Week 5, 29 March

Mechanics in Energy Resources Engineering - Ch.3 Torsion (2)

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1st exam

- 31 March   09:00 – 10:45
- If you can solve the home assignment with confidence, you will do a good job.
- More than 50% from the home assignments.
- ~90% from the examples and the problems from the textbook.
- Try to interpret the problem in terms of physical behaviour. You will be required to explain your answer physically.

Q & A Session

- Date and Time: 29 March 16:00 – 19:00 (?)
- Location: Seok Jeong Seminar Room (38-118)
- Teaching Assistant will be available for discussion.

Chapter 3 Torsion (비틀림)

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- Introduction
- Torsional Deformations of a circular bar (원형봉의 비틀림 변형)
- Circular bars of linearly elastic materials (선형탄성 원형봉)
- Nonuniform torsion (불균질 비틀림)
- Stresses and Strains in Pure Shear (순수전단에서의 응력과 변형율)
- Relationship Between Moduli of Elasticity E and G (탄성계수 E와 G의 관계)
- Transmission of Power by Circular Shafts (원형축에 의한 동력전달)
- Statically Indeterminate Torsional Members (부정정 비틀림 부재)
- Strain Energy in Torsion and Pure Shear (비틀림과 순수전단에서의 변형에너지)

Preview

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- Introduction
- Changes in Lengths of Axially Loaded Members (축하중을 받는 부재의 길이변화)
- Changes in Lengths Under Nonuniform Conditions (균일봉 길이변화)
- Statically Indeterminate Structures (부정정 구조물)
- Thermal Effects, Misfits, and Prestrains (열효과, 어긋남 및 사전변형)
- Stresses on Inclined Sections (경사면에서의 응력)
- Strain Energy (변형율 에너지)
- Impact Loading (충격하중)
- Stress Concentrations* (응력집중)

- Introduction
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Introduction

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- Torsion (비틀림):
  - Twisting of a straight bar when it is loaded by moments (or torques) that tend to produce rotation about the longitudinal axis of the bar

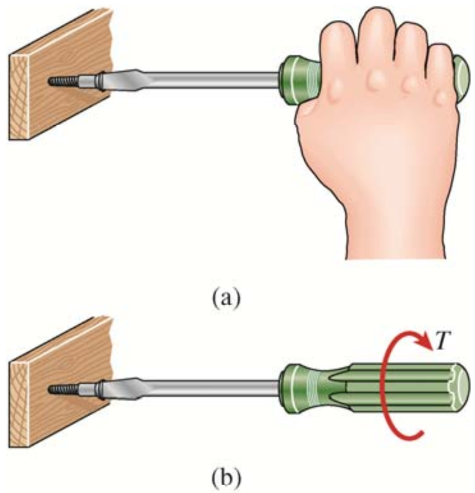

FIG. 3-1 Torsion of a screwdriver due to a torque T applied to the handle

Introduction

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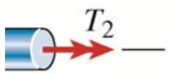
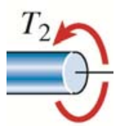
- Circular bar subjected to torsion.
  - Each pair of forces forms a couple (consists of two parallel forces that are equal in magnitude and opposite in direction).
  - Moments that produce twisting of a bar (T_1 and T_2) → torques or twisting moments

$$T_1 = P_1 d_1 \qquad T_2 = P_2 d_2$$

  - Representation:  or 

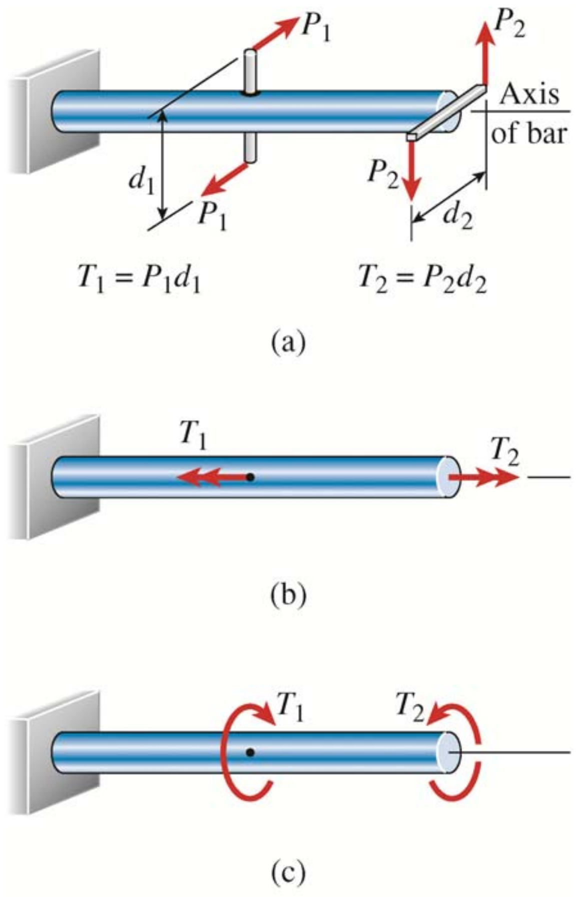
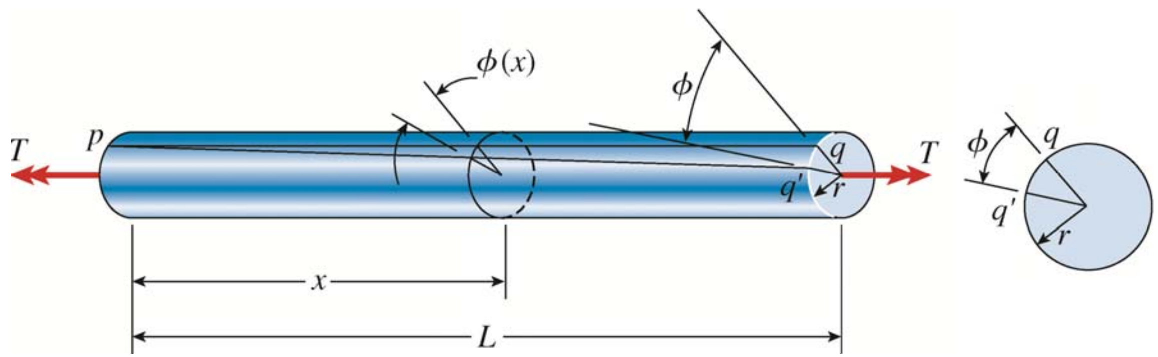


FIG. 3-2 Circular bar subjected to torsion by torques T_1 and T_2

Torsional Deformations of a Circular Bar

- Pure torsion
  - same internal torque T in every cross section
  - Cross section do not change in shape → all cross sections remain plane and circular and all radii remain straight. Why?



