Week 10, 3 May
Week 11, 10 &12 May

Mechanics in Energy Resources Engineering - Chapter 7 Analysis of Stress and Strain

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1st exam

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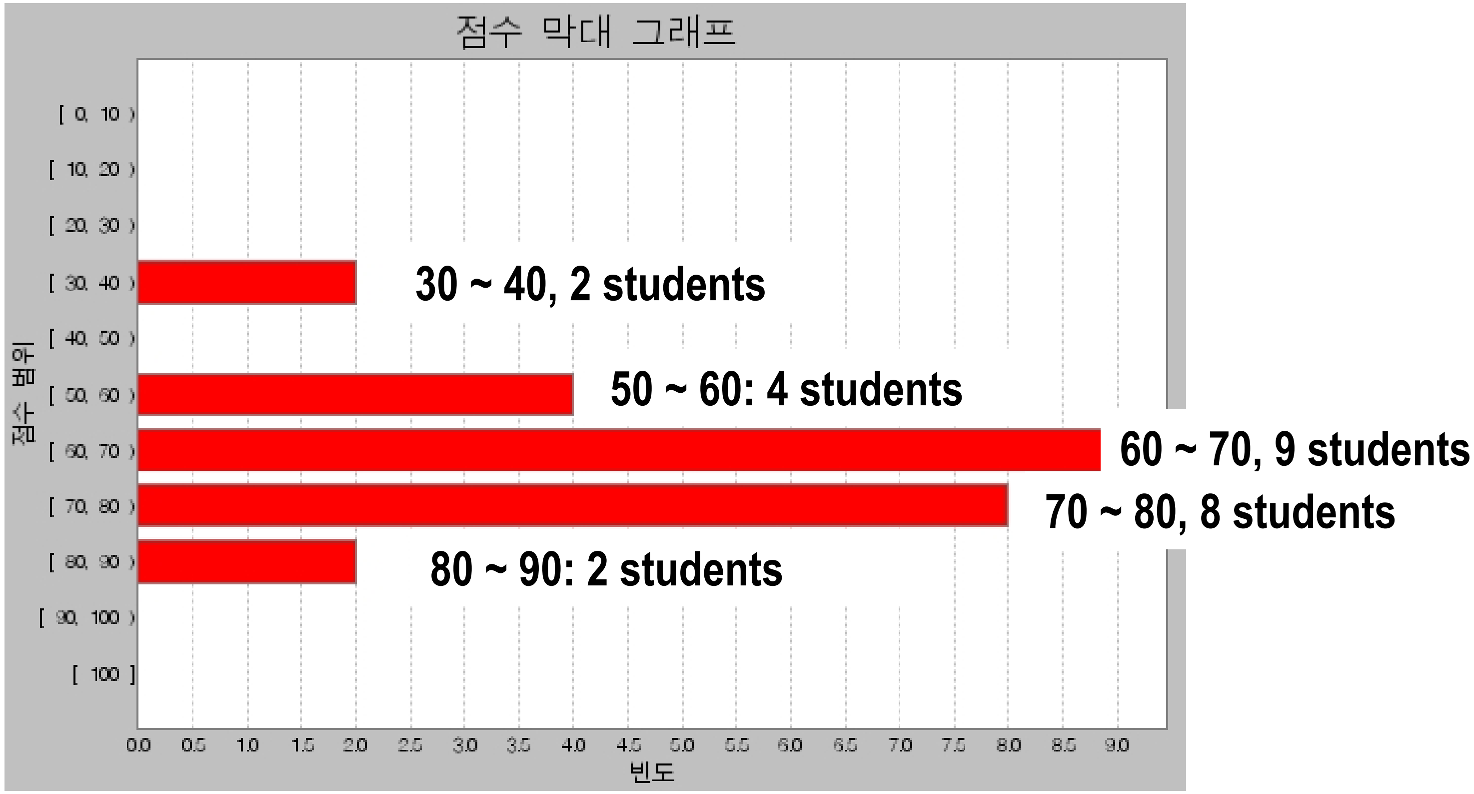
- Mean: 65.3, standard deviation: 12.9
- Max: 86.0, Min: 30.0



2nd Exam

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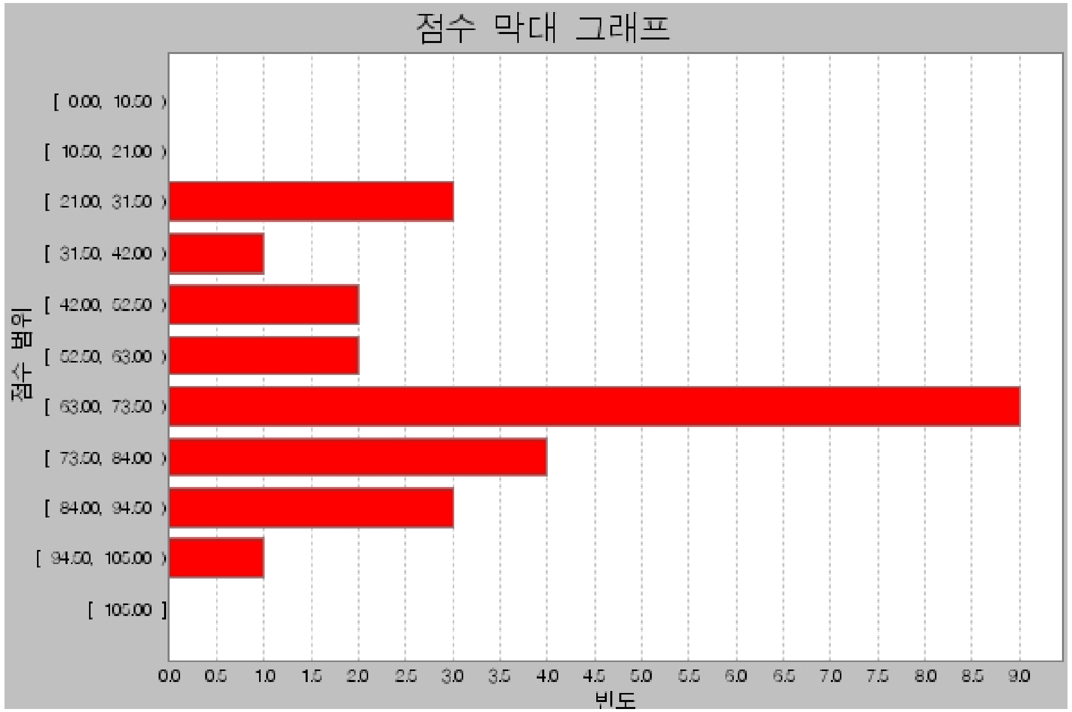
- Mean: 63.8, standard deviation: 20.79
- Max: 98.0, Min: 21.0



schedule

- Ch.7 Analysis of Stress and Strain
  – 3 May, 10 May, 12 May
- Ch.8 Application of Plane Stress
  – 17 May, 19 May
- Ch.9 Deflection of Beams
  – 24 May, 26 May, 31 May
- Ch.10 Statically Indeterminate Beams
  – 2 June, 7 June
- Final Exam: 9 June

Outline

- Introduction
- Plane Stress
- Principal Stresses and Maximum Shear Stresses
- Mohr's Circle for Plane Stress
- Hooke's Law for Plane Stress
- Triaxial Stress
- Plane Strain

Introduction

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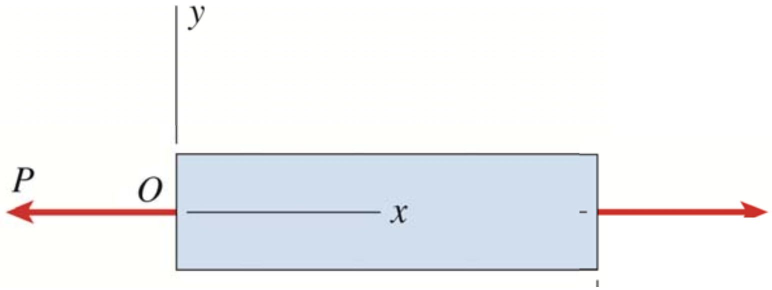
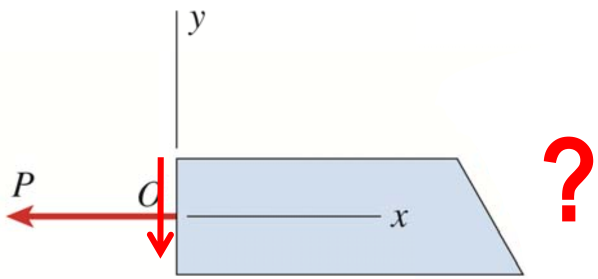
- Stresses in cross section



- Stresses in inclined section: larger stresses may occur
  - Finding the normal and shear stresses acting on inclined section is necessary
  - Main content of Ch.5!



Introduction

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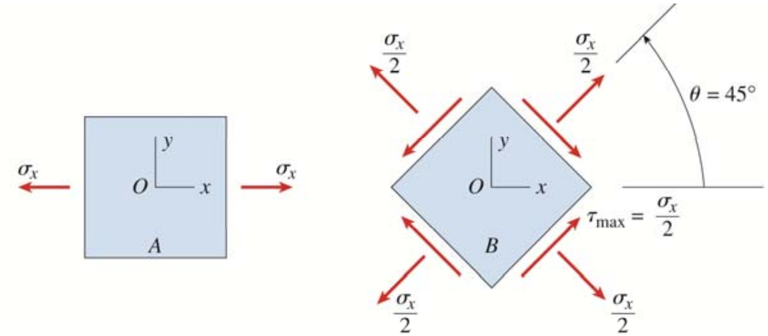
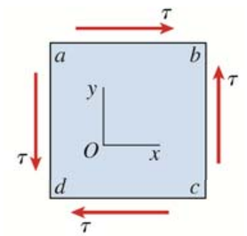
- We have already learned this!
  - Uniaxial Stress & Stresses in inclined section

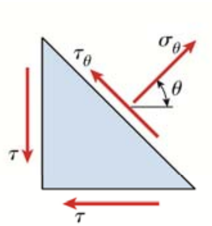


$$\sigma_\theta = \sigma_x \cos^2\theta = \frac{1}{2}\sigma_x\left(1+\cos 2\theta\right)$$

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta = -\frac{\sigma_x}{2}\sin 2\theta$$

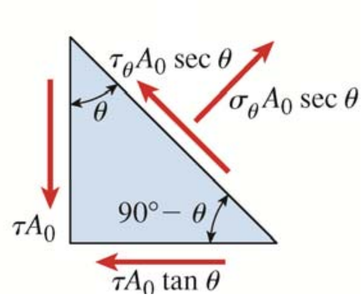
  - Pure Shear & Streses in inclined section





$$\sigma_\theta = \tau \sin 2\theta$$

$$\tau_\theta = \tau \cos 2\theta$$

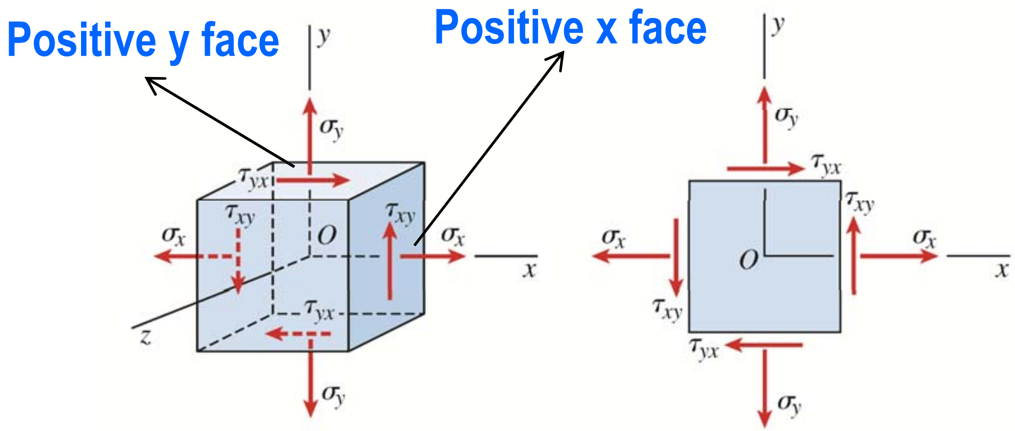


Introduction

- ONE instrinsic state of stress can be expressed in many many different ways depending on the reference axis (or orientation of element).
  - Similarity to force: One intrinsic state of force (vector) can be expressed similarly depending on the reference axis.
  - Difference from force: we use different transformation equations from those of vectors
  - Stress is NOT a vector BUT a (2nd order) tensor → they do not combine according to the parallelogram law of addition

Plane Stress
Definition

- Plane Stress: Stresses in 2D plane
- Normal stress, σ : subscript identify the face on which the stress act. Ex) σ_x
- Shear stress, τ : 1st subscript denotes the face on which the stress acts, and the 2nd gives the direction on that face. Ex) τ_{xy}



