# CHAPTER 9. Waveguides and Cavity Resonators

***Reading assignments:*** Cheng Ch.9, Hayt Ch.14

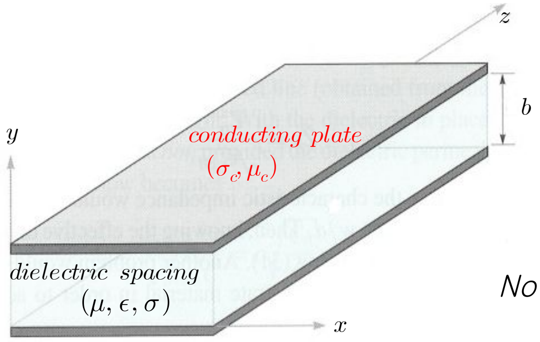
## 1. General Wave Behaviors along Guiding Structures

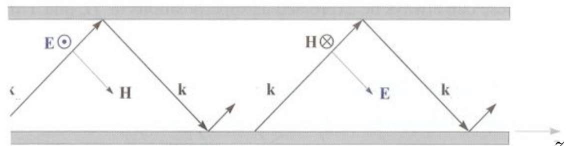
### *A. Types of Waveguides*

Waveguides = EM wave transmission devices in higher modes (TE and TM) which typically have the forms of hollow metal pipes, but come many different forms that depend on the purpose of the guide and on the wave frequencies [$f \gtrsim f_{UHF}$ ($\sim 1GHz$), $\exists$ cutoff ($f < f_c$, $\lambda > \lambda_c$ : no propagation)]

1) Parallel-plate waveguide

for high-frequency waves of $f \gtrsim f_{MW}$ (UHF, SHF, EHF) and $\lambda < b,\ l$



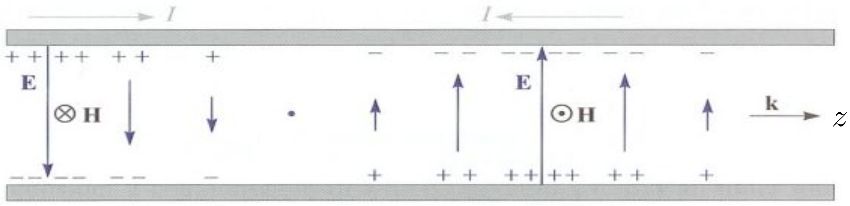

*TE* mode *TM* mode

*Note)* $\alpha \propto R \propto R_s = \sqrt{\pi f \mu_c / \sigma_c} \propto \sqrt{f}$

$\Rightarrow$ Considerable power loss along the waveguide at high frequencies

*Note)* Parallel-plate transmission line

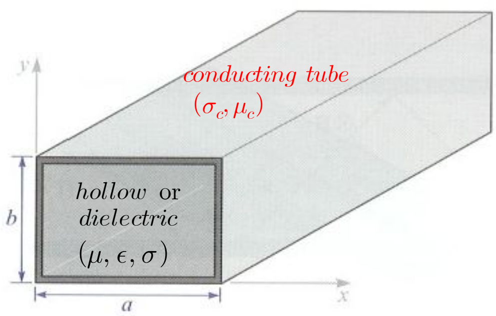
for low-frequency waves of $f < f_{MW}$ and $\lambda \gg b,\ l$

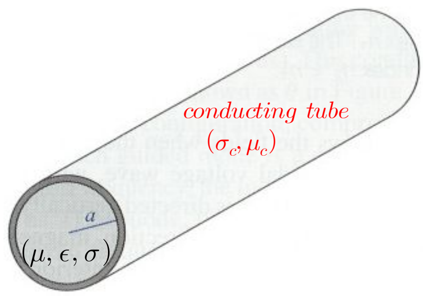



*TEM* mode

2) Single-conductor hollow (or dielectric-filled) waveguide

for high-frequency waves of $f \gtrsim f_{MW}$ (UHF, SHF, EHF) and $\lambda < b,\ a,\ l$

