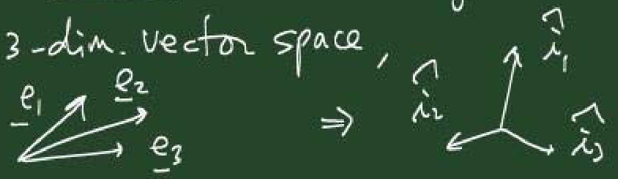
노트 제목 2010-03-04

- Gram-Schmidt Orthonormalization

Given any basis ($\underline{e}_i$) ($i = 1, 2, \cdots, n$), we can generate an orthonormal basis ($\hat{i}_i$) of same dimension.

For 3-dim. vector space,



$$\hat{i}_1 = \frac{\underline{e}_1}{|\underline{e}_1|}$$

$$\hat{i}_2 = \frac{\underline{e}_2 - \alpha \hat{i}_1}{|\underline{e}_2 - \alpha \hat{i}_1|}$$ choose α such that $\hat{i}_1 \cdot \hat{i}_2 = 0$

$\rightarrow \alpha = \underline{e}_2 \cdot \hat{i}_1$

$$\therefore \hat{i}_2 = \frac{\underline{e}_2 - (\underline{e}_2 \cdot \hat{i}_1)\hat{i}_1}{|\quad \text{"} \quad|}$$

$$\hat{i}_3 = \frac{\underline{e}_3 - \gamma \hat{i}_1 - \beta \hat{i}_2}{|\quad \text{"} \quad|}$$ choose γ and β such that $\hat{i}_3 \cdot \hat{i}_1 = 0$, $\hat{i}_3 \cdot \hat{i}_2 = 0$

$\rightarrow \gamma = \underline{e}_3 \cdot \hat{i}_1$, $\beta = \underline{e}_3 \cdot \hat{i}_2$

$$\therefore \hat{i}_3 = \frac{\underline{e}_3 - (\underline{e}_3 \cdot \hat{i}_1)\hat{i}_1 - (\underline{e}_3 \cdot \hat{i}_2)\hat{i}_2}{|\quad \text{"} \quad|}$$

- Vector product : $\underline{c} = \underline{a} \times \underline{b}$

$$\underline{a} \times \underline{b} = (a_x \hat{i} + a_y \hat{j} + a_z \hat{k}) \times (b_x \hat{i} + b_y \hat{j} + b_z \hat{k})$$
$$= (a_y b_z - a_z b_y)\hat{i} + (a_z b_x - a_x b_z)\hat{j} + (a_x b_y - a_y b_x)\hat{k}$$

$$= \epsilon_{ijk}\, a_i b_j\, \hat{i}_k$$

$$\underline{a} \times \underline{b} = a_i \hat{i}_i \times b_j \hat{i}_j = a_i b_j\, \hat{i}_i \times \hat{i}_j$$

$$\rightarrow \boxed{\begin{aligned} \hat{i}_i \times \hat{i}_j &= \epsilon_{ijk}\, \hat{i}_k \\ \hat{i}_i \cdot \hat{i}_j &= \delta_{ij} \end{aligned}}$$

- Triple scalar product

$$\underline{a} \times \underline{b} \cdot \underline{c} = (a_i \hat{i}_i) \times (b_j \hat{i}_j) \cdot (c_k \hat{i}_k)$$

$$= a_i b_j c_k\, \hat{i}_i \times \hat{i}_j \cdot \hat{i}_k$$

$$= a_i b_j c_k\, \epsilon_{ijl}\, \hat{i}_l \cdot \hat{i}_k = a_i b_j c_k\, \epsilon_{ijl}\, \delta_{lk}$$

$$= a_i b_j c_k\, \epsilon_{ijk} = \epsilon_{ijk}\, a_i b_j c_k$$

- Triple vector product : $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\, \underline{b} - (\underline{a} \cdot \underline{b})\, \underline{c}$

$$\boxed{\epsilon_{ijk}\, \epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}}$$