Torsional Deformations of a Circular Bar

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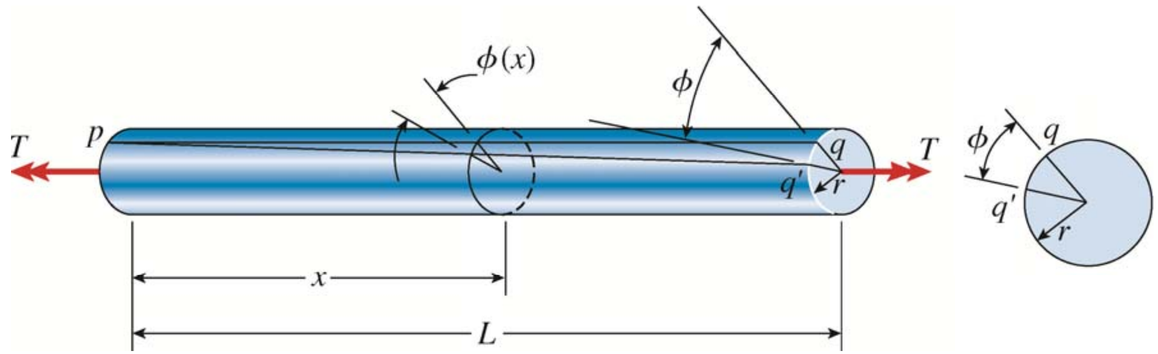
- Angle of Twist (Angle of rotation, 비틂각, Φ)
  - A small angle which the right-hand end will rotate through
  - The angle of twist changes along the axis of the bar. Φ(x) at intermediate cross sections . Why?



Torsional Deformations of a Circular Bar
Shear strain at outer surface

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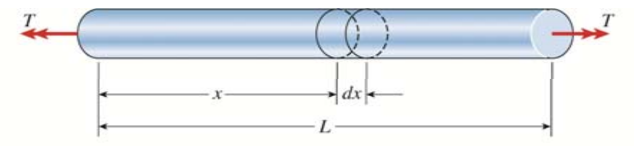
- An element of a bar between two cross sections
  - Magnitude of shear strain at outer surface

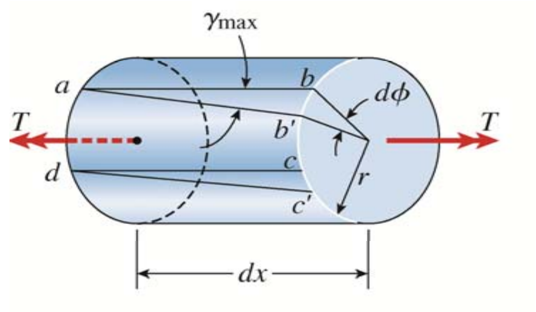
$$\gamma_{\max} = \frac{bb'}{ab} = \frac{rd\phi}{dx}$$



- Rate of twist, θ (비틀림 변화율, angle of twist per unit length)
  - Rate of change of Φ with respect to the distance x measured from the axis of the bar

$$\theta = \frac{d\phi}{dx}$$

  - Shear strain at outer surface



$$\gamma_{\max} = \frac{rd\phi}{dx} = r\theta$$

$$\gamma_{\max} = \frac{r\phi}{L}$$

Pure torsion

Torsional Deformations of a Circular Bar
Shear strain within the bar / Circular tube

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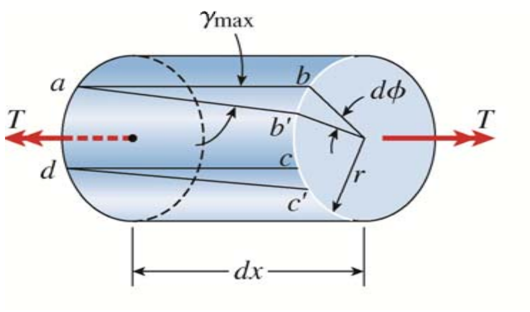
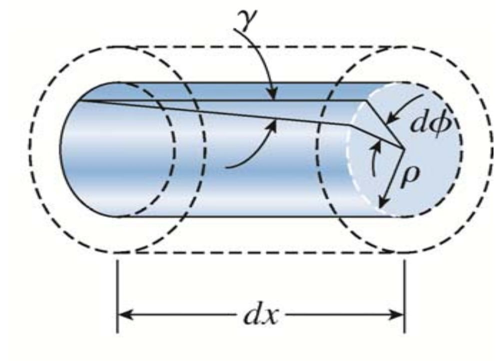


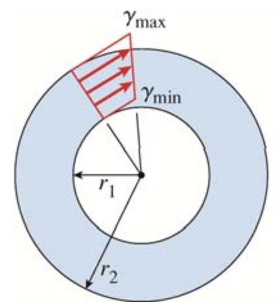


- Shear strain *within* the interior can be found similarly.
  - γ: shear strain within the bar, ρ: radius of interior

$$\gamma_{\max} = \frac{r d\phi}{dx} = r\theta \longrightarrow \boxed{\gamma = \rho\theta = \frac{\rho}{r}\gamma_{\max}}$$

- Circular tubes

$$\gamma_{\max} = \frac{r_2\phi}{L} \qquad \gamma_{\min} = \frac{r_1}{r_2}\gamma_{\max} = \frac{r_1\phi}{L}$$



Circular Bars of Linearly Elastic Materials
Shear Stress

- Corresponding shear stress?
  - Direction of shear stress
  - Magnitude of shear stress

$$\tau = G\gamma$$

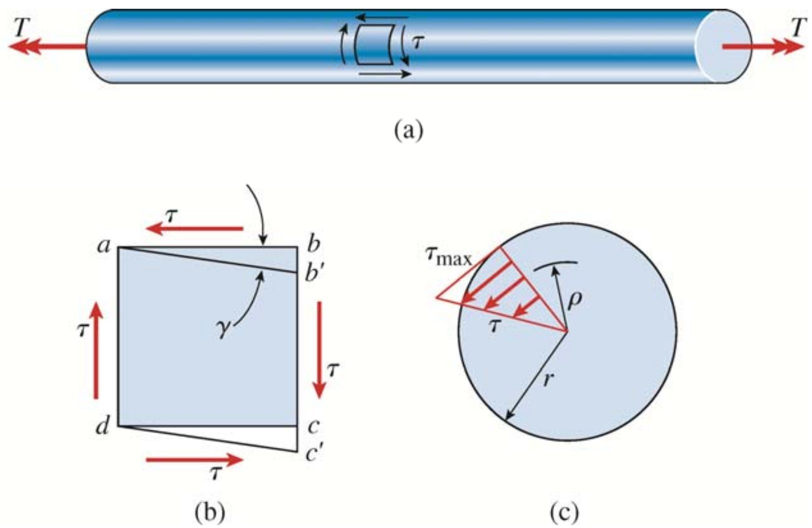
$$\tau_{\max} = Gr\theta$$

Maximum shear stress at the outer surface of the bar

$$\tau = G\rho\theta = \frac{\rho}{r}\tau_{\max}$$

shear stress at an interior point

  - Shear stress vary linearly with r.



