노트 제목 2010-03-11

- Conservation of energy ($\alpha = \rho e + \frac{1}{2}\rho \underline{u}\cdot\underline{u}$)

  e: internal energy

  total work done by $\underline{P} \rightarrow \int_s \underline{u}\cdot\underline{P}\, ds$

  $\underline{f} \longrightarrow \int_V \underline{u}\cdot\rho\underline{f}\, dV$

  $\underline{q}$: conductive heat flux leaving the C.V

  the quantity of heat leaving the fluid mass per unit time per unit surface area $= \underline{q}\cdot\underline{n}$

  $\therefore$ net amount of heat leaving the fluid per unit time $= \int_s \underline{q}\cdot\underline{n}\, dS$

$$\underbrace{\frac{D}{Dt}\int_V (\rho e + \tfrac{1}{2}\rho\underline{u}\cdot\underline{u})\, dV}_{\text{RTT}} = \int_s \underline{u}\cdot\underline{P}\, ds + \int_V \underline{u}\cdot\rho\underline{f}\, dV - \int_s \underline{q}\cdot\underline{n}\, ds$$

$$\int_s \underline{u}\cdot\underline{P}\, ds = \int_s u_j \sigma_{ij} n_i\, ds = \int_V \frac{\partial}{\partial x_i}(u_j\sigma_{ij})\, dV$$

$$\int_V \underline{u}\cdot\rho\underline{f}\, dV = \int_V u_j \rho f_j\, dV$$

$$\int_s \underline{q}\cdot\underline{n}\, ds = \int_s q_j n_j\, ds = \int_V \frac{\partial q_j}{\partial x_j}\, dV$$

$$\therefore \frac{\partial}{\partial t}(\rho e + \tfrac{1}{2}\rho u_j u_j) + \frac{\partial}{\partial x_k}(\rho e + \tfrac{1}{2}\rho u_j u_j)u_k$$

$$= \frac{\partial}{\partial x_i}(u_j\sigma_{ij}) + u_j\rho f_j - \frac{\partial q_j}{\partial x_j}$$

After some algebra,

$$\boxed{\rho\frac{\partial e}{\partial t} + \rho u_k\frac{\partial e}{\partial x_k} = \sigma_{ij}\frac{\partial u_j}{\partial x_i} - \frac{\partial q_j}{\partial x_j}}$$ // energy eq.

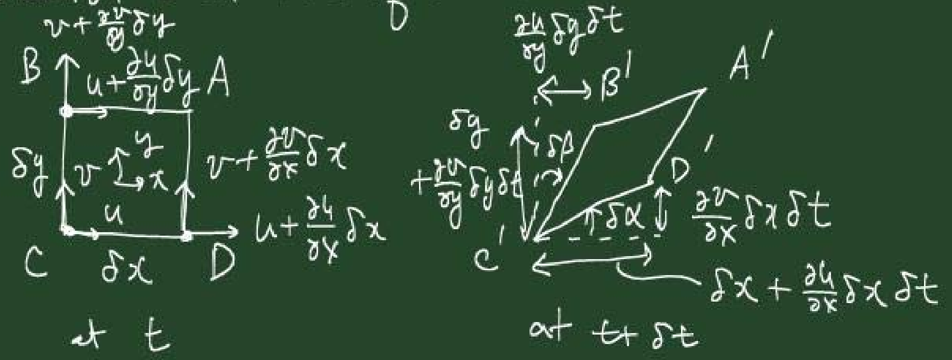
- Discussion of conservation eqs.

  σ_{ij} and q_j should be further specified

  → constitutive eqs. $\sigma_{ij} \sim f\left(\frac{\partial u_i}{\partial x_j}\right)$

  $q_j \sim g\left(\frac{\partial T}{\partial x_j}\right)$

- rotation and rate of shear



$\rightarrow \frac{\delta\alpha}{\delta t} = \frac{\partial v}{\partial x}$, $\left(\delta y + \cancel{\frac{\partial v}{\partial y}\delta y\delta t}\right)\delta\beta = \frac{\partial u}{\partial y}\delta y\delta t$

α: counterclockwise $\rightarrow \frac{\delta\beta}{\delta t} = \frac{\partial u}{\partial y}$ β: clockwise

rate of counterclockwise rotation

$$= \frac{1}{2}\left(\frac{\delta\alpha}{\delta t} - \frac{\delta\beta}{\delta t}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

rate of shear $= \frac{1}{2}\left(\frac{\delta\alpha}{\delta t} + \frac{\delta\beta}{\delta t}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$

In general,

rate of rotation $= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ anti-symmetric

rate of shear $= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ symmetric

Deformation-rate tensor e_{ij}

$$e_{ij} = \frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- Constitutive eq.

4 conditions that the stress tensor is supposed to satisfy

① fluid at rest → stress is hydrostatic pressure

thermodynamic press.

② σ_{ij} is linearly related to e_{kl} and depends only on that tensor.

③ for solid-body rotation of the fluid

→ no shearing action → no shear stress during such a motion.

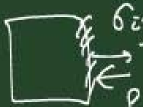
④ no preferred direction in the fluid

→ fluid properties are point fts. (i.e. isotropic)

① → $\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij}$

thermo. press

τ_{ij}: depends on the fluid motion. called shear stress tensor.



$\sigma_{ij} > 0$

$p > 0$

② → $\sigma_{ij} \sim e_{kl} \Rightarrow \tau_{ij} = \alpha_{ijkl}\dfrac{\partial u_k}{\partial x_l}$

$$\tau_{ij} = \alpha_{ijkl}\cdot\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) + \alpha_{ijkl}\cdot\frac{1}{2}\underline{\left(\frac{\partial u_k}{\partial x_l} - \frac{\partial u_l}{\partial x_k}\right)}$$

for ③ → $\tau_{ij} = 0$ but $\neq 0$

So, this term should go away!

$$\therefore \tau_{ij} = \frac{1}{2} \underbrace{\beta_{ijkl}}_{\text{fluid property}} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)$$

cond. ④ → isotropic

$$\rightarrow \beta_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right)$$

$$+ \underbrace{\gamma \left(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \right)}_{\text{is zero when multiplied by shear rate.}} \qquad \text{(App. B)}$$

$$\therefore \tau_{ij} = \frac{1}{2} \left[\lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \right] \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)$$

$$\lambda \delta_{ij} \cdot 2 \frac{\partial u_k}{\partial x_k} \qquad 2\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$\boxed{\tau_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)}$$

$$\boxed{\sigma_{ij} = -p \delta_{ij} + \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)}$$

σ_{ij} : second-order symmetric tensor.

τ_{ij} : " " "

$$\boxed{\sigma_{ij} \sim O\, e_{kl}} \; a_1 e_{kl} + a_1 f_r$$

$$+ O \boxed{e_{kl}^2} + O\, e_{kl}^3$$

coeffs. λ and μ → determined empirically

$$\cdot \; q_j = -k \frac{\partial T}{\partial x_j} \qquad k: \text{thermal conductivity}$$

Fourier law

- Viscosity coefficients

Consider a simple shear flow $u = u(y)$

$v = w = 0$

$\rightarrow \sigma_{12} = \sigma_{21} = \mu \frac{du}{dy} \Rightarrow$ μ : (dynamic) viscosity

$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p$ $\quad \nu \equiv \mu / \rho$: kinematic viscosity

$\sigma_{13} = \sigma_{31} = \sigma_{23} = \sigma_{32} = 0$ $\quad \rightarrow$ thermo. press.