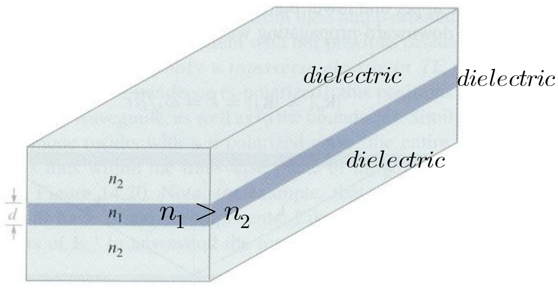


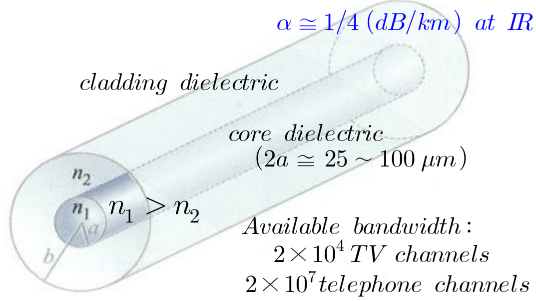

Rectangular waveguide Cylindrical waveguide

### 3) Dielectric waveguides

primarily for optical frequencies (IR, visible light)(TEM mode, no cutoff)




Dielectric slab waveguide



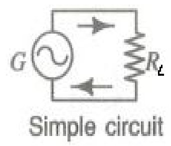

Optical fiber waveguide

*(cf)* $\alpha = 1\,(Np/m) = 8.69\,(dB/m)$

$\alpha \cong \sim 30$ (dB/km) for holl. metal waveguides

$\alpha \cong$ few hundreds (dB/km) for coaxial cables

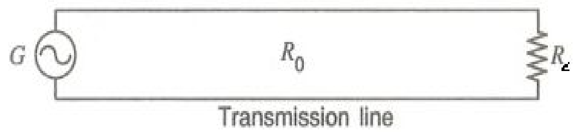
*(cf)* Comparison of transmission systems:




for $f \ll f_{MW}$ and $\lambda \gg L_s$

$\Rightarrow$ Lumped circuit theory

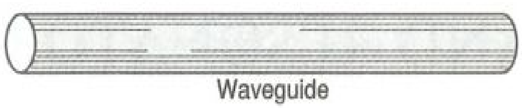
( $V,\ I;\ R,\ L,\ C$ )




for $f < f_{MW}$ and $\lambda > L_s$

$\Rightarrow$ Distributed circuit theory

( $V,\ I;\ R/l,\ L/l,\ G/l,\ C/l$ )




for $f \gtrsim f_{MW}$ and $\lambda < L_s$

$\Rightarrow$ EM field theory

$(\boldsymbol{E},\ \boldsymbol{H};\ \sigma_c,\ \mu,\ \epsilon,\ \sigma)$

## *B. EM Field Equations for Uniform Straight Guiding Structures*

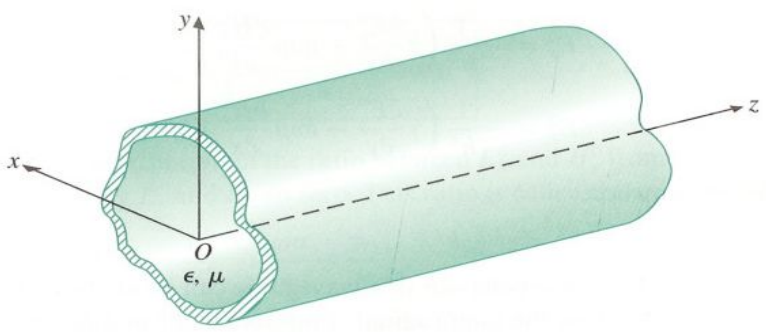



*FIGURE 9−1 A uniform waveguide with an arbitrary cross section*

Time-harmonic wave fields propagating along +z direction:

$$\boldsymbol{E}(x,y,z;t) = Re\left[\boldsymbol{E}^o(x,y)\,e^{(j\omega t - \gamma z)}\right] = Re\left[\boldsymbol{E}^o(x,y)\,e^{-\alpha z}\,e^{j(\omega t - \beta z)}\right] \qquad \text{(9-1, 2)}$$

$$\boldsymbol{H}(x,y,z;t) = Re\left[\boldsymbol{H}^o(x,y)\,e^{(j\omega t - \gamma z)}\right] = Re\left[\boldsymbol{H}^o(x,y)\,e^{-\alpha z}\,e^{j(\omega t - \beta z)}\right] \qquad \text{(9-1, 2)*}$$

