NO. WRK DATE.

· Heat & mass transfer

$\sigma_t \simeq 0.9$ near-wall flows

0.5 plane jets and mixing layers

0.7 round jets.

major shortcoming of the mixing-length model :

$\nu_t$ and $\Gamma \simeq 0$ when $\partial \bar{U}_i / \partial x_j = 0$.

not true in reality $\Rightarrow$ $\nu_t \simeq 0.8\, \nu_{t\,max}$ at centerline of pipe channel.

For vel. it is ok $\because$ $\overline{uv} = 0$ there.

but for temp. $\Gamma = 0 \rightarrow$ unrealistic conseq.


there must be heat transport.

mixing-length model implies that turb. is in a state of local equil. (see §2.5)

$P \simeq \varepsilon$.

there can be no influence from other pts via transport of turb. quantities.

this inability to account for transport of turb. quantities is the main reason for the shortcomings of the mixing-length model.

channel flow : turb is produced near the wall and is transported to centerline by diffusion

mixing-length model neglects this diffusive transport and thus predicts zero turb. at centerline.

The model also neglects the conv. transport and thus predicts zero $\nu_t$ and $\Gamma$ in grid turb. and unrealistically low values in recirculating flows.

$\therefore$ The mixing-length model is <u>not</u> suitable when processes of convective or diffusive transport of turb. are important.

c) Prandtl's free-shear-layer model

$\nu_t = c\,\delta\,|U_{max} - U_{min}|$ : quite popular & works well.

$c$ ← empirical const.

| Flow | plane mix. layer | plane jet | Round jet | Radial jet | Plane wake |
|---|---|---|---|---|---|
| c | 0.01 | 0.014 | 0.011 | 0.019 | 0.026 |

2.5 One-equation models

- give up the direct link bet. the flu. vel. scale and the mean-vel. gradients
- determine this scale from a transport eq.

a) Models using the eddy-viscosity concept

vel. scale ? → k ! → $\sqrt{k}$

$\therefore \nu_t = c_\mu' \sqrt{k}\, L$ : Kolmogorov ('42) – Prandtl ('45) expression

$c_\mu'$ ← empirical const

determine k by solving a transport eq.

- $$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = -\frac{\partial}{\partial x_i}\left[\overline{u_i\left(\frac{1}{2}u_j u_j + \frac{p}{\rho}\right)}\right] - \overline{u_i u_j}\frac{\partial U_i}{\partial x_j} - \beta g_i \overline{u_i \varphi} - \nu \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}$$

conv. transport, diff. transport, P production, G buoyant prod./destruction, $\varepsilon$

- Modelled form of the k-eq.

$$-\overline{u_i\left(\frac{1}{2}u_j u_j + \frac{p}{\rho}\right)} = \frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}$$

$\sigma_k$ ← emp. const.

$$\varepsilon = c_D \frac{k^{\frac{3}{2}}}{L}$$ from dimensional argument

$c_D$ ← emp. const.

determined by large scales, large scale

i.e. $\varepsilon = f(k, L)$

$$-\overline{u_i\varphi} = \Gamma \frac{\partial \phi}{\partial x_i} = \frac{\nu_t}{\sigma_t}\frac{\partial \phi}{\partial x_i}, \quad -\overline{u_i u_j} = \nu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}k\delta_{ij}$$

$$\Rightarrow \frac{\partial k}{\partial t} + U_j\frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + \nu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} + \beta g_i \frac{\nu_t}{\sigma_t}\frac{\partial \phi}{\partial x_i} - c_D\frac{k^{\frac{3}{2}}}{L}.$$

high Re # form.

$c_\mu' C_D \simeq 0.08$ , $\sigma_k \simeq 1$

the model is not applicable to the viscous sublayer.

when ~~the rate~~ $\frac{\partial k}{\partial t} \simeq 0$, conv. & diffusive transports are negligible,

$P \simeq \epsilon$. For non-buoyant shear layers,

state of local equil. $\nu_t \left(\frac{\partial U}{\partial y}\right)^2 = C_D \frac{k^{\frac{3}{2}}}{L}$

$\Rightarrow \nu_t = c_\mu' \sqrt{k} L$ $\quad \nu_t = \underbrace{\left(\frac{c_\mu'^3}{C_D}\right)^{\frac{1}{2}} L^2}_{= l_m^2} \left|\frac{\partial U}{\partial y}\right|$ : mixing-length formula

mixing-length model is suitable only for flows where turb. is in local equil.

