Ch.2. Flow Kinematics

노트 제목 2010-03-23

2.1 Flow lines

- Streamlines : lines whose tangents are everywhere parallel to the velocity vector

  instantaneous streamlines in the case of unsteady flows.

$$\frac{dy}{dx} = \frac{v}{u}, \quad \frac{dz}{dx} = \frac{w}{u}, \quad \frac{dz}{dy} = \frac{w}{v} \quad @\ t$$

$$\rightarrow \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = ds$$

$$\rightarrow \frac{dx_i}{ds} = u_i(\underset{\sim}{x}, t) \qquad t \text{ fixed}$$

$$\rightarrow x_i = x_i(x_0, y_0, z_0, t, s)$$

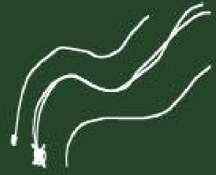
(x_0, y_0, z_0) ex) $u = x(1+2t), \ v = y, \ w = 0$ $\quad$ $x=1, \ y=1 \ @ \ t=0, \ s=0$

$$\frac{dx}{ds} = u = x(1+2t) \rightarrow x = c_1 e^{(1+2t)s} = e^{(1+2t)s}$$

$$\frac{dy}{ds} = v = y \rightarrow y = c_2 e^{s} = e^{s}$$

$$\Rightarrow x = y^{1+2t}$$

- Pathlines : lines which are traced out in time by given fluid particles as they flow.



$$\frac{dx_i}{dt} = u_i(\underset{\sim}{x}, t) \rightarrow x_i = x_i(x_0, y_0, z_0, t)$$

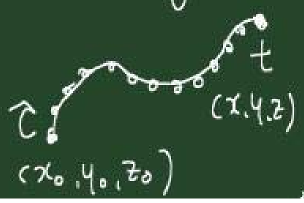
ex) $u = x(1+2t)$, $v = y$, $w = 0$ $\quad x = y = 1$ @ $t = 0$

$$\frac{dx}{dt} = u = x(1+2t) \rightarrow x = c_1 e^{t(1+t)} = e^{t(1+t)}$$

$$\frac{dy}{dt} = v = y \longrightarrow y = c_2 e^t = e^t$$

$$\underline{x = y^{1+\log y}}.$$

- Streaklines: lines which are traced out by neutrally buoyant marker fluids which are continuously injected into a flow field at a fixed pt. in space



A particle at (x, y, z) at t must have passed through the injection point (x_0, y_0, z_0) at $t = \tau$ $(\tau \leq t)$

$$\rightarrow x_i = x_i(x_0, y_0, z_0, t, \tau)$$

ex) $u = x(1+2t)$, $v = y$, $w = 0$

$$\frac{dx}{dt} = u = x(1+2t) \longrightarrow x = c_1 e^{t(1+t)}$$

$$\frac{dy}{dt} = v = y \quad \rightarrow \quad y = c_2 e^t$$

@ $t = \tau$, $x = y = 1 \rightarrow x = e^{t(1+t) - \tau(1+\tau)}$

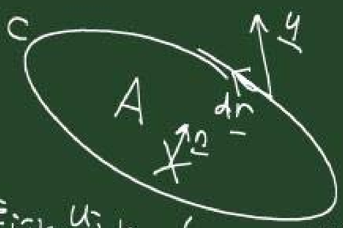
$$y = e^{t-\tau}$$

@ $t = 0$, $x = e^{-\tau(1+\tau)}$, $y = e^{-\tau}$

$$\hookrightarrow \underline{x = y^{1-\log y}}$$

streamline $\neq$ pathline $\neq$ streakline for unsteady flow

2.2 Circulation and vorticity



Circulation $\Gamma \equiv \oint_C \underline{u}\cdot d\underline{r}$

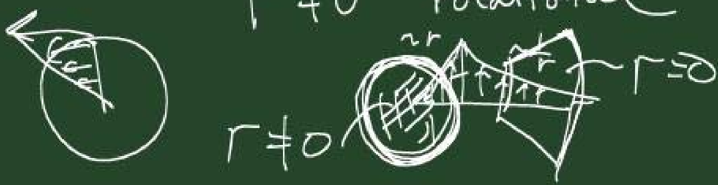
vorticity $\underline{\omega} = \nabla\times\underline{u}$, $\omega_i = -\epsilon_{ijk}\, u_{j,k}$ $\left(u_{j,k} = \frac{\partial u_j}{\partial x_k}\right)$

Stokes theorem:

$$\Gamma = \oint_C \underline{u}\cdot d\underline{r} = \int_A (\nabla\times\underline{u})\cdot\underline{n}\,dA = \boxed{\int_A \underline{\omega}\cdot\underline{n}\,dA}$$

If $\underline{\omega} = 0 \rightarrow \Gamma = 0$ irrotational

$\Gamma \neq 0$ rotational



2.3 Stream tubes and vortex tubes

vortex lines: lines whose tangents are everywhere parallel to the vorticity vector

stream filament: streamtube whose cross section is infinitesimally small.