EM wave equations in the source-free ($\rho_v = 0,\ \boldsymbol{J} = 0$) dielectric region

$\Rightarrow$ Homogeneous vector Helmholtz's equations (6-98, 99) :

$$\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = \left(\nabla_{xy}^2 + \frac{\partial^2}{\partial z^2} + k^2\right)\boldsymbol{E} = (\nabla_{xy}^2 + \gamma^2 + k^2)\boldsymbol{E} = \mathbf{0} \qquad \text{(9-3, 6)}$$

$$\Rightarrow \quad \nabla_{xy}^2 \boldsymbol{E} + (\gamma^2 + k^2)\boldsymbol{E} = \mathbf{0} \qquad \text{(9-7)}$$

Similarly, $\nabla_{xy}^2 \boldsymbol{H} + (\gamma^2 + k^2)\boldsymbol{H} = \mathbf{0}$ (9-8)

where $k = \omega\sqrt{\mu\epsilon} = \omega/u = 2\pi/\lambda$ (9-5)

Interrelationship among $E_x$, $E_y$, $E_z$, $H_x$, $H_y$, and $H_z$ are given by two source-free Maxwell's equations (6-80a, 80b) in Cartesian coordinates by replacing $\partial/\partial z$ with $-\gamma$ and by omitting the common $e^{-\gamma z}$ factor:

| From $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$: ($Faraday's\ law$) | From $\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$: ($Ampere's\ law$) |
|---|---|
| $\dfrac{\partial E_z^0}{\partial y} + \gamma E_y^0 = -j\omega\mu H_x^0$ (9-9a) | $\dfrac{\partial H_z^0}{\partial y} + \gamma H_y^0 = j\omega\epsilon E_x^0$ (9-10a) |
| $-\gamma E_x^0 - \dfrac{\partial E_z^0}{\partial x} = -j\omega\mu H_y^0$ (9-9b) | $-\gamma H_x^0 - \dfrac{\partial H_z^0}{\partial x} = j\omega\epsilon E_y^0$ (9-10b) |
| $\dfrac{\partial E_y^0}{\partial x} - \dfrac{\partial E_x^0}{\partial y} = -j\omega\mu H_z^0$ (9-9c) | $\dfrac{\partial H_y^0}{\partial x} - \dfrac{\partial H_x^0}{\partial y} = j\omega\epsilon E_z^0$ (9-10c) |

Rearrangements of the above equations yield the expressions of the transverse field components ($H_x^o, H_y^o, E_x^o, E_y^o$) in terms of the longitudinal field components ($E_z^o, H_z^o$) as follows:

$$H_x^0 = -\frac{1}{h^2}\left(\gamma\frac{\partial H_z^0}{\partial x} - j\omega\epsilon\frac{\partial E_z^0}{\partial y}\right) \qquad \text{(9-11)}$$

$$H_y^0 = -\frac{1}{h^2}\left(\gamma\frac{\partial H_z^0}{\partial y} + j\omega\epsilon\frac{\partial E_z^0}{\partial x}\right) \qquad \text{(9-12)}$$

$$E_x^0 = -\frac{1}{h^2}\left(\gamma\frac{\partial E_z^0}{\partial x} + j\omega\mu\frac{\partial H_z^0}{\partial y}\right) \qquad \text{(9-13)}$$

$$E_y^0 = -\frac{1}{h^2}\left(\gamma\frac{\partial E_z^0}{\partial y} - j\omega\mu\frac{\partial H_z^0}{\partial x}\right) \qquad \text{(9-14)}$$

where $h^2 = \gamma^2 + k^2$ (9-15)

*Notes)* i) Solutions of (9-7, 8) for $E_z^o, H_z^o$ subject to BCs determine other transverse fields ($H_x^o, H_y^o, E_x^o, E_y^o$ ) by using (9-11) ~ (9-14).