Plane Stress
Definition

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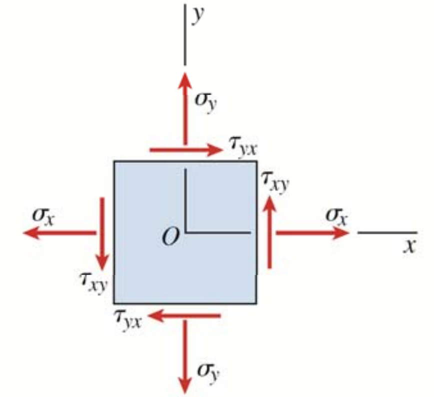
- Sign convention
  - Normal stress: tension (+), compression (-)
  - Shear stress:
    - acts on a positive face of an element in the positive direction of an axis (+) : plus-plus or minus-minus
    - acts on a positive face of an element in the negative direction of an axis (-): plus-minus or minus-plus



Positive normal & shear stresses

Plane Stress
Definition

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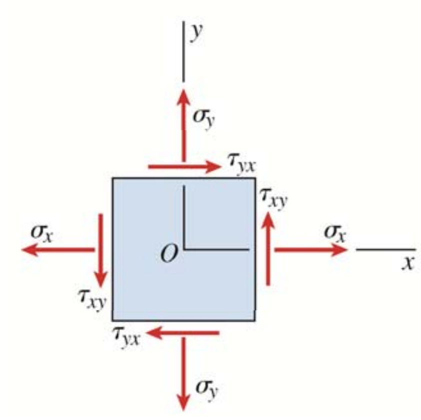
- Shear stresses in perpendicular planes are equal in magnitude and directions shown in the below.
  - Derived from the moment equilibrium

$$\tau_{xy} = \tau_{yx}$$



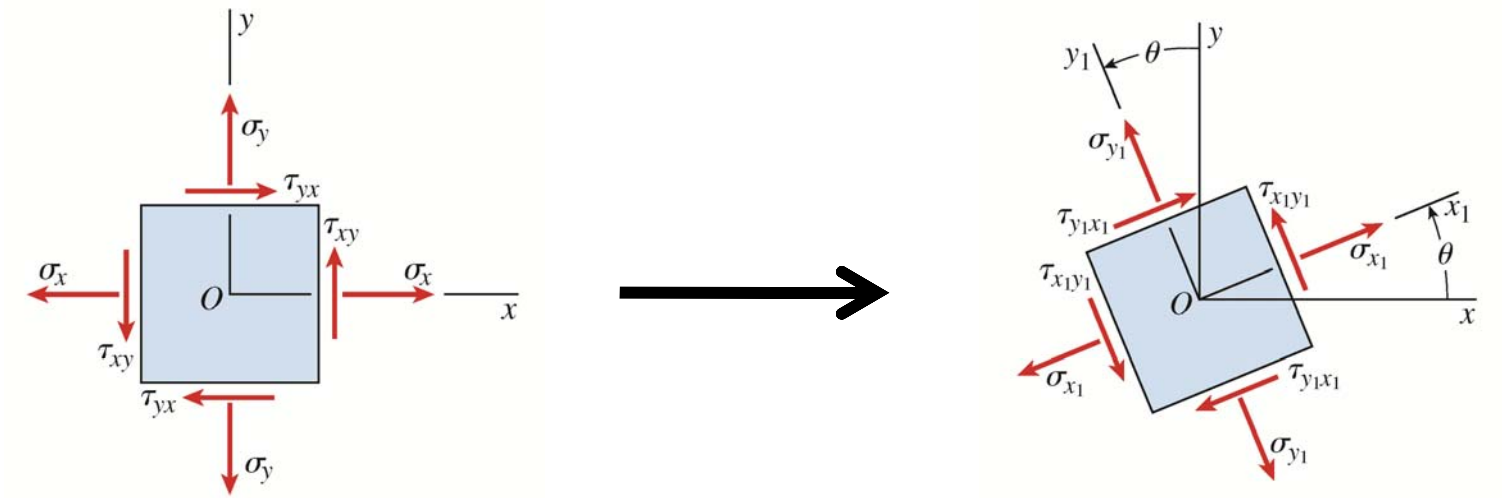
- In 2D (plane stress), we need three (independent) components to describe a complete state of stress

σ_x $\quad$ σ_y $\quad$ τ_{xy}

$$\begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix}$$

Plane Stress
Stresses on inclined sections

- Stresses acting on inclined sections assuming that σ_x, σ_y, τ_{xy} are known.
  - x_1y_1 axes are rotated counterclockwise through an angle θ
  - Strategy??? →
  - wedge shaped stress element



Plane Stress
Stresses on inclined sections

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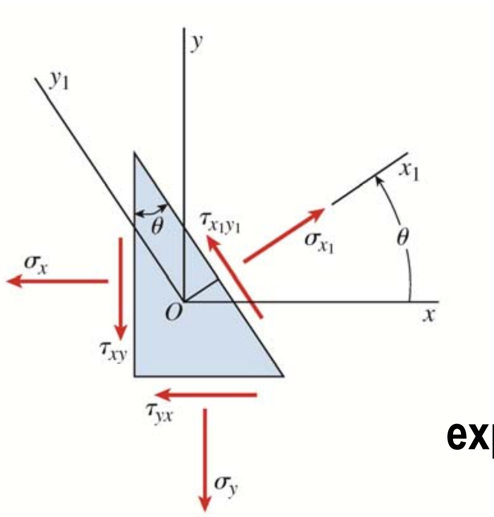
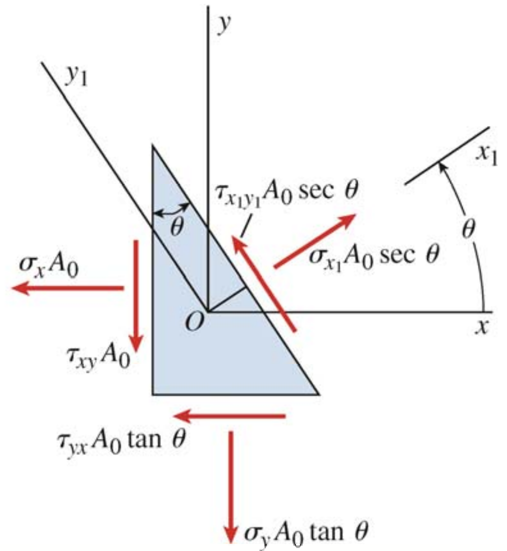


Free Body Diagram

express in terms of "Force"



- Force Equilibrium Equations in x_1 and y_1 directions

$$\sum F_{x_1} = \sigma_{x_1} A_0 \sec\theta - \sigma_x A_0 \cos\theta - \tau_{xy} A_0 \sin\theta$$
$$- \sigma_y A_0 \tan\theta \sin\theta - \tau_{yx} A_0 \tan\theta \cos\theta = 0$$

$$\sum F_{y_1} = \tau_{x_1y_1} A_0 \sec\theta + \sigma_x A_0 \sin\theta - \tau_{xy} A_0 \cos\theta$$
$$- \sigma_y A_0 \tan\theta \cos\theta + \tau_{yx} A_0 \tan\theta \sin\theta = 0$$

Plane Stress
Stresses on inclined sections

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- Using $\tau_{xy} = \tau_{yx}$ and simplifying

$$\sigma_{x_1} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta$$

$$\tau_{x_1y_1} = -\left(\sigma_x - \sigma_y\right)\sin\theta\cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

  - When θ = 0,

$$\sigma_{x_1} = \sigma_x \qquad \tau_{x_1y_1} = \tau_{xy}$$

  - When θ = 90,

$$\sigma_{x_1} = \sigma_y \qquad \tau_{x_1y_1} = -\tau_{xy}$$

Plane Stress
Transformation Equations

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- From half angle and double angle formulas

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \qquad \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

- Transformation equations for plane stress

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad \tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

  – Intrinsic state of stress is the same but the reference axis are different

  – Derived solely from equilibrium→ applicable to stresses in any kind of materials (linear or nonlinear or elastic or inelastic)

Plane Stress
Transformation Equations

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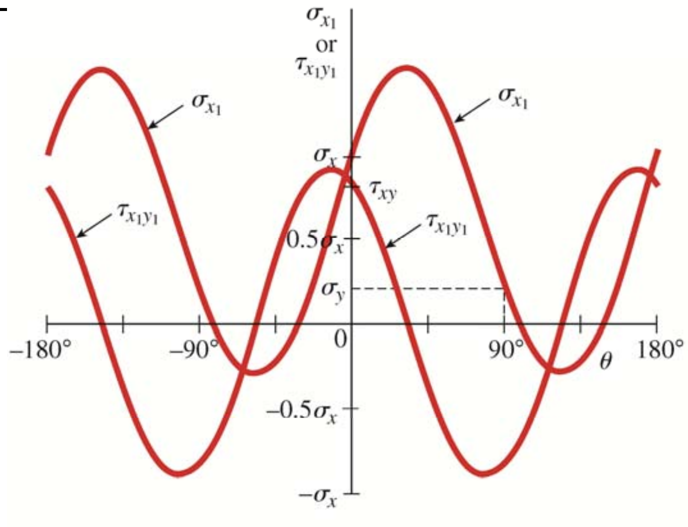


With σ_y=0.2σ_x & τ_{xy}=0.8 σ_x

- For σ_{y1}, $\theta \rightarrow \theta + 90$,

$$\sigma_{y_1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

  - Making summations

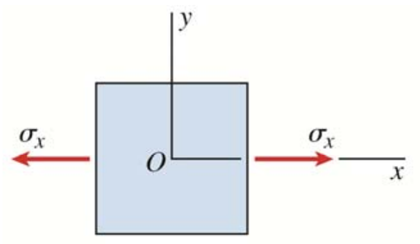
$$\sigma_{x_1} + \sigma_{y_1} = \sigma_x + \sigma_y$$

  - Sum of the normal stresses acting on perpendicular faces of plane stress elements is constant and independent of θ

Plane Stress
Special Cases of Plane Stress

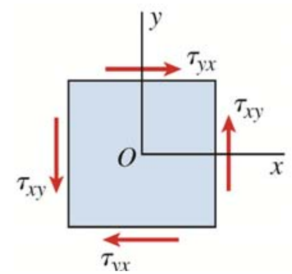
- Uniaxial stress

$$\sigma_{x_1} = \frac{\sigma_x}{2}(1+\cos 2\theta) \qquad \tau_{x_1y_1} = -\frac{\sigma_x}{2}\sin 2\theta$$



- Pure Shear

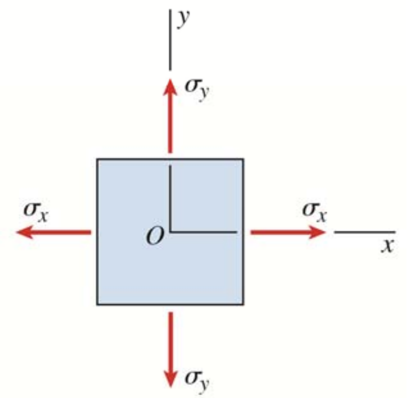
$$\sigma_{x_1} = \tau_{xy}\sin 2\theta \qquad \tau_{x_1y_1} = \tau_{xy}\cos 2\theta$$



- Biaxial Stress

$$\sigma_{x_1} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta$$

$$\tau_{x_1y_1} = -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta$$



Plane Stress
Example 7-1

- Determine the stress acting on an element inclined at an angle θ = 45°

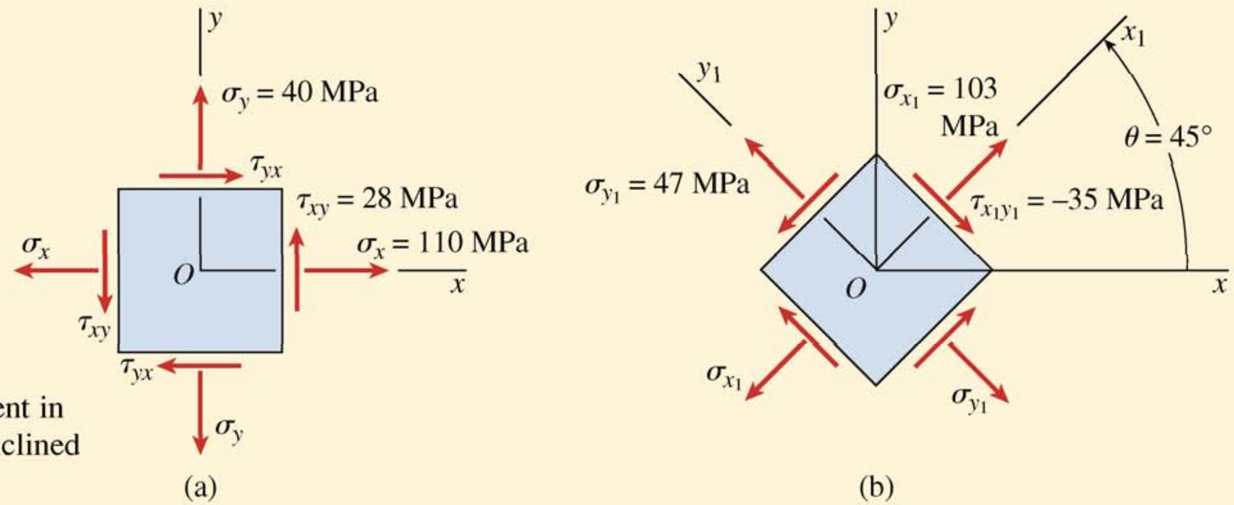


FIG. 7-7 Example 7-1. (a) Element in plane stress, and (b) element inclined at an angle $\theta = 45°$