- Length-scale determination

  L needs to be specified to complete turb. model.

  In most models, L is determined from simple emp. relations similar to those given for lm.

  L is no easier to prescribe than lm!

b) Bradshaw et al. ('67, '73)'s model

- does not employ the eddy-viscosity concept.
- they solve a transport eq. for $\overline{uv}$
- Original model for wall bdry layers

$$\frac{\overline{uv}}{k} = a_1 \simeq \text{const} = 0.3 \rightarrow k = \frac{\overline{uv}}{a_1}$$

$\frac{-\overline{uv}}{u_{rms} v_{rms}} \simeq 0.4$

k-eq $\Rightarrow$ $\overline{uv}$ eq.:

$$U \frac{\partial(\overline{uv}/a_1)}{\partial x} + V \frac{\partial(\overline{uv}/a_1)}{\partial y} = -\frac{\partial}{\partial y}\left[G \overline{uv} (\overline{uv}_{max})^{\frac{1}{2}}\right] - \overline{uv}\frac{\partial U}{\partial y} - \frac{\overline{uv}^{\frac{3}{2}}}{L}$$

$$G = \left(\frac{\overline{uv}_{max}}{u_\infty^2}\right)^{\frac{1}{2}} f_1\left(\frac{y}{\delta}\right), \quad \frac{L}{\delta} = f_2\left(\frac{y}{\delta}\right).$$

↑ emp. ft.

Bradshaw's model was applied with success in many wall-bdry-layer calculations.

In shear flows, } $k > 0$ but $\overline{uv}$ changes sign ⇐ problem!

Some one : $|\overline{uv}| \propto k$

or $\frac{\overline{uv}}{k} = f\left(\int_0^x \frac{\partial U}{\partial y} \frac{dx}{U}\right)$

c) Heat & mass transfer calculations and buoyancy effects

d) Assessment

- One-eq models account for conv. & diff. transport of turb. vel. scale and therefore are superior to the mixing-length model, when this transport is important : e.g. non-equil bdry layers w/ rapidly changing free-stream conditions, bdry layers w/ freestream turb, and heat transfer across planes w/ $\partial U/\partial y = 0$ and in recirculating flows.
- the eddy-viscosity models are more generally applicable than Bradshaw's model.
- Application of 1-eq. modes was restricted mainly to shear layes.

→ move on to 2-eq models which determine the length scale from a transport eq.

## 2.6 Two-Equation Models

equation for L is needed.

a) Length scale eqs.

Dependent variable used for L in the literature

- $\varepsilon \propto k^{\frac{3}{2}}/L$
- $kL$
- $k^{\frac{1}{2}}/L$
- $k/L^2$

eqs for these variables are very similar

prod. of k

$$\underbrace{\frac{\partial Z}{\partial t} + U_j \frac{\partial Z}{\partial x_j}}_{\text{conv.}} = \underbrace{\frac{\partial}{\partial x_j}\left(\frac{\sqrt{k}L}{\sigma_Z}\frac{\partial Z}{\partial x_j}\right)}_{\text{diffusion}} + \underbrace{c_{Z1}\frac{Z}{k}P}_{\text{prod.}} - \underbrace{c_{Z2} Z \frac{\sqrt{k}}{L}}_{\text{destruction}} + S$$

differs depending on the choice of Z. important near the wall

Near the wall, the gradient assumption for diffusion works better for $Z = \varepsilon$.

Also, $\varepsilon$ - eq. does not require $S$.