Circular Bars of Linearly Elastic Materials
Shear Stress

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- τ acting on a cross-sectional plane are accompanied by τ of the same magnitude on longitudinal planes ← equal τ always exist at mutually perpendicular planes.
- Stresses on an element oriented at an angle of 45°

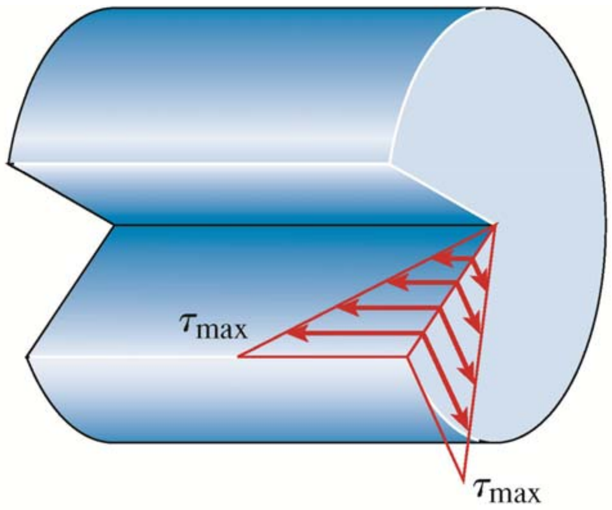

FIG. 3-7 Longitudinal and transverse shear stresses in a circular bar subjected to torsion

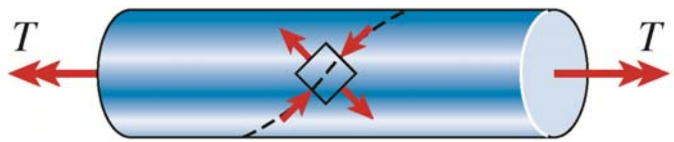

FIG. 3-8 Tensile and compressive stresses acting on a stress element oriented at 45° to the longitudinal axis

Circular Bars of Linearly Elastic Materials
Torsional Formula

- <u>Relationship between shear stress & Torque (T)</u>
- In a small element,
  - Shear stress resultant = Applied torque

$$dM = \tau \rho dA = \frac{\tau_{\max}}{r} \rho^2 dA$$

$$T = \int_{dA} dM = \frac{\tau_{\max}}{r} \int_{dA} \rho^2 dA = \frac{\tau_{\max}}{r} I_p$$

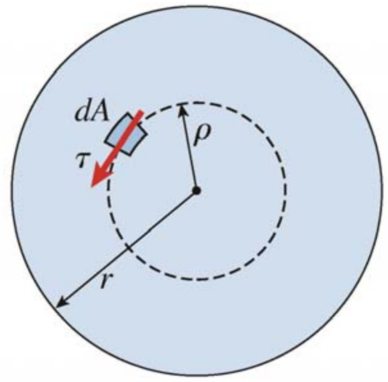

FIG. 3-9 Determination of the resultant of the shear stresses acting on a cross section

  - Polar moment of inertia of the circular cross section

$$I_p = \int_A \rho^2 dA$$

  - For a circle of rardius r and diameter d,

$$I_p = \frac{\pi r^4}{2} = \frac{\pi d^4}{32}$$

Circular Bars of Linearly Elastic Materials
Torsional Formula

- Torsional Formula (비틀림 공식)

$$\tau_{\max} = \frac{Tr}{I_p}$$

  - Maximum shear stress is proportional to the applied torque T and inversely proportional to the polar moment of inertia I_p.
  - For a solid circular cross section with diameter d,

$$\tau_{\max} = \frac{16T}{\pi d^3}$$

when diameter (d) is doubled → shear stress?

  - Generalized torsional formula at distance ρ from the center

$$\tau = \frac{\rho}{r}\tau_{\max} = \frac{T\rho}{I_p}$$

Circular Bars of Linearly Elastic Materials
Angle of twist

- Rate of twist

$$\tau = \frac{T\rho}{I_p}$$

$$\tau = G\rho\theta$$

$$\longrightarrow \quad \theta = \frac{T}{GI_p}$$

GI_p: torsional rigidity

- Total Twist (in pure torsion)

$$\phi = \theta L \quad \longrightarrow \quad \phi = \frac{TL}{GI_p}$$

GI_p/L: torsional stiffness

  – Compare with $\delta = \dfrac{PL}{EA}$

Circular Bars of Linearly Elastic Materials
Circular tubes

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- Circular tubes are more *efficient* than solid bars in resisting torsional loads.
  - Maximum shear stress at the outer boundary
  - Most of the material in solid bar is stressed significantly below the maximum shear stress
  - Polar moment of inertia of the cross-sectional area of a tubge

$$I_p = \frac{\pi}{2}\left(r_2^4 - r_1^4\right) = \frac{\pi}{32}\left(d_2^4 - d_1^4\right)$$

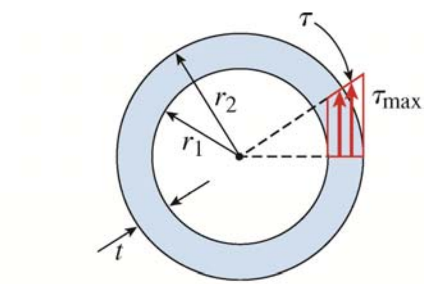

FIG. 3-10 Circular tube in torsion

Circular Bars of Linearly Elastic Materials
limitations

- Linearly elastic material
- Away from stress concentration
- Equations for circular bars and tubes cannot be used for bars of other shapes.
  - Ex) rectangular bars or I shaped cross sections
  - More advanced method needed for other shapes

Example 3-1

- G=80 GPa, T=340 Nm
  - Maximum shear stress? Angle of twist between the ends?
  - If allowable shear stress is 42 MPa, allowable angle of twist is 2.5 °, what is the maximum permissible torque?

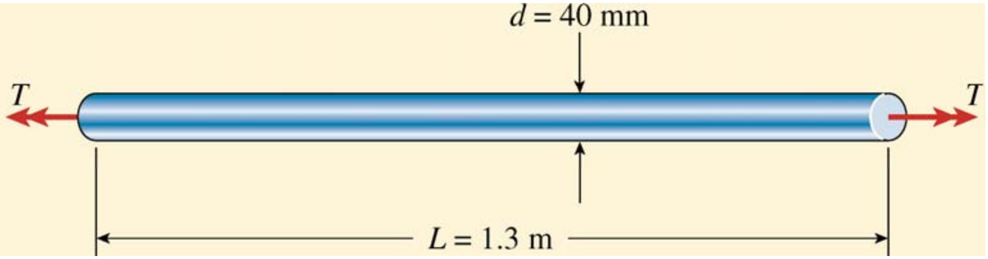

FIG. 3-11 Example 3-1. Bar in pure torsion

Nonuniform Torsion

- Nonuniform Torsion
  - Bar need not be prismatic
  - Applied torques may vary along the axis
- Approach
  - Divide into the segment with constant torque
  - Find torque in each segment by FBD

$$T_{AB} = -T_1 \qquad T_{BC} = -T_1 - T_2 \qquad T_{CD} = -T_1 - T_2 + T_3$$

  - Make a summation

$$\boxed{\phi = \sum_{i=1}^{n} \phi_i = \sum_{i=1}^{n} \frac{T_i L_i}{G_i \left(I_p\right)_i}} \xleftarrow{\text{analogy}} \delta = \sum_{i=1}^{n} \frac{N_i L_i}{E_i A_i}$$

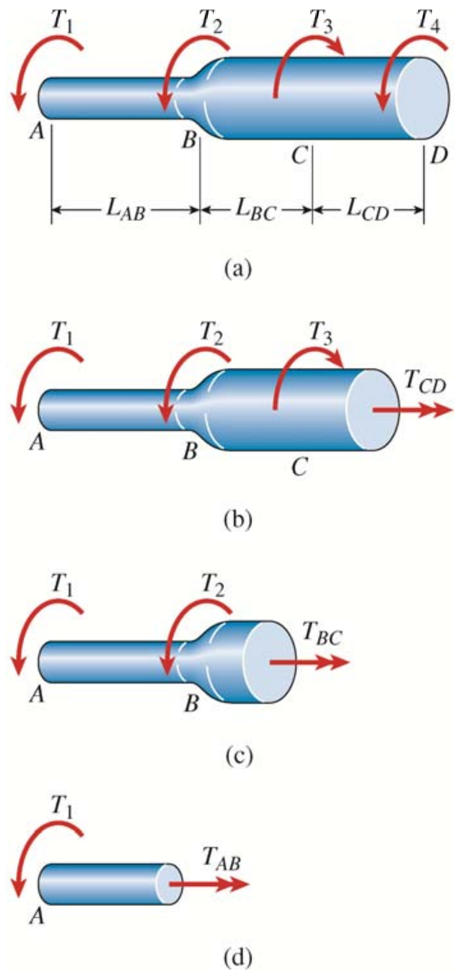

FIG. 3-14 Bar in nonuniform torsion (Case 1)