Plane Stress
Example 7-1

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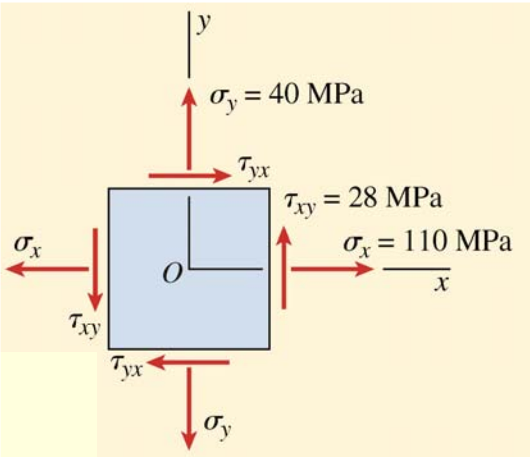
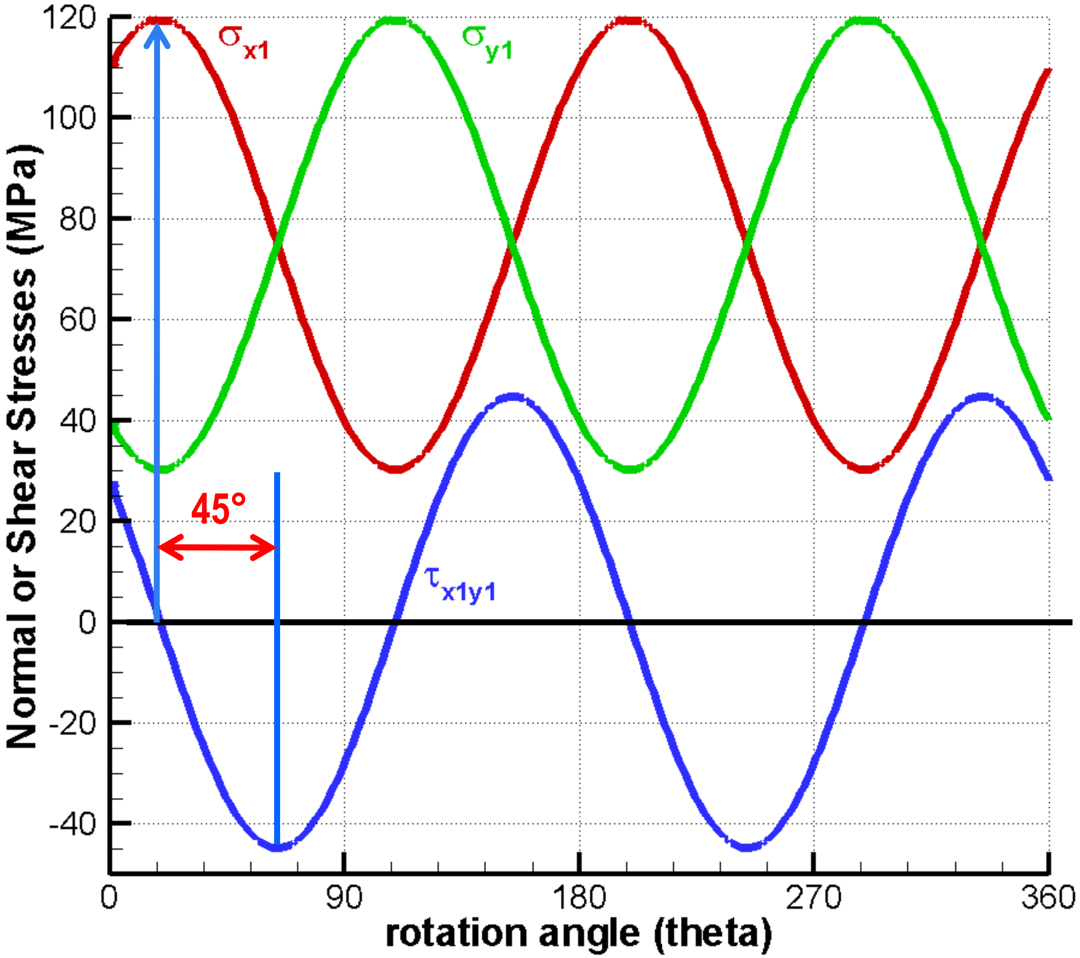




Outline

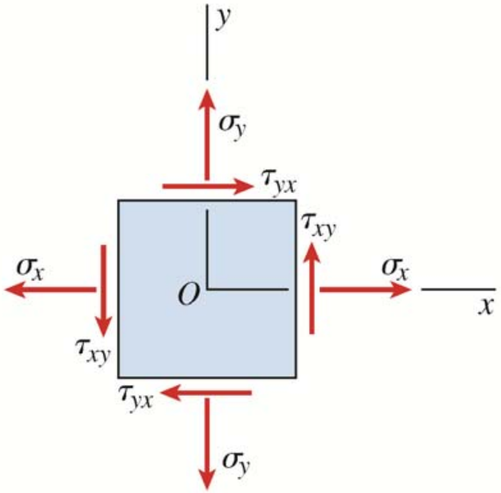
- Introduction
- Plane Stress (Transformation Equation for Plane Stress)
- Principal Stresses and Maximum Shear Stresses
- Mohr's Circle for Plane Stress
- Hooke's Law for Plane Stress
- Triaxial Stress
- Plane Strain

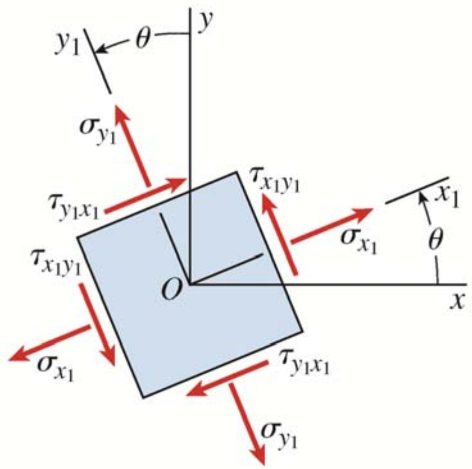
Plane Stress
Stresses on inclined sections

- Stresses acting on inclined sections assuming that σ_x, σ_y, τ_{xy} are known.
  - x_1y_1 axes are rotated counterclockwise through an angle θ

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$





Plane Stress
Stresses on inclined sections

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- A different way of obtaining transformed stresses
  - For vector

$$\begin{pmatrix} F_{x1} \\ F_{y1} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

  - For tensor (stress)

$$\begin{pmatrix} \sigma_{x1} & \tau_{x1y1} \\ \tau_{x1y1} & \sigma_{y1} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^T$$

=

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Outline

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- Introduction
- Plane Stress
- Principal Stresses and Maximum Shear Stresses
- Mohr's Circle for Plane Stress
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- Triaxial Stress
- Plane Strain

Plane Stress
Example 7-1

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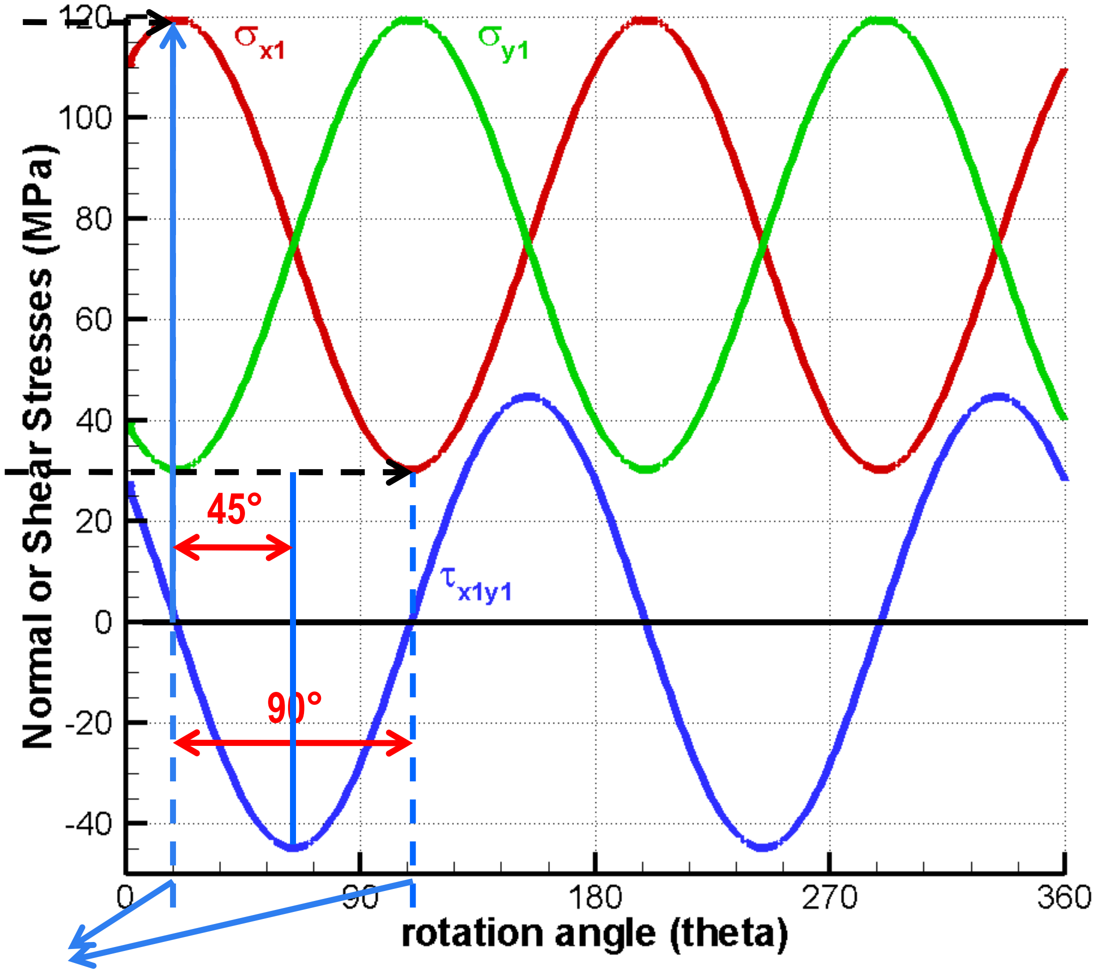
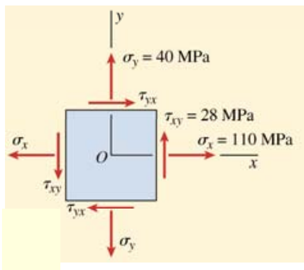
Principal stresses



Principal stresses



Principal angle

Principal stresses

- Principal Stresses (주응력)
  - Maximum normal stress & Minimum normal stress
  - Strategy?
  - Taking derivatives of normal stress with respect to θ

$$\frac{d\sigma_{x1}}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\longrightarrow \quad \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

  - θ_p:orientation of the principal planes (planes on which the principal stresses act)

- Principal stresses can be obtained by substituting θ_p

Principal stresses

- Two values of angle $2\theta_p$: 0 °~ 360 °
  - One : 0 °~ 180 °
  - The other (differ by 180°) : 180 °~ 360 °
- Two values of angle θ_p: 0 °~ 180 °→ Principal angles
  - One : 0 °~ 90 °
  - The other (differ by 90°) : 90 °~ 180 °

→ <u>principal stresses occur on mutually perpendicular planes</u>

Principal stresses

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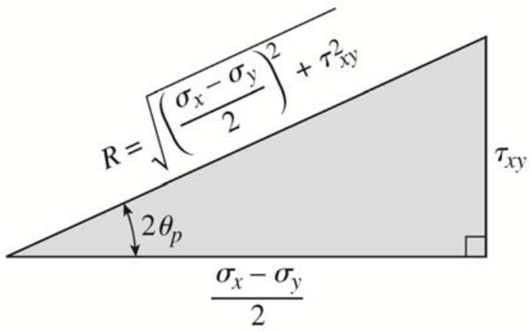
- Calculation of principal stresses

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

? ?

