# 4. Plane EM Waves in Plasmas

## A. Waves in Unmagnetized Cold Plasmas

### 1) Wave equations

· Plasma model (Initial equilibrium state unperturbed by waves)

① homogeneous plasma ($N = const$), ② no external EM fields( $E_{ext}$ = 0, $B_{ext}$ = 0),

③ cold plasma ($T=0$), ④ no drift motion ($v_o = 0$), ⑤ fixed ions

· Governing equations (Elec. eqn. of motion, Maxwell's equations)

$$m\frac{\partial v}{\partial t} = -eE \quad (m\text{=elec. mass, } -e\text{=elec. charge}) \qquad (13)$$

$$\nabla\times E = -\frac{\partial B}{\partial t} \qquad (6\text{-}94a)$$

$$\nabla\times B = \mu_o J + \mu_o\varepsilon_o\frac{\partial E}{\partial t} \qquad (6\text{-}94b)*$$

$$\nabla\cdot E = \rho_v/\varepsilon_o \qquad (6\text{-}94c)*$$

$$\nabla\cdot B = 0 \qquad (6\text{-}94d)*$$

$$J = -eNv \quad (N = \text{plasma density } m^{-3}) \qquad (14)$$

· Wave equations in plasmas

$$\nabla\times(6-94a):\quad \nabla\times(\nabla\times E) = -\frac{\partial}{\partial t}(\nabla\times B)$$

$$\Rightarrow \quad \nabla\times(\nabla\times E) = -\mu_o\frac{\partial J}{\partial t} - \mu_o\varepsilon_o\frac{\partial^2 E}{\partial t^2} \qquad (15)$$

(13) (14) in (15) using $\nabla\times\nabla\times E = \nabla(\nabla\cdot E) - \nabla^2 E$:

$$\nabla(\nabla\cdot E) - \nabla^2 E = -\mu_o\frac{Ne^2}{m}E - \mu_o\varepsilon_o\frac{\partial^2 E}{\partial t^2}$$

$$\Rightarrow \quad \nabla^2 \boldsymbol{E} - \nabla(\nabla\cdot \boldsymbol{E}) - \omega_p^2\mu_o\epsilon_o\boldsymbol{E} - \mu_o\epsilon_o\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0 \qquad (15)*$$

where $\omega_p = 2\pi f_p \equiv \sqrt{\dfrac{Ne^2}{m\epsilon_o}} \cong 2\pi\times 9\sqrt{N}$ (Hz): plasma frequency (7-131, 133)

*Notes)* i) (15) becomes $\nabla^2 E - \mu\sigma\dfrac{\partial E}{\partial t} - \mu\varepsilon\dfrac{\partial^2 E}{\partial t^2} = 0$ (Problem 7-1a)

in source-free( $\nabla\cdot E = 0$) conducting( $J = \sigma E$) medium.

ii) (15)* becomes $\nabla^2 E - \mu_o\varepsilon_o\dfrac{\partial^2 E}{\partial t^2} = 0$ (6-96)

in no plasma (source-free, nonconducting → free space).

For time-harmonic fields $E(r,t) = Re[E_s(r)\,e^{j\omega t}] = Re[E_o(r)\,e^{j(\omega t - k\cdot r)}]$,( 6-79, 7-23 )

$\nabla \rightarrow -jk,\ \nabla^2 \rightarrow -k^2;\ \partial/\partial t \rightarrow j\omega,\ \partial^2/\partial t^2 \rightarrow -\omega^2$ in (15)* :

$$\nabla^2\boldsymbol{E} + \mu_o\epsilon_p\omega^2\boldsymbol{E} = \nabla(\nabla\cdot\boldsymbol{E}):\ \text{Helmholtz's equation} \qquad (15)**[(6\text{-}98)*]$$

where $\epsilon_p = \epsilon_o\left(1 - \dfrac{\omega_p^2}{\omega^2}\right) = \epsilon_o\left(1 - \dfrac{f_p^2}{f^2}\right)$ (F/m) : effective permittivity (7-130) or $$\left(k^2 - \frac{\omega^2 - \omega_p^2}{c^2}\right)\boldsymbol{E} = \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E}) \qquad (15)^{***}$$

where $c = \sqrt{1/\mu_o\epsilon_o}$ = speed of light

*Note)* (6-94d)* $\Rightarrow$ $\boldsymbol{k}\cdot\boldsymbol{B} = 0$ : Transverse magnetic wave ( $\boldsymbol{k}\perp\boldsymbol{B}$ )