Nonuniform Torsion

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- More general case with continuously varying cross sections and torques

$$\phi = \int_0^L d\phi = \int_0^L \frac{T(x)dx}{GI_p(x)}$$

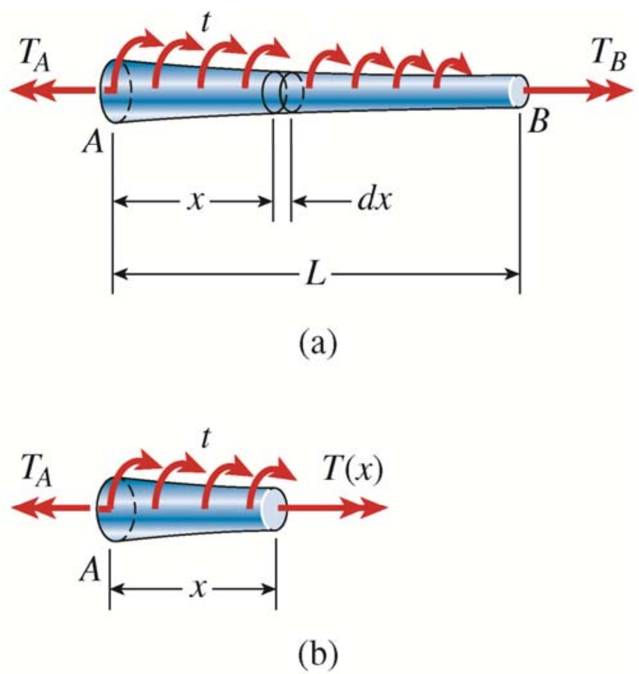

FIG. 3-16 Bar in nonuniform torsion (Case 3)

Q & A

- Moment of Inertia (or rotational Inertia) ← as defined in 'Physics'
  - a measure of the resistance of an object to a *change* in its *rotational* motion)

$$I = \sum m_i r_i^2 \qquad I = \int \rho^2 dm$$

- Polar moment of inertia of plane area

$$I_p = \int \rho^2 dA$$

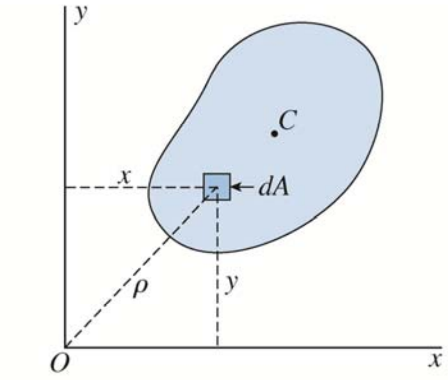

FIG. 12-17 Plane area of arbitrary shape

Stresses and Strains in Pure Shear
Stresses and strains during twisting?

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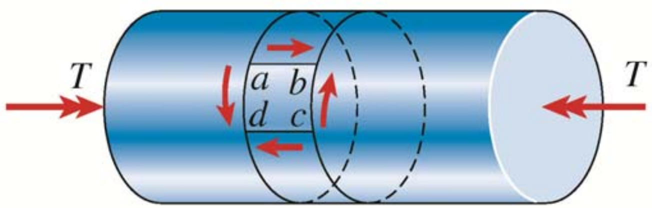
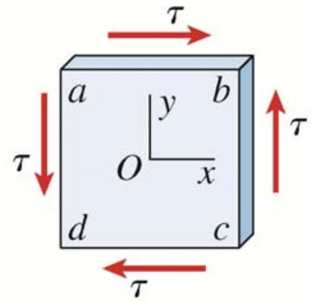
- A bar in torsion
  - The only stresses acting on the stress element abcd is shear stresses on four side faces → Pure Shear





- Sign convention of shear
  - (+) face of an element :
    - (+) if it acts in the (+) direction of one of the coordinate axes,
    - (-) if it acts in the (-) direction of one of the coordinate axes,
  - (-) face of an element
    - (+) → if it acts in the (-) direction of one of the coordinate axes.
    - (-) → if it acts in the (+) direction of one of the coordinate axes.

All four are (+)

Stresses and Strains in Pure Shear
stresses on inclined planes

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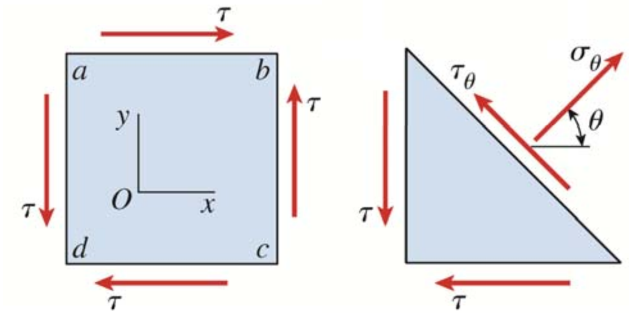
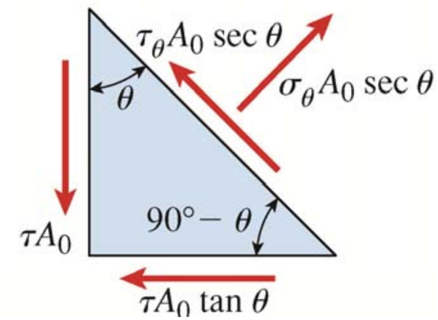
- Strategy?
  - We follow the same techniques used for stresses on inclined sections

$$\sum F_{\sigma_\theta\, direction} = 0 \qquad \sum F_{\tau_\theta\, direction} = 0$$



  - Sign convention:
    - σ_θ: (+) for tension
    - τ_θ: (+) for counterclockwise rotation



Free Body Diagram

Stresses and Strains in Pure Shear
stresses on inclined planes

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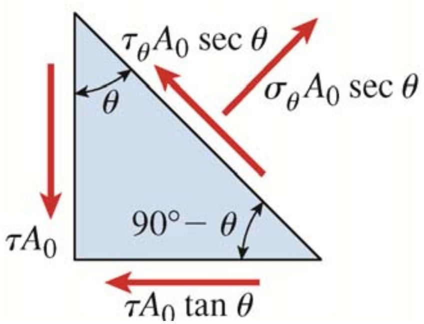
- Force Equilibrium on FBD

$$\sum F_{\sigma_\theta\, direction} = 0 \longrightarrow \sigma_\theta A_0 \sec\theta = \tau A_0 \sin\theta + \tau A_0 \tan\theta \cos\theta$$

$$\sigma_\theta = 2\tau \sin\theta \cos\theta \longrightarrow \boxed{\sigma_\theta = \tau \sin 2\theta}$$

$$\sum F_{\tau_\theta\, direction} = 0 \longrightarrow \tau_\theta A_0 \sec\theta = \tau A_0 \cos\theta - \tau A_0 \tan\theta \sin\theta$$