$\Rightarrow$ $\varepsilon$-eq. has become popular.

b) k-$\varepsilon$ eq. model

- $\varepsilon$ - eq. — $\varepsilon = \nu \overline{(\partial u_i/\partial x_j)^2}$

eq. and derivation are very complicated.

we use grad. assump.

$$\nu_t = c_\mu \frac{k^2}{\varepsilon}, \qquad \Gamma = \frac{\nu_t}{\sigma_t}$$

$$\frac{\partial k}{\partial t} + U_j\frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right) + \overbrace{\nu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}}^{P} + \overbrace{\beta g_j \frac{\nu_t}{\sigma_t}\frac{\partial \Phi}{\partial x_j}}^{G} - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + U_j\frac{\partial \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right) + c_{1\varepsilon}\frac{\varepsilon}{k}(P+G)(1+c_{3\varepsilon}R_f) - c_{2\varepsilon}\frac{\varepsilon^2}{k}$$

buoyancy correction

$c_\mu = 0.09$, $c_{1\varepsilon} = 1.44$, $c_{2\varepsilon} = 1.92$

$\sigma_k = 1.0$, $\sigma_\varepsilon = 1.3$

$\Leftarrow$ determined from extensive examination of free shear flows.

~~Constants in wall flows~~?

Constants $C_{1\epsilon}$ and $C_{2\epsilon}$ affect sols.

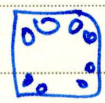
there have been many modifications in the constants.

c) non-isotropic eddy viscosity

Standard k-ε model : eddy viscosity is the same for all the Reynolds stresses

i.e. isotropic eddy viscosity

this is not valid for square duct flow



secondary motion is not produced by standard k-ε model.

cure? non-isotropic $\nu_t$ → algebraic stress model § 2.7

d) Heat/mass transfer and buoyancy

e) k-ε model for depth-average calculations

f) Assessment

Two-eq. models are the simplest models that promise success for flows for which the length-scale cannot be prescribed empirically in an easy way.

most ~~widley~~ widely tested and successfully applied.

2.7 Turbulent stress / Flux-equation models.

So far, local state of turb ~~can~~ is assumed to be characterized by one-vel. scale and the individual Reynolds stresses are related to this scale. ∴ transport of the individual stresses is <u>not</u> adequately accounted. ⇒ transport eqs. for $\overline{u_iu_j}$.

a) Reynolds-stress eqs.

$$\frac{\partial \overline{u_i u_j}}{\partial t} + \overbrace{U_\ell \frac{\partial \overline{u_i u_j}}{\partial x_\ell}}^{\text{conv.}} = \overbrace{-\frac{\partial}{\partial x_\ell}\left(\overline{u_\ell u_i u_j}\right) - \frac{1}{\rho}\left(\frac{\partial \overline{u_j p}}{\partial x_i} + \frac{\partial \overline{u_i p}}{\partial x_j}\right)}^{\text{diffusive transport}} \underbrace{- \overline{u_i u_\ell}\frac{\partial U_j}{\partial x_\ell} - \overline{u_j u_\ell}\frac{\partial U_i}{\partial x_\ell}}_{P_{ij} = \text{stress production}}$$

$$\underbrace{-\beta\left(g_i \overline{u_j\varphi} + g_j\overline{u_i\varphi}\right)}_{\text{buoyancy prod.}} + \underbrace{\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}}_{\Pi_{ij}\text{ : press-strain corr.}} \underbrace{- 2\nu\, \overline{\frac{\partial u_i}{\partial x_\ell}\frac{\partial u_j}{\partial x_\ell}}}_{\varepsilon_{ij}\text{ : vis. diss.}}$$

- At high Re, turb is locally isotropic.

  $\overline{\frac{\partial u_i}{\partial x_\ell}\frac{\partial u_j}{\partial x_\ell}} = 0$ for $i \neq j$ when local isotropy prevails.

  $\Rightarrow \varepsilon_{ij} = \frac{2}{3}\varepsilon\,\delta_{ij}$

- press-strain corr. modelling : eliminate $p$ via a Poisson eq.

  $\Pi_{ij,1} = -c_1 \frac{\varepsilon}{k}\left(\overline{u_i u_j} - \frac{2}{3}\delta_{ij}k\right)$ : interaction of flu. velocities

  $\Pi_{ij,2} = -\frac{c_2+8}{11}\underbrace{\left(P_{ij} - \frac{2}{3}\delta_{ij}P\right)}_{\text{dominant}} - \frac{30c_2-2}{55}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)k$ : interaction of mean strain & flu. vel.

