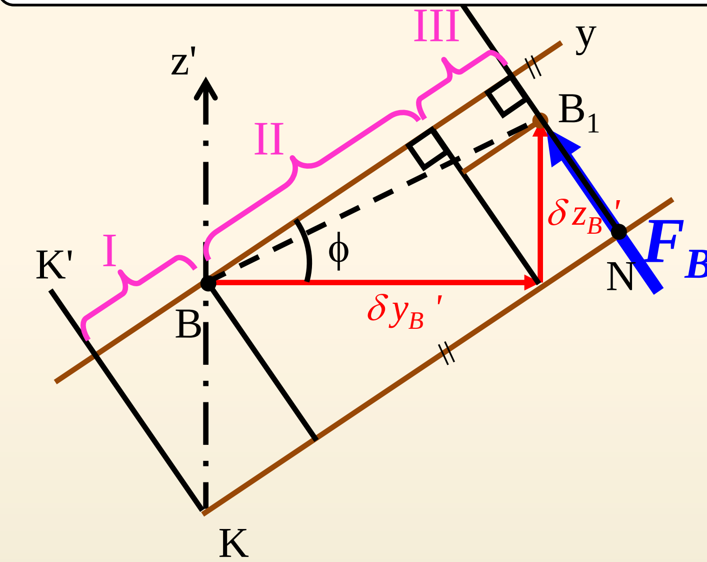


$O'x'y'z'$: *Body fixed frame*
$Oxyz$: *Waterplane fixed frame*

B_1: Changed center of buoyancy , K : Keel
M : The intersection of the line of buoyant force through B1 with the centerline of the ship
Z : The intersection of the line of buoyant force through B1 with the transverse line through G
N : The intersection of the line of buoyant force through B1 with the transverse line through K

G: Center of mass
B: Center of buoyancy
$\mathbf{F_G}$: Weight of ship (=**W**)
$\mathbf{F_B}$: Buoyant force (=ρg▽)

Righting moment arm(KN) about origin K about KK' axis is as follows



Calculation of KN

Method ① $= I + II + III$

in body Fixed frame. in body Fixed frame. in body Fixed frame.

$$= KB\sin\phi + \delta y'_B \cos\phi + \delta z'_B \sin\phi$$

Rotational Transformation

Waterplane fixed frame

Method ②

$$= KG\sin\phi + GZ$$

(Righting arm)

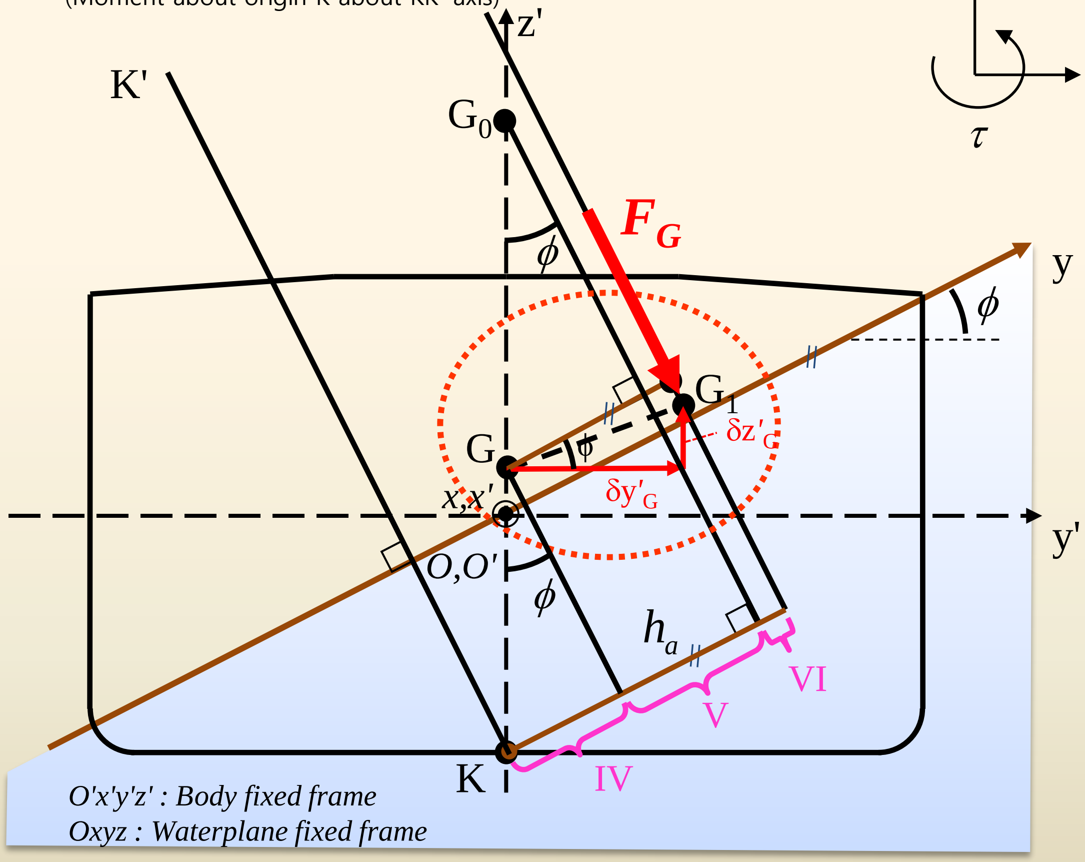
Calculation of Heel Angle due to the shift of center of mass

- Heeling Arm due to Shift of Center of Mass (h_a)

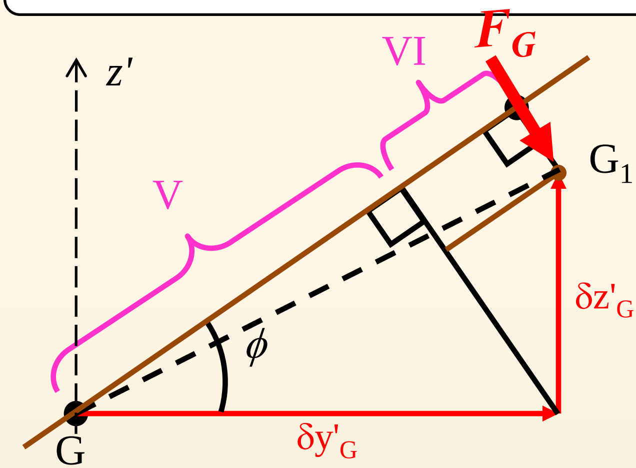
- Given : Changed center of mass $G_1(y'_G, z'_G)$
- Find: Heeling arm ha → An angle of heel ϕ

Heeling arm due to shift of center of mass h_a

(Moment about origin K about KK' axis)



Heeling moment arm(h_a) about origin K about KK' axis is as follows



$$h_a(\phi) = IV + V + VI$$

in body Fixed frame. in body Fixed frame.

$$= KG\sin\phi + \boxed{\delta y'_G \cos\phi} + \boxed{\delta z'_G \sin\phi}$$

Rotational Transformation

Waterplane fixed frame

$$= KG\sin\phi + GG_0\tan\phi\cos\phi + \delta z_G{}'\sin\phi$$

$$\boxed{= (KG + GG_0)\sin\phi + \delta z'_G\sin\phi}$$

G: Center of mass
G_1: Changed center of mass
F_G : Weight of ship (=**W**)

K : Keel

Calculation of Heel Angle due to the Shift of Center of Mass

$$\tau_{restoring} = F_B \cdot GZ$$
$$GZ \approx GM \sin\phi$$
$$GM = KB + BM - KG$$

- Given : Changed center of mass $G_1(y'_G, z'_G)$
- Find: Heeling arm ha → An angle of heel ϕ

Calculation of Heel Angle due to the Shift of Center of Mass

Heeling Moment = Righting Moment

(Moment about origin K about KK' axis)

$$F_B \cdot KN(\phi) = F_G \cdot h_a(\phi) \Rightarrow KN(\phi) = h_a(\phi)$$

Static Equilibrium of Force: Buoyant force = Gravitational force

$$F_B = F_G$$

$$KN(\phi) = h_a(\phi)$$

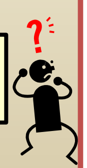
$$\Downarrow \quad (KN = KG\sin\phi + GZ)$$

$$\cancel{KG\sin\phi} + GZ = \cancel{KG\sin\phi} + \delta y'_G \cos\phi + \delta z'_G \sin\phi$$

$$GZ = \delta y'_G \cos\phi + \delta z'_G \sin\phi$$

$$\Downarrow \quad GZ \approx GM\sin\phi \quad (\text{ if } \phi \text{ is small})$$

$$\boxed{GM\sin\phi = \delta y'_G \cos\phi + \delta z'_G \sin\phi}$$



Meaning of equation ?

① If we know angle of heel ϕ and $\delta y'_G$, $\delta z'_G$, we can calculate GM. (= 'Inclining Test')

② After calculating GM, if another shift of center of mass ($\delta y'_G$, $\delta z'_G$) happens, we can calculate an angle of heel ϕ.