ii) Classification of waves

① Transverse electromagnetic (TEM): $E_z = 0$ and $H_z = 0$

② Transverse magnetic (TM): $E_z \neq 0\ but\ H_z = 0$

③ Transverse electric (TE): $H_z \neq 0\ but\ E_z = 0$

## C. General Properties of TEM Waves ($E_z = 0$ and $H_z = 0$)

For both $E_z = 0$ and $H_z = 0$, the condition for the existence of nontrivial solutions of (9-11) ~ (9-14) : $h^2 = \gamma_{TEM}^2 + k^2 = 0$ (9-16)

$$\Rightarrow \quad \gamma_{TEM} = jk = j\omega\sqrt{\mu\epsilon} = j\beta \qquad (9\text{-}17)$$

: same for a plane wave in unbounded medium
and a TEM wave on a lossless transmission line

Phase velocity:

$$u_{p(TEM)} = \frac{\omega}{\beta} = \frac{\omega}{k} \overset{(9\text{-}17)}{=} \frac{1}{\sqrt{\mu\epsilon}} \quad \text{(m/s)} : \text{ ind. of } f \qquad (9\text{-}18)$$

Wave impedance from (9-9b) and (9-10a):

$$Z_{TEM} \equiv \frac{E_x^o}{H_y^o}\left(= -\frac{E_y^o}{H_x^o}\right) = \frac{j\omega\mu}{\gamma_{TEM}} = \frac{\gamma_{TEM}}{j\omega\epsilon} \overset{(9\text{-}17)}{=} \sqrt{\frac{\mu}{\epsilon}} = \eta : \text{ ind. of } f \qquad (9\text{-}19,\ 20)$$

intrinsic impedance (7-14)

In single-conductor hollow (or dielectric-filled) wave guides of any shape,

$$E_z = 0 \quad \Rightarrow \quad \frac{\partial E_z}{\partial t} = 0 \quad \Rightarrow \quad J_{Dz} = 0 \text{ (no displacement current)}$$

and $\boldsymbol{J} = \boldsymbol{0}$ (no conduction current)

$\Rightarrow$ no closed $\boldsymbol{B}$ lines in any transverse plane

$\Rightarrow$ TEM waves cannot exist.

## D. General Properties of TM Waves ($E_z \neq 0$ but $H_z = 0$)

For $E_z \neq 0$ but $H_z = 0$, from (9-7)

$$\nabla_{xy}^2 \boldsymbol{E}_z^o + h^2 \boldsymbol{E}_z^o = \boldsymbol{0} \qquad (9\text{-}22)$$

subject to BCs.

$\Rightarrow$ BVP to find the characteristic properties (eigenmodes) of TM modes for the eigenvalues *h* (= discrete values of *h*)

Wave impedance from (9-10a) and (9-10b) :

$$Z_{TM} \equiv \frac{E_x^o}{H_y^o} = -\frac{E_y^o}{H_x^o} = \frac{\gamma}{j\omega\epsilon} \quad (\Omega) \qquad (9\text{-}23)$$

From (9-15), $\gamma = \sqrt{h^2 - k^2} = \sqrt{h^2 - \omega^2\mu\epsilon}$ (9-24)

Cutoff frequency defined at which $\gamma = 0$ for the eigenvalue $h$ of a particular mode: (9-24) for $\gamma = 0$ velocity of light in the medium

$$f_c = \frac{\omega_c}{2\pi} = \frac{h}{2\pi\sqrt{\mu\epsilon}} = \frac{hu}{2\pi} \qquad (9\text{-}26)$$

*Notes)* $k^2 = \omega^2\mu\epsilon$, $h^2 = \omega_c^2\mu\epsilon$

Then, from (9-24) $\gamma = h\sqrt{1 - \left(\frac{f}{f_c}\right)^2}$ (9-27)

1) Propagating modes in the waveguide (if $f > f_c$ )

For $f > f_c$ , from (9-24, 26)

$$\gamma = jk\sqrt{1-(h/k)^2} = jk\sqrt{1-(f_c/f)^2} = j\beta \qquad (\alpha = 0) \qquad \text{(9-28)}$$

$$\Rightarrow \qquad \beta = k\sqrt{1-(f_c/f)^2} \ \text{(rad/m)} : \text{dispersion relation} \qquad \text{(9-29)}$$

$\Rightarrow \quad e^{-\gamma z} = e^{-j\beta z}$ : propagating wave along $+z$ direction