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$



$$\cos 2\theta_p = \frac{\sigma_x - \sigma_y}{2R}$$

$$\sin 2\theta_p = \frac{\tau_{xy}}{R}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{2}}$$

- By substituting,

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\left(\frac{\sigma_x - \sigma_y}{2R}\right) + \tau_{xy}\left(\frac{\tau_{xy}}{R}\right)$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{2}}$$

Larger of two principal stresses
= Maximum Principal Stress

Principal stresses

- The smaller of the principal stresses (= minimum principal stress)

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y \longrightarrow \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

- Putting into shear stress transformation equation

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \qquad 0 = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

  - Shear stresses are zero on the principal stresses

Same equation for principal angles

- Principal stresses

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

Principal stresses

- Alternative way of finding the smaller of the principal stresses (= minimum principal stress)

$$\cos(2\theta_p + 180) = -\frac{\sigma_x - \sigma_y}{2R} \qquad \sin(2\theta_p + 180) = -\frac{\tau_{xy}}{R}$$

- By substituting into the transformation equations

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

Principal Angles

- Principal angles correspond to principal stresses

$$\theta_{p1} \longrightarrow \sigma_1$$

$$\theta_{p2} \longrightarrow \sigma_2$$

  – Both angles satisfy $\tan 2\theta_p = 0$
  – Procedure to distinguish θ_{p1} from θ_{p2}
    1) Substitute these into transformation equations → tell which is σ_1.
    2) Or find the angle that satisfies

$$\cos 2\theta_p = \frac{\sigma_x - \sigma_y}{2R} \qquad \sin 2\theta_p = \frac{\tau_{xy}}{R}$$

Special cases

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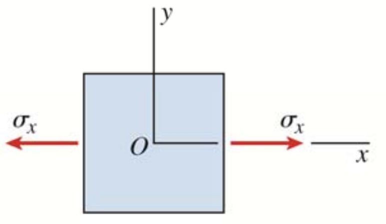
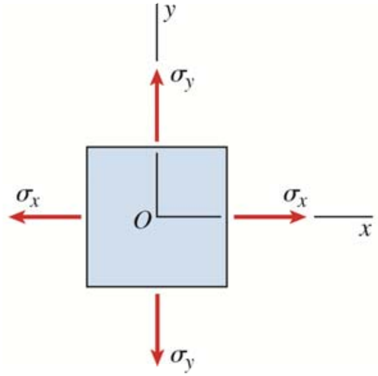
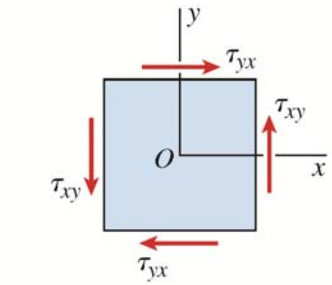
- Uniaxial stress & Biaxial stress

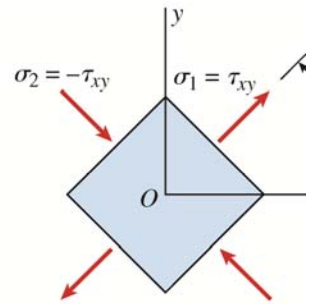




  - Principal planes? $\tan 2\theta_p = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}$
  - θ_p = 0 ° and 90° → how do we get this?



- Pure Shear
  - Principal planes?
  - θ_p = 45 ° and 135° → how do we get this?
  - If τ_{xy} is positive, $\sigma_1 = \tau_{xy}$ & $\sigma_2 = -\tau_{xy}$



Principal Stresses and Maximum Shear Stresses
The Third Principal Stress

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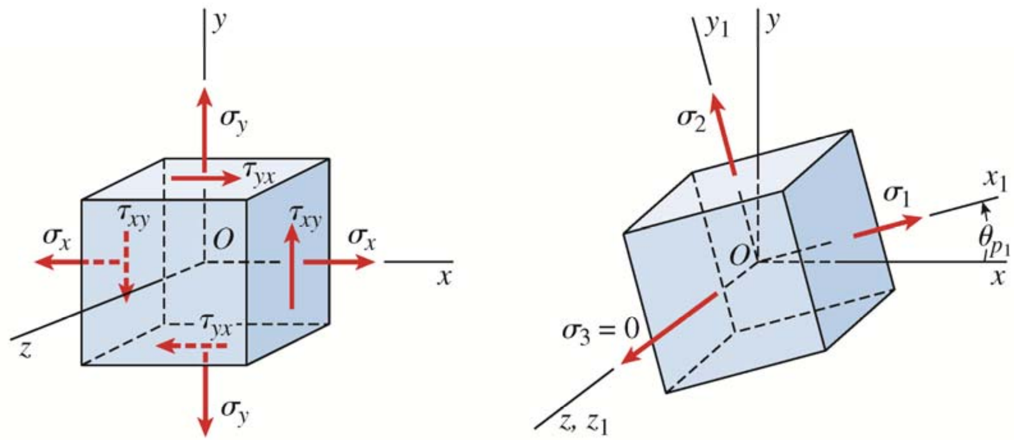
- Stress element is three dimensional
  - Three principal stresses (σ_1 ,σ_2 and σ_3) on three mutually perpendicular planes



Maximum Shear Stress

- Maximum Shear Stress?
  - Strategy?
  - Taking derivatives of normal stress with respect to θ

$$\frac{d\tau_{x1y1}}{d\theta} = -\left(\sigma_x - \sigma_y\right)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

$$\longrightarrow \quad \tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

  - θ_s:orientation of the planes of the maximum positive and negative shear stresses
    - One : 0 °~ 90 °
    - The other (differ by 90°) : 90 °~ 180 °

-- Maximum positive and maximum negative shear stresses differ only in sign. Why???

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Maximum Shear Stress

- Relationship between Principal angles, θ_p and angle of the planes of maximum positive and negative shear stresses, θ_s

$$\tan 2\theta_s = -\frac{1}{\tan 2\theta_p} = -\cot 2\theta_p$$

$$\frac{\sin 2\theta_s}{\cos 2\theta_s} + \frac{\cos 2\theta_p}{\sin 2\theta_p} = 0 \qquad \sin 2\theta_s \sin 2\theta_p + \cos 2\theta_s \cos 2\theta_p = 0$$

$$\cos\left(2\theta_s - 2\theta_p\right) = 0 \qquad 2\theta_s - 2\theta_p = \pm 90°$$

$$\boxed{\theta_s = \theta_p \pm 45°}$$

- The planes of maximum shear stress occur at 45° to the principal planes

Principal Stresses and Maximum Shear Stresses

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Maximum Shear Stress

- sin2θ$_s$ & cos2θ$_s$?

$$R=\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^{\ 2}}$$

$$\cos 2\theta_{s1}=\frac{\tau_{xy}}{R}\qquad \sin 2\theta_{s1}=-\frac{\sigma_x-\sigma_y}{2R} \longleftarrow \boxed{\tan 2\theta_s=-\frac{\sigma_x-\sigma_y}{2\tau_{xy}}}$$

$$\tau_{\max}=\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^{\ 2}} \longrightarrow \theta_{s1}=\theta_{p1}-45°$$

$$\cos 2\theta_{s1}=-\frac{\tau_{xy}}{R}\qquad \sin 2\theta_{s1}=\frac{\sigma_x-\sigma_y}{2R}$$

$$\tau_{\max}=-\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^{\ 2}} \longrightarrow \theta_{s2}=\theta_{p1}+45°$$

- Maximum (positive or negative) shear stress, τ_{max}

$$\boxed{\tau_{\max}=\pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^{\ 2}}}\qquad \boxed{\tau_{\max}=\pm\frac{\sigma_1-\sigma_2}{2}}$$

Maximum positive shear stress is equal to one-half the difference of the principal stress

Principal Stresses and Maximum Shear Stresses
Maximum Shear Stress

- Normal stress at the plane of τ_{max}?

$$\boxed{\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta} \longleftarrow \begin{array}{l} \cos 2\theta_{s1} = \dfrac{\tau_{xy}}{R} \\ \sin 2\theta_{s1} = -\dfrac{\sigma_x - \sigma_y}{2R} \end{array}$$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} = \sigma_{aver} \quad \boxed{= \sigma_{y_1}}$$

$\longleftarrow$ **From Mr. Ahn's observation**

  – Normal stress acting on the planes of maximum positive shear stresses equal to the average of the normal stresses on the x and y planes.

  – And same normal stress acts on the planes of maximum negative shear stress

- Uniaxial, biaxial or pure shear?

Principal Stresses and Maximum Shear Stresses
In-Plane and Out-of-Plane Shear Stresses

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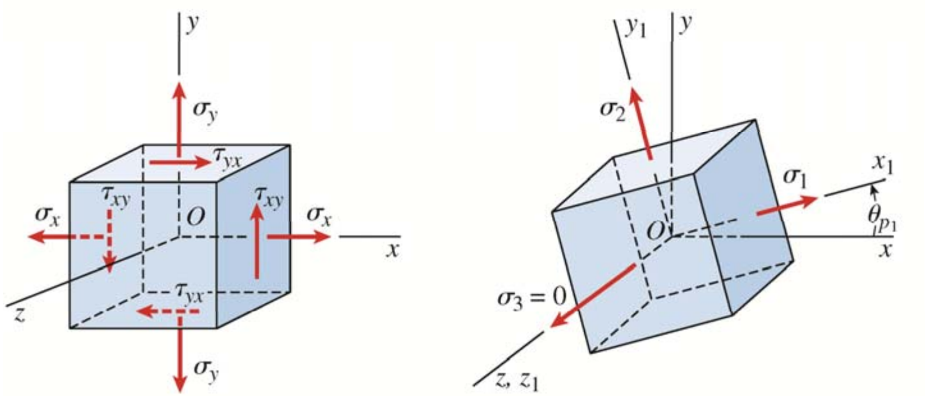
- So far we have dealt only with in-plane shear stress acting in the xy plane.
  - Maximum shear stresses by 45° rotations about the other two principal axes

$$(\tau_{\max})_{x1} = \pm\frac{\sigma_2}{2} \qquad (\tau_{\max})_{y1} = \pm\frac{\sigma_1}{2} \qquad (\tau_{\max})_{z1} = \pm\frac{(\sigma_1 - \sigma_2)}{2}$$

  - The stresses obtained by rotations about the x_1 and y_1 axes are 'out-of-plane shear stresses'

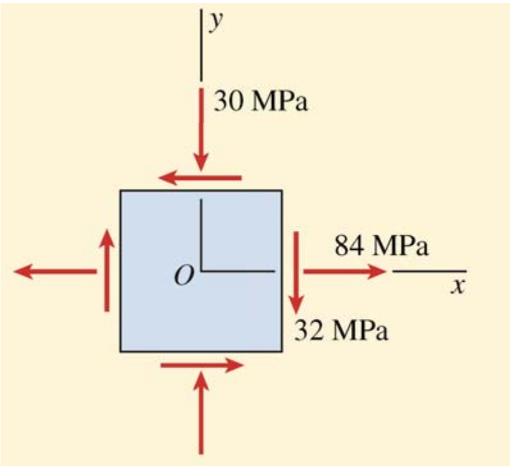


Principal Stresses and Maximum Shear Stresses

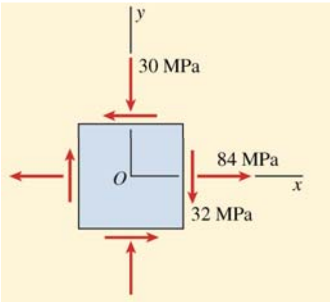
Example 7-3

1) Determine the principal stresses and show them on a sketch of a properly oriented element

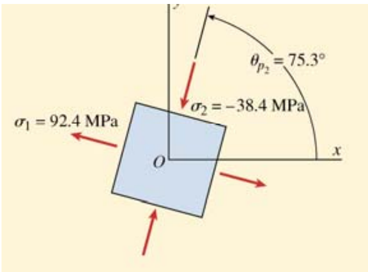
2) Determine the maximum shear stresses and show them on a properly oriented element.