Calculation of Heel Angle

Calculation of Heel Angle due to the Shift of Center of Mass

Heeling Moment = Righting Moment

(K점을 지나고 yz평면에 수직인 축에 대한 모멘트)

$$F_B \cdot KN(\phi) = F_G \cdot h_a(\phi)$$

Static Equilibrium of Force: Buoyant force = Gravitational force

$$F_B = F_G$$

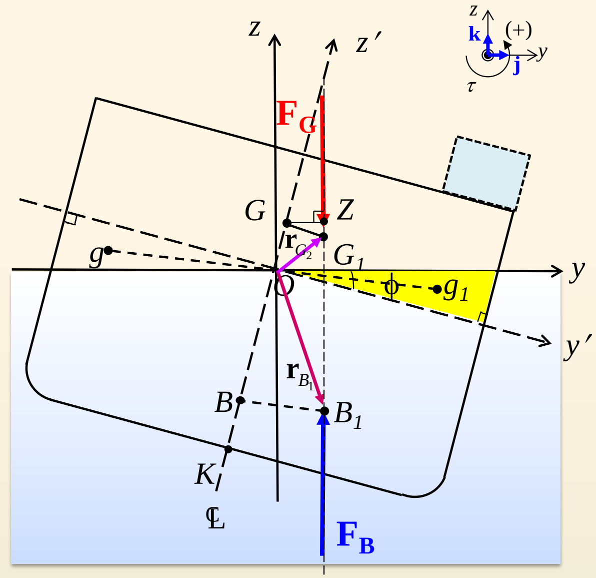
$$KN(\phi) = h_a(\phi)$$

$$GM \sin\phi = \delta y_G{}' \cos\phi + \delta z_G{}' \sin\phi$$

Is it related with transverse righting moment due to movement of cargo ?

(Review) Transverse Righting Moment due to Movement of Cargo

Case2 : Considering Weight of Cargo is included in Weight of Ship



$$GG_1 \cos\phi = GZ$$

By geometric shape

$$GG_1 \cos\phi = GM \sin\phi$$

- $\overline{GG_1}$ is a shift of center of mass in y' direction $\delta y'_G$

$$GM \sin\phi = \delta y'_G \cos\phi$$

$$\overline{GG_1} = \delta y_G{}'$$

'Transverse Righting Moment due to Movement of Cargo' considered shift of center of mass in **only y' direction**.
If we consider a shift of center of mass in z' direction, we may derive same equation.

References

Movement of Center caused by Movement of Area (1)

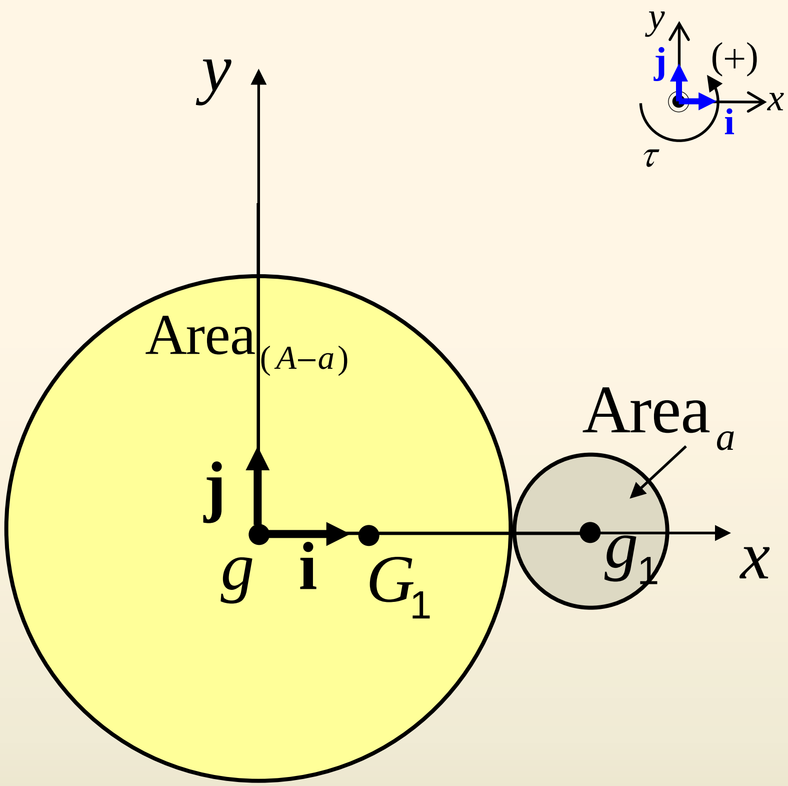
1) Gere, Mechanics of Materials, 6th ,Ch.12.3, 2006

First Moment of Composite Area(Q_x)[1)]

$$Q_x = \sum_{n=1}^{n} A_i \cdot \bar{x}_i$$

$$A \cdot \bar{x} = \sum_{n=1}^{n} A_i \cdot \bar{x}_i$$

Q_x : 1st Moment
A_i : Each Area
A : Total Area
$\bar{x}$: Coordinate of Centroid



<1st moment of area>

Let's consider 1st moment of area about origin g about y axis,

$$gG_1 \cdot \text{Area}_A = \cancel{gg} \cdot \text{Area}_{(A-a)} + gg_1 \cdot \text{Area}_a$$

$$,(gg = 0)$$

$$gG_1 \cdot \text{Area}_A = gg_1 \cdot \text{Area}_a$$

$$\frac{gG_1}{gg_1} = \frac{\text{Area}_a}{\text{Area}_A} \quad \cdots ①$$

G_1 : Centroid of total area, Area_A : Total area
g : Centroid of large circle, $\text{Area}_{A\text{-}a}$: Area of large circle
g_1 : Centroid of small circle, Area_a : Area of small circle

Movement of Center caused by Movement of Area (2)

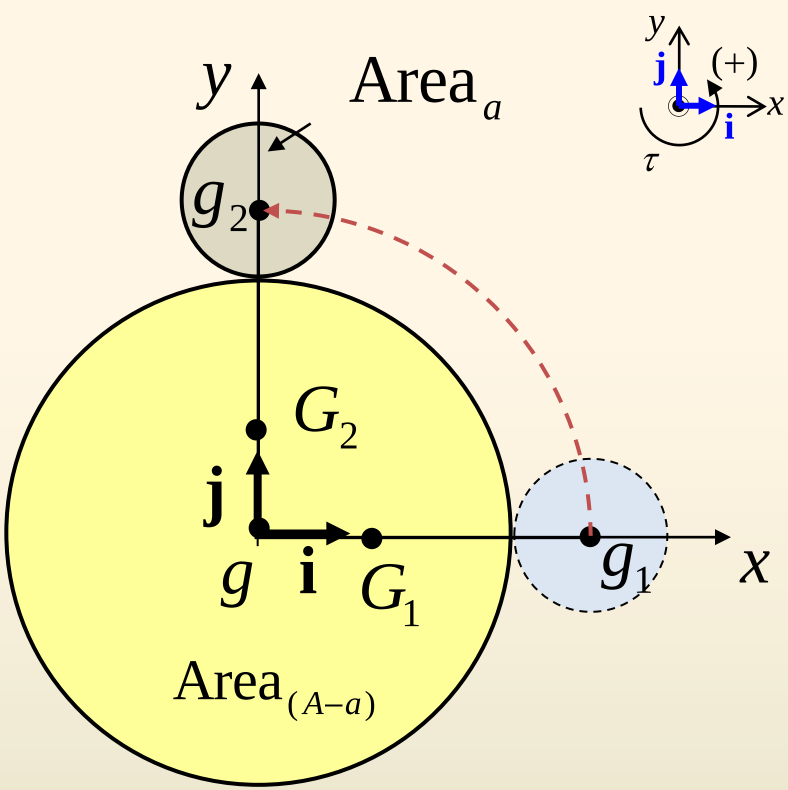
1) Gere, Mechanics of Materials, 6th ,Ch.12.3, 2006

First Moment of Composite Area(Q_x)[1]

$$Q_x = \sum_{n=1}^{n} A_i \cdot \bar{x}_i$$

$$A \cdot \bar{x} = \sum_{n=1}^{n} A_i \cdot \bar{x}_i$$

Q_x : 1st Moment
A_i : Each Area
A : Total Area
$\bar{x}$: Coordinate of Centroid



<면적a가 g_1에서 g_2로 이동했을 때, 면적 모멘트>

중심 g를 통하여 그 면에 수직한 축에 대한 1차 면적 모멘트을 고려하면,

$$gG_2 \cdot \text{Area}_A = \cancel{gg} \cdot \text{Area}_{(A-a)} + gg_2 \cdot \text{Area}_a$$

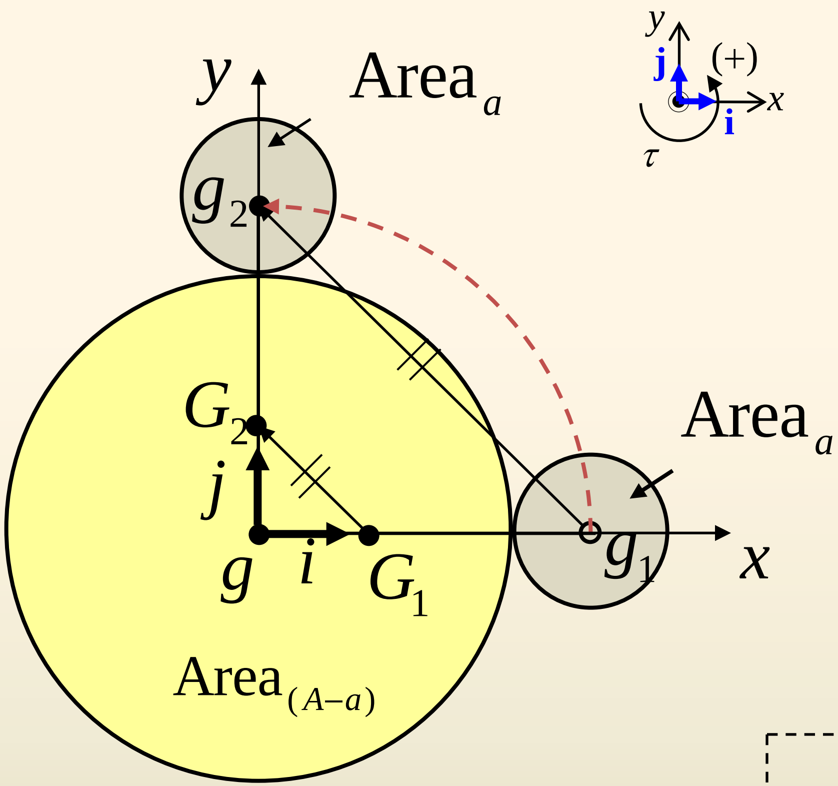
$$,(gg = 0)$$

$$gG_2 \cdot \text{Area}_A = gg_2 \cdot \text{Area}_a$$

$$\frac{gG_2}{gg_2} = \frac{\text{Area}_a}{\text{Area}_A} \cdots ②$$

G_1 : Centroid of total area, A : Total area
g : Centroid of large circle, A-a : Area of large circle
g_1 : Centroid of small circle, a : Area of small circle