Wavelength in the waveguide: wavelength in unbounded medium

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1-(f_c/f)^2}} > \lambda = \frac{2\pi}{k} = \frac{1}{f\sqrt{\mu\epsilon}} = \frac{u}{f} \qquad \text{(9-30, 31)}$$

$$\Rightarrow \qquad \frac{1}{\lambda^2} = \frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2} \qquad \text{(9-32)}$$

Phase velocity propagating in the guide:

$$u_p = \frac{\omega}{\beta} = \frac{u}{\sqrt{1-(f_c/f)^2}} \quad : \text{dep. on } f \ \text{(dispersive)} \qquad \text{(9-33)}$$

$$= \frac{\lambda_g}{\lambda} u > u \qquad : \text{faster than that in unbounded medium}$$

(9-28) in (9-23):

$$Z_{TM} = \eta\sqrt{1-(f_c/f)^2} \ (\Omega) \quad < \eta \qquad \text{(9-34)}$$

: purely resistive and less than intrinsic one

2) Evanescent modes in the waveguide (if $f < f_c$ )

For $f < f_c$ , from (9-27)

$$\gamma = h\sqrt{1-(f/f_c)^2} = \alpha \qquad (j\beta = 0) \qquad \text{(9-35)}$$

$\Rightarrow e^{-\gamma z} = e^{-\alpha z}$ : attenuating (evanescent) wave along $+z$ direction

$\Rightarrow$ Waveguides act as high-pass filters

(Only waves with $f > f_c$ can propagate in the guide)

(9-35) in (9-23):

$$Z_{TM} = -j\frac{h\sqrt{1-(f/f_c)^2}}{\omega\epsilon} \quad : \text{purely reactive}$$

(no power flow of evanescent waves)

## E. General Properties of TE Waves ($H_z \neq 0$ *but* $E_z = 0$ )

For $H_z \neq 0$ *but* $E_z = 0$, from (9-8)

$$\nabla_{xy}^2 \boldsymbol{H_z^o} + h^2 \boldsymbol{H_z^o} = \mathbf{0} \qquad (9\text{-}36)$$

subject to BCs. $\Rightarrow$ BVP for TE modes

Wave impedance from (9-9a) and (9-9b) :

$$Z_{TE} \equiv \frac{E_x^o}{H_y^o} = -\frac{E_y^o}{H_x^o} = \frac{j\omega\mu}{\gamma} \quad (\Omega) \qquad (9\text{-}37)$$

$\gamma, f_c, \beta, \lambda_g, \lambda, u_p$ expressed in (9-24) ~ (9-33) also apply to TE waves.

### 1) Propagating modes in the waveguide (if $f > f_c$ )

For $f > f_c$ , from (9-24, 26)

$$\gamma = jk\sqrt{1-(f_c/f)^2} = j\beta \qquad (\alpha = 0) \qquad (9\text{-}38)$$

$\Rightarrow \quad \beta = k\sqrt{1-(f_c/f)^2}$ (rad/m) : dispersion relation (9-29)

$\Rightarrow \quad e^{-\gamma z} = e^{-j\beta z}$ : propagating wave along $+z$ direction

Wavelength in the waveguide:

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1-(f_c/f)^2}} > \lambda \qquad (9\text{-}30)$$

$$\Rightarrow \quad \frac{1}{\lambda^2} = \frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2} \qquad (9\text{-}32)$$

Phase velocity propagating in the guide:

$$u_p = \frac{\omega}{\beta} = \frac{u}{\sqrt{1-(f_c/f)^2}} \quad : \text{dep. on } f \text{ (dispersive)} \qquad (9\text{-}33)$$

$= (\lambda_g/\lambda)u > u$ : faster than that in unbounded medium

(9-38) in (9-37):

$$Z_{TE} = \frac{\eta}{\sqrt{1-(f_c/f)^2}} \quad (\Omega) > \eta \qquad (9\text{-}39)$$

: purely resistive and larger than intrinsic one

### 2) Evanescent modes in the waveguide (if $f < f_c$ )

For $f < f_c$ , from (9-27)

$$\gamma = h\sqrt{1-(f/f_c)^2} = \alpha \qquad (j\beta = 0) \qquad (9\text{-}40)$$