$$\underline{a} \times (\underline{b} \times \underline{c}) = a_i \hat{i}_i \times (b_j \hat{i}_j \times c_k \hat{i}_k)$$

$$= a_i b_j c_k\, \hat{i}_i \times (\hat{i}_j \times \hat{i}_k)$$

$$= \quad '' \quad \hat{i}_i \times \epsilon_{jkl}\, \hat{i}_l$$

$$= \quad '' \quad \epsilon_{jkl}\, \epsilon_{ilm}\, \hat{i}_m$$

$$= \epsilon_{ljk}\, \epsilon_{lmi}\, a_i b_j c_k\, \hat{i}_m$$

$$= (\delta_{jm}\delta_{ki} - \delta_{ji}\delta_{km})\, a_i b_j c_k\, \hat{i}_m$$

$$= a_k b_m c_k\, \hat{i}_m - a_i b_i c_k\, \hat{i}_k$$

$$= (\underline{a} \cdot \underline{c})\, \underline{b} - (\underline{a} \cdot \underline{b})\, \underline{c}$$

- Linear vector function $\underline{t}$

$$\underline{u} \longrightarrow \boxed{\underline{t}} \longrightarrow \underline{t}(\underline{u}) = \underset{\sim}{T}\underline{u}$$

vector input — vector output

dyadic product : $\underline{a} \otimes \underline{b} = \underline{a}\,\underline{b}$

tensor $\quad (\underline{a} \otimes \underline{b})\underline{u} \equiv \underline{a}\,(\underline{b}\cdot\underline{u}) = (\underline{b}\cdot\underline{u})\,\underline{a}$

$\underset{\sim}{T} \equiv \underline{\underline{T_{ij}}}\,\hat{i}_i \otimes \hat{i}_j$: linear transf.

2nd order tensor

↳ components of $\underset{\sim}{T}$ (9 comps.)

$$\underset{\sim}{T}\,\hat{i}_m = (T_{ij}\,\hat{i}_i \otimes \hat{i}_j)\,\hat{i}_m = T_{ij}\,\hat{i}_i\,(\hat{i}_j \cdot \hat{i}_m) = T_{im}\,\hat{i}_i$$

$$\underline{(\underset{\sim}{T}\,\hat{i}_m)\cdot\hat{i}_n = T_{im}\,\hat{i}_i\cdot\hat{i}_n = T_{nm}.}$$

$$t_i = \underline{t}(\underline{u})\cdot\hat{i}_i = (\underset{\sim}{T}\underline{u})\cdot\hat{i}_i = (T_{pj}\,\overline{\hat{i}_p \otimes \hat{i}_j\,\underline{u}})\cdot\hat{i}_i$$

$$\hat{i}_p\,(\hat{i}_j\cdot\underline{u}) = \hat{i}_p\,u_j$$

$$= T_{pj}\,u_j\,\hat{i}_p\cdot\hat{i}_i = T_{ij}\,u_j$$

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

- A quantity $\underset{\sim}{T}$ is a second-order tensor if it is the most general linear transformation of vectors to vector, i.e.

(i) $\underset{\sim}{T}(\underline{u}+\underline{v}) = \underset{\sim}{T}(\underline{u}) + \underset{\sim}{T}(\underline{v})$ $\qquad \underset{\sim}{T} = T_{ij}\,\hat{i}_i \otimes \hat{i}_j$

(ii) $\underset{\sim}{T}(\alpha\underline{u}) = \alpha\,\underset{\sim}{T}(\underline{u})$

$(\underset{\sim}{T}+\underset{\sim}{S})\underline{u} = \underset{\sim}{T}\underline{u} + \underset{\sim}{S}\underline{u}$

$(\alpha\underset{\sim}{T})\underline{u} = \alpha(\underset{\sim}{T}\underline{u})$

zero tensor: $\underset{\sim}{0}\,\underline{u} = \underline{0}$

identity tensor: $\underset{\sim}{1}\,\underline{u} = \underline{u}$ $\quad \underset{\sim}{1} = \hat{i}_i \otimes \hat{i}_i \quad \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

