Ch4. Mean-flow equations

노트 제목 2010-03-23

⊙ Reynolds eqs.

· Reynolds decomposition

$$\underline{U}(\underline{x},t) = \langle \underline{U}(\underline{x},t)\rangle + \underline{u}(\underline{x},t)$$

instantaneous velocity — mean velocity — velocity fluctuation

$$\langle \underline{u} \rangle = 0$$

· continuity

$$\nabla\cdot\underline{U} = 0$$

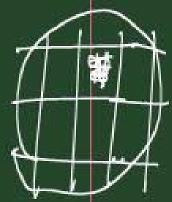
$$\langle \nabla\cdot(\langle\underline{U}\rangle + \underline{u}) = 0 \rangle \to \nabla\cdot\langle\underline{U}\rangle + \cancelto{0}{\langle\nabla\cdot\underline{u}\rangle} = 0$$

$$\to \nabla\cdot\langle\underline{U}\rangle = 0 \to \nabla\cdot\underline{u} = 0$$

· mtm eq.

$$\left\langle \frac{DU_j}{Dt} = \underline{\frac{\partial U_j}{\partial t}} + \frac{\partial}{\partial x_i}(U_iU_j) = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \nu\nabla^2 U_j \right\rangle$$

$$\to \frac{\partial}{\partial t}\langle U_j\rangle + \frac{\partial}{\partial x_i}\langle U_iU_j\rangle = -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_j} + \nu\nabla^2\langle U_j\rangle$$



$$\langle U_iU_j\rangle = \langle(\langle U_i\rangle + u_i)(\langle U_j\rangle + u_j)\rangle$$

$$= \langle\langle U_i\rangle\langle U_j\rangle + u_iu_j + \underline{u_i\langle U_j\rangle} + u_j\langle U_i\rangle\rangle$$

$$= \langle U_i\rangle\langle U_j\rangle + \langle u_iu_j\rangle + \cancelto{0}{\langle u_i\rangle}\langle U_j\rangle + \cancelto{0}{\langle u_j\rangle}\langle U_i\rangle$$

$$\to \frac{\partial\langle U_j\rangle}{\partial t} + \frac{\partial}{\partial x_i}\langle U_i\rangle\langle U_j\rangle = -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_j} + \nu\nabla^2\langle U_j\rangle - \frac{\partial}{\partial x_i}\langle u_iu_j\rangle$$

(RANS) Reynolds-averaged Navier-Stokes eq.

⊙ Reynolds stresses

- $\rho \dfrac{\partial \langle U_j \rangle}{\partial t} + \rho \dfrac{\partial}{\partial x_i} \langle U_i \rangle \langle U_j \rangle$

$$= \frac{\partial}{\partial x_i}\Big[\underbrace{-\langle p \rangle \delta_{ij}}_{\text{isotropic stress}} + \underbrace{\mu\Big(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i}\Big)}_{\text{viscous stress}} \underbrace{- \rho \langle u_i u_j \rangle}_{\text{Reynolds stress}} \Big]$$

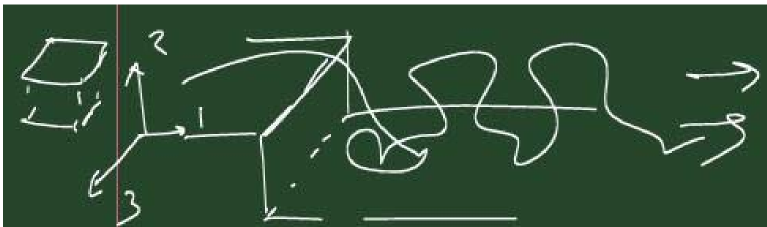
Reynolds stress (stress from fluctuating velocity)

- eq. for $\langle U_j \rangle \rightarrow \langle u_i u_j \rangle$

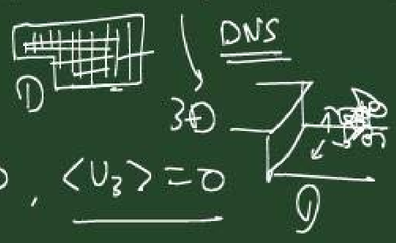
  unknowns : $\overset{3}{\langle U_j \rangle}$, $\overset{1}{\langle p \rangle}$ and $\overset{6}{\langle u_i u_j \rangle} \rightarrow 10$

  equations : cont + mtm $\longrightarrow 4$

  $\rightarrow$ closure problem $\rightarrow$ turbulence modeling



$\longrightarrow$ RANS $\rightarrow$ 2D simulation



mean quantity $\langle U_1 \rangle, \langle U_2 \rangle \neq 0$, $\underline{\langle U_3 \rangle = 0}$