$\Rightarrow e^{-\gamma z} = e^{-\alpha z}$ : attenuating (evanescent) wave along $+z$ direction

$\Rightarrow$ Waveguides act as high-pass filters

(Only waves with $f > f_c$ can propagate in the guide)

(9-40) in (9-37):

$$Z_{TE} = j\frac{\omega\mu}{h\sqrt{1-(f/f_c)^2}} \quad : \text{purely reactive} \qquad (9\text{-}41)$$

(no power flow of evanescent waves)

## F. *Dispersion Diagram* ($\omega-\beta$ graph) : *(e.g.9-1)*

Dispersion relation (9-29) for TM and TE propagating modes ($f > f_c$):

$$\beta = k\sqrt{1-(f_c/f)^2} \quad \Rightarrow \quad \frac{\beta^2}{k^2} = 1 - \left(\frac{\omega_c^2}{\omega^2}\right) \quad \Rightarrow \quad \frac{\omega^2}{k^2} = \frac{\omega^2 - \omega_c^2}{\beta^2}$$

$$\Rightarrow \quad \omega^2 = \omega_c^2 + u^2\beta^2 = \frac{u^2\beta^2}{1-\omega_c^2/\omega^2} \qquad (9\text{-}43)$$

$u = \omega/k = 1/\sqrt{\mu\epsilon}$

*(cf)* Ch.7, p.31, (17) $\omega^2 = \omega_c^2 + c^2k^2$ for plasma waves

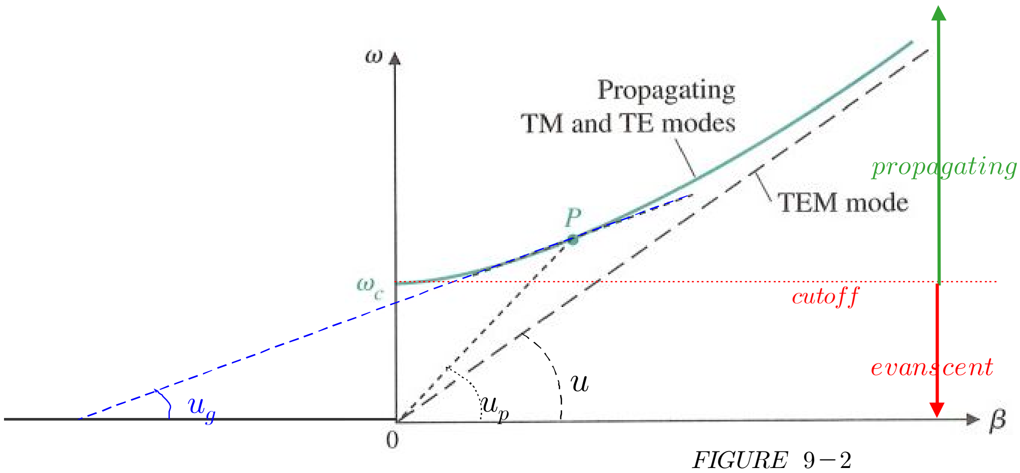



*FIGURE* 9−2

Phase velocity:

$$u_p = \frac{\omega}{\beta} = \frac{u}{\sqrt{1-(f_c/f)^2}} = \frac{u}{\sqrt{1-(\omega_c/\omega)^2}} > u \qquad (9\text{-}33)$$

*(cf)* Ch.7, p.31, (18) $u_p = \frac{\omega}{k} = \frac{c}{\sqrt{1-(\omega_p/\omega)^2}} = \frac{c}{\sqrt{\epsilon_r}}$ for plasma waves

Group velocity:

$$u_g = \frac{d\omega}{d\beta} \underset{(9-43)}{=} \frac{\beta}{\omega}u^2 \underset{(9-33)}{=} \frac{u^2}{u_p} < u \qquad (9\text{-}33)*$$

$$\Rightarrow \quad u_g u_p = u^2 \qquad (9\text{-}43)$$

When $\omega \to \infty$, $u_p = u_g = u$

Cutoff: $\omega = \omega_c$ when $\beta = 0$ ($i.e.$, $\gamma = 0$)

