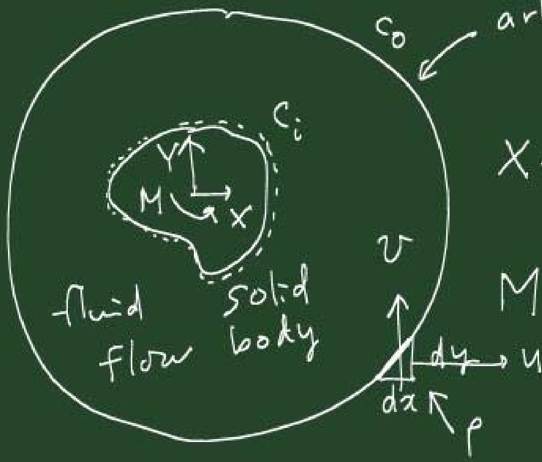
노트 제목 2010-04-08

- Blasius integral laws



$$X - iY = i\frac{\rho}{2}\int_{c_o} W^2\,dz$$

$$M = -\frac{\rho}{2}\,\mathrm{Re}\left(\int_{c_o} z W^2\,dz\right)$$

$$\Sigma F_x = -X - \int_{c_o} p\,dy = \int_{c_o} \rho(u\,dy - v\,dx)\,u$$

$$\Sigma F_y = -Y + \int_{c_o} p\,dx = \int_{c_o} \rho(u\,dy - v\,dx)\,v$$

$$\rightarrow \quad X = \int_{c_o}(-p\,dy - \rho u^2 dy + \rho uv\,dy)$$

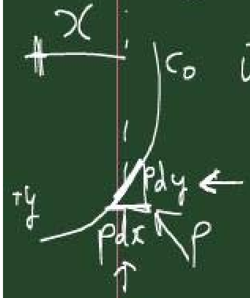
$$Y = \int_{c_o}(p\,dx - \rho uv\,dy + \rho v^2 dx)$$

Bernoulli eq. $\quad p + \frac{1}{2}\rho(u^2+v^2) = B$

$$\rightarrow \quad X = \rho\int_{c_o}\left[uv\,dx - \frac{1}{2}(u^2-v^2)\,dy\right] \qquad \left(\int_{c_o} B\,dy = 0\right)$$

$$Y = -\rho\int_{c_o}\left[uv\,dy + \frac{1}{2}(u^2-v^2)\,dx\right]$$

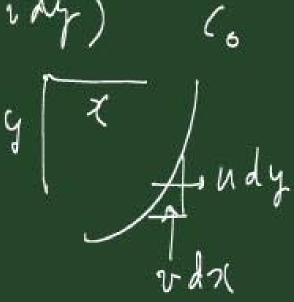
On the other hand,



$$i\frac{\rho}{2}\int_{C_0} W^2 dz = i\frac{\rho}{2}\int_{C_0}(u-iv)^2(dx+i\,dy)$$

$$= - - - -$$

$$= X - iY$$



$$\circlearrowright\ \Sigma M = M - \int_{C_0} px\,dx - \int_{C_0} py\,dy$$

$$= \int_{C_0} \rho(u\,dy - v\,dx)(uy - vx)$$

$$M = \int_{C_0}\left(px\,dx + py\,dy + \rho(u^2 y\,dy + v^2 x\,dx - uvy\,dx - uvx\,dy)\right)$$

(Bernoulli eq. $p + \frac{1}{2}\rho(u^2+v^2) = B$)

$$= \rho\int_{C_0}\left[-\frac{1}{2}(u^2+v^2)(x\,dx + y\,dy) + (u^2 y\,dx + v^2 x\,dx) - (uvy\,dx + uvx\,dy)\right]$$

$$\left(\begin{array}{l}\int_{C_0} Bx\,dx = \int_{C_0} B\,d\left(\frac{1}{2}x^2\right) = 0 \\ \int_{C_0} By\,dy = 0\end{array}\right.$$

$$= \cdots$$

$$= -\frac{\rho}{2}\int_{C_0}\left[(u^2-v^2)(x\,dx - y\,dy) + 2uv(x\,dy + y\,dx)\right]$$

On the other hand,

$$Re\left[\frac{\rho}{2}\int_{C_0} zW^2 dz\right] = Re\left[\frac{\rho}{2}\int_{C_0}(x+iy)(u-iv)^2(dx+i\,dy)\right]$$

$$= \cdots = -M$$

$$\therefore \begin{cases} X - iY = i\dfrac{\rho}{2}\displaystyle\int_{c_0} W^2 dz \\ M = -\dfrac{\rho}{2}\,\mathrm{Re}\left[\displaystyle\int_{c_0} zW^2 dz\right] \end{cases}$$

Blasius integral laws.