- $\langle u_i u_j \rangle = \langle u_j u_i \rangle$ sym. tensor

  $\langle u_1^2 \rangle, \langle u_2^2 \rangle, \langle u_3^2 \rangle$ normal stresses (Reynolds)

  $\langle u_1 u_2 \rangle, \langle u_1 u_3 \rangle, \langle u_2 u_3 \rangle$ shear stresses

  $k \equiv \frac{1}{2} \langle u_i u_i \rangle$ turbulent kinetic energy

• Anisotropy $\qquad a_{ii} = \langle u_i u_i \rangle - \frac{2}{3}k\cdot 3 = 0$

$$a_{ij} \equiv \langle u_i u_j \rangle - \frac{2}{3} k \delta_{ij} \quad : \text{anisotropic part.}$$

$$b_{ij} \equiv a_{ij}/2k = \frac{\langle u_i u_j \rangle}{\langle u_\ell u_\ell \rangle} - \frac{1}{3}\delta_{ij}$$

$\rho \frac{\partial}{\partial x_j}\left(\frac{2}{3}k\right)$

$$\underline{\rho \frac{\partial \langle u_i u_j \rangle}{\partial x_i}} + \underline{\frac{\partial \langle p \rangle}{\partial x_j}} = \rho \frac{\partial a_{ij}}{\partial x_i} + \rho \frac{\partial}{\partial x_i}\left(\frac{2}{3}k\delta_{ij}\right) + \frac{\partial \langle p \rangle}{\partial x_j}$$

$$= \rho \frac{\partial a_{ij}}{\partial x_i} + \frac{\partial}{\partial x_j}\Big[\underbrace{\langle p \rangle + \frac{2}{3}\rho k}_{\equiv \langle p_1 \rangle}\Big]$$

∴ only the anisotropic part is effective in transporting momentum because isotropic one is absorbed in a modified pressure.

$$\begin{cases} \dfrac{\partial \langle U_i \rangle}{\partial x_i} = 0 \\ \rho \dfrac{\partial \langle U_j \rangle}{\partial t} + \rho \dfrac{\partial}{\partial x_i}\langle U_i \rangle \langle U_j \rangle = -\dfrac{\partial \langle p \rangle}{\partial x_j} - \rho \dfrac{\partial \langle u_i u_j \rangle}{\partial x_i} + \nu \nabla^2 \langle U_j \rangle \end{cases}$$

$$\rho \;\; '' \; + \; '' \; = -\frac{\partial \langle p_1 \rangle}{\partial x_j} - \rho \frac{\partial a_{ij}}{\partial x_i} + \; ''$$

• Irrotational motion

consider <u>irrotational</u> random velocity field

$$\rightarrow \underline{\omega} = 0 \rightarrow \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = 0$$

$$\left\langle u_i \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) = 0 \right\rangle$$

$$\rightarrow \frac{\partial}{\partial x_j}\left(\frac{1}{2}\langle u_i u_i\rangle\right) - \frac{\partial}{\partial x_i}\langle u_i u_j\rangle = 0$$

$$\rightarrow \frac{\partial}{\partial x_i}\langle u_i u_j\rangle = \frac{\partial}{\partial x_j} k \quad \text{for irrotational flow}$$

$\because \nabla\times \underline{u} = 0 \rightarrow \underline{u} = \nabla\phi$, $\nabla\cdot\underline{u} = 0$

Then, $$\rho\frac{\partial}{\partial x_i}\langle u_i u_j\rangle + \frac{\partial\langle p\rangle}{\partial x_j} = \frac{\partial}{\partial x_j}\underbrace{(\rho k + \langle p\rangle)}_{\langle P_1\rangle}$$

Reynolds stress → absorbed in a modified press.

