Where do we stand?

25 May 2009

10 March

Equation of Equilibrium and fluid flow.

$$\tau_{ij} = 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk}\delta_{ij} + \alpha P_p \delta_{ij}$$
$$= 2\mu\left(\frac{1}{2}(u_{i,j}+u_{j,i})\right) + \lambda \varepsilon_{kk}\delta_{ij} + \alpha P_p \delta_{ij}$$
$$\tau_{ij,j} + f_i = \mu(u_{i,jj}+u_{j,ij}) + \lambda u_{k,kj}\delta_{ij} + \alpha P_{p,j}\delta_{ij} = 0$$
$$= \mu u_{i,jj} + \mu u_{j,ji} + \lambda u_{k,ki} + \alpha P_{p,i} + f_i = 0.$$
$$\underline{\mu u_{i,jj} + (\lambda+\mu) u_{j,ji} + (f_i + \alpha P_{p,i}) = 0.}$$

Biot coefficient $\left(\alpha \frac{\partial P_p}{\partial x_1},\ \alpha\frac{\partial P_p}{\partial x_2},\ \alpha\frac{\partial P_p}{\partial x_3}\right)$

looks like body force.

(mathematically) play the role of additional body force.

P_p $\quad$ P_p $\quad$ if you forget about blood it makes sense.

hw

in terms of excess fluid content, ζ

$$\mu u_{i,jj} + (\lambda+\mu+\alpha^2 M) u_{j,ji} + f_i + \alpha M \zeta_{,i} = 0.$$

$$\frac{BK}{\alpha(1-\alpha B)}$$

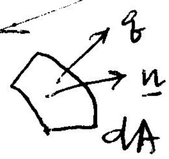
3 eqn & 4 unknowns.

We need one more eqn.

Start with fluid flow vector, q_i (m/s)

we are not talking about individual parts.

REV



$q \cdot n \, dA$ = volumetric flux through dA.

tangential component q_t is just circulating in the surface of dA.

$$q' = R q R' = \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix}$$

Total Flow out of this region R ∂R

$$= \iint_{\partial R} q\cdot n \, dA = \iint_{\partial R} q_i n_i \, dA = -\frac{\partial}{\partial t}\iiint_R \zeta \, dV = -\iiint_R \frac{\partial \zeta}{\partial t} dV.$$

(we already subtracted the compressibility term).

$\frac{\partial}{\partial t}\iiint \zeta \, dV$: mass transfer across the outer boundary

$\iint q \cdot n \, dA$: total volumetric flux of fluid leaving this region per unit time!

div theorem

$$\iiint_R q_{i,i}\, dV = -\iiint \frac{\partial \xi}{\partial t}\, dV$$

$$\iiint_R \left(q_{i,i} + \frac{\partial \xi}{\partial t}\right) dV = 0$$

$$q_{i,i} + \frac{\partial \xi}{\partial t} = 0 .$$

How about $q_{i,i}$?

→ use Darcy's law.

$$q = -\frac{1}{\mu}\underline{\underline{k}}\ \mathrm{grad}\, P_p \Rightarrow \underline{q_i = -\frac{1}{\mu} k_{ij} P_{p,j}}$$

$$q_1 = -\frac{1}{\mu}\left(k_{11}\frac{\partial P}{\partial x_1} + k_{12}\frac{\partial P}{\partial x_2} + k_{13}\frac{\partial P}{\partial x_3}\right)$$

similarly for q_2 & q_3.

$$\underline{\underline{k}} = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}$$

theorem. k is symmetric, ie, $k_{ij} = k_{ji}$ ex) $K_{12} = K_{21}$.

For isotropic

Robert think there is no proof about this
empiricism

$$\underline{k} = k\,\delta_{ij} = \begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix} .$$

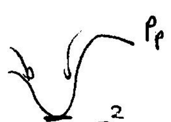
1 Darcy $\fallingdotseq 10^{-12}\ m^2$

don't use 'K', hydraulic conductivity.

$$q_i = -\frac{k}{\mu}\delta_{ij} P_{p,j} = -\frac{k}{\mu} P_{p,i} = -\frac{k}{\mu}\ \mathrm{grad}\, P$$

$$-\frac{k}{\mu} \cdot P_{p,ii} + \frac{\partial \xi}{\partial t} = 0 \qquad \frac{\partial \xi}{\partial t} = \frac{k}{\mu} P_{p,ii} = \frac{k}{\mu}\nabla^2 P_p$$

$$\frac{\partial \xi}{\partial t} = \frac{k}{\mu} \cdot \nabla^2 P_p$$



$\nabla^2 P_p = \oplus \quad \frac{\partial \xi}{\partial t} > 0$.