Movement of Center caused by Movement of Area (3)



$$\angle G_1 g G_2 = \angle g_1 g g_2 \quad \cdots ③$$

By ①, ②, ③

Triangle $\triangle G_1 g G_2$ and $\triangle g_1 g g_2$ are similar.
(SAS(Side-Angle-Side) similarity)

$$G_1 G_2 \,//\, g_1 g_2$$

$$\frac{G_1 G_2}{g_1 g_2} = \frac{\text{Area}_a}{\text{Area}_A} \Rightarrow G_1 G_2 = \frac{\text{Area}_a}{\text{Area}_A} \times g_1 g_2$$

Using ratio of similitude

$$\frac{gG_1}{gg_1} = \frac{\text{Area}_a}{\text{Area}_A} \quad \cdots ① \qquad \frac{gG_2}{gg_2} = \frac{\text{Area}_a}{\text{Area}_A} \quad \cdots ②$$

G_1 : Centroid of total area, A : Total area

g : Centroid of large circle, A-a : Area of large circle

g_1 : Centroid of small circle, a : Area of small circle

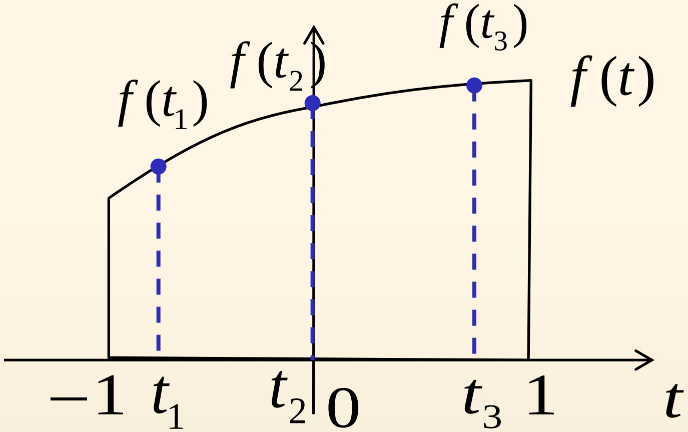
In case that partial area is moved, moving distance of total area can be calculated by using ① areas of each shape and ② moving distance of partial area.

Path of the total area is parallel to path of the partial area.

(Ref.) Gaussian quadrature

Given: function $f(t)$

Find: Integral of $f(t)$ over [-1,1]



$$\int_{-1}^{1} f(t)dt \approx \sum_{j=1}^{n} A_j \cdot f(t_j)$$

In case of 3rd order Gaussian quadrature

$$\int_{-1}^{1} f(t)dt \approx A_1 \cdot f(t_1) + A_2 \cdot f(t_2) + A_3 \cdot f(t_3)$$

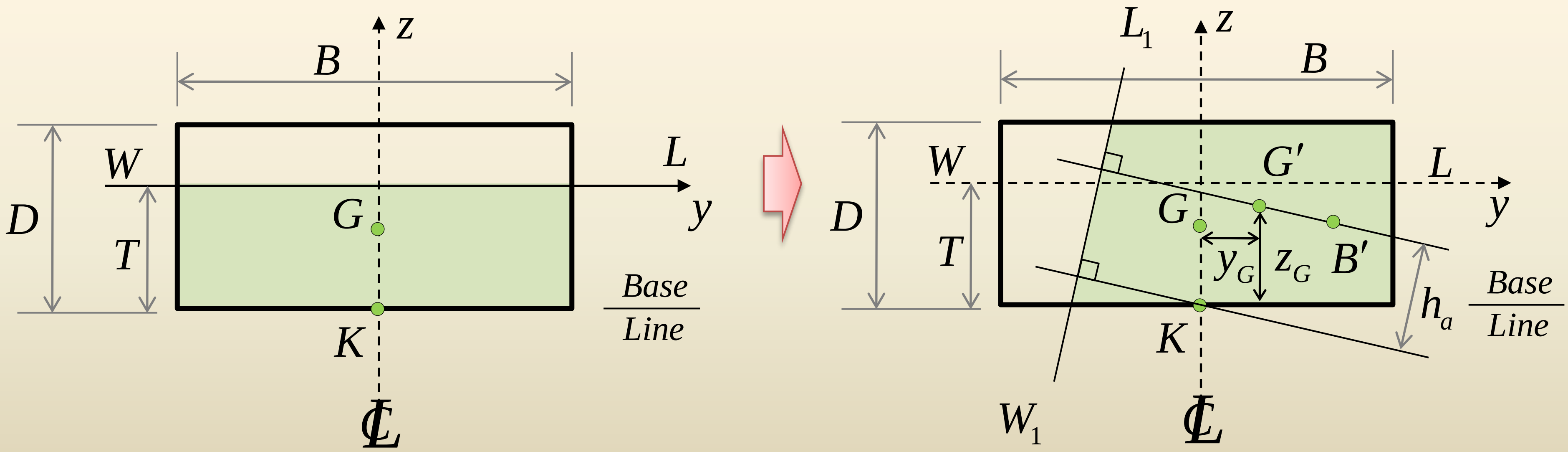
n	Coefficients A_j	Node t_j
3	A_1 = 0.5555555556	t_1 = -0.7745966692
	A_2 = 0.8888888889	t_2 = 0
	A_3 = 0.5555555556	t_3 = 0.7745966692

Examples

Example 1> Heeling Moment caused by Fluid in Tanks(1)

Question) A ship with section of breadth B, depth D is afloat in water, and fluid is partially filled in tank with draft T (Waterline WL). Waterline of a fluid in tank is shifted from WL to W_1L_1 due to heel. Calculated heeling moment arm about origin K about the axis which is perpendicular to waterline W_1L_1.

- Given : B, D, T, ϕ
- Find : Heeling arm h_a about the axis which is perpendicular to waterline W_1L_1



G : Center of mass,
G' : Changed cemter of mass,
B': Changed center of buoyancy

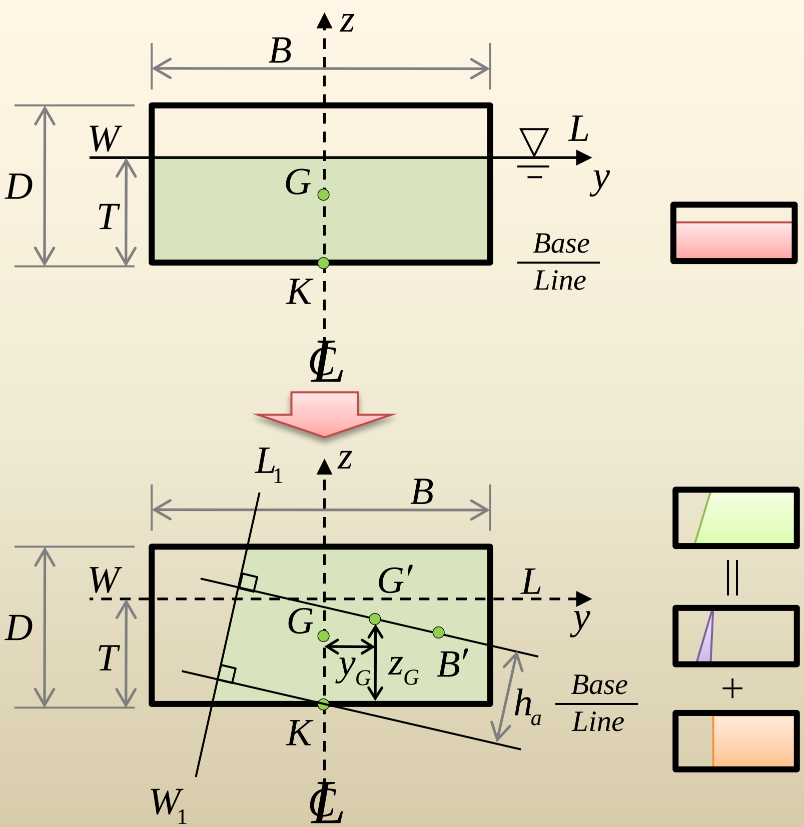
Example 1> Heeling Moment caused by Fluid in Tanks(2)