## 2) Plasma oscillation and TEM Wave

· Dispersion equations $[\omega = \omega(k)]$

Let $\boldsymbol{E} = \boldsymbol{E}_\parallel + \boldsymbol{E}_\perp$ with respect to $\boldsymbol{k}$-direction (propagating direction), then longitudinal $E$ wave part ( $\boldsymbol{k}\parallel\boldsymbol{E}_\parallel$, i.e., $\boldsymbol{k}\cdot\boldsymbol{E} = kE$) :

$$(15)^{***}_{\parallel} : \left(\cancel{k^2} - \frac{\omega^2 - \omega_p^2}{c^2}\right)E = \cancel{k^2}E$$

$\Rightarrow$ $\omega^2 = \omega_p^2$ : *plasma oscillation* = natural oscillation in plasmas (16)

caused by balancing between inertial and electrostatic restoring forces

Transverse *E* wave part ( $\boldsymbol{k}\perp\boldsymbol{E}_\perp$, i.e., $\boldsymbol{k}\cdot\boldsymbol{E} = 0$) :

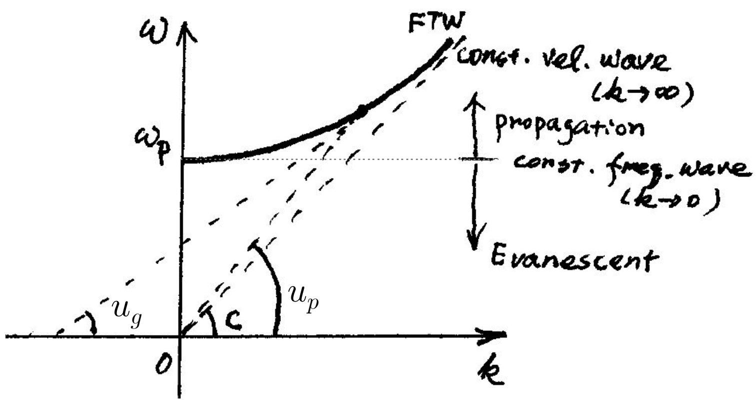
$$(15)^{***}_{\perp} : \left(k^2 - \frac{\omega^2 - \omega_p^2}{c^2}\right)E = 0$$

$$\Rightarrow \quad \omega^2 = \omega_p^2 + k^2c^2 = k^2c^2/(1 - \omega_p^2/\omega^2) \qquad (17)$$

$$\text{or} \quad \frac{\omega^2}{k^2} = \frac{c^2}{(1 - \omega_p^2/\omega^2)} \qquad \left(u_p^2 = \frac{1}{\mu\epsilon_p}\right) \qquad (17)^*$$

Similarly, we get the same dispersion equations for magnetic wave(*B*).

## 3) Discussions on TEM wave characteristics and applications




a) Zero density (free space) or very high freq. limit $[\omega_p = 0$ in (17)]

$\omega = kc$ : TEM wave propagating with $c$ in free space

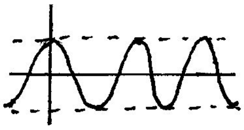
b) $$u_p = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \omega_p^2/\omega^2}} = \frac{1}{\sqrt{\mu\varepsilon_p}} = \frac{1}{\sqrt{\mu_o\varepsilon_o\varepsilon_r}} = \frac{c}{\sqrt{\varepsilon_r}} \qquad (18)$$

$$u_g = \frac{d\omega}{dk} = \frac{k}{\omega}c^2 = \frac{c^2}{u_p} < c \qquad (18)^*$$

c) Cutoff $(k \to 0)$

From $\omega = \pm \dfrac{kc}{\sqrt{1-\omega_p^2/\omega^2}} = \pm \dfrac{kc}{\sqrt{1-f_p^2/f^2}}$ (17)**

$f > f_p$: $k = real$

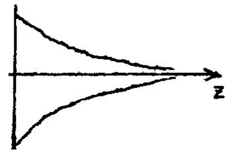


$\Rightarrow \; e^{j(\omega t - kz)}$: $FTW \; (\omega k > 0)$, $BTW \; (\omega k < 0)$ $\Rightarrow$ *propagating wave*

$f < f_p$: $k = \pm j|k| = pure \;\; imag.(= k_c = -j\gamma)$

$\Rightarrow \; e^{j(\omega t - kz)} = e^{j\omega t}e^{-|k|z}\,(= e^{j\omega t}e^{-\gamma z} = e^{j\omega t}e^{-\alpha z}e^{-j\beta z})$

$\Rightarrow$ *attenuating wave (evanescent wave)*




*Note)* $\gamma \equiv \alpha + j\beta = jk$ by (7-42)

(17)** $\searrow$ $= j\dfrac{\omega}{c}\sqrt{1-\omega_p^2/\omega^2} = j\omega\sqrt{\mu_o\epsilon_o}\sqrt{1-(f_p/f)^2}$ (7-132)