$\nabla^2 P_p = \ominus \quad \frac{\partial \xi}{\partial t} < 0$

k : a definition at a steady state
ability to transmit fluid.

a measure of how quickly a material can carry flow away from a source.

$$d\zeta = -\phi\left[C_{pc}\, dP_c - (C_{pp} + C_f)\, dP_p\right]$$
→ uncoupled case.

$\frac{k}{\rho C}$ = α — specific heat thermal capacity diffusivity.

special case $P_p^c = 0$.

$$d\zeta = \phi\,(C_{pp} + C_f)\, dP_p .$$

$$\phi\,(C_{pp} + C_f)\frac{\partial P_p}{\partial t} = \frac{k}{\mu}\left(\frac{\partial^2 P_p}{\partial x^2} + \frac{\partial^2 P_p}{\partial y^2} + \frac{\partial^2 P_p}{\partial z^2}\right)$$

$$\frac{\partial P_p}{\partial t} = \frac{k}{\phi\,\mu\,(C_{pp} + C_f)}\nabla^2 P_p .$$
→ important for petroleum

rigid → $C_{pp} = 0$ $(C_{pc} = 0)$ → $\frac{\partial P_p}{\partial t} = \frac{k}{\phi \mu C_f}\nabla^2 P_p$. $\frac{k}{\mu S}$.

→ Hydraulic diffusivity.

For nonrigid case,

recall $P_p = M\zeta + \alpha M \varepsilon_b$.

$$\zeta = \frac{1}{M}P_p - \alpha\,\varepsilon_b$$

$\varepsilon_b = \frac{1}{K}(P_c - \alpha P_p)$

$\zeta = -\frac{\alpha}{K}P_c + \frac{\alpha}{KB}P_p$.

$$\frac{1}{M}\frac{\partial P_p}{\partial t} - \alpha\frac{\partial \varepsilon_b}{\partial t} = \frac{k}{\mu}\nabla^2 P_p$$

diffusivity for constant strain

$$\frac{\partial P_p}{\partial t} = \alpha M\frac{\partial \varepsilon_b}{\partial t} + \frac{M}{\mu}k\,\nabla^2 P_p .$$
→ fourth Equation.

express in terms of pressure/stress

$$\varepsilon_b = \frac{1}{K}(P_c - \alpha P_p)$$

$$\frac{\partial P_p}{\partial t} = \frac{\alpha M}{K}\frac{\partial P_c}{\partial t} - \frac{\alpha^2 M}{K}\frac{\partial P_p}{\partial t} + \frac{Mk}{\mu}\nabla^2 P_p .$$

$$\frac{\partial P_p}{\partial t}\left(1 + \frac{\alpha^2 M}{K}\right) = \frac{\alpha M}{K}\frac{\partial P_c}{\partial t} + \frac{Mk}{\mu}\nabla^2 P_p . \qquad M = \frac{BK}{\alpha(1-\alpha B)} .$$

$$\frac{\partial P_p}{\partial t}\underbrace{\left(1 + \frac{\alpha B}{1-\alpha B}\right)}_{\frac{1}{1-\alpha B}} = \frac{B}{1-\alpha B}\frac{\partial P_c}{\partial t} + \frac{BKk}{\alpha\mu(1-\alpha B)}\nabla^2 P_p .$$

$$\frac{\partial P_p}{\partial t} = B\cdot\frac{\partial \sigma_m}{\partial t} + \frac{BKk}{\mu\alpha}\nabla^2 P_p$$
→ diffusivity for constant stress.

if we don't consider fluid, we see Skempton.

Special case 1, τ_m = constant.

$$\frac{\partial p_p}{\partial t} = \frac{BKk}{\mu\alpha}\nabla^2 p_p .$$

$\hookrightarrow$ diffusivity coeff.

Special case 2, ε_b = constant

$$\frac{\partial p_p}{\partial t} = \frac{BK}{\alpha(1-\alpha B)}\frac{k}{\mu}\cdot\nabla^2 p_p .$$

$\hookrightarrow$ replace K with $\frac{K}{1-\alpha B}$.