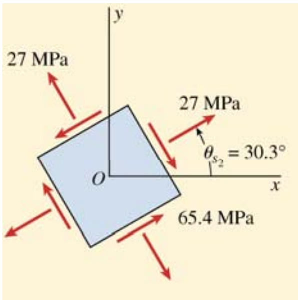
Example7.3



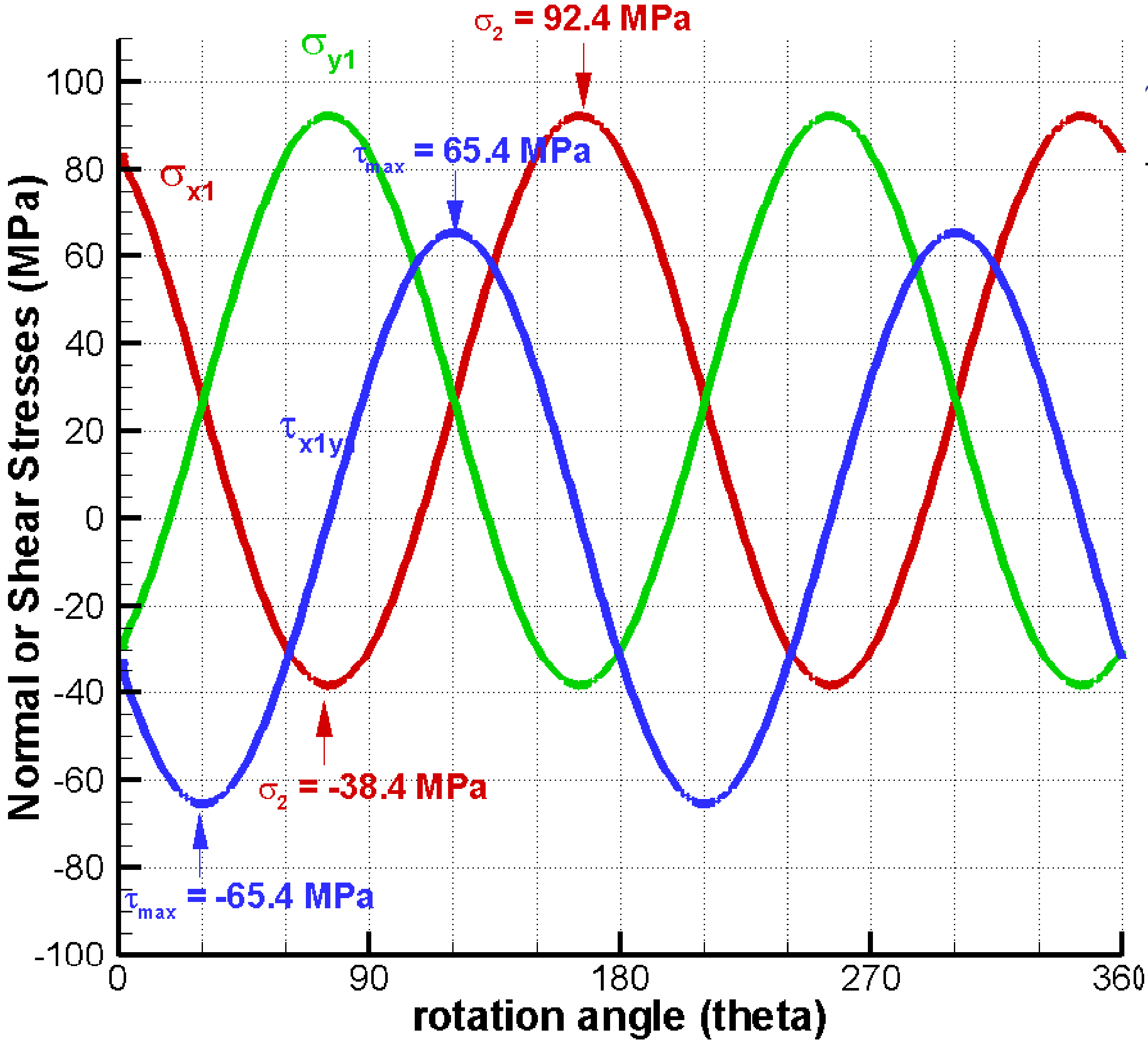
y
30 MPa
84 MPa
O
x
32 MPa



θ_{p_2} = 75.3°
σ_2 = −38.4 MPa
σ_1 = 92.4 MPa
O
x



y
27 MPa
27 MPa
θ_{s_2} = 30.3°
O
x
65.4 MPa



Normal or Shear Stresses (MPa)
rotation angle (theta)
100
80
60
40
20
0
-20
-40
-60
-80
-100
0
90
180
270
360
σ_{x1}
σ_{y1}
τ_{x1y1}
σ_2 = 92.4 MPa
τ_{max} = 65.4 MPa
σ_2 = -38.4 MPa
τ_{max} = -65.4 MPa

Mohr's Circle for Plane Stress

- Mohr's Circle
  - Graphical representation of the transformation equation for stress
  - Extremely useful to visualize the relationship between σ_x and τ_{xy}
  - Also used for calculating principal stresses, maximum shear stresses, and stresses on inclined sections
  - Also used for other quantities of similar nature such as strain.

Mohr's Circle for Plane Stress
Equations of Mohr's Circle

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- The transformation Equations for plane stress

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

  – Rearranging the above equations

$$\sigma_{x1} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

  – Square both sides of each equation and sum the two equations

$$(\sigma_{x1} - \frac{\sigma_x + \sigma_y}{2})^2 + \tau_{x1y1}^2 = (\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2$$

  – Equation of a circle in standard algebraic form

$$(x - x_0)^2 + y^2 = R^2$$

Mohr's Circle for Plane Stress
Equations of Mohr's Circle

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$$(\sigma_{x_1} - \frac{\sigma_x + \sigma_y}{2})^2 + \tau_{x_1 y_1}^2 = (\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2$$

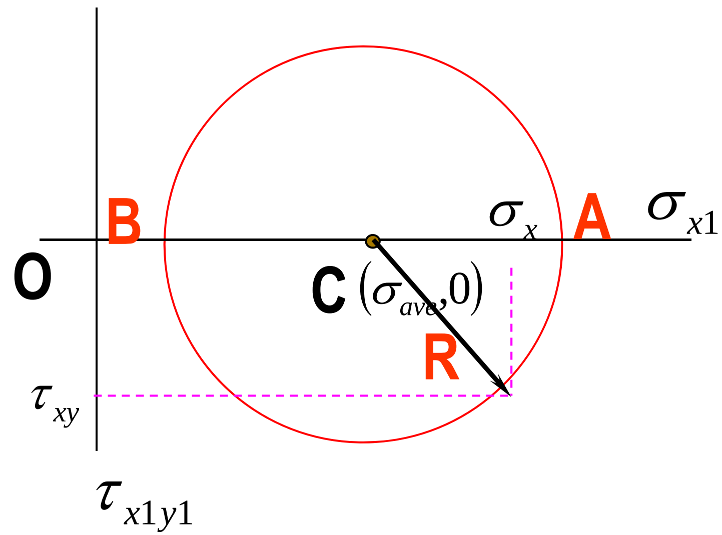
Centre (σ_{ave}, 0)

Radius^2 of a circle

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

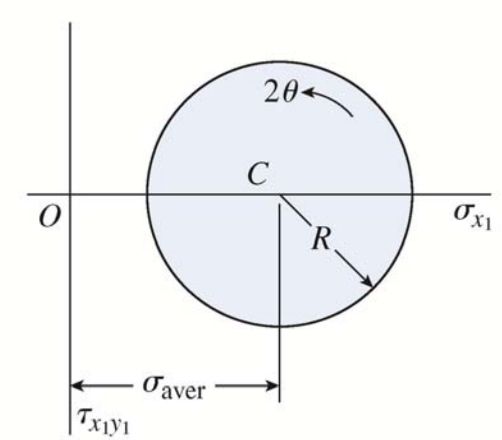
$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$(\sigma_{x_1} - \sigma_{ave})^2 + \tau_{x_1 y_1}^2 = R^2$$

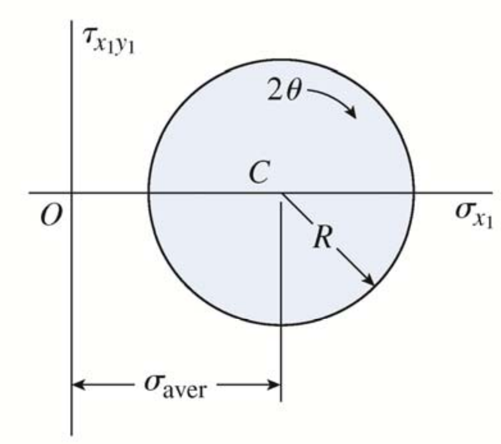


Recognized by Mohr in 1882

Mohr's Circle for Plane Stress
Two forms of Mohr's Circle



- Shear stress (+) ↓ θ (+) counterclockwise
  - Chosen for this course!



- Shear stress (+) ↑ θ (+) clockwise

Mohr's Circle for Plane Stress
Construction of Mohr's Circle

- If stresses σ_x, σ_y and τ_{xy} acting on the x and y faces of a stress element are known, the Mohr's circle can be constructed in the following steps:

  1. Draw a set of coordinate axes with σ_{x1} on the x-axis and τ_{x1y1} on the y-axis
  2. Locate the center C of the circle at the point having $\sigma_{x1}=\sigma_{ave}$ and $\tau_{x1y1}= 0$
  3. Locate point A, representing the stress conditions on the x face of the element by plotting $\sigma_{x1}= \sigma_x$ and $\tau_{x1y1}= \tau_{xy}$. Point A corresponds to $\theta=0°$
  4. Locate point B, representing the stress condition on the y face of the element by plotting $\sigma_{x1}= \sigma_y$ and $\tau_{x1y1}=-\tau_{xy}$. Point B corresponds to $\theta=90°$
  5. Draw a line from point A to point B. This line is a diameter and passes through the center C. Points B and B, representing the stresses on planes 90° to each other, are at the opposite ends of the diameter, and therefore are 180° apart on the circle.
  6. Using point C as the center, draw Mohr's circle through point A and B.

Mohr’s Circle for Plane Stress
Construction of Mohr’s Circle

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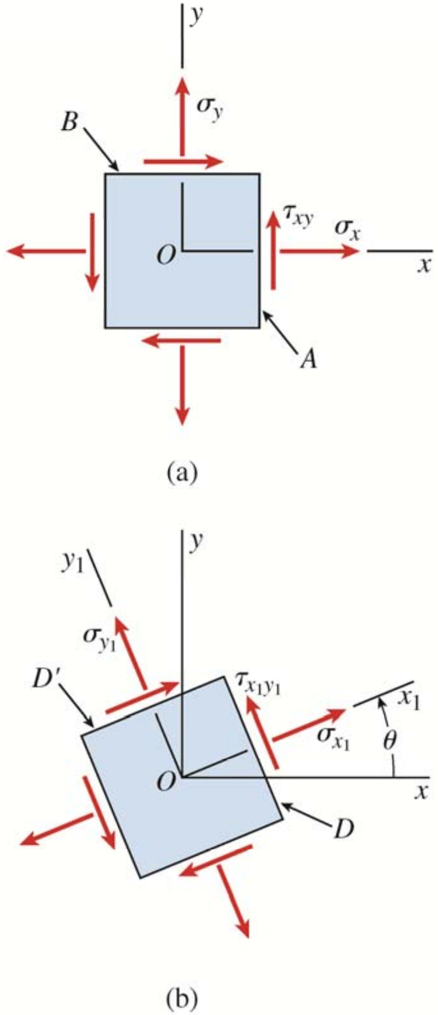
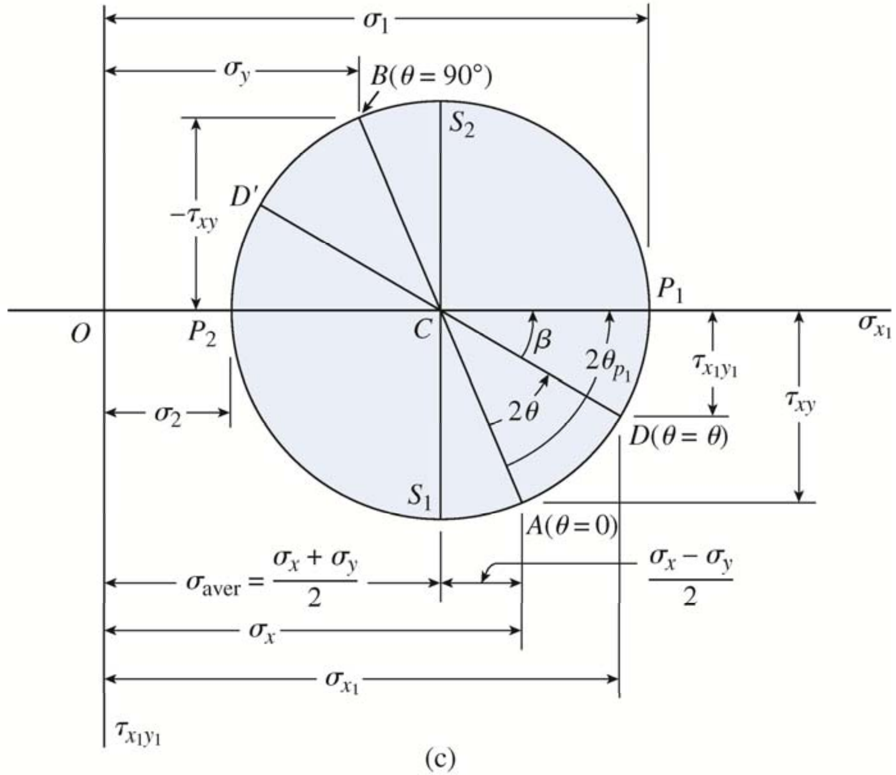