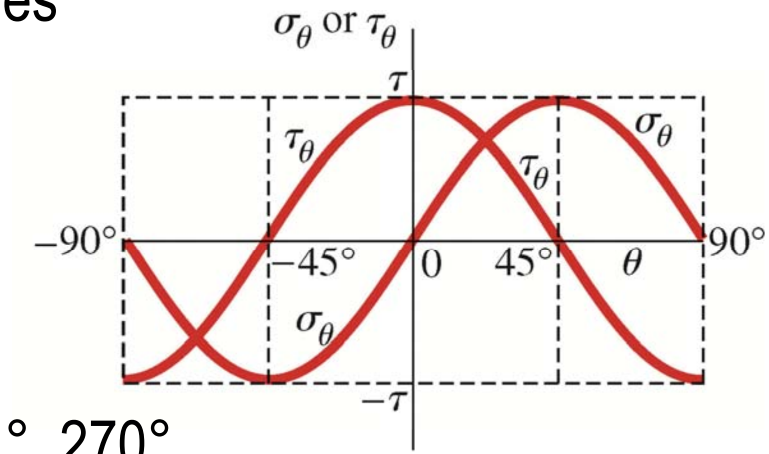
$$\tau_\theta = \tau\left(\cos^2\theta - \sin^2\theta\right) \longrightarrow \boxed{\tau_\theta = \tau \cos 2\theta}$$



Stresses and Strains in Pure Shear
stresses on inclined planes

– Normal and shear stresses acting on any inclined plane in terms of shear stress τ acting on x and y planes

$$\sigma_\theta = \tau \sin 2\theta \qquad \tau_\theta = \tau \cos 2\theta$$



– Top face of the element (θ = 90),

$$\sigma_\theta = 0 \qquad \tau_\theta = -\tau$$

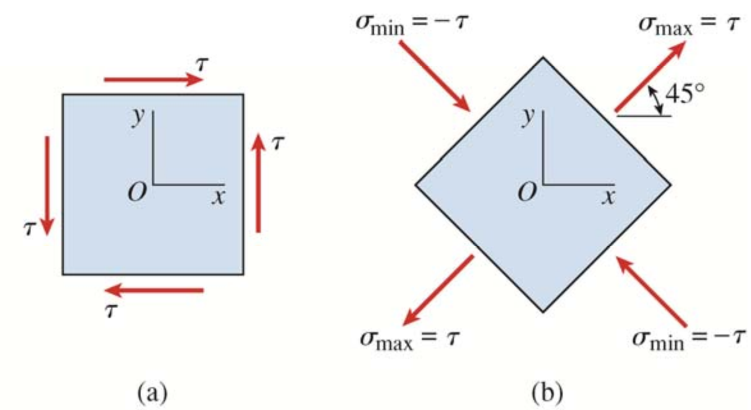
– τ: absolute maximum: 0 °, 90 °, 180 °, 270°

– σ: maximum at 45 °

$$\sigma_\theta = \tau \qquad \tau = 0$$

– σ: minimum at -45 °

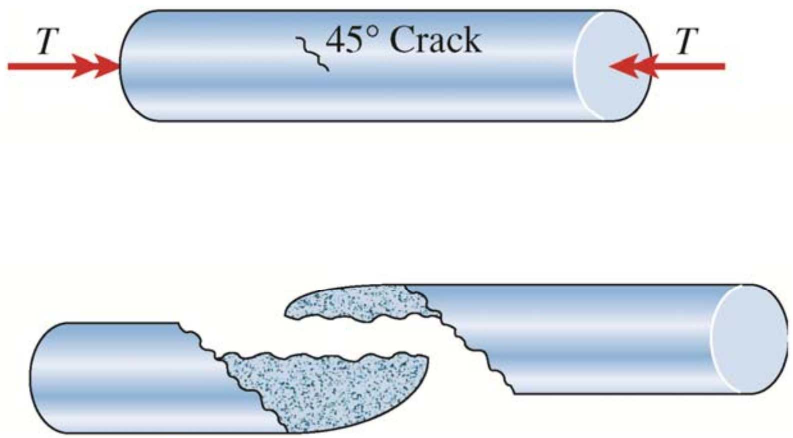
$$\sigma_\theta = -\tau \qquad \tau = 0$$



(a) (b)

Stresses and Strains in Pure Shear
stresses on inclined planes

– An experiment



– Why?

Stresses and Strains in Pure Shear
strains in pure shear

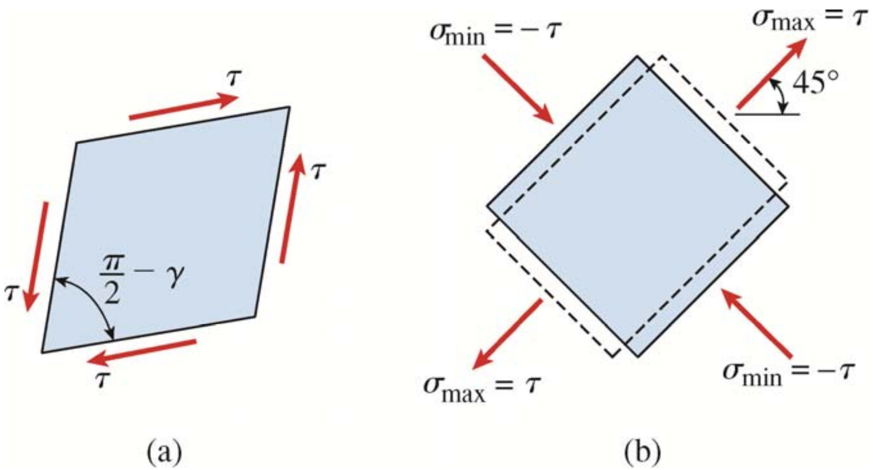
- Shear Strains for the element oriented at θ = 0 °

$$\gamma = \frac{\tau}{G}$$

- Normal strain for the element oriented at θ = 45°

$$\varepsilon_{max} = \frac{\tau}{E} + \nu\frac{\tau}{E}$$

$$\varepsilon_{45} = \frac{\sigma_{max}}{E}$$



Produce positive strain at 45 ° direction

$$\varepsilon_{45} = -\nu\frac{\sigma_{min}}{E}$$

Also produce positive strain at 45 ° direction

Example 3-6

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- Maximum shear, tensile and compressive stresses? Show these on sketches of properly oriented stress element.
- Maximum shear and normal strains? Show these on sketches of the deformed elements.