  $\qquad - \frac{8c_2-2}{11}\left(D_{ij} - \frac{2}{3}\delta_{ij}P\right), \quad D_{ij} = -\left(\overline{u_i u_\ell}\frac{\partial U_\ell}{\partial x_j} + \overline{u_j u_\ell}\frac{\partial U_\ell}{\partial x_i}\right)$

  $\Rightarrow \Pi_{ij,2} = -\gamma\left(P_{ij} - \frac{2}{3}\delta_{ij}P\right)$

- diffusive transport $= c_s \frac{\partial}{\partial x_\ell}\left(\frac{k}{\varepsilon}\overline{u_k u_\ell}\frac{\partial \overline{u_i u_j}}{\partial x_k}\right)$

- $\varepsilon$ ? $\rightarrow$ $\varepsilon$-eq.

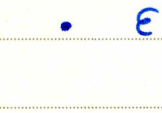
  diffusion term in $\varepsilon$-eq $\Rightarrow$ $c_\varepsilon \frac{\partial}{\partial x_\ell}\left(\frac{k}{\varepsilon}\overline{u_\ell u_k}\frac{\partial \varepsilon}{\partial x_k}\right)$.

- Wall & free-surface effect

  wall-normal stress is significantly damped by the wall.

  so, correction is needed.

  $\Pi'_{ij,1} = c'_1\frac{\varepsilon}{k}\left(\overline{u_n^2}\,\delta_{ij} - \frac{3}{2}\overline{u_n u_i}\,\delta_{nj} - \frac{3}{2}\overline{u_n u_j}\,\delta_{ni}\right) f\left(\frac{L}{x_n}\right)$




  $\Pi'_{ij,2} = c'_2\left(\Pi_{nn,2}\,\delta_{ij} - \frac{3}{2}\Pi_{ni,2}\,\delta_{nj} - \frac{3}{2}\Pi_{nj,2}\,\delta_{ni}\right) f\left(\frac{L}{x_n}\right)$

  $f = k^{\frac{3}{2}}/(x_n\varepsilon)$.

  empirical constants $\rightarrow$ Table 4

- Launder-Reece-Rodi model ~~ok~~ good ( wall jets, turb-driven secondary flow, extra strain rates caused by wall curvature

  bad - round jet, swirl, wake ...

Same type of modelling to $\overline{u_j\phi}$.

c) Algebraic stress / flux models.

$\overline{u_i u_j} \to 6$ pdes $\to$ too much.

neglect the rate of change and transport terms

$\to$ sufficiently accurate approx. in many cases

Rodi. . . . $\to$ $\overline{u_i u_j} = k\left[\frac{2}{3}\delta_{ij} + \frac{(1-\gamma)\left(\frac{P_{ij}}{\epsilon} - \frac{2}{3}\delta_{ij}\frac{P}{\epsilon}\right) + (1-C_3)\left(\frac{G_{ij}}{\epsilon} - \frac{2}{3}\delta_{ij}\frac{G}{\epsilon}\right)}{c_1 + \frac{P+G}{\epsilon} - 1}\right]$

use this eq. together with k-ε eqs.

For non-buoyant thin shear layers, this model ~~is basically~~ turns to an eddy-viscosity relationship.

$$-\overline{uv} = \underbrace{\frac{2}{3}\,\frac{1-\gamma}{c_1}\,\frac{c_1 - 1 + \gamma P/\epsilon}{(c_1 - 1 + P/\epsilon)^2}}_{c_\mu \,=\, \text{fn of } P/\epsilon \text{ not const.!}}\,\frac{k^2}{\epsilon}\,\frac{\partial U}{\partial y}$$

d) Assessment

- Stress/flux-eq models have great potential and are the only promising models when transport becomes important.
- Computationally expensive
- ε-eq. has problems. $\to$ using RSM does not cure.

2.8 Boundary conditions

- Wall boundaries

at wall, $\overline{u_i u_j} = 0$ but computationally not economical

also not good for high Re # k-ε model.

resultant vel. ← parallel to the wall

$\therefore \quad \frac{U_{res}}{U_\tau} = \frac{1}{\kappa}\ln(y^+ E)$ for $30 < y^+ < 100$

E : roughness parameter

E = 9 for smooth wall

At $30 < y^+ < 100$, Rey. stresses are nearly const.