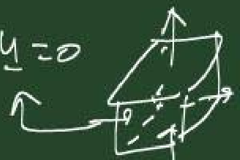
vortex filament : vortex tube whose area is infinitesimally small

2.4 Kinematics of vortex lines

$\underline{\omega} = \nabla \times \underline{u} \rightarrow \nabla \cdot \underline{\omega} = \nabla \cdot (\nabla \times \underline{u}) = 0$

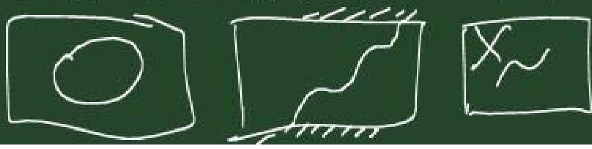
$\therefore$ vorticity is divergence free.

$\nabla \cdot \underline{u} = 0$



no source or no sink of vorticity in the fluid itself

Vortex lines either must form closed loops or must terminate on the boundaries of fluid.



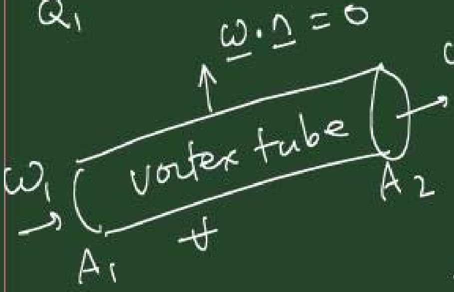
Helmholtz theorem of vorticity

$\nabla \cdot \underline{u} = 0$

$\rightarrow \int_V \nabla \cdot \underline{u} = 0$

$\rightarrow \int_S \underline{u} \cdot \underline{n} \, ds = 0$

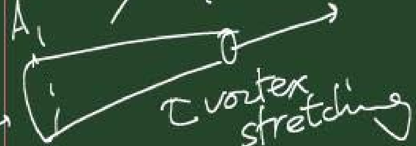
$\rightarrow Q_1 = Q_2$



$\boxed{\nabla \cdot \underline{\omega} = 0}$

$\rightarrow \int_V \nabla \cdot \underline{\omega} = 0 \rightarrow \int_S \underline{\omega} \cdot \underline{n} \, ds = 0$

$A_1 > A_2 \quad \omega_2 > \omega_1 \rightarrow \Gamma_1 = \Gamma_2$



$\omega_1 A_1 = \omega_2 A_2$

averaged vorticity

Ch. 3 Special Forms of the Governing Eqs.

- Kelvin's theorem : irrotational flow
- Bernoulli equation :
- Crocco's equation : entropy - vorticity
- Vorticity equation

⊝ Kelvin's theorem

$\nu = 0$ $\quad p = p(\rho)$ $\quad \rho = \text{const}$

" For an inviscid fluid in which the density is constant, or in which the pressure depends on the density alone, and for which any body forces which exist are $\boxed{\text{conservative}}$ the vorticity of each fluid particle will be preserved. "

body force $f_i \xrightarrow[\text{conservative}]{} f_i = \dfrac{\partial G}{\partial x_i}$

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \rho \frac{\partial G}{\partial x_i} \qquad (\nu = 0)$$

$$\rightarrow \quad \frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial G}{\partial x_i}$$

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint u_i dx_i = \oint \left[\frac{Du_i}{Dt} dx_i + u_i \frac{D dx_i}{Dt} \right]$$

$$\frac{D dx_i}{Dt} = d \frac{D x_i}{Dt} = d\left(\frac{\partial x_i}{\partial t} + u_j \frac{\partial x_i}{\partial x_j} \right) = du_i$$

$$= \oint \left[\underline{\frac{Du_i}{Dt} dx_i} + u_i du_i \right]$$

$$= \oint \left[\underline{-\frac{1}{\rho}\frac{\partial p}{\partial x_i} dx_i + \frac{\partial G}{\partial x_i} dx_i} + u_i du_i \right]$$

$$= \oint \left[-\frac{1}{\rho}dp + dG + \frac{1}{2}d(u_i u_i) \right] = \oint \left[-\frac{1}{\rho}dp \right]$$

if $\rho = \text{const}$, $\frac{D\Gamma}{Dt} = 0$

if $p = p(\rho) = g(\rho)$, $\frac{D\Gamma}{Dt} = \oint \left[-\frac{1}{\rho} g'(\rho)\, d\rho \right] = \oint d f(\rho)$

$f' d\rho$

Kelvin theorem $\boxed{\frac{D\Gamma}{Dt} = 0}$

If we follow a given contour as it flows, the vorticity inside the contour will not change.

$\Gamma_1 \xrightarrow{w/\ \text{flow}} \Gamma_2 = \Gamma_1$

if $\nu = 0$,
ρ = const or $p = p(\rho)$,
conservative body force

uniform flow

$\Gamma = 0 \longrightarrow \Gamma = 0 \longrightarrow \nabla \times \underline{u} = 0$

$\Gamma \neq 0$ near the body