$(\underset{\sim}{1}$ or $\underset{\sim}{I})$ $\quad (\hat{i}_i \otimes \hat{i}_i)\,\underline{u} = \hat{i}_i(\hat{i}_i \cdot \underline{u}) = \hat{i}_i u_i = \underline{u}$

$(\underset{\sim}{T}\underset{\sim}{S})\underline{u} = \underset{\sim}{T}(\underset{\sim}{S}\underline{u})$

$(\underset{\sim}{T}\underset{\sim}{S})\underset{\sim}{R} = \underset{\sim}{T}(\underset{\sim}{S}\underset{\sim}{R})$

$\underset{\sim}{T}\underset{\sim}{S} \neq \underset{\sim}{S}\underset{\sim}{T}$ unless $\underset{\sim}{T}$ and $\underset{\sim}{S}$ commute.

- Components of product $\underset{\sim}{T}\underset{\sim}{S}$

$$(\underset{\sim}{T}\underset{\sim}{S})_{ij} = [(\underset{\sim}{T}\underset{\sim}{S})\hat{i}_j]\cdot\hat{i}_i$$

$$= [\underset{\sim}{T} S_{mp} \underbrace{\hat{i}_m \otimes \hat{i}_p\, \hat{i}_j}_{\hat{i}_m \delta_{pj}}]\cdot\hat{i}_i$$

$$= (\underset{\sim}{T} S_{mj}\, \hat{i}_m)\cdot\hat{i}_i = (T_{pq}\, \hat{i}_p \otimes \underbrace{\hat{i}_q\; S_{mj}\, \hat{i}_m}_{\hat{i}_p \delta_{qm}})\cdot\hat{i}_i$$

$$= T_{pm} S_{mj}\, \hat{i}_p \cdot \hat{i}_i = \underline{T_{im} S_{mj}}$$

- non-singular tensor

$\underset{\sim}{T}$ is non-singular if $\underset{\sim}{T}\underline{u} = 0 \Leftrightarrow \underline{u} = 0$

- Inverse tensor $\underset{\sim}{T}^{-1}$: $\underset{\sim}{T}\underset{\sim}{T}^{-1} = \underset{\sim}{T}^{-1}\underset{\sim}{T} = \underset{\sim}{1}$

$(\underset{\sim}{T}\underset{\sim}{S})^{-1} = \underset{\sim}{S}^{-1}\underset{\sim}{T}^{-1}$ if $\underset{\sim}{T}$ & $\underset{\sim}{S}$ are not singular.

- Transpose of a tensor $\underset{\sim}{T}^T$: $(\underset{\sim}{T}^T \underline{u}) \cdot \underline{v} \equiv \underline{u} \cdot \underset{\sim}{T} \underline{v}$

$$(\underset{\sim}{T}^T)_{ij} = (\underset{\sim}{T}^T \hat{i}_j) \cdot \hat{i}_i = \hat{i}_j \cdot \underset{\sim}{T} \hat{i}_i = \hat{i}_j \cdot (T_{pm} \underbrace{\hat{i}_p \otimes \hat{i}_m) \hat{i}_i}_{\hat{i}_p \delta_{mi}}$$

$$= \hat{i}_j \cdot (T_{pi} \hat{i}_p) = T_{ji}.$$

$$\underset{\sim}{T}^T = \begin{pmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{pmatrix}$$

$$(\underset{\sim}{T} \underset{\sim}{S})^T = \underset{\sim}{S}^T \underset{\sim}{T}^T$$

$$(\underline{a} \otimes \underline{b})^T = \underline{b} \otimes \underline{a} \qquad (\underline{a}\,\underline{b})^T = \underline{b}\,\underline{a}$$

- Symmetric tensor : $\underset{\sim}{T}$ is symmetric if $\underset{\sim}{T} = \underset{\sim}{T}^T$
  Skew (anti-symmetric) tensor ; $\underset{\sim}{T}$ is skew if $\underset{\sim}{T} = -\underset{\sim}{T}^T$
- Every tensor $\underset{\sim}{T}$ admits the unique decomposition into symmetric and skew tensors.