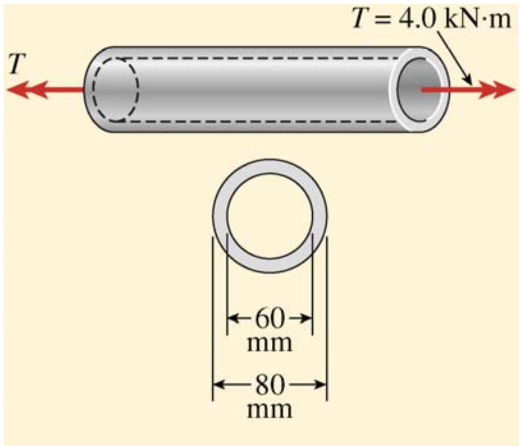

FIG. 3-26 Example 3-6. Circular tube in torsion

Relationship between moduli of elasticity E and G

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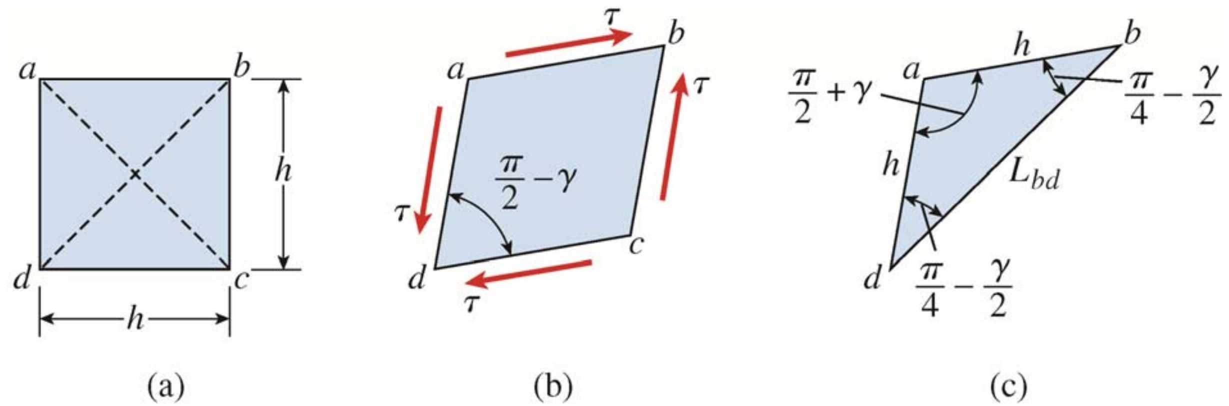
- E, v (Poisson's ratio) and G are not independent!
  - The relationship can be obtained from the geometry during the pure shear

$$G = \frac{E}{2(1+\nu)}$$

  - If any two of them are known, the third can be calculated.



Statically Indeterminate Torsional Members

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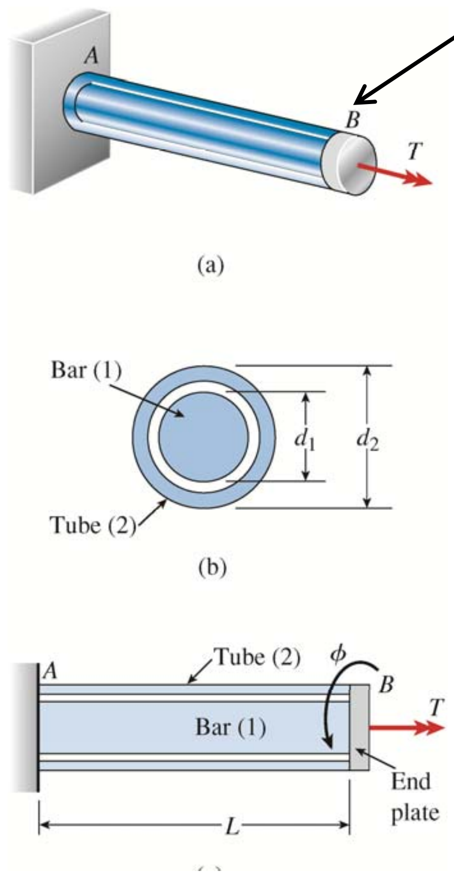
- Statically indeterminate ← internal torques and all reactions can not be obtained from Equilibrium Equations alone.

Solid bar and the tube joined to a rigid end plate at B



Statically Indeterminate Torsional Members

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- General methodology
  - 1) Equations equilibrium

$$T_1 + T_2 = T$$

  - 2) Compatibility equation

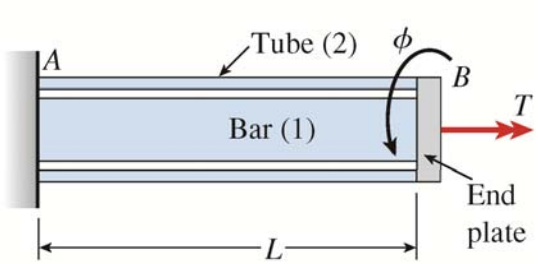
$$\phi_1 = \phi_2$$

  - 3) torque-displacement relations

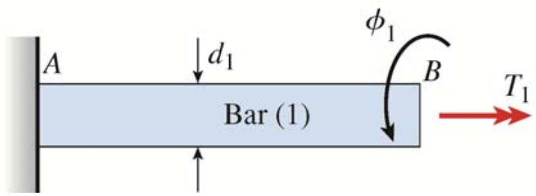
$$\phi_1 = \frac{T_1 L}{G_1 I_{p1}} \qquad \phi_2 = \frac{T_2 L}{G_2 I_{p2}}$$

  - Combining 1), 2) & 3)

$$T_1 = T\left(\frac{G_1 I_{p1}}{G_1 I_{p1} + G_2 I_{p2}}\right) \qquad T_2 = T\left(\frac{G_2 I_{p2}}{G_1 I_{p1} + G_2 I_{p2}}\right)$$

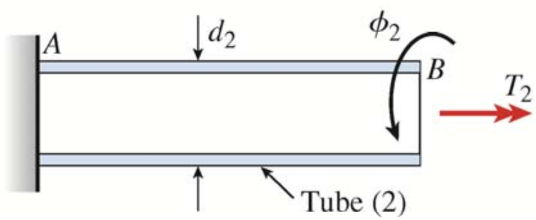


(c)



T_1 developed in the solid bar

Φ_1 angle of trist in the solid bar



T_2 developed in the tube

Φ_2 angle of trist in the solid bar

Statically Indeterminate Torsional Members
Example 3-9

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- General methodology
  - 1) Equations equilibrium $T_1 + T_2 = T$
  - 2) Compatibility equation $\phi_1 = \phi_2$
  - 3) torque-displacement relations

$$\phi_1 = \frac{T_0 L_A}{GI_{PA}} \qquad \phi_2 = -\frac{T_B L_A}{GI_{PA}} - \frac{T_B L_B}{GI_{PB}}$$

  - Combining 1), 2) & 3)

$$\frac{T_0 L_A}{GI_{PA}} - \frac{T_B L_A}{GI_{PA}} - \frac{T_B L_B}{GI_{PB}} = 0$$

$$T_A = T_0 \left(\frac{L_B I_{PA}}{L_B I_{PA} + L_A I_{PB}} \right) \qquad T_B = T_0 \left(\frac{L_A I_{PB}}{L_B I_{PA} + L_A I_{PB}} \right)$$

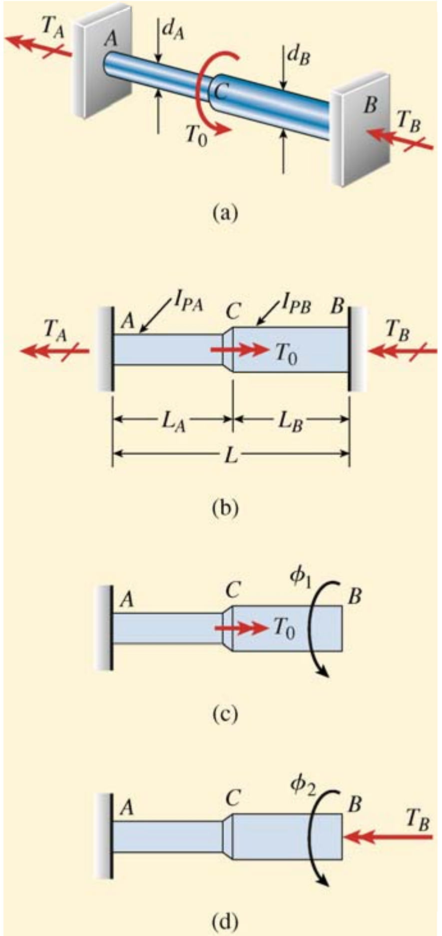

FIG. 3-33 Example 3-9. Statically indeterminate bar in torsion

Statically Indeterminate Torsional Members
Example 3-9

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- Reactive torques, max shear stress & rotation?
- Equation of Equilibrium $T_1 + T_2 = T$
- Compatibility Equation $\phi_1 = \phi_2$
- Torque-displacement relations
  - Angle of twist at the end B due to T_0 alone