$f > f_p \Rightarrow \beta = k \Rightarrow \gamma = pure \;\; imaginary$

$f < f_p \Rightarrow \alpha = |k| \Rightarrow \gamma = real$

$f = f_p$, i.e., $\omega = \omega_p$ : Cutoff frequency

d) Spatial attenuation

$\omega < \omega_p$, $\quad e^{jkz} = e^{-|k|z} = e^{-\alpha z} \overset{(7-56)}{=} e^{-x/\delta}$

where $\delta \equiv \dfrac{1}{\alpha} = \dfrac{1}{|k|} = \dfrac{c}{\omega}\dfrac{1}{\sqrt{\omega_p^2/\omega^2 - 1}}$ : *skin depth*

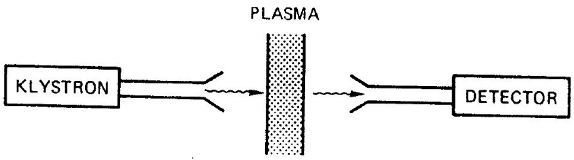
e) Applications

① Measurement of plasma density n

* TEM wave transmission method

$\omega^2 = \omega_p^2 + k^2c^2 = \dfrac{e^2N}{\varepsilon_o m} + k^2c^2$

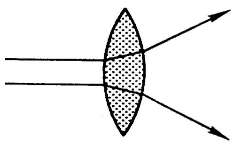
Cutoff density : $N_c = \dfrac{\varepsilon_o m\omega_p^2}{e^2}$




* TEM wave interferometer

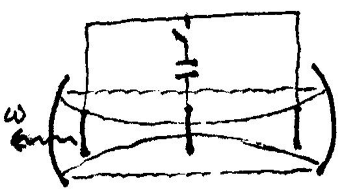
② Plasma lenses

Index of refraction : $n_r = \dfrac{c}{u_p} = \dfrac{ck}{\omega} = \sqrt{1-\omega_p^2/\omega^2} < 1$

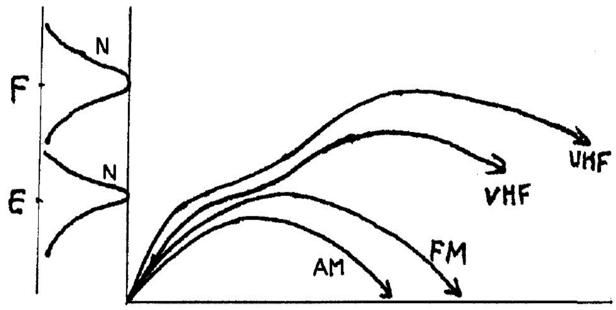


*Convex plasma lenses*

$\Rightarrow$ HCN gas laser $(f = 10^{12}\text{Hz},\;\; 337\mu)$



③ Radio communication by the ionosphere [plasma layer produced by UV from the sun in the earth's upper atmosphere from 50~500 km in altitude; $N = 10^{10} - 10^{12}\ m^{-3}$, $T_e = 300 - 10^3\ K$ ($10^{-1}\ eV$) ]




$$f_p = 9\sqrt{N} \quad (7\text{-}133)$$

$$\approx 0.9 \sim 9\ MHz$$

$N = 10^{10} \sim 10^{12}\ m^{-3}$

④ Communication blackout of a reentry spacecraft

(e.g. 7-11) $N = 2\times10^{14}\ m^{-3} \Rightarrow f_p = 9\sqrt{2\times10^{14}} = 127$ (MHz)

$\Rightarrow f < 127$ MHz : comm. blackout

## B. Summary of Other Plasma Waves

Plasma model (Equil. conditions)

① Homogeneous (quasi-neutral, infinite) plasma : $N_e = N_i \equiv N = const$

② Field-free (unmagnetized) plasma : $E_{ext} = 0$ and $B_{ext} = 0$

$_\parallel$②$_\parallel$ Magnetized plasma : $k \parallel B_{ext} \parallel E$

$_\parallel$②$_\perp$ Magnetized plasma : $k \parallel B_{ext} \perp E$

$_\perp$②$_\parallel$ Magnetized plasma : $k \perp B_{ext} \parallel E$

$_\perp$②$_\perp$ Magnetized plasma : $k \perp B_{ext} \perp E$

③ Cold plasma : $T_e = T_i = 0$ (no thermal motions)

③$^e$ Warm electrons : $T_e \neq 0$ (electron thermal motions)

③$_i$ Warm ions : $T_i \neq 0$ (ion thermal motions)