FIG. 7-16 Construction of Mohr’s circle for plane stress

Calculation of R from geometry

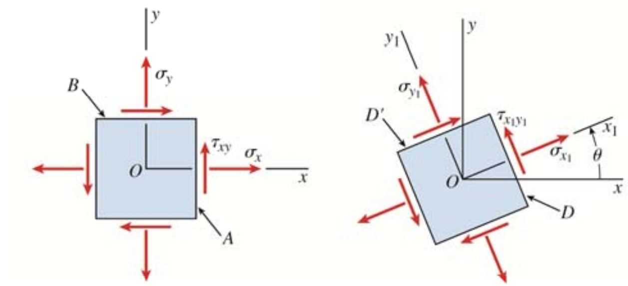


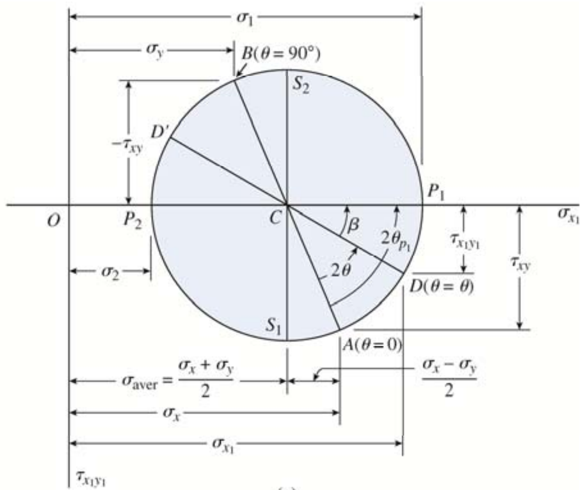
Mohr's Circle for Plane Stress
Stresses on an Inclined Element

- Stresses acting on the faces oriented at an angle θ from the x-axis.
  - Measure an angle 2 θ ctw from radius CA

$$D = (\sigma_{x_1}, \tau_{x1y1})$$

  - Angle 2θ in Mohr's Circle corresponds to an angle θ on a stress element
  - We need to show that D is indeed given by the stress-transformation equations





(c)

Mohr's Circle for Plane Stress
Stresses on an Inclined Element

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– From the geometry, $\sigma_{x1} = \dfrac{\sigma_x + \sigma_y}{2} + R\cos\beta$ $\quad \tau_{x1y1} = R\sin\beta$

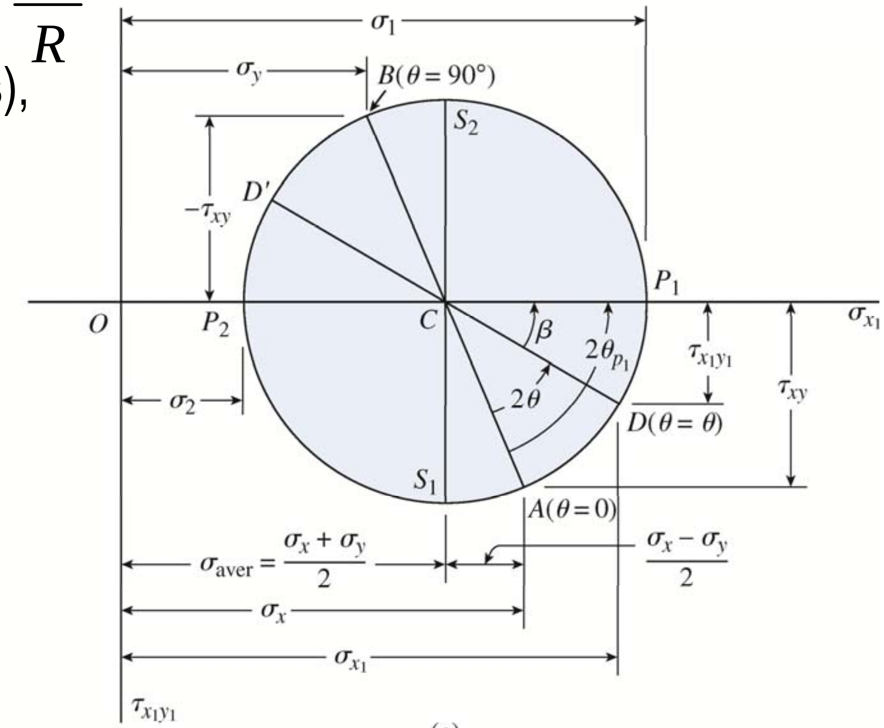
– Considering the angle between the radius CA and horizontal axis,

$$\cos(2\theta + \beta) = \frac{\sigma_x - \sigma_y}{2R} \qquad \sin(2\theta + \beta) = \frac{\tau_{xy}}{R}$$

– Expanding this (using addition formulas),

$$\cos 2\theta \cos\beta - \sin 2\theta \sin\beta = \frac{\sigma_x - \sigma_y}{2R}$$

$$\sin 2\theta \cos\beta + \cos 2\theta \sin\beta = \frac{\tau_{xy}}{R}$$



(c)

Mohr's Circle for Plane Stress
Stresses on an Inclined Element

– Multiplying first by cos2θ, the second by sin2θ, and then adding

$$\cos\beta = \frac{1}{R}\left(\frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right)$$

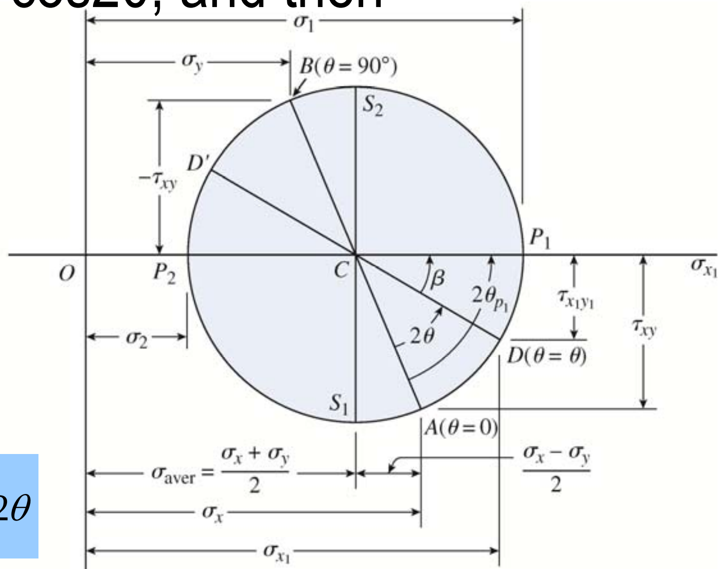
– Multiplying first by sin2θ, the second by cos2θ, and then substracting

$$\sin\beta = \frac{1}{R}\left(-\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta\right)$$

– Putting these into

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + R\cos\beta \qquad \tau_{x1y1} = R\sin\beta$$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad \tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$



• Point D on Mohr's circle, defined by the angle 2θ, represents the stress conditions on the x_1 face defined by the angle θ

Mohr's Circle for Plane Stress
Principal Stresses

- Principal stresses

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + R \qquad \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - R$$

- Cosine and sine of angle $2\theta_{p1}$ can be obtained by inspection

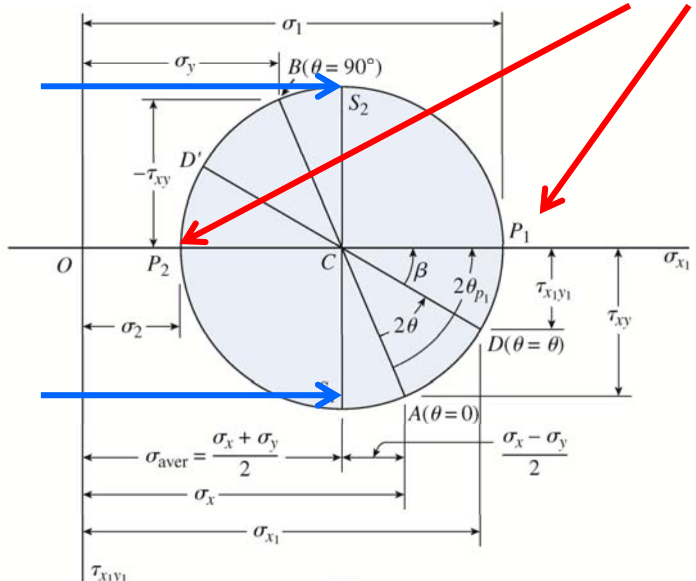
$$\cos 2\theta_{p1} = \frac{\sigma_x - \sigma_y}{2R} \qquad \sin 2\theta_{p1} = \frac{\tau_{xy}}{R}$$

$$\theta_{p2} = \theta_{p1} + 90°$$

Principal stresses & Principal planes

Maximum (-) shear stress

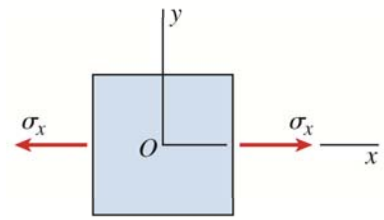
Maximum (+) shear stress

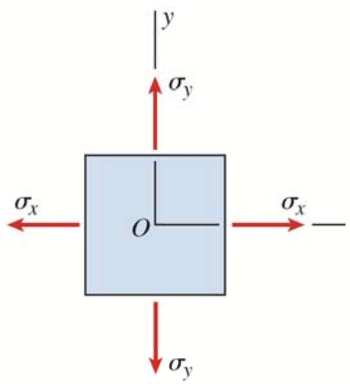


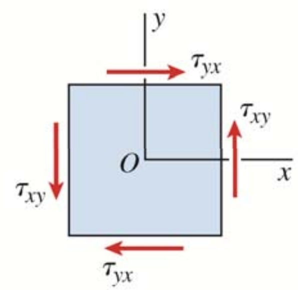
(c)

Mohr’s Circle for Plane Stress
General Comments

- We can find the stresses acting on any inclined plane, as well as principal stresses and maximum shear stresses from Mohr’s Circle.
- All stresses on Mohr’s Circle in this course are in-plane stresses ← rotation of axes in the xy plane
- Special cases of
  - Uniaxial stresses
  - Biaxial stresses
  - Pure shear







Mohr's Circle for Plane Stress

Example 7-4 (<u>when principal stresses were given</u>)

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- Using Mohr's Circle, determine the stresses acting on an element inclined at an angle θ = 30°.

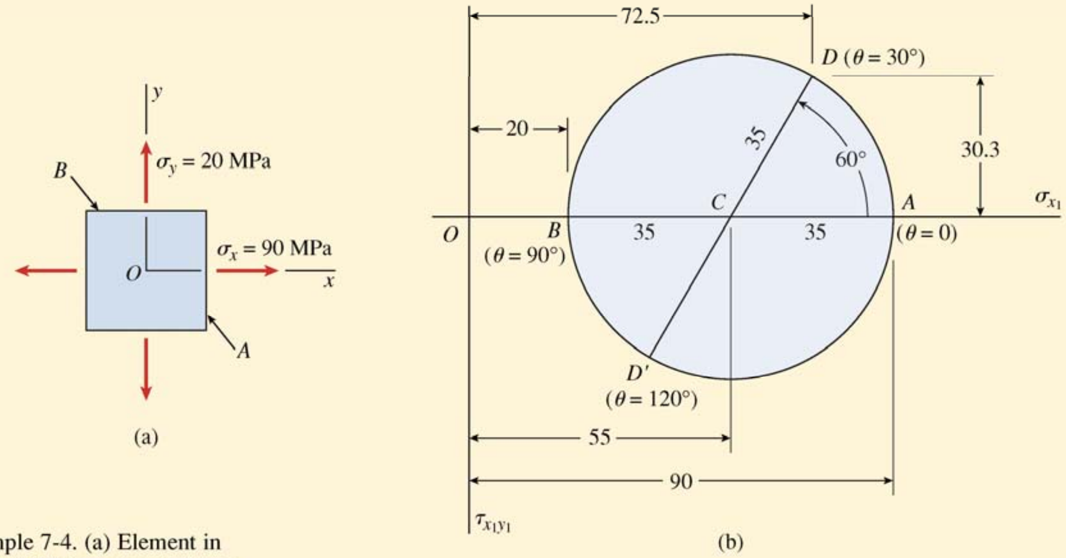
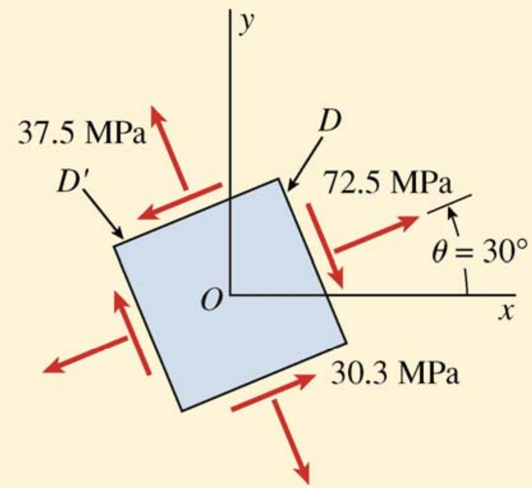


FIG. 7-18 Example 7-4. (a) Element in plane stress, and (b) the corresponding Mohr's circle. (*Note:* All stresses on the circle have units of MPa.)