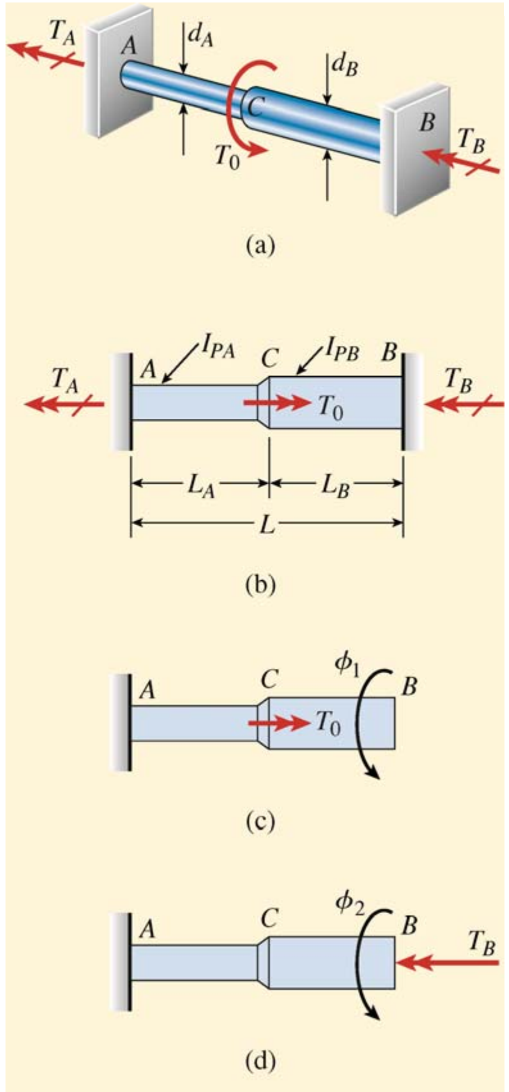
$$\phi_1 = \frac{T_0 L_A}{G I_{PA}}$$

  - Angle of twist at the end B due to T_B alone

$$\phi_2 = -\frac{T_B L_A}{G I_{PA}} - \frac{T_B L_B}{G I_{PB}}$$

- Solution of equations

$$T_A = T_0 \left(\frac{L_B I_{PA}}{L_B I_{PA} + L_A I_{PB}} \right) \qquad T_B = T_0 \left(\frac{L_A I_{PB}}{L_B I_{PA} + L_A I_{PB}} \right)$$



Statically Indeterminate Torsional Members
Example 3-9

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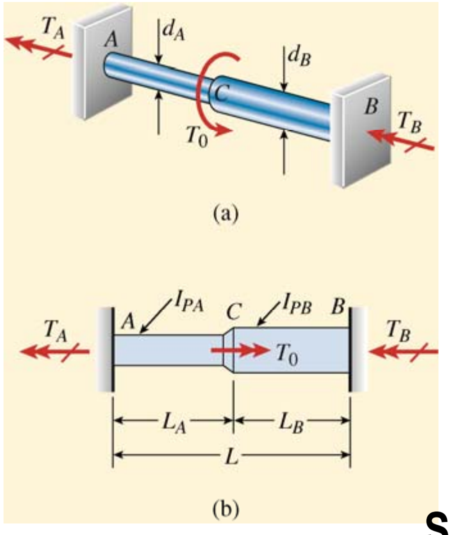
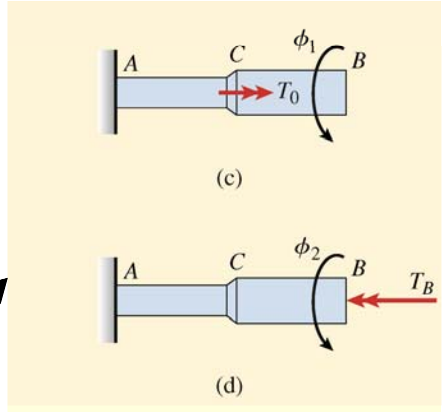
- Two options



(a)
(b)

textbook

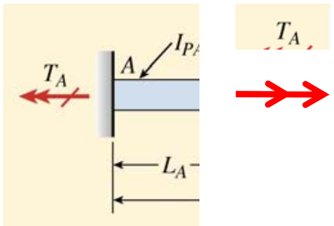


(c)
(d)

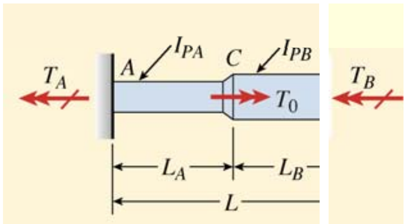
$$\phi_1 = \frac{T_0 L_A}{GI_{PA}}$$

$$\phi_2 = -\frac{T_B L_A}{GI_{PA}} - \frac{T_B L_B}{GI_{PB}}$$

Similar to p.107-108



$$\phi_1 = \frac{T_A L_A}{GI_{PA}}$$



$$\phi_2 = -\frac{T_B L_B}{GI_{PB}}$$

Both approaches should result in the same results

Statically Indeterminate Torsional Members
Example 3-9

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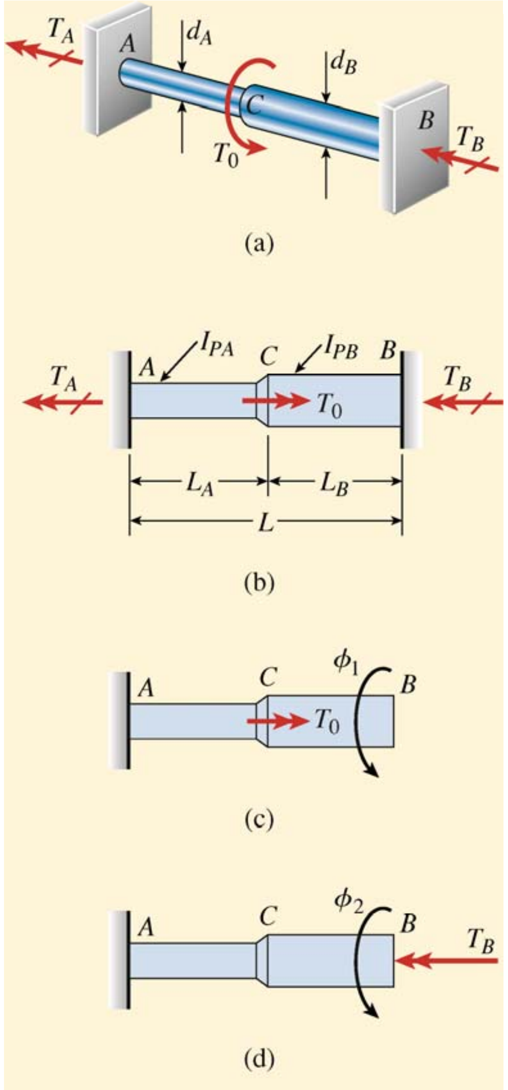
– Maximum shear stress

$$\boxed{\tau_{\max} = \frac{Tr}{I_p}} \longrightarrow \tau_{AC} = \frac{T_A d_A}{2I_{PA}} \qquad \tau_{CB} = \frac{T_B d_B}{2I_{PB}}$$

$$\tau_{AC} = \frac{T_0 L_B d_A}{2\left(L_B I_{PA} + L_A I_{PB}\right)} \qquad \tau_{CB} = \frac{T_0 L_A d_B}{2\left(L_B I_{PA} + L_A I_{PB}\right)}$$

– Angle of rotation

$$\phi_C = \frac{T_A L_A}{G I_{PA}} = \frac{T_B L_B}{G I_{PB}} = \frac{T_0 L_A L_B}{G\left(L_B I_{PA} + L_A I_{PB}\right)}$$



Strain Energy in Torsion and Pure Shear

- Work is performed by the load applied to a structure and strain energy is developed in the structure.