• Given : B, D, T, ϕ

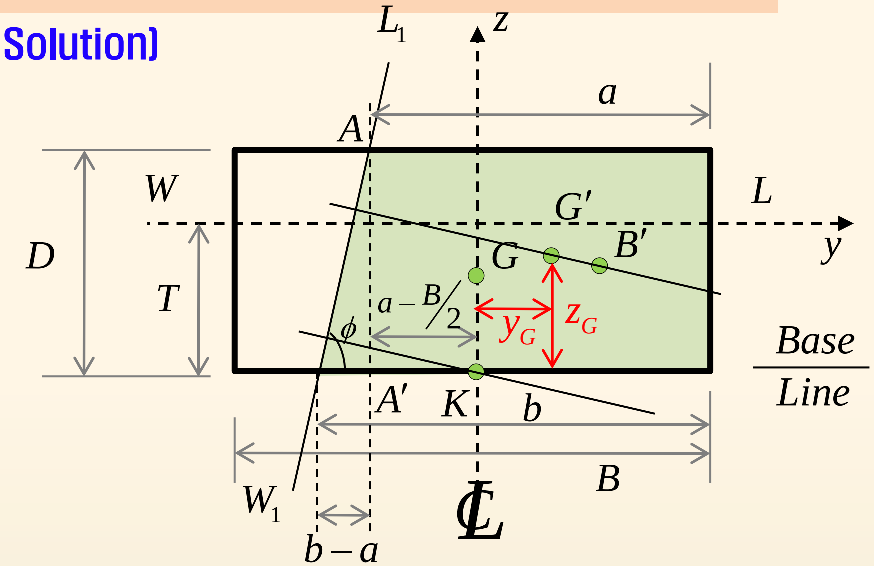
• Find : Heeling arm h_a about the axis which is perpendicular to waterline W_1L_1

$$h_a = y_G \cos\phi + z_G \sin\phi$$

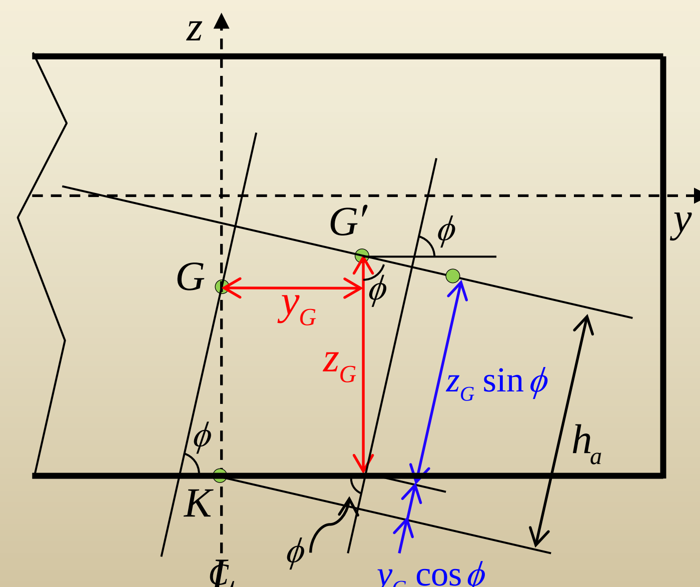
Question) A ship with section of breadth B, depth D is afloat in water, and fluid is partially filled in tank with draft T (Waterline WL). Waterline of a fluid in tank is shifted from WL to W_1L_1 due to heel. Calculated heeling moment arm about origin K about the axis which is perpendicular to waterline W_1L_1.



Solution)



Heeling arm about new waterline ha : $h_a = y_G \cos\phi + z_G \sin\phi$



G : Center of mass,

G' : Changed cemter of mass,

B': Changed center of buoyancy

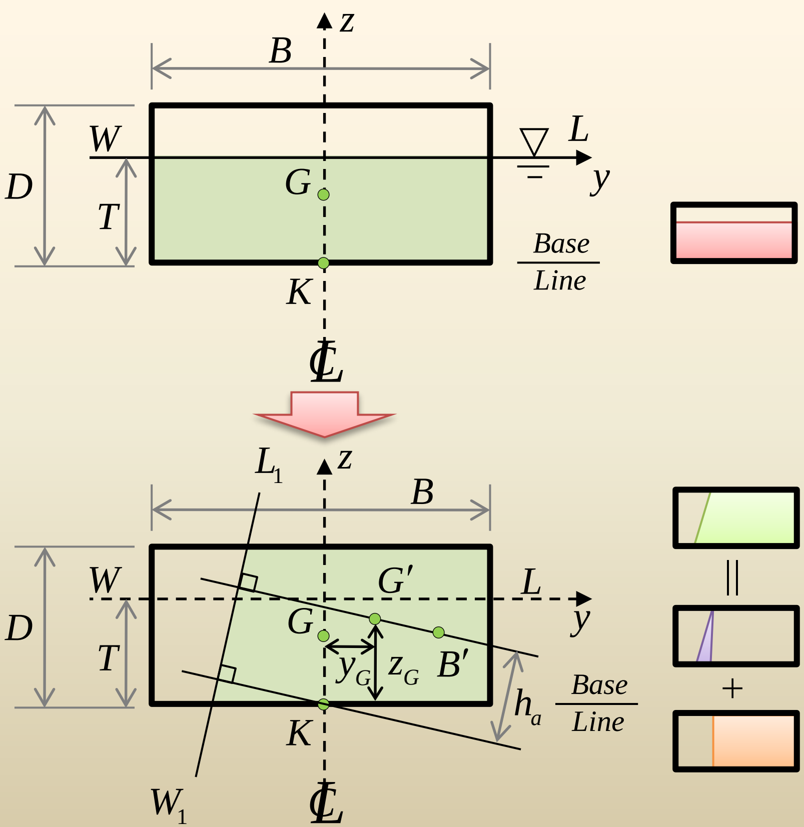
Example 1> Heeling Moment caused by Fluid in Tanks(3)

• Given : B, D, T, ϕ

• Find : Heeling arm h_a about the axis which is perpendicular to waterline W_1L_1

$$h_a = y_G \cos\phi + z_G \sin\phi$$

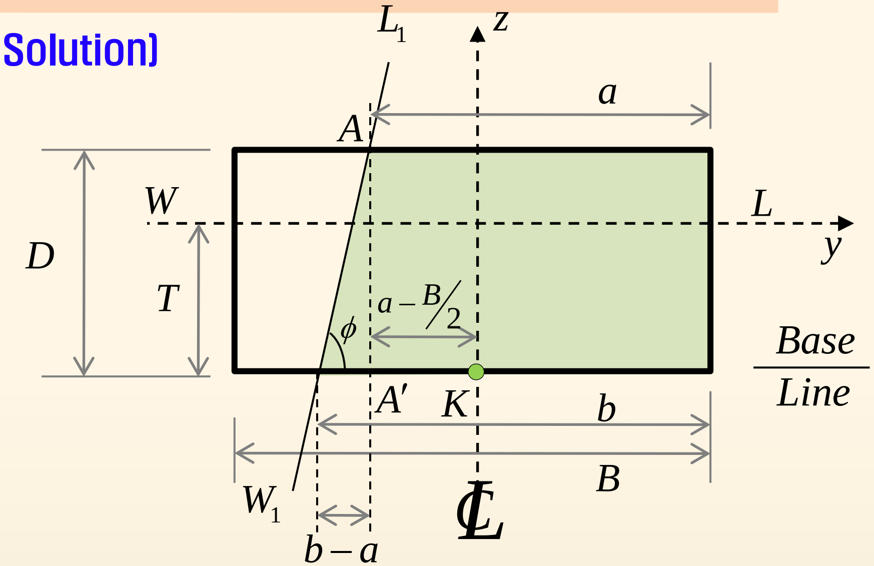
Question) A ship with section of breadth B, depth D is afloat in water, and fluid is partially filled in tank with draft T (Waterline WL). Waterline of a fluid in tank is shifted from WL to W_1L_1 due to heel. Calculated heeling moment arm about origin K about the axis which is perpendicular to waterline W_1L_1.



G : Center of mass,

G' : Changed cemter of mass, B': Changed center of buoyancy

Solution)



Heeling arm ha : $h_a = y_G \cos\phi + z_G \sin\phi$

① Calculation of b

$= BT$

$$\frac{1}{2}\frac{D}{\tan\phi}D + (b - \frac{D}{\tan\phi})D = BT$$

$$b = \frac{BT}{D} + \frac{D}{2\tan\phi}$$

② Calculation of a

From the equation, $b - a = \dfrac{D}{\tan\phi}$

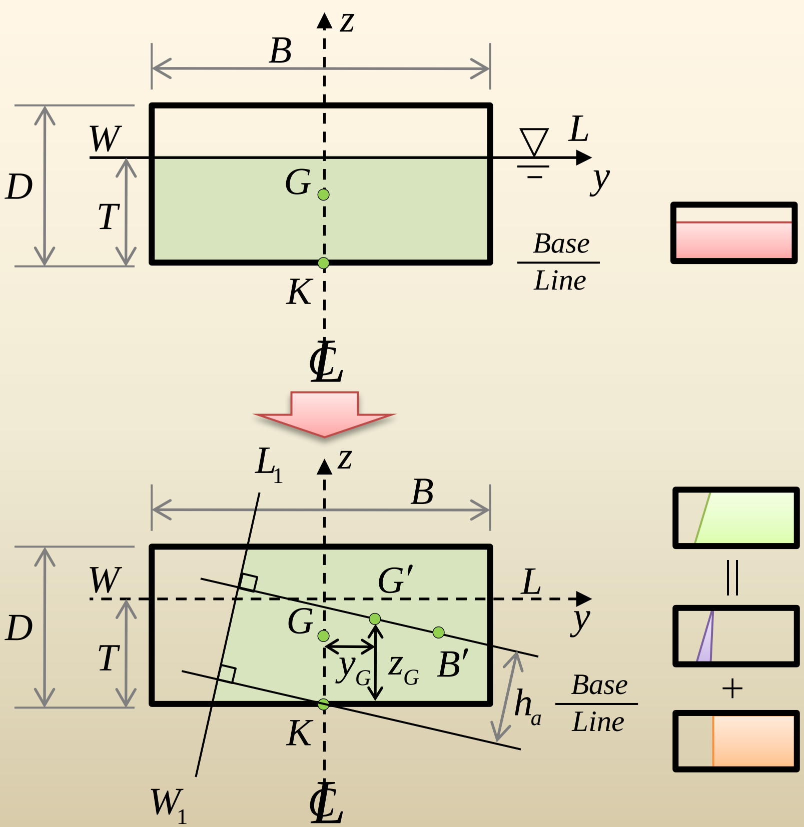
$$a = b - \frac{D}{\tan\phi} = \frac{BT}{D} - \frac{D}{2\tan\phi}$$