Mohr's Circle for Plane Stress

Example 7-5 (<u>when both normal and shear stresses were given</u>)

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- Using Mohr's Circle, determine
  - The stresses acting on an element inclined at an angle θ = 40°
  - The principal stresses, and maximum shear stresses

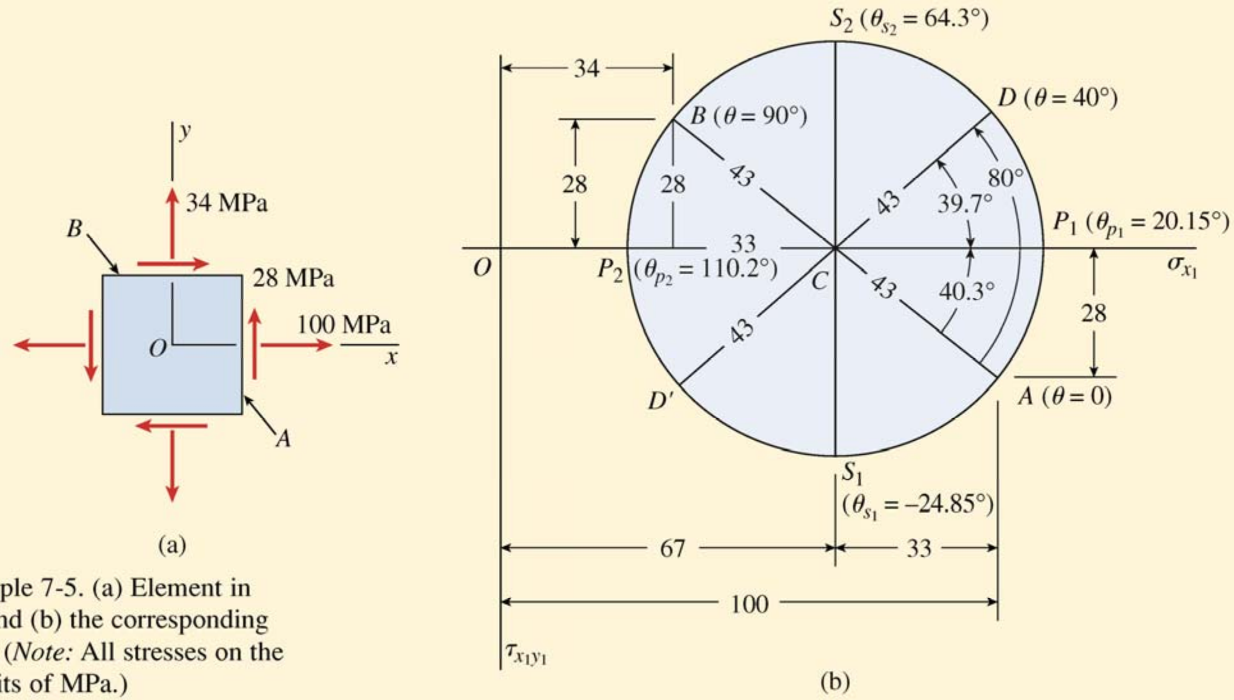

FIG. 7-20 Example 7-5. (a) Element in plane stress, and (b) the corresponding Mohr's circle. (*Note:* All stresses on the circle have units of MPa.)

Mohr's Circle for Plane Stress
Example 7-6

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- Using Mohr's Circle, determine
  - The stresses acting on an element inclined at an angle θ = 45°
  - The principal stresses, and maximum shear stresses

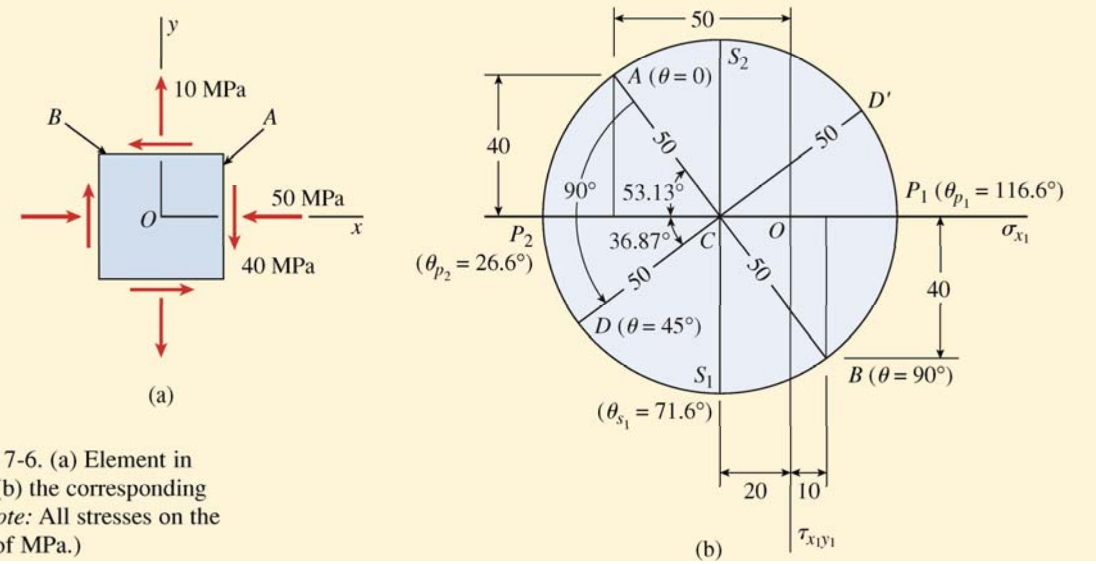

FIG. 7-22 Example 7-6. (a) Element in plane stress, and (b) the corresponding Mohr's circle. (*Note:* All stresses on the circle have units of MPa.)

Mohr's Circle for Plane Stress
Alternative way of understanding

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- The transformation Equations for plane stress

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad \tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

  – In terms of principal stresses (shear stress becomes zero)

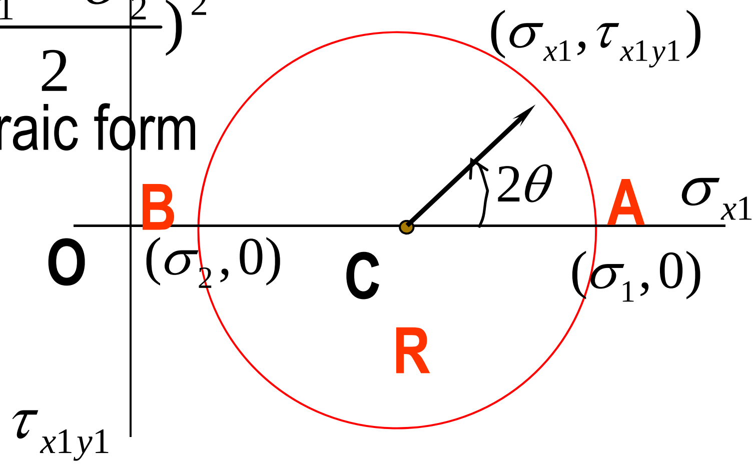
$$\sigma_{x_1} - \frac{\sigma_1 + \sigma_2}{2} = \frac{\sigma_1 - \sigma_2}{2}\cos 2\theta \qquad \tau_{x_1 y_1} = -\frac{\sigma_1 - \sigma_2}{2}\sin 2\theta$$

  – Square both sides of each equation and sum the two equations

$$(\sigma_{x_1} - \frac{\sigma_x + \sigma_y}{2})^2 + \tau_{x_1 y_1}^2 = (\frac{\sigma_1 - \sigma_2}{2})^2$$

  – Equation of a circle in standard algebraic form

$$(x - x_0)^2 + y^2 = R^2$$



Hooke's Law for Plane Stress

- Stresses on inclined planes?
  - Subject of previous sections
  - Properties (E, G or v) were not needed
- Strain or deformation?
  - Knowledge of material properties are ne
  - Assumption:
    - Isotropic
    - Homogeneous
    - Linearly elastic (follows Hooke's law)

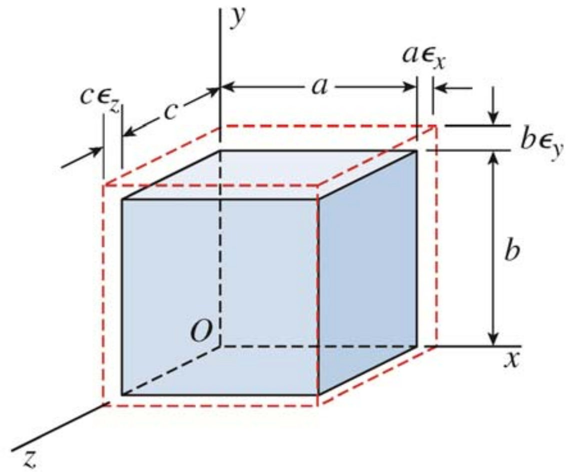


FIG. 7-25 Element of material subjected to normal strains ϵ_x, ϵ_y, and ϵ_z

Hooke's Law for Plane Stress

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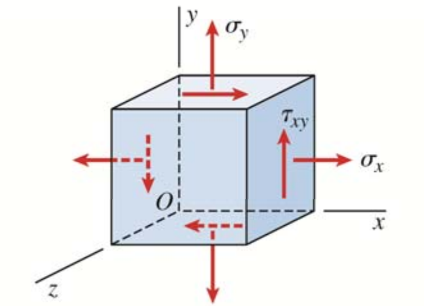
- Normal strains under plane stress

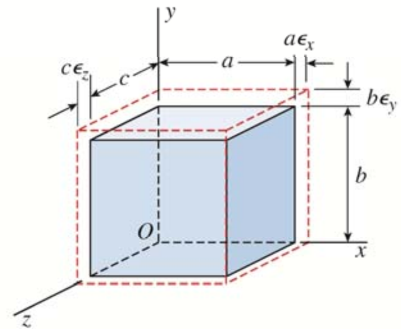
Normal strain, ε_x $=$ $\frac{1}{E}\sigma_x$ $+$ $-\frac{\nu}{E}\sigma_y$



$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right)$$

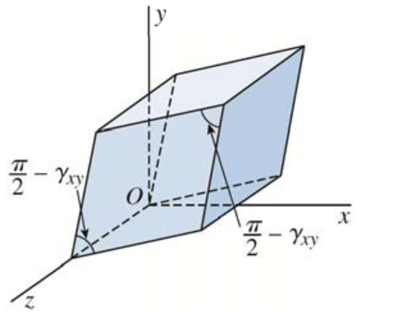
E: Elastic Modulus or Young's Modulus
v: Poisson's ratio

  - Similarly



$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \qquad \varepsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right)$$

- Shear strains under plane stress
  - Shear strain is the decrease of angle
  - σ_x and σ_y has no effect



$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

G: Shear Modulus

Hooke's Law for Plane Stress

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- Hooke's Law for Plane Stress
  - Strains in terms of stresses (plane stress)

Normal strain in z-direction can be non-zero

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \qquad \varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \qquad \varepsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right) \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

  - Stresses in terms of strains (plane stress)

Normal stress in z-direction is non-zero

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) \qquad \sigma_y = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right) \qquad \sigma_z = 0 \qquad \tau_{xy} = G\gamma_{xy}$$

  - They contain three material properties, but only two are independent.

$$G = \frac{E}{2(1+\nu)}$$

Hooke's Law for Plane Stress
Special cases

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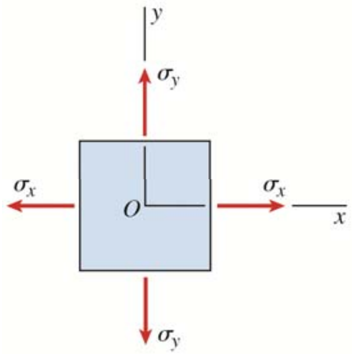
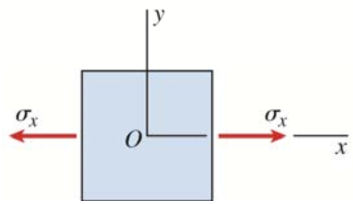
– Biaxial Stress $\sigma_x \neq 0,\ \sigma_y \neq 0,\ \tau_{xy} = 0$



$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \qquad \varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \qquad \varepsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right) \qquad \gamma_{xy} = 0$$

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) \qquad \sigma_y = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right) \qquad \sigma_z = 0 \qquad \tau_{xy} = 0$$