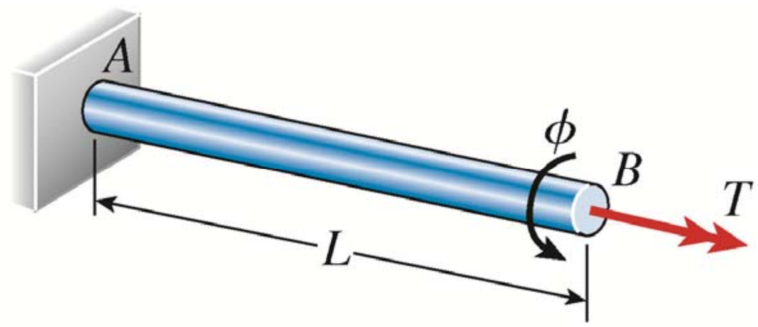


FIG. 3-34 Prismatic bar in pure torsion

Strain Energy in Torsion and Pure Shear

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- Strain energy U due to torsion (assuming linearly elastic)

$$U = W = \frac{T\phi}{2}$$

$$U = \frac{T^2 L}{2GI_P} = \frac{GI_P \phi^2}{2L}$$

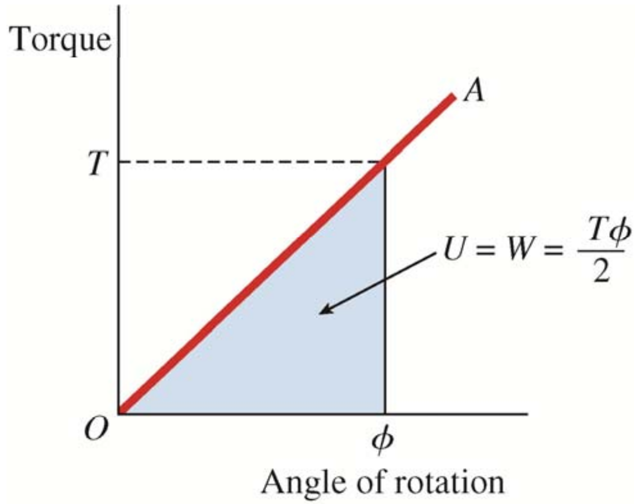
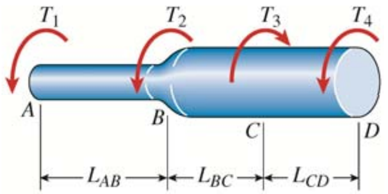
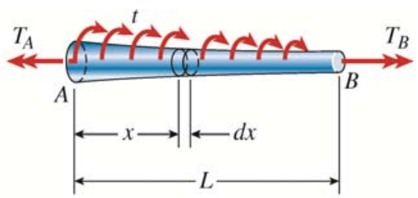


FIG. 3-35 Torque-rotation diagram for a bar in pure torsion (linearly elastic material)

- Nonuniform torsion



$$U = \sum_{i=1}^{n} \frac{T_i^2 L_i}{2G_i \left(I_P\right)_i}$$



$$U = \int_0^L \frac{\left[T(x)\right]^2 dx}{2GI_P(x)}$$

Strain Energy in Torsion and Pure Shear
Strain energy density in pure shear

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– Shear force V acting on the side of the element

$$V = \tau h t$$

– Displacement at the top face

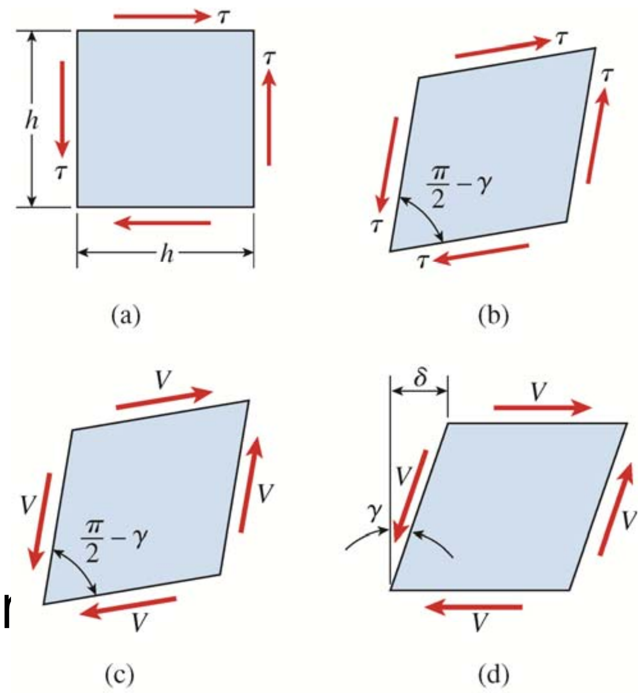
$$\delta = \gamma h$$

– Strain energy stored in the element

$$U = W = \frac{V\delta}{2} \longrightarrow U = \frac{\tau\gamma h^2 t}{2}$$

– Strain energy density divided by the volur

$$\boxed{u = \frac{\tau\gamma}{2} = \frac{\tau^2}{2G} = \frac{G\gamma^2}{2}}$$



Strain Energy in Torsion and Pure Shear

Example 3-11

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- Derive a formula for the strain energy of the bar
- Evaluate the strain energy of a hollow shaft used for drilling into the earth if the data are as follows.
  T=2100 Nm/m, L=3.7m, G=80GPa, I_p=7.15x10^{-6} m^4

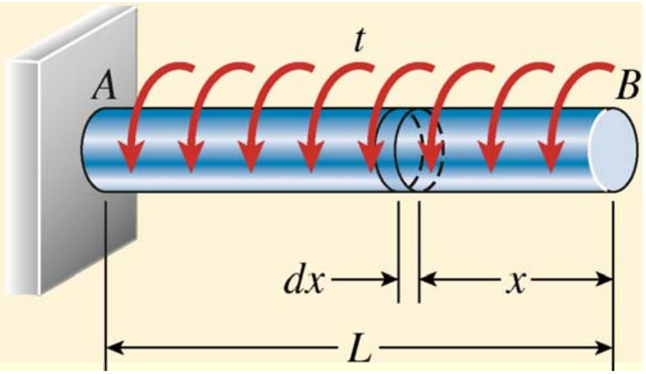

FIG. 3-38 Example 3-11. Strain energy produced by a distributed torque