$$\underset{\sim}{T} = \frac{1}{2}(\underset{\sim}{T} + \underset{\sim}{T}^T) + \frac{1}{2}(\underset{\sim}{T} - \underset{\sim}{T}^T)$$

$$= \underset{\sim}{S} + \underset{\sim}{W}$$

$$\underset{\sim}{S} = \underset{\sim}{S}^T, \qquad \underset{\sim}{W} = -\underset{\sim}{W}^T$$

- Orthogonal tensor
  $\underset{\sim}{Q}$ is orthogonal if it is not singular and $\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}^T$

$$\underline{\underset{\sim}{Q} \underset{\sim}{Q}^{-1} = \underset{\sim}{Q} \underset{\sim}{Q}^T = 1}, \quad \underline{\underset{\sim}{Q}^{-1} \underset{\sim}{Q} = \underset{\sim}{Q}^T \underset{\sim}{Q} = 1}$$

- Trace of $\underset{\sim}{T}$ = $\mathrm{tr}\,\underset{\sim}{T}$ is defined
  (i) $\mathrm{tr}(\underset{\sim}{T} + \underset{\sim}{S}) = \mathrm{tr}\,\underset{\sim}{T} + \mathrm{tr}\,\underset{\sim}{S}$
  (ii) $\mathrm{tr}(\alpha \underset{\sim}{T}) = \alpha\, \mathrm{tr}\,\underset{\sim}{T}$
  (iii) $\mathrm{tr}(\underline{a} \otimes \underline{b}) = \underline{a} \cdot \underline{b}$

$\underset{\sim}{T} = T_{ij}\, \hat{i}_i \otimes \hat{i}_j$

component of tr $\underset{\sim}{T}$

$\mathrm{tr}\,\underset{\sim}{T} = \mathrm{tr}\left(T_{ij}\, \hat{i}_i \otimes \hat{i}_j\right) = T_{ij}\, \mathrm{tr}\left(\hat{i}_i \otimes \hat{i}_j\right) = T_{ij}\, \hat{i}_i \cdot \hat{i}_j$

$= T_{ij}\,\delta_{ij} = T_{ii}$

- Inner product of tensors $\underset{\sim}{T}$ and $\underset{\sim}{S}$ $= \mathrm{tr}(\underset{\sim}{T}\underset{\sim}{S}^T) = \underset{\sim}{T} \cdot \underset{\sim}{S}^T$

  $\mathrm{tr}(\underset{\sim}{T}\underset{\sim}{S}^T) = \mathrm{tr}(T_{im} \underbrace{S_{jm}\,)}_{\hat{i}_i \otimes \hat{i}_j} = T_{im} S_{jm}\, \mathrm{tr}(\hat{i}_i \otimes \hat{i}_j) = T_{im} S_{im}$

  $\mathrm{tr}(\underset{\sim}{T}\underset{\sim}{T}^T) = T_{ij} T_{ij} = T_{11}^2 + T_{12}^2 + T_{13}^2 + T_{21}^2 + T_{22}^2 + T_{23}^2$

  $+ T_{31}^2 + T_{32}^2 + T_{33}^2 \geq 0$ positive definite

  $|\underset{\sim}{T}| = \sqrt{\mathrm{tr}(\underset{\sim}{T}\underset{\sim}{T}^T)}$: magnitude of $\underset{\sim}{T}$

- Handedness of an ordered basis $(\underset{-}{e}_1, \underset{-}{e}_2, \underset{-}{e}_3)$

  the ordered basis $(\underset{-}{e}_1, \underset{-}{e}_2, \underset{-}{e}_3)$ is right-handed

  if $\underset{-}{e}_3$ forms an acute angle with the right handed

  screw of rotating $\underset{-}{e}_1$ into $\underset{-}{e}_2$.

- Determinant of $\underset{\sim}{T} \equiv \det \underset{\sim}{T}$