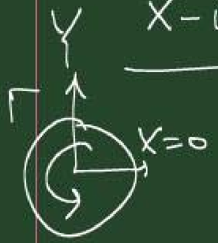
Contour integral is usually evaluated by use of the residue theorem.

- Force and moment on a circular cylinder

$$F(z) = U\left(z + \frac{a^2}{z}\right) + i\frac{\Gamma}{2\pi}\ln\frac{z}{a}$$

$$W(z) = \frac{dF}{dz} = U\left(1 - \frac{a^2}{z^2}\right) + i\frac{\Gamma}{2\pi z}$$

$$W^2 = U^2 - \frac{2U^2a^2}{z^2} + \frac{U^2a^4}{z^4} + \frac{iU\Gamma}{\pi z} - \frac{iU\Gamma a^2}{\pi z^3} - \frac{\Gamma^2}{4\pi^2 z^2}$$



$$X - iY = i\frac{\rho}{2}\int_{c_0} W^2 dz \qquad \left(\oint \frac{1}{z^n}dz = 0 \text{ if } n>1\right)$$

$$= i\frac{\rho}{2}\left[2\pi i\,\Sigma\,(\text{residues of } W^2 \text{ inside } c_0)\right]$$

$$= i\frac{\rho}{2}\cdot 2\pi i\cdot\frac{iU\Gamma}{\pi} = -i\rho U\Gamma$$

$$\therefore\ \underset{\substack{\uparrow\\ \text{drag}}}{X = 0} \text{ and } \underset{\substack{\uparrow\\ \text{lift}}}{\boxed{Y = \rho U\Gamma}}$$ Kutta-Joukowski Law.

$$zW^2 = U^2z - \frac{2U^2a^2}{z} + \frac{U^2a^4}{z^3} + \frac{iU\Gamma}{\pi} - \frac{iU\Gamma a^2}{\pi z^2} - \frac{\Gamma^2}{4\pi^2 z}$$

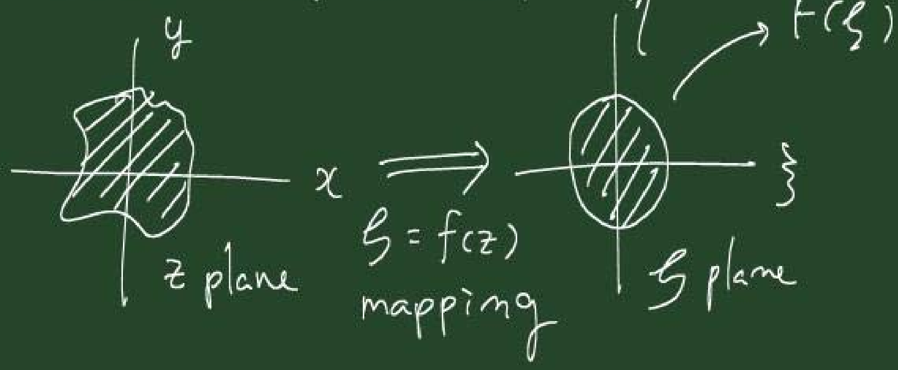
$$M = -\frac{\rho}{2}\,\mathrm{Re}\left[\int_{c_0} zW^2 dz\right]$$

$$= -\frac{\rho}{2}\,\mathrm{Re}\left[2\pi i \,\Sigma \text{ residues of } zW^2\right]$$

$$= -\frac{\rho}{2}\,\mathrm{Re}\left[2\pi i\left(-2U^2a^2 - \frac{\Gamma^2}{4\pi^2}\right)\right]$$

$$= 0 \quad \therefore \text{no hydrodynamic moment.}$$

- Conformal transformation



If the transformation is conformal (i.e. a function that preserves both angle size and orientation is said to be conformal), the mapping ft. f will be analytic and the real and imaginary parts of the new variable ζ will be harmonic.

ϕ and ψ satisfy the Laplace eq.

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial x}\right)$$

$$= \left(\frac{\partial \xi}{\partial x}\right)^2 \frac{\partial^2 \phi}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{\partial^2 \phi}{\partial \eta^2} + 2\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}\frac{\partial^2 \phi}{\partial \xi \partial \eta}$$
$$+ \frac{\partial^2 \xi}{\partial x^2}\frac{\partial \phi}{\partial \xi} + \frac{\partial^2 \eta}{\partial x^2}\frac{\partial \phi}{\partial \eta}$$

Similarly,

$$\frac{\partial^2 \phi}{\partial y^2} = \left(\frac{\partial \xi}{\partial y}\right)^2 \frac{\partial^2 \phi}{\partial \xi^2} + \left(\frac{\partial \eta}{\partial y}\right)^2 \frac{\partial^2 \phi}{\partial \eta^2} + 2\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}\frac{\partial^2 \phi}{\partial \xi \partial \eta}$$
$$+ \frac{\partial^2 \xi}{\partial y^2}\frac{\partial \phi}{\partial \xi} + \frac{\partial^2 \eta}{\partial y^2}\frac{\partial \phi}{\partial \eta}$$