Example 1> Heeling Moment caused by Fluid in Tanks(4)

• Given : B, D, T, ϕ • Find : Heeling arm h_a about the axis which is perpendicular to waterline W_1L_1

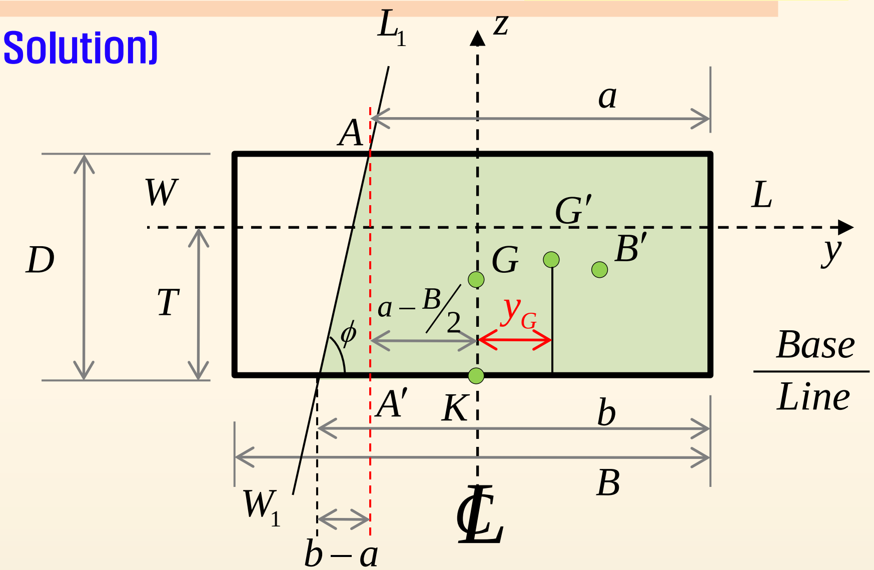
$$h_a = y_G \cos\phi + z_G \sin\phi$$

Question) A ship with section of breadth B, depth D is afloat in water, and fluid is partially filled in tank with draft T (Waterline WL). Waterline of a fluid in tank is shifted from WL to W_1L_1 due to heel. Calculated heeling moment arm about origin K about the axis which is perpendicular to waterline W_1L_1.

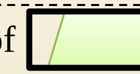


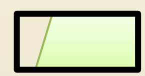
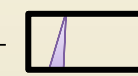
G : Center of mass,
G' : Changed cemter of mass, B': Changed center of buoyancy

Solution)



Heeling arm ha : $h_a = \boxed{y_G}\cos\phi + z_G \sin\phi$

③ The transverse moment of  about A-A' axis

$M_y =$  $\cdot y_G =$  $\cdot y_{G1} +$  $\cdot y_{G2}$

$$= aD\frac{a}{2} + \frac{(b-a)}{2}D(-1)\frac{(b-a)}{3}$$

$$= \frac{(BT)^2}{2D} - \frac{BT\,D}{2\tan\phi} + \frac{D^3}{8\tan^2\phi} - \frac{D^3}{6\tan^2\phi}$$

④ Transverse center of mass about center line

$$y_G = \frac{M_y}{BT} - (a - \frac{B}{2})$$

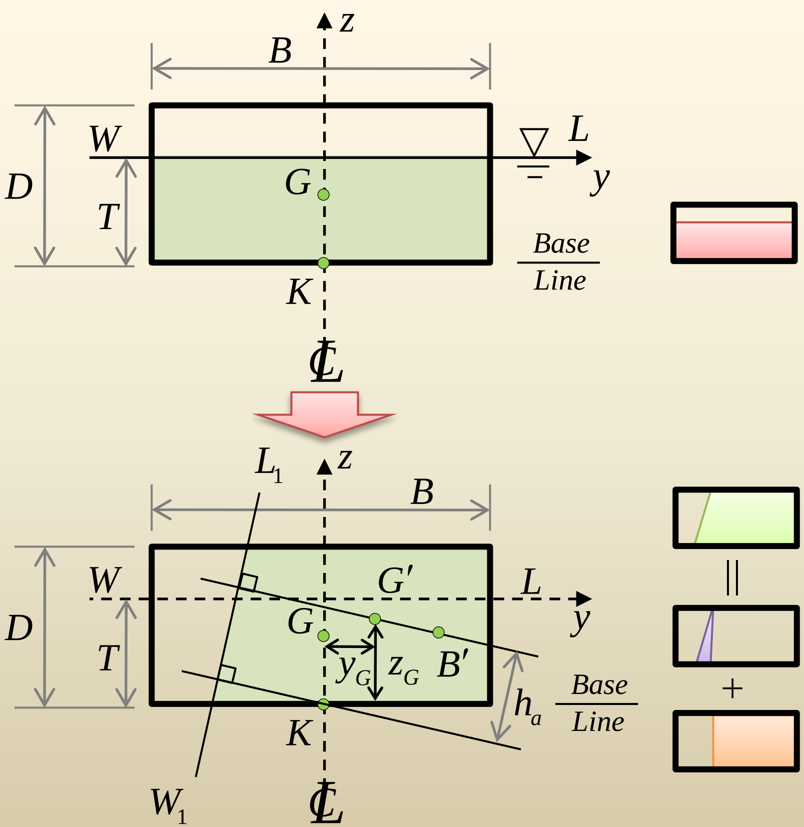
$$= \frac{B}{2} - \frac{BT}{2D} - \frac{D^3}{24\tan^2\phi\ BT}$$

Example 1> Heeling Moment caused by Fluid in Tanks(5)

• Given : B, D, T, ϕ • Find : Heeling arm h_a about the axis which is perpendicular to waterline W_1L_1

$$h_a = y_G \cos\phi + z_G \sin\phi$$

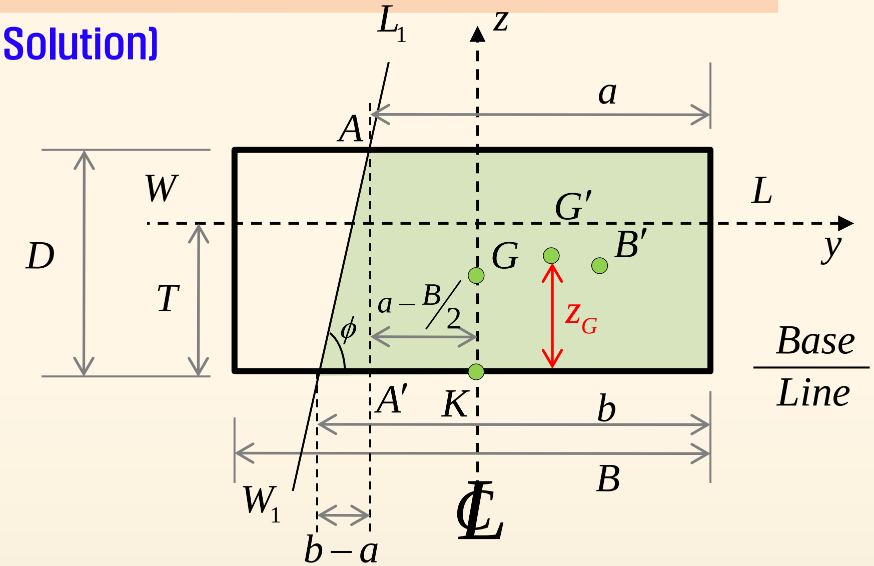
Question) A ship with section of breadth B, depth D is afloat in water, and fluid is partially filled in tank with draft T (Waterline WL). Waterline of a fluid in tank is shifted from WL to W_1L_1 due to heel. Calculated heeling moment arm about origin K about the axis which is perpendicular to waterline W_1L_1.