↳ local equil. $P = \varepsilon$.

$\nu_t = C_\mu \frac{k^2}{\varepsilon}$ → $\frac{k}{u_\tau^2} = \frac{1}{\sqrt{C_\mu}}$ : bdry cond. for k.

$\varepsilon = P = u_\tau^2 \frac{\partial U}{\partial y}$ & $\frac{U_{res}}{u_\tau} = \frac{1}{\kappa} \ln(y^+ E)$

→ $\varepsilon = \frac{u_\tau^3}{\kappa y}$ : bdry cond. for $\varepsilon$

- free boundary : $\overline{u_i u_j} = 0$ , $\varepsilon = 0$ ...
- Symmetry cond.

$u_\tau = \sqrt{\frac{\tau_w}{\rho}}$

$= \sqrt{\frac{\mu}{\rho} \frac{\partial U}{\partial y}}$

$\frac{\partial U}{\partial y} = \frac{u_\tau^2}{\nu}$

$P = \nu_t \left(\frac{\partial U}{\partial y}\right)^2 = \varepsilon$

$-\overline{uv} \frac{\partial U}{\partial y} = \nu_t \left(\frac{\partial U}{\partial y}\right)^2 \rightarrow \nu_t = \frac{u_\tau^2}{\partial U / \partial y}$

$= u_\tau^2 \frac{\partial U}{\partial y}$

$\frac{1}{u_\tau} \frac{\partial U}{\partial y} = \frac{1}{\kappa y}$

$\rightarrow \frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y}$

$\varepsilon = u_\tau^2 \frac{\partial U}{\partial y} = u_\tau^3 / \kappa y$

$\varepsilon = \frac{C_\mu k^2}{\nu_t}$

$u_\tau^2 \frac{\partial U}{\partial y} = \frac{C_\mu k^2}{\nu_t}$

$\cancel{u_\tau^2 \frac{\partial U}{\partial y} \cdot \frac{u_\tau^2}{\partial U / \partial y} = C_\mu k^2}$

$\frac{k}{u_\tau^2} = \frac{1}{\sqrt{C_\mu}}$

NO.

- Low Reynolds number $k-\epsilon$ model
  by Launder - Sharma (Letters in Heat & Mass Transfer Vol. 1, 1974, 131-138)

- $k-\omega$ shear stress transport (SST) model
  by Menter (AIAA J. Vol. 32, 1994, 1598-1605)

- $k-\epsilon-v^2-f$ model (V2F)
  by Durbin (theoret. Comput. Fluid Dynamics Vol. 3, 1991, 1-13)

$$\rho u_j \frac{\partial k}{\partial x_j} = P - \epsilon + \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]$$

$$\rho u_j \frac{\partial \epsilon}{\partial x_j} = (C_{\epsilon_1} P - C_{\epsilon_2}\epsilon)/T + \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_\epsilon}\right)\frac{\partial \epsilon}{\partial x_j}\right]$$

$$\rho u_j \frac{\partial \overline{v^2}}{\partial x_j} = kf - \overline{v^2}\frac{\epsilon}{k} + \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial \overline{v^2}}{\partial x_j}\right]$$

$$L^2 \nabla^2 f - f = (1 - C_1)\left[\frac{2}{3} - \frac{\overline{v^2}}{k}\right]/T - C_2 \frac{P}{k}$$

$$P = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_j}{\partial x_i}$$

$$\mu_t = \rho \nu_t \,, \quad \nu_t = C_\mu \overline{v^2}\, T$$

$$\mu = \rho \nu$$

$$L = C_L\, l \,, \quad l^2 = \max\left[\frac{k^3}{\epsilon^2}, C_\eta^2 \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{2}}\right]$$

$$T = \max\left[\frac{k}{\epsilon}, 6\left(\frac{\nu}{\epsilon}\right)^{\frac{1}{2}}\right]$$

- Spalart - Allmaras model (1994) - 1-eq. model
  Recherche Aérospatiale Vol. 1, pp 5-21.
  quite successful for aerodynamic flows.

$$\frac{D\nu_T}{Dt} = \nabla \cdot \left(\frac{\nu_T}{\sigma_T}\nabla \nu_T\right) + S_\nu .$$

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