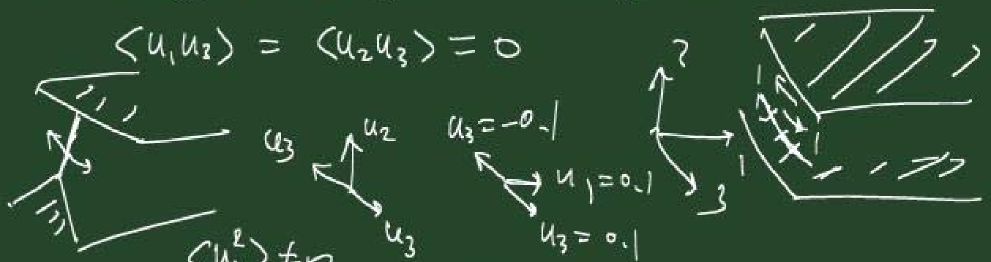
⇒ The Reynolds stresses arising from an irrotational field have no effect on the mean velocity field.

- Symmetries

Consider a statistically 2D flow in which statistics are indep. of x_3, and which is statistically invariant under reflections of x_3 coord.

$$\rightarrow \langle U_3\rangle = -\langle U_3\rangle \rightarrow \langle U_3\rangle = 0$$

$$\langle u_1 u_3\rangle = \langle u_2 u_3\rangle = 0$$



$\langle u_3^2\rangle \neq 0$

$$\begin{bmatrix} \langle u_1^2\rangle & \langle u_1 u_2\rangle & 0 \\ \langle u_1 u_2\rangle & \langle u_2^2\rangle & 0 \\ 0 & 0 & \langle u_3^2\rangle \end{bmatrix}$$

⊙ Mean scalar eq.

$$\left\langle \frac{\partial \phi}{\partial t} + \nabla\cdot(\underline{U}\phi) = \Gamma \nabla^2 \phi \right\rangle, \quad \phi = \langle\phi\rangle + \phi'$$

$$\rightarrow \frac{\partial\langle\phi\rangle}{\partial t} + \underbrace{\frac{\partial}{\partial x_j}\langle U_j\rangle\langle\phi\rangle = \frac{\partial}{\partial x_j}\left(\Gamma \frac{\partial\langle\phi\rangle}{\partial x_j} - \langle u_j\phi'\rangle\right)}$$

again closure problem $\qquad$ scalar flux

⊙ Gradient-diffusion and turbulent-viscosity hypotheses

· $\langle \underline{u}\phi'\rangle = -\Gamma_T \nabla\langle\phi\rangle \qquad \Gamma_T$: turbulent diffusivity

gradient-diffusion hypothesis

$$\rightarrow \frac{\partial\langle\phi\rangle}{\partial t} + \frac{\partial}{\partial x_j}\langle U_j\rangle\langle\phi\rangle = \frac{\partial}{\partial x_j}\left(\Gamma_{eff}\frac{\partial\langle\phi\rangle}{\partial x_j}\right)$$

$\Gamma_{eff} = \Gamma + \Gamma_T(\underline{x}, t)$: effective diffusivity

fluid property $\qquad$ flow property

· $-\rho\langle u_i u_j\rangle + \frac{2}{3}\rho k\delta_{ij} = \rho\nu_T\left(\frac{\partial\langle U_j\rangle}{\partial x_i} + \frac{\partial\langle U_i\rangle}{\partial x_j}\right)$

turbulent-viscosity hypothesis $\qquad = 2\rho\nu_T\langle S_{ij}\rangle$

ν_T : turbulent (or eddy) viscosity

$\quad$ ~ ft$(\underline{x}, t)$: flow property

(Boussinesq 1877)

$$\rightarrow \frac{\partial\langle U_j\rangle}{\partial t} + \frac{\partial}{\partial x_i}\langle U_i\rangle\langle U_j\rangle = \frac{\partial}{\partial x_i}\left[\nu_{eff}\left(\frac{\partial\langle U_i\rangle}{\partial x_j} + \frac{\partial\langle U_j\rangle}{\partial x_i}\right)\right] - \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\langle p\rangle + \frac{2}{3}\rho k\right).$$

$\nu_{eff} = \nu + \nu_T(x, t)$: effective viscosity