④ No drifts : $v_{oe} = v_{oi} = 0$

⑤ Fixed ions

### 1) Electrostatic electron waves ( $E \neq 0$, $B = 0$, $k \parallel E$, ①④⑤)

: High-frequency branch of longitudinal electrostatic waves

| Waves | Model | Dispersion relation | Characteristics |
|---|---|---|---|
| a) Plasma osc. (Langmuir) | ② ③ or $_\parallel$②$_\parallel$ | $\omega^2 = \omega_p^2 \equiv \dfrac{Ne^2}{m_e \varepsilon_o}$ | *High-freq. osc. (no prop.)* |
| b) Electron wave (Bohm-Gross) | ② ③$^e$ or $_\parallel$②$_\parallel$ | $\omega^2 = \omega_p^2 + k^2 v_{se}^2 = \omega_p^2 + k^2\left(\dfrac{\gamma K T_e}{m_e}\right)$ | *High-freq. longit. wave* |
| c) Upper hybrid osc. | $_\perp$② ③ | $\omega^2 = \omega_p^2 + \omega_{ec}^2 = \omega_p^2 + \left(\dfrac{eB_{ext}}{m_e}\right)^2 \equiv \omega_h^2$ | *High freq. hybrid osc.* |

### 2) Electrostatic ion waves ( $E \neq 0$, $B = 0$, $k \parallel E$, ①③$_i^e$④ )

: Low-frequency branch of longitudinal electrostatic waves

| Waves | Model | Dispersion relation | Characteristics |
|---|---|---|---|
| a) Ion wave (Ion acoustic) | ②( $_\parallel$②$_\parallel$ ) | $\omega^2 = k^2 v_s^2 \equiv k^2 \sqrt{\dfrac{KT_e}{m_i}}$ | *Low freq longit sonic wave with const.* $v_s \approx v_{si}$ |
| b) Electrostatic ion cyclotron wave | $_\perp$② | $\omega^2 = \omega_{ic}^2 + k^2 v_s^2$ | *Propagating low freq. longit. wave* |
| or lower hybrid osc. | $_\perp$② | $\omega^2 = \omega_{ic}\,\omega_{ec} \equiv \omega_l^2$ | *Low-freq hybrid osc.* |

### 3) EM electron waves ( $E \neq 0$, $B \neq 0$, $k \perp B$, ①③④⑤)

: High-frequency branch of TEM waves

a) EM wave ②

$$\omega^2 = \omega_p^2 + k^2c^2$$
$$\left[k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)\right]$$

*High freq. TEM wave with cutoff* $\omega_p$

b) O wave $_{\perp}$②$_{\parallel}$

$$\omega^2 = \omega_p^2 + k^2c^2$$
$$\left[k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)\right]$$

*High freq. TEM wave with cutoff* $\omega_p$

c) X wave $_{\perp}$②$_{\perp}$

$$k^2 = \frac{1}{c^2}\frac{(\omega^2 - \omega_R^2)(\omega^2 - \omega_L^2)}{\omega^2 - \omega_h^2}$$
$$\omega_{R,L} \equiv \frac{1}{2}\left(\pm\omega_{ec} + \sqrt{\omega_{ec}^2 + 4\omega_p^2}\right) > 0$$

*Not a pure transverse wave*
*Cutoff :* $\omega_R$, $\omega_L$
*Resonance :* $\omega_h$ $(\omega_R)$

d) R, L waves $_{\parallel}$②$_{\perp}$

$$k_{R,L}^2 = \frac{\omega^2}{c^2}\frac{(\omega - \omega_{R,L}^*)(\omega - \omega_{R,L})}{\omega(\omega \mp \omega_{ec})}$$
$$\omega_{R,L}^* \equiv \frac{1}{2}\left(\pm\omega_{ec} - \sqrt{\omega_{ec}^2 + 4\omega_p^2}\right) < 0$$

*Circul. pol. transv. waves*
*Cutoff :* $\omega_R$, $\omega_L$
*Resonance :* $\omega_c (\Omega_c)$
*Faraday rot.* $(\omega \gg \omega_p)$
*Whistler mode* $(\omega < \omega_c/2)$

### 4) EM ion (MHD) waves ( $E \neq 0$, $B \neq 0$, $k \perp B$, ①④⑤)

: Low-frequency branch of TEM waves

a) Alfven wave $_{\parallel}$②$_{\perp}$③

$$\omega^2 = k^2 v_A^2 \equiv k^2 \frac{B_{ext}^2}{\mu_o \rho}$$

*Low-freq. incompressible trans. waves with const.* $v_A$

b) Magnetosonic wave $_{\perp}$②$_{\perp}$③$_i^e$

$$\omega^2 = k^2(v_s^2 + v_A^2)$$

*Low-freq. compressional sonic and transv. e.m. wave*

## Homework Set 3

1) P.7-15 2) P.7-17 3) P.7-19 4) P.7-27 5) P.7-30 2) P.7-33