See, p. ~~437~~-439 188

$$\Rightarrow \frac{\partial \zeta}{\partial t} = \frac{k}{\mu S}\nabla^2 \zeta , \quad S = \frac{1}{M} + \frac{\alpha^2}{\lambda+2\mu} = \frac{1}{M} + \frac{\alpha^2}{K+\frac{4}{3}\mu}$$

no coupling terms? But coupled in the Boundary condition

$$= (C_f - C_m)\phi + \left[1 + \frac{2(1-\nu)\alpha}{3(1-\nu)}\right]\alpha C_{BC}$$

(Zimmerman)

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undrained moduli $(\mu_u = \mu)$

$$K_u = \frac{K}{1-\alpha B} . \qquad G_u = G$$

recall, $\varepsilon^{dev} = \frac{1}{2\mu}\tau_m$

$$\nu = \frac{3K-2\mu}{6K+2\mu} . \qquad \nu_u = \frac{3K_u-2\mu}{6K_u+2\mu} .$$

$$\nu_u = \frac{3\cdot\frac{K}{1-\alpha B} - 2\mu}{6\cdot\frac{K}{1-\alpha B} + 2\mu} = \frac{3K-2\mu(1-\alpha B)}{6K+2\mu(1-\alpha B)} .$$

$\nu_u > \nu$.

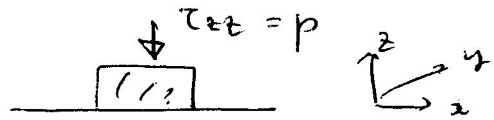
$0 < \nu < \nu_u < 0.5$.

0.15–0.25   0.25–0.35

Soil $\to$ 1/2 not so different

But rock is different.

Terzaghi's problem. (consolidation)



$T_m > 0$.

time to equilibriate $p_p \uparrow$.

$$t = \frac{L^2}{D} = \frac{L^2}{k}$$

function of both size (L) & k .

General expression

$$\frac{\partial \zeta}{\partial t} = \frac{k}{\mu} \nabla^2 p_p$$

① uncoupled case

$$\frac{\partial P_p}{\partial t} = \frac{k}{\phi \mu (C_{pp} + C_f)} \nabla^2 p_p ,$$

② uncoupled case – rigid rock

$$\frac{\partial P_p}{\partial t} = \frac{k}{\phi \mu C_f} \nabla^2 p_p .$$

③ $$\frac{\partial P_p}{\partial t} = \alpha M \frac{\partial \varepsilon_b}{\partial t} + \left(\frac{M}{\mu} k\right) \nabla^2 p_p$$

coupled case – expressed in terms of strain.

diffusivity under constant strain.

④ coupled case – expressed in terms of stress.

$$\frac{\partial P_p}{\partial t} = B \frac{\partial \tau_m^{P_c}}{\partial t} + \left(\frac{BKk}{\mu \alpha}\right) \nabla^2 p_p$$

diffusivity for constant stress.

★ Specific storage coefficient, S, $1/\mathrm{Pa}$

ratio of the change in the volume of water added to the storage per unit aquifer volume divided by the change in pore pressure.

$$\frac{\zeta}{P_p} \qquad \frac{\frac{\Delta V_f}{V}}{P_p}.$$

① rigid rock

$$S = \phi C_f \rightarrow \frac{\partial P_p}{\partial t} = \frac{k}{\phi \mu C_f}\nabla^2 P_p, \quad c = \frac{k}{\phi \mu C_f}.$$

② under constant stress, S_σ $\left|\frac{\zeta}{P_p}\right|_{\sigma=0}$.

$$\zeta = -\frac{\alpha}{K}P_c + \frac{\alpha}{KB}P_p$$

$$S_\sigma = \frac{\alpha}{KB}$$

$$B = \frac{C_{pc}}{C_{pp}+C_f} \qquad K - C_{Bc}$$

$$\cancel{\frac{\partial P_p}{\partial t} = \frac{k}{\mu S_\sigma}\nabla^2 P_p - \frac{\partial P_c}{\partial t}}$$

$$C_{Bc} - C_{Bp} = C_m \qquad C_{Bp} = \phi C_{pc}.$$

$$C_{Bc} - C_{pp} = C_m$$

③ under constant strain, S_ε $\left|\frac{\zeta}{P_p}\right|_{\varepsilon=0}$.