G : Center of mass,

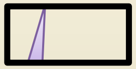
G' : Changed cemter of mass, B': Changed center of buoyancy

Solution)



Heeling arm ha : $h_a = y_G \cos\phi + \boxed{z_G} \sin\phi$

⑤ Vertical moment of  about base line

$M_z =$  $\cdot z_G =$  $\cdot z_{G1} +$  $\cdot z_{G2}$

$$= a\,D\frac{D}{2} + \frac{(b-a)}{2}D\frac{D}{3}$$

$$= \frac{D}{2}BT - \frac{D^3}{12\tan\phi\; BT}$$

⑥ Vertical center of mass about baseline

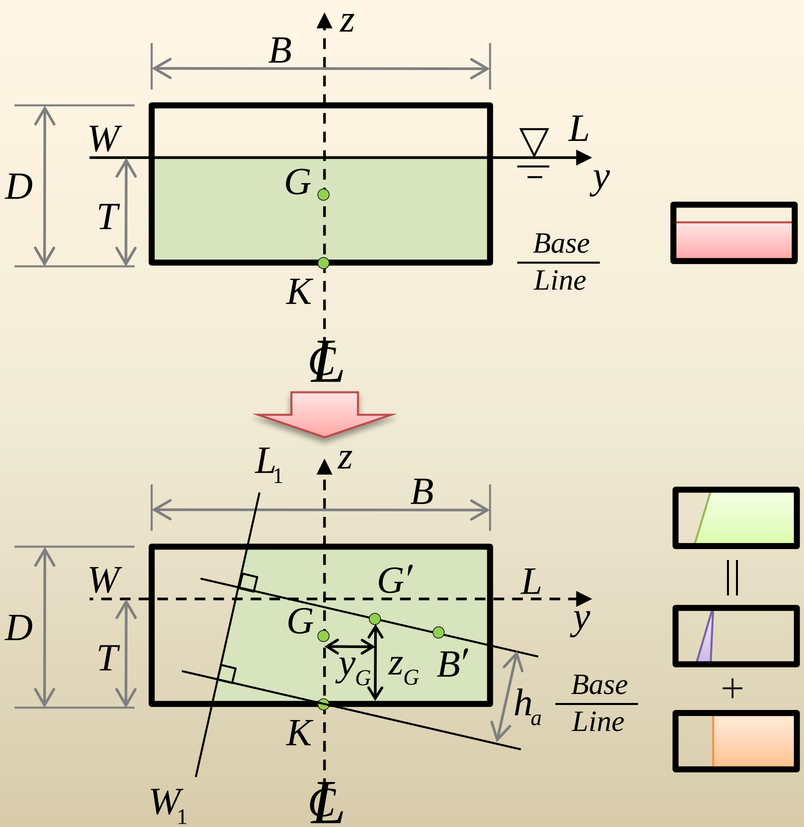
$$z_G = \frac{M_z}{BT} = \frac{D}{2} - \frac{D^3}{12\tan\phi\, BT}$$

Example 1> Heeling Moment caused by Fluid in Tanks(6)

• Given : B, D, T, ϕ • Find : Heeling arm h_a about the axis which is perpendicular to waterline W_1L_1

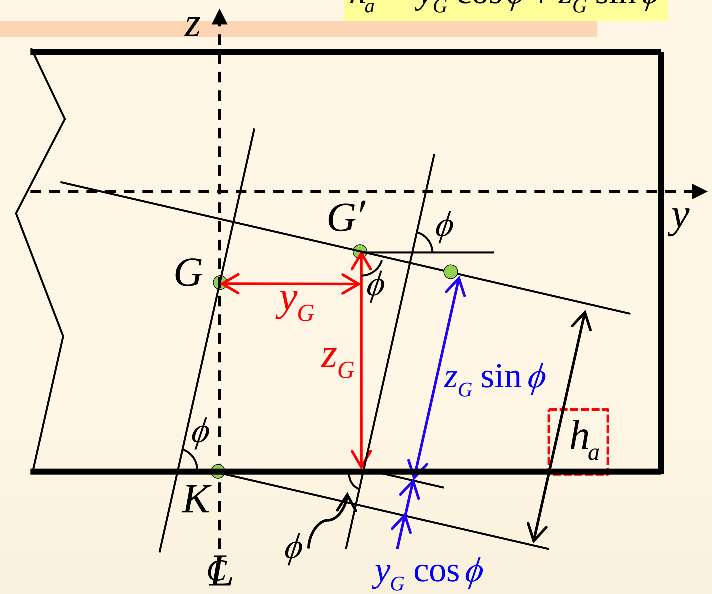
$h_a = y_G \cos\phi + z_G \sin\phi$

Question) A ship with section of breadth B, depth D is afloat in water, and fluid is partially filled in tank with draft T (Waterline WL). Waterline of a fluid in tank is shifted from WL to W_1L_1 due to heel. Calculated heeling moment arm about origin K about the axis which is perpendicular to waterline W_1L_1.



G : Center of mass,
G' : Changed cemter of mass, B': Changed center of buoyancy

Solution)



Heeling arm ha : $\boxed{h_a} = y_G \cos\phi + z_G \sin\phi$

⑤ When center of mass is shifted from G to G1(y_G,z_G), then heeling arm ha is

$$y_G = \frac{B}{2} - \frac{BT}{2D} - \frac{D^3}{24\tan^2\phi\, BT}$$

$$z_G = \frac{D}{2} - \frac{D^3}{12\tan\phi\, BT}$$

$$h_a = y_G \cos\phi + z_G \sin\phi$$

$$= \left[\frac{B}{2} - \frac{BT}{2D} - \frac{D^3}{12BT}\left(\frac{1}{2\tan^2\phi} + 1\right)\right]\cos\phi + \frac{D}{2}\sin\phi$$

Example2>Heel Angle caused by Movement of Passengers in Ferry (1)

- Given : KB, KG, I_T, Heeling moment M_h
- Find : An angle of heel ϕ
- GZ of wall sided ship

$$GZ = \left(GM + \frac{1}{2} BM \tan^2 \phi \right) \sin \phi$$

Question) Emergency circumstance happens in Ferry with displacement (mass) 102.5 ton. Heeling moment of 8 ton·m occurs due to passengers moving to the right of the ship. What will be an angle of heel?
Assume that wall sided ship with KB=0.6m, KG=2.4m, I_T=200m^4.

Solution) If it is in static equilibrium at an angle of heel ϕ

Righting moment in wall sided ship(M_r) = Heeling moment (M_h)

$$\Delta \left(GM + \frac{1}{2} BM \tan^2 \phi \right) \sin \phi = 8\, ton \cdot m$$

① Calculation of BM

$$\Delta = 102.5 \text{ ton} \quad \rightarrow \quad \nabla = \Delta / 1.025 = 100 \text{ m}^3$$

$$BM = \frac{I_T}{\nabla} = \frac{200}{100} = 2m$$

② Calculation of GM

$$GM = KB + BM - KG$$
$$= 0.6 + 2 - 2.4 = 0.2 \ m$$

$$\left(0.2 + \tan^2 \phi\right) \sin \phi = \frac{8}{102.5}$$

Non linear equation about ϕ ?

Example2>Heel Angle caused by Movement of Passengers in Ferry (2)

- Given : KB, KG, I_T, Heeling moment M_h
- Find : An angle of heel ϕ
- GZ of wall sided ship

$$GZ = \left(GM + \frac{1}{2} BM \tan^2 \phi \right) \sin \phi$$

Question) Emergency circumstance happens in Ferry with displacement (mass) 102.5 ton. Heeling moment of 8 ton·m occurs due to passengers moving to the right of the ship. What will be an angle of heel?
Assume that wall sided ship with KB=0.6m, KG=2.4m, I_T=200m^4.

Solution) If it is in static equilibrium at an angle of heel ϕ

Righting moment in wall sided ship(M_r) = Heeling moment (M_h)

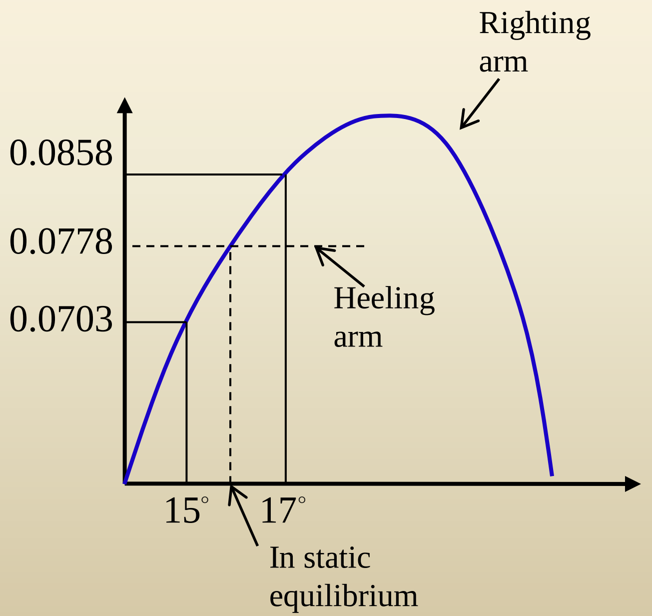
$$\Delta \left(GM + \frac{1}{2} BM \tan^2 \phi \right) \sin \phi = 8\, ton \cdot m$$

$$\left(0.2 + \tan^2 \phi\right) \sin \phi = 0.078$$

Because of nonlinear equation, solve it by numerical method.

Result of calculation is about ϕ=16.0˚.