## G. *Infinite Parallel-Plate Waveguide* : *(e.g.9-2)*

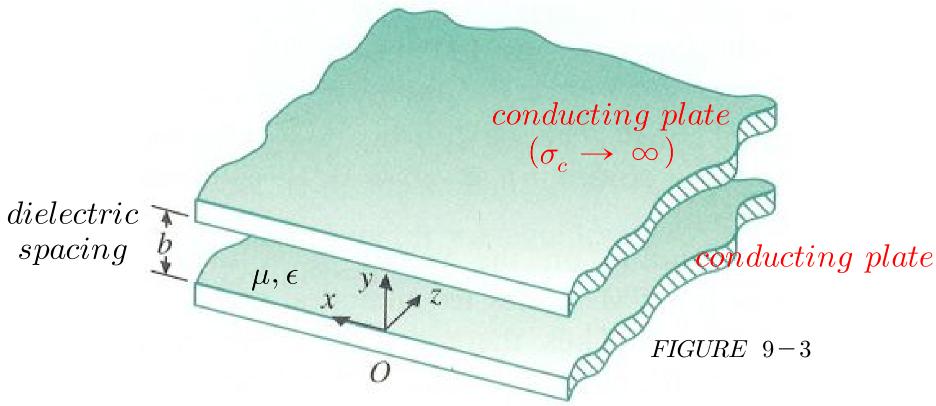


*FIGURE* 9−3

For TM modes ($H_z = 0$), $E_z(y) = E_z^o(y)e^{-\gamma z}$ since $-\infty < x < \infty$.

Wave equation from (9-22):

$$\frac{d^2 E_z^o(y)}{dy^2} + h^2 E_z^o(y) = 0 \qquad (9\text{-}44)$$

BCs ($\hat{n} \times \boldsymbol{E} = 0$):

$$E_z^o(y)|_{y=0} = 0 \qquad (9\text{-}44)_0$$

$$E_z^o(y)|_{y=b} = 0 \qquad (9\text{-}44)_b$$

General solution:

$$E_z^o(y) = A_n \sin(hy) + B_n \cos(hy) \qquad (9\text{-}45)$$

$(9\text{-}44)_0$ in (9-45) $\Rightarrow$ $B_n = 0$

$(9\text{-}44)_b$ in (9-45) $\Rightarrow$ $\sin(hb) = 0$

$\Rightarrow$ $h = n\pi/b$, $(n = 1, 2, 3, .....)$ : eigenvalues (9-46)

Hence, (9-45) becomes

$$E_z^o(y) = A_n \sin\left(\frac{n\pi y}{b}\right) \quad \text{: eigenmodes} \qquad (9\text{-}47)$$

where $A_n$ is to be usually determined by an initial condition at $t = 0$ for the excitation strength of the particular TM wave.

$H_z = 0$ and $\partial E_z^o / \partial x = 0$ in (9-11) ~ (9-14):

$$H_x^o(y) = \frac{j\omega\epsilon}{h} A_n \cos\left(\frac{n\pi y}{b}\right), \qquad H_y^o(y) = 0 \qquad (9\text{-}48)$$

$$E_x^o(y) = 0, \qquad E_y^o(y) = -\frac{\gamma}{h} A_n \cos\left(\frac{n\pi y}{b}\right) \qquad (9\text{-}49)$$

where $\gamma = \sqrt{(n\pi/b)^2 - \omega^2\mu\epsilon}$ (9-50)

$\gamma = 0$ in (9-50) $\Rightarrow$ Cutoff frequency:

$$f_c = \frac{n}{2b\sqrt{\mu\epsilon}} = \frac{hu}{2\pi} \quad \text{(Hz)} \quad (n = 1, 2, 3, .....) \qquad (9\text{-}51)$$

For TM$_1$ mode ($n = 1$), $(f_c)_1 = 1/2b\sqrt{\mu\epsilon} = u/2b$, $(\lambda_c)_1 = 2b$ (dominant mode)

For TM$_2$ mode ($n = 2$), $(f_c)_1 = 1/b\sqrt{\mu\epsilon} = u/b$, $(\lambda_c)_2 = b$ .....

For $n = 0$, $E_z = 0$ in (9-47) and only $H_x$ and $E_y$ exist in (9-48, 49).

$\Rightarrow$ TEM mode with $f_c = 0$ (no cutoff)