– Uniaxial Stress $\sigma_x \neq 0,\ \sigma_y = 0,\ \tau_{xy} = 0$



$$\varepsilon_x = \frac{1}{E}\sigma_x \qquad \varepsilon_y = \varepsilon_z = -\nu\frac{\sigma_x}{E} \qquad \gamma_{xy} = 0$$

$$\sigma_x = E\varepsilon_x \quad \sigma_y = \sigma_z = \tau_{xy} = 0$$

– Pure Shear $\sigma_x = 0,\ \sigma_y = 0,\ \tau_{xy} \neq 0$

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = 0 \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\sigma_x = \sigma_y = \sigma_z = 0 \qquad \tau_{xy} = G\gamma_{xy}$$

Hooke's Law for Plane Stress
Volume Change

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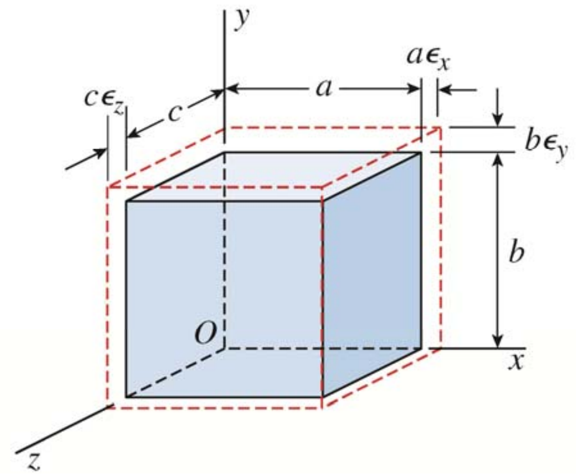
- When a solid undergoes strains, its volume will change
  - The original volume $V_0 = abc$
  - Final volume after deformation $V_1 = (a + a\varepsilon_x)(b + b\varepsilon_y)(c + c\varepsilon_z) = abc(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)$
    $= V_0(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)$
  - Upon expanding the terms in the right hand side

$$V_1 = V_0(1 + \varepsilon_x + \varepsilon_y + \varepsilon_z + \varepsilon_x\varepsilon_y + \varepsilon_x\varepsilon_z + \varepsilon_y\varepsilon_z + \varepsilon_x\varepsilon_y\varepsilon_z)$$

  - With small strains $V_1 = V_0(1 + \varepsilon_x + \varepsilon_y + \varepsilon_z)$
  - Volume change $\Delta V = V_1 - = V_0(\varepsilon_x + \varepsilon_y + \varepsilon_z)$
    - Does not have to be linearly elastic
    - General 3D (not confined to 2D)
    - Shear strain produce no change in volume



Hooke's Law for Plane Stress
Volume Change

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- The unit volume change (= dilatation).

$$e = \frac{\Delta V}{V_0} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

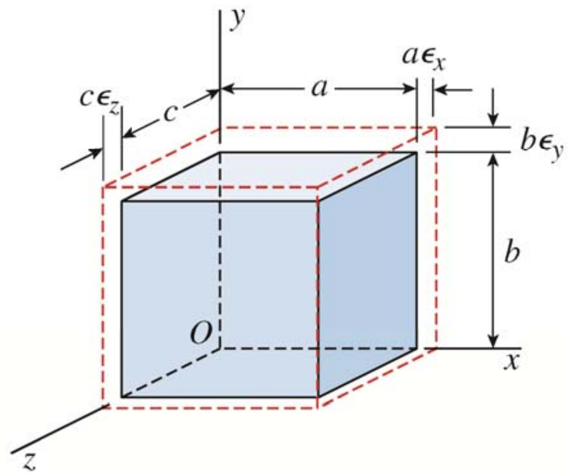
$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$
$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$
$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

  – (+) expansion, (-) contraction
  – Unit volume change in terms of stress
    - plane stress or biaxial

$$e = \frac{\Delta V}{V_0} = \frac{1-2\nu}{E}(\sigma_x + \sigma_y)$$

    - uniaxial

$$e = \frac{\Delta V}{V_0} = \frac{\sigma_x}{E}(1-2\nu)$$



Hooke's Law for Plane Stress
Strain-Energy Density in Plane Stress

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- Strain Energy Density, u, in Plane Stress
  - Strain energy stored in a unit volume of the material

$$u = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy})$$

  - Strain energy density in terms of stresses alone

$$u = \frac{1}{2E}(\sigma_x^{\ 2} + \sigma_y^{\ 2} - 2\nu\sigma_x\sigma_y) + \frac{\tau_{xy}^{\ 2}}{2G}$$

  - Strain energy density in terms of strains alone

$$u = \frac{E}{2(1-\nu^2)}(\varepsilon_x^{\ 2} + \varepsilon_y^{\ 2} + 2\nu\varepsilon_x\varepsilon_y) + \frac{G\gamma_{xy}^{\ 2}}{2}$$

  - Strain energy density in uniaxial stress $\quad u = \frac{\sigma_x^{\ 2}}{2E} \qquad u = \frac{E\varepsilon_x^{\ 2}}{2}$
  - Strain energy density in pure shear $\quad u = \frac{\tau_{xy}^{\ 2}}{2G} \qquad u = \frac{G\gamma_{xy}^{\ 2}}{2}$

Triaxial Stress

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- Triaxial stress:
  - three normal stresses in three mutually perpendicular direction
  - Shear stress exist in inclined section
- Maximum shear stress

$$(\tau_{max})_z = \pm\frac{(\sigma_x - \sigma_y)}{2} \qquad (\tau_{max})_x = \pm\frac{(\sigma_y - \sigma_z)}{} \qquad (\tau_{max})_y = \pm\frac{(\sigma_x - \sigma_z)}{2}$$

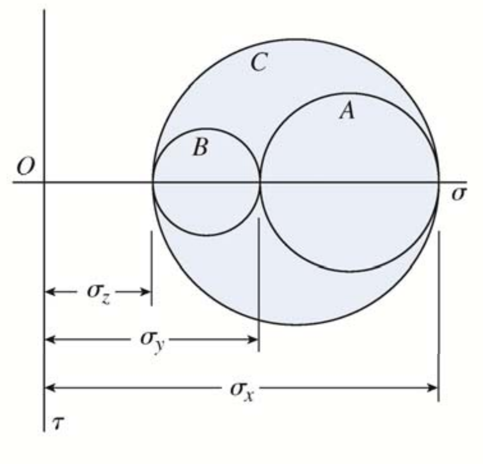
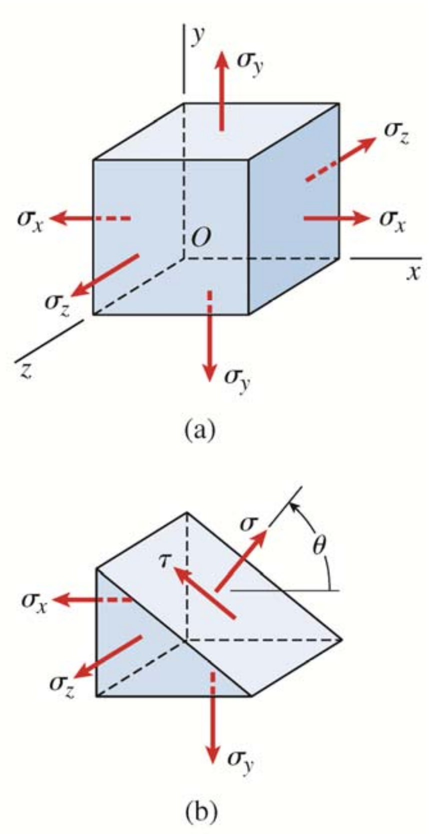


FIG. 7-28 Mohr's circles for an element in triaxial stress



Triaxial Stress

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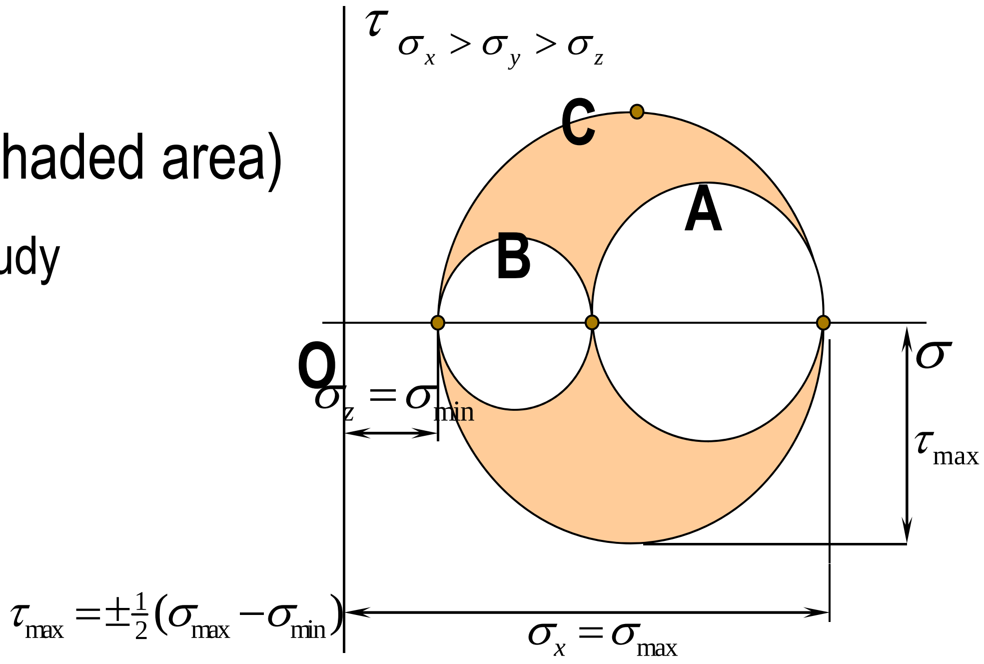
- Mohr's Circles for 3D
  - Rotation about z-axis (A)
  - Rotation about x-axis (B)
  - Rotation about y-axis (C)
  - Rotation about skew axis (shaded area)
    - Subject of more advanced study



Triaxial Stress
Hooke's Law for Triaxial Stress

- Strains in terms of Triaxial Stress

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z)$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x)$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y)$$

- Unit Volume Change

$$e = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$$

- Stresses in terms of strains

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)\right]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)\right]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)\right]$$

Triaxial Stress
Strain Energy Density

- Strain Energy Density, u, in Triaxial Stress (no shear stress)

$$u = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z)$$

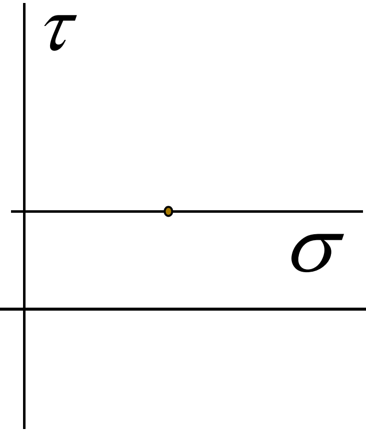
  – Strain Energy Density in terms of stresses

$$u = \frac{1}{2E}(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2) - \frac{\nu}{E}(\sigma_x \sigma_y + \sigma_x \sigma_z + \sigma_y \sigma_z)$$

  – Strain Energy Density in terms of strains

$$u = \frac{E}{2(1+\nu)(1-2\nu)}\left[(1-\nu)(\varepsilon_x{}^2 + \varepsilon_y{}^2 + \varepsilon_z{}^2) + 2\nu(\varepsilon_x \varepsilon_y + \varepsilon_x \varepsilon_z + \varepsilon_y \varepsilon_z)\right]$$

Triaxial Stress
Spherical Stress



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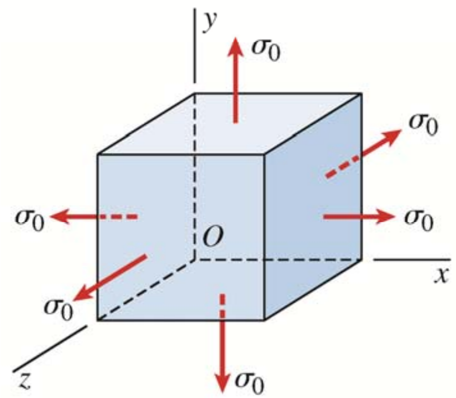
- Spherical Stress :
  - when three normal stresses are equal $\sigma_x = \sigma_y = \sigma_z = \sigma_0$
  - Any plane cut through the element will be subjected to the same normal stress σ_0
  - Normal Strain $\varepsilon_0 = \frac{\sigma_0}{E}(1-2\nu)$
  - Unit volume change

$$e = 3\varepsilon_0 = \frac{3\sigma_0}{E}(1-2\nu) = \frac{\sigma_0}{K}$$



  - Bulk modulus (of elasticity), K $K = \frac{E}{3(1-2\nu)} = \frac{1}{\beta}$ **Compressibility, β**
  - Uniform pressure in all directions: **<u>Hydrostatic</u>**
    - An object submerged in water or deep rock within the earth

Plane Strain

Outline

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- Introduction
- Plane Stress
- Principal Stresses and Maximum Shear Stresses
- Mohr's Circle for Plane Stress
- Hooke's Law for Plane Stress
- Triaxial Stress
- Plane Strain