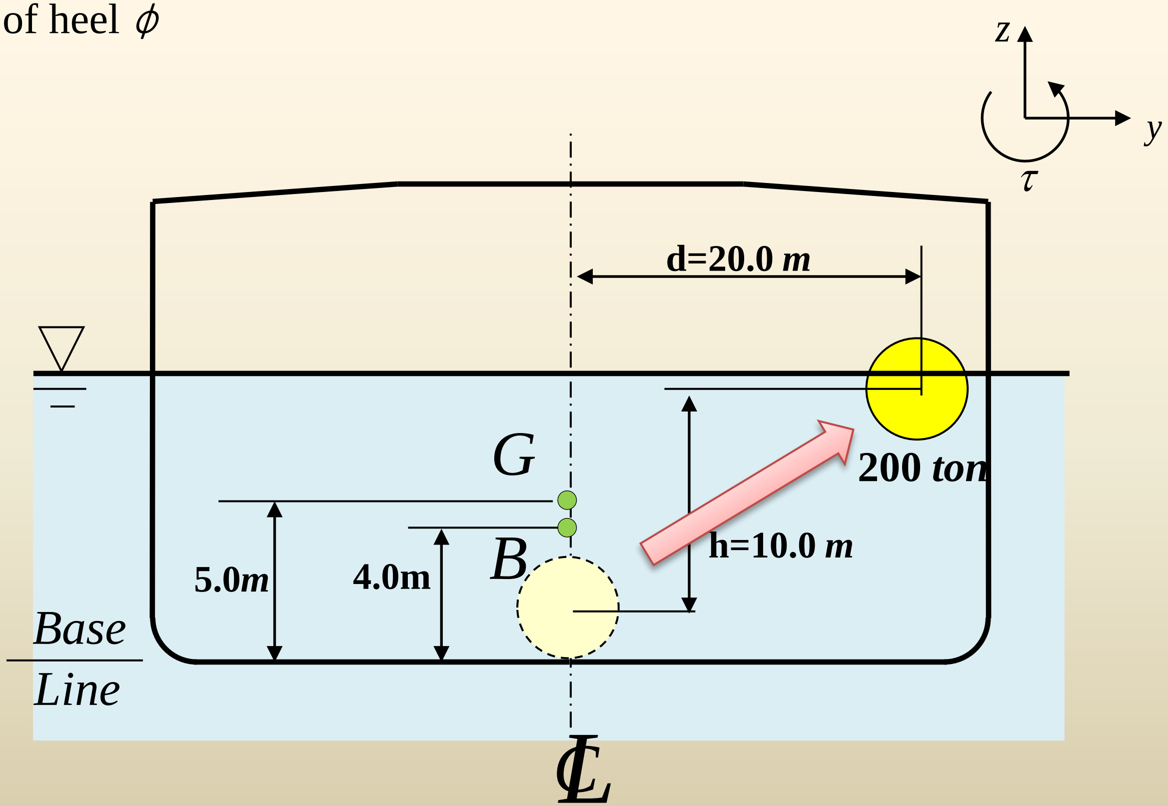
ϕ	LHS (Righting arm)	RHS (Heeling arm)
15˚	0.0703	0.0780
16˚	0.0778	0.0780
17˚	0.0858	0.0780



Example3> Heel Angle caused by Movement of Cargo (1)

Question) A cargo carrier of 10,000 ton displacement is floating. KB=4.0m, BM=2.5m, KG=5.0m. Cargo in hold of cargo carrier is shifted in vertical direction through a 10 meter, and shifted in transverse direction through a 20 meters. Find an angle of heel.

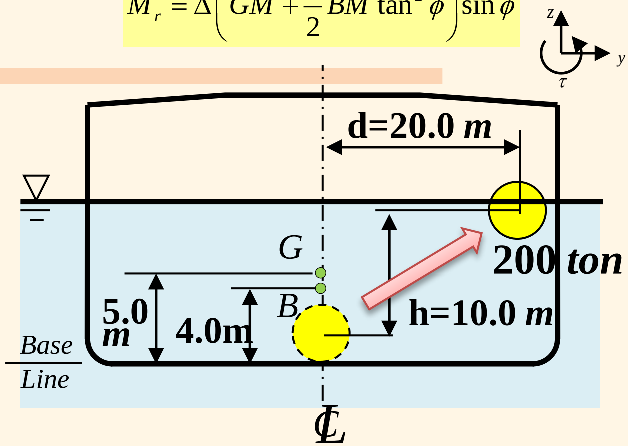
- Given : displacement (Δ), KB, BM, KG, weight of cargo(w) and moving distance
- Find : angle of heel ϕ



Example3> Heel Angle caused by Movement of Cargo (2)

• GZ of wall sided ship

$$M_r = \Delta\left(GM + \frac{1}{2}BM\tan^2\phi\right)\sin\phi$$



Question) A cargo carrier of 10,000 ton displacement is floating. KB=4.0m, BM=2.5m, KG=5.0m. Cargo in hold of cargo carrier is shifted in vertical direction through a 10 meter, and shifted in transverse direction through a 20 meters. Find an angle of heel.

- Given : displacement (Δ), KB, BM, KG, weight of cargo(w) and moving distance
- Find : angle of heel ϕ

Solution) If it is in static equilibrium at an angle of heel ϕ

Heeling moment due to shift of center of mass	=	Righting moment
$W(KG\sin\phi + \boxed{\delta y_G}\cos\phi + \boxed{\delta z_G}\sin\phi)$	=	$\Delta \cdot KN$

① Shift of center of mass of the ship

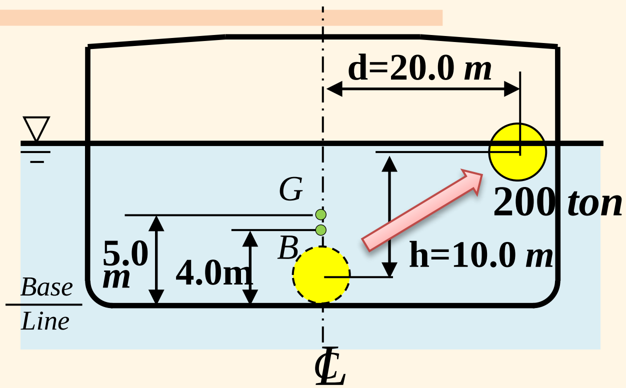
$$\delta y_G = \frac{w \cdot d}{\Delta} = \frac{200 \cdot 20}{10,000} = 0.4$$

$$\delta z_G = \frac{w \cdot h}{\Delta} = \frac{200 \cdot 10}{10,000} = 0.2$$

Example3> Heel Angle caused by Movement of Cargo (3)

•수직 측벽선의 GZ

$$M_r = \Delta\left(GM + \frac{1}{2}BM\tan^2\phi\right)\sin\phi$$



Question) A cargo carrier of 10,000 ton displacement is floating. KB=4.0m, BM=2.5m, KG=5.0m. Cargo in hold of cargo carrier is shifted in vertical direction through a 10 meter, and shifted in transverse direction through a 20 meters. Find an angle of heel.

- Given : displacement (Δ), KB, BM, KG, weight of cargo(w) and moving distance
- Find : angle of heel ϕ

Solution) If it is in static equilibrium at an angle of heel ϕ

Heeling moment due to shift of center of mass	=	Righting moment
$W(KG\sin\phi + \delta y_G\cos\phi + \delta z_G\sin\phi)$	=	$\Delta \cdot KN$

② From the equation of moment equilibrium

$$W(KG\sin\phi + \delta y_G\cos\phi + \delta z_G\sin\phi) = \Delta \cdot KN$$
$$= \Delta \cdot (KG\sin\phi + GZ)$$
$$\therefore \delta y_G\cos\phi + \delta z_G\sin\phi = \boxed{GZ}$$

Case1) Assumption of small angle of heel (ϕ<10°)

$$GZ = GM\cdot\sin\phi$$

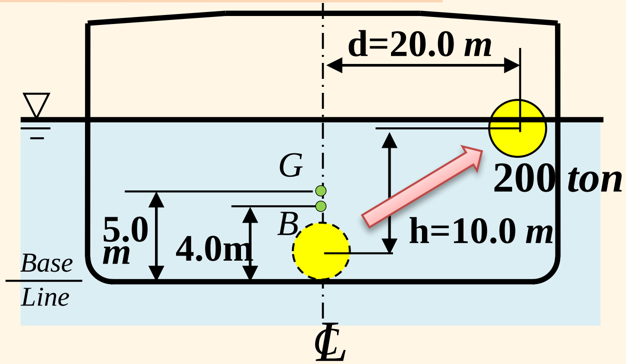
Case1) Assumption of large angle of heel (ϕ<10°)

$$GZ = \left(GM + \frac{1}{2}BM\tan^2\phi\right)\sin\phi$$

Example3> Heel Angle caused by Movement of Cargo (4)

•수직 측벽선의 GZ

$$M_r = \Delta\left(GM + \frac{1}{2}BM\tan^2\phi\right)\sin\phi$$



Question) A cargo carrier of 10,000 ton displacement is floating. KB=4.0m, BM=2.5m, KG=5.0m. Cargo in hold of cargo carrier is shifted in vertical direction through a 10 meter, and shifted in transverse direction through a 20 meters. Find an angle of heel.

- Given : displacement (Δ), KB, BM, KG, weight of cargo(w) and moving distance
- Find : angle of heel ϕ

Solution) If it is in static equilibrium at an angle of heel ϕ

Heeling moment due to shift of center of mass	=	Righting moment
$W(KG\sin\phi + \delta y_G\cos\phi + \delta z_G\sin\phi)$	=	$\Delta \cdot KN$

$$\therefore \delta y_G\cos\phi + \delta z_G\sin\phi = \boxed{GZ}$$

Case1) Assumption of small angle of heel (ϕ<10°) $\quad GZ = GM\cdot\sin\phi$

Also small angle of heel follows, $\quad \delta z_G\cdot\sin\phi \approx 0$

$$\delta y_G\cos\phi = GM\cdot\sin\phi$$

$$\frac{\delta y_G}{GM} = \tan\phi$$

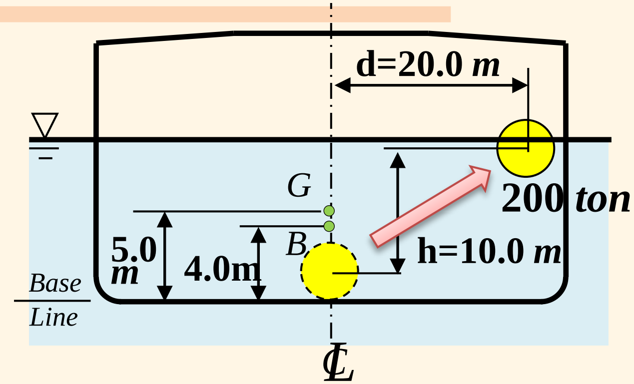
$$\phi = \tan^{-1}\frac{\delta y_G}{GM} = \tan^{-1}\frac{0.4}{1.5} = 0.2666\ \text{rad}\ (=14.93^\circ)$$

Because of ϕ>10°
This result contradicts the assumption

Example3> Heel Angle caused by Movement of Cargo (5)

•수직 측벽선의 GZ

$$M_r = \Delta\left(GM + \frac{1}{2}BM\tan^2\phi\right)\sin\phi$$



Question) A cargo carrier of 10,000 ton displacement is floating. KB=4.0m, BM=2.5m, KG=5.0m. Cargo in hold of cargo carrier is shifted in vertical direction through a 10 meter, and shifted in transverse direction through a 20 meters. Find an angle of heel.