$\nabla^2 \phi = 0$:

$$\left[\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2\right]\frac{\partial^2 \phi}{\partial \xi^2} + \left[\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2\right]\frac{\partial^2 \phi}{\partial \eta^2}$$

$$+ 2\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}\right)\frac{\partial^2 \phi}{\partial \xi \partial \eta}$$
$$+ \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2}\right)\frac{\partial \phi}{\partial \xi} + \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right)\frac{\partial \phi}{\partial \eta} = 0$$

Since the transf. is conformal, ζ is harmonic

$$\zeta = \xi + i\eta$$

$$\rightarrow \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} = 0 , \quad \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = 0$$

$$\rightarrow \frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} , \quad \frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x}$$; Cauchy-Riemann eqs.

Hence, $\left[\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2\right]\frac{\partial^2 \phi}{\partial \xi^2} + \left[\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2\right]\frac{\partial^2 \phi}{\partial \eta^2} = 0$

Use C-R eqs $\rightarrow$ $\boxed{\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2} = 0}$

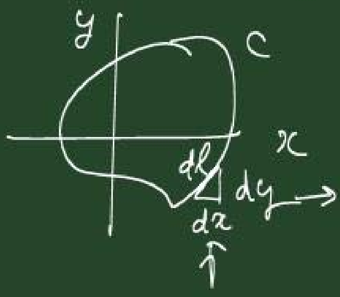
Laplace eq. in the z plane transforms into Laplace eq. in the ζ plane, provided these two planes are related by a conformal transf.

→ If the sol. for some simple body is known in ζ plane, then the sol. for more complex body is obtained by substituting $\zeta = f(z)$ to $F(\zeta)$.

- $W(z) = \frac{dF(z)}{dz} = \frac{dF(\zeta)}{d\zeta}\frac{d\zeta}{dz} = \frac{d\zeta}{dz} W(\zeta)$

$$\therefore \boxed{W(z) = \frac{d\zeta}{dz} W(\zeta)} \quad \leftarrow W(z) \neq W(\zeta)$$

- Source, sink, vorticity



$$m = \int_c \underline{u}\cdot\underline{n}\, dl = \int_c (u\,dy - v\,dx)$$

$$\Gamma = \int_c \underline{u}\cdot d\underline{l} = \int_c (u\,dx + v\,dy)$$

$$\int_c W(z)dz = \int_c (u - iv)(dx + i dy)$$

$$= \int_c (u\,dx + v\,dy) + i\int_c (u\,dy - v\,dx)$$

$$= \Gamma + im$$

$$\therefore \Gamma_z + im_z = \int_{c_z} w(z)\,dz = \int_{c_z} w(\zeta)\frac{d\zeta}{\cancel{dz}}\cancel{dz}$$

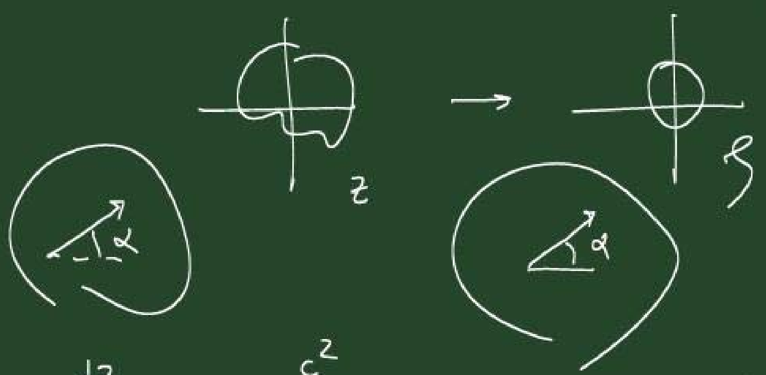
$$= \int_{c_\zeta} w(\zeta)\,d\zeta = \Gamma_\zeta + im_\zeta$$

$$\rightarrow \boxed{\Gamma_z = \Gamma_\zeta, \quad m_z = m_\zeta}$$

Source/sink & vortices map into
" " " of the same strength
under a conformal transf.

- Joukowski transf. : $z = \zeta + \dfrac{c^2}{\zeta}$

  . for large values of $|\zeta|$, $z \to \zeta$: identity mapping far from origin



  . $\dfrac{dz}{d\zeta} = 1 - \dfrac{c^2}{\zeta^2}$

  $\zeta = 0$: singular pt. does not cause any difficulty because $\zeta = 0$ corresponds to the inner pt. of the body.

$\zeta = \pm c$; critical pts. $\left(\frac{dz}{d\zeta} = 0\right)$