- Given : displacement (Δ), KB, BM, KG, weight of cargo(w) and moving distance
- Find : angle of heel ϕ

Solution) If it is in static equilibrium at an angle of heel ϕ

Heeling moment due to shift of center of mass	=	Righting moment
$W(KG\sin\phi + \delta y_G\cos\phi + \delta z_G\sin\phi)$	=	$\Delta\cdot KN$

$$\therefore \delta y_G\cos\phi + \delta z_G\sin\phi = \boxed{GZ}$$

Case1) Assumption of large angle of heel (ϕ<10°)

Also assumption of wall sided ship $GZ = (GM + \frac{1}{2}BM\tan^2\phi)\sin\phi$

$$\delta y_G\cos\phi + \delta z_G\sin\phi = (GM + \frac{1}{2}BM\tan^2\phi)\sin\phi$$

Because of nonlinear equation, solve it by numerical method.

Initial value? Start with value 14.9° from case 1

If we divide $\cos\phi$ for both side, we can get an cubic equation about $\tan\phi$.

$$\delta y_G + \delta z_G\tan\phi = (GM + \frac{1}{2}BM\tan^2\phi)\tan\phi$$

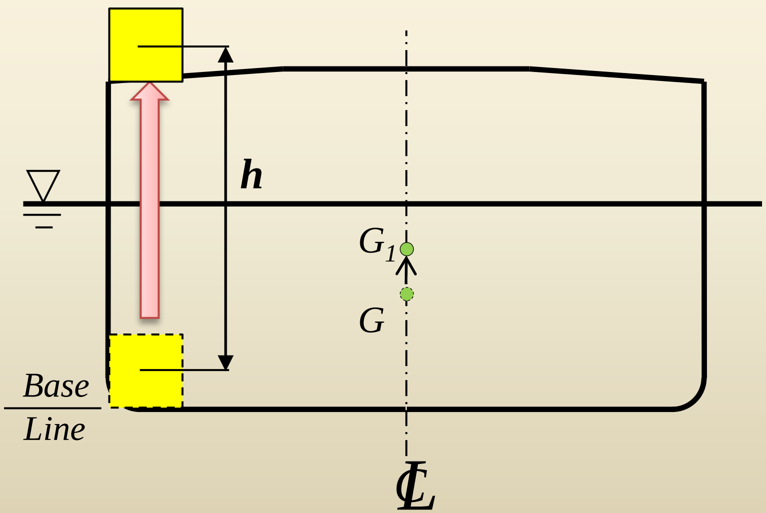
	LHS	RHS	Ratio (LHS/RHS)
14.9	0.4532	0.4227	107%
15.1	0.4540	0.4293	106%
15.3	0.4547	0.4359	104%
15.5	0.4555	0.4426	103%
15.7	0.4562	0.4494	102%
15.9	0.4570	0.4562	100%
16.1	0.4577	0.4630	99%

$$\therefore \phi \approx 15.9^\circ$$

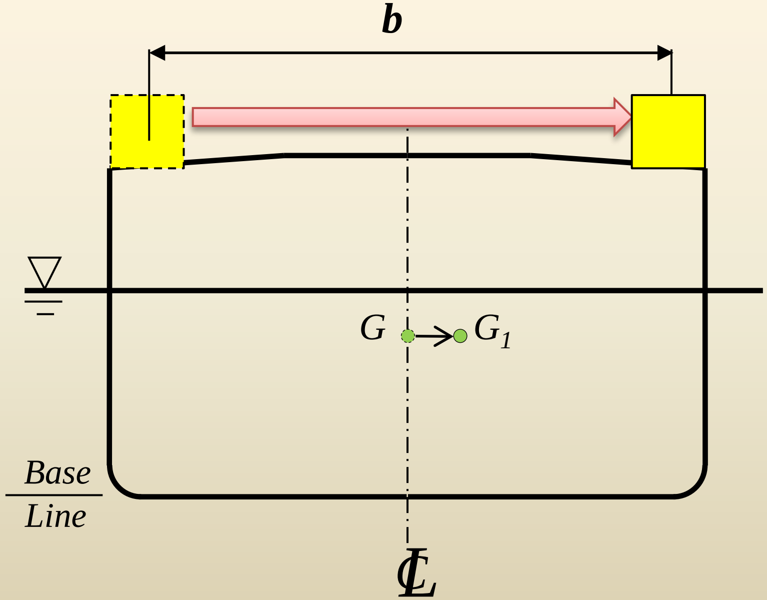
Example4> Change of Center caused by Movement of Cargo (5)

Question) As below cases partial weight w of the ship is shifted. What is the shift distance of center of mass of the ship?

Case1) Vertical shift of the partial weight



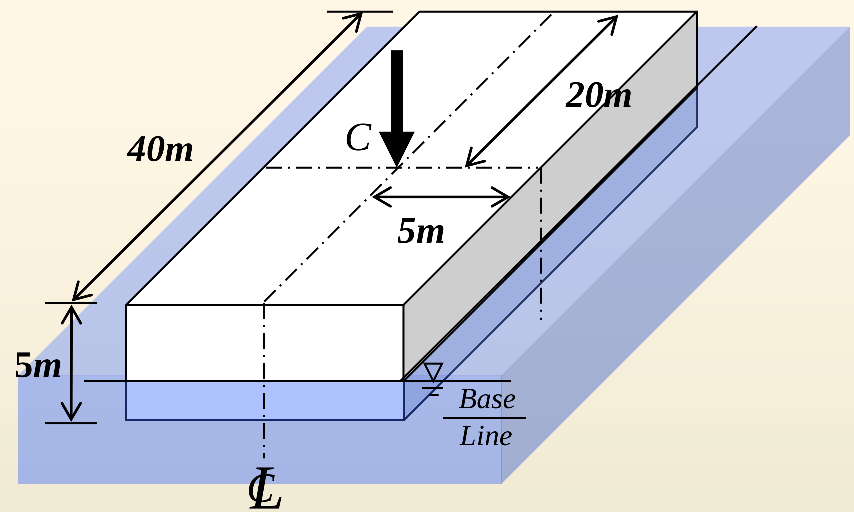
Case2) Horizontal shift of the partial weight



Example5> Calculation of Deadweight of Barge

Question)

A barge is 40m length, 10m breadth, 5m depth, and is floating at 1 m draft. The vertical center of mass of the ship is located in 2 m from the baseline. A cargo is supposed to be loaded in center of the deck. Find the maximum loadable weight that keeps the stability of ship.



Problem to calculated position of the ship when external force are applied.

Example6> Calculation of Position of Ship when Cargo is moved by Crane

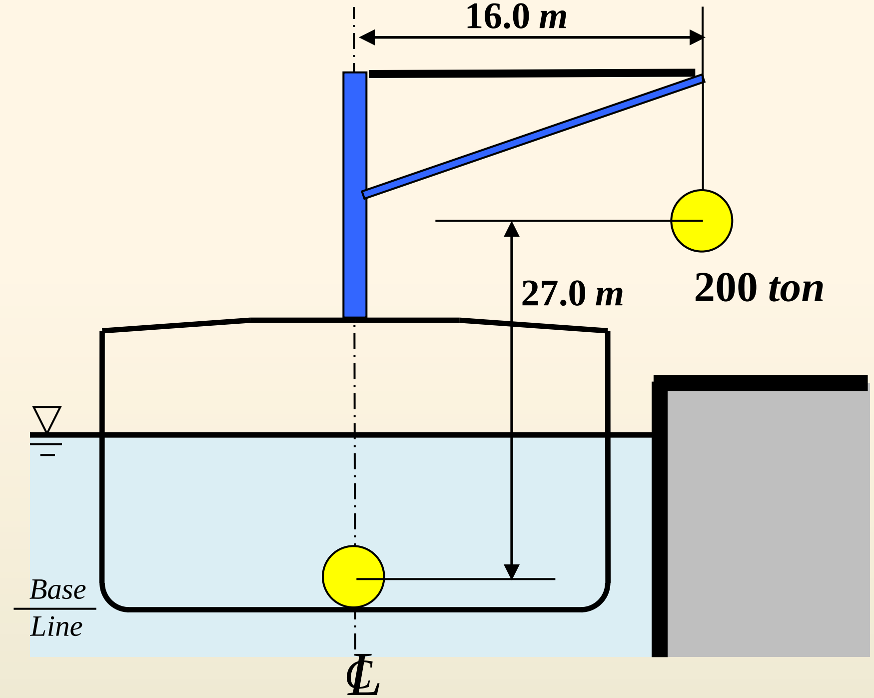
Question)

A Cargo carrier of 18,000 ton displacement is afloat and has GM = 1.5m. And we want to transfer the cargo of 200 ton weight from bottom of the ship to land.

A lifting height of cargo is 27.0 m from the original position.

After lifting the cargo, turn the cargo to the right through a distance of 16.0 m from the centerline.

What will be the angle of heel of the ship?



Problem to calculated position of the ship when external force are applied.