$$\varepsilon = \frac{1}{K}P_c - \frac{\alpha}{K}P_p \rightarrow P_c = \alpha P_p.$$

$$\zeta = -\frac{\alpha}{K}P_c + \frac{\alpha}{KB}P_p. \quad = -\frac{\alpha^2}{K}P_p + \frac{\alpha}{KB}P_p = \frac{\alpha - \alpha^2 B}{KB}P_p$$

$$\frac{\zeta}{P_p} = \frac{\alpha(1-\alpha B)}{KB} = \frac{\alpha}{K_u B}. \qquad \frac{KB}{\alpha(1-\alpha B)} \quad \frac{\alpha(1-\alpha B)}{KB}$$

$$\frac{\partial P_p}{\partial t} = \frac{k}{\mu S_\varepsilon}\nabla^2 P - \alpha M\frac{\partial \varepsilon}{\partial t}.$$

④ 'hydrologic' definition = "the volume of water released ~~from~~ per unit decline of head per unit bulk volume while maintaining the REV in a state of zero lateral strain and constant vertical stress"

$$S_s = (\rho_f g)\cdot\left.\frac{\zeta}{P}\right|_{\varepsilon_{xx}=\varepsilon_{yy}=0,\ \tau_{zz}=0} \qquad \frac{\partial P_p}{\partial t} = \frac{k}{\mu S_{hydro}}\nabla^2 P_p.$$

Specific storage coefficient for hydrogeologist, S_{hydro}

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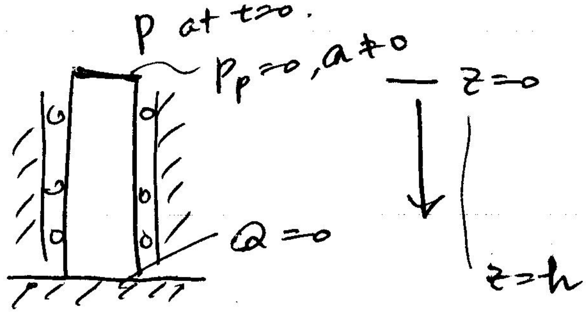
$$S_{hydro} = \frac{S_s}{\rho_f g} = S_\sigma \left(1 - \frac{4\eta B}{3}\right)$$

$$\eta = \frac{1-2\nu}{2(1-\nu)}\alpha$$

1D consolidation,



at $t=0$, $\sigma = p$, porepressure $= B \cdot p$.
displacement = undrained E.
w

$$\tau_{zz}^{\circ} = p, \quad \tau_{xx}^{\circ} = \tau_{yy}^{\circ} = \nu p (1-\nu).$$

$$\varepsilon_{zz}^{\circ} = \frac{p}{(\lambda + 2G + \alpha^2 M)}. \quad \varepsilon_{xx}^{\circ} = \varepsilon_{yy}^{\circ} = 0,$$

shear stress & strains are zero.

$$P_p^{\circ} = \frac{\alpha M}{(\lambda + 2G + \alpha^2 M)} p.$$

$$w^{\circ} = \frac{p}{\lambda + 2G + \alpha^2 M}(z-h)$$

satisfy Navier(P) & diffusion equation.
with $\qquad \varepsilon_b$ or σ.

$$G\nabla^2 \underline{\underline{u}} + (\lambda+G)\nabla(\nabla\cdot\underline{\underline{u}}) = -f - \alpha\nabla P_p.$$

$$(\lambda+2G)\frac{\partial^2 w(z,t)}{\partial z^2} = -\alpha\frac{\partial P_p(z,t)}{\partial z}$$

by integrating,

$$(\lambda+2G)\frac{\partial w(z,t)}{\partial z} + \alpha P_p(z,t) = g(t)$$

$$\tau_{zz}(z,t) = g(t). \quad z=0 \rightarrow \tau_{zz} = p.$$

$$\tau_{zz}(z,t) = p.$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}. \quad \varepsilon_{zz}(z,t) = \frac{1}{\lambda+2G}\left[p - \alpha P_p(z,t)\right] \quad \text{integrali}$$

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in uniaxial strain $\varepsilon_b = \varepsilon_{zz}$.

$$\frac{\partial \varepsilon_b}{\partial t} = \frac{-\alpha}{\lambda+2G}\frac{\partial P_p}{\partial t}$$

$$+ \frac{\partial P_p}{\partial t} = \frac{kM}{\mu}\nabla^2 P_p + \alpha M \frac{\partial \varepsilon_b}{\partial t}$$

$$\frac{k}{\mu}\nabla^2 P_p = \left(\frac{1}{M} + \frac{\alpha^2}{\lambda+2G}\right)\frac{\partial P_p}{\partial t} \equiv S\frac{\partial P_p}{\partial t}.$$

diffusivity $D = \frac{k}{\mu S}$: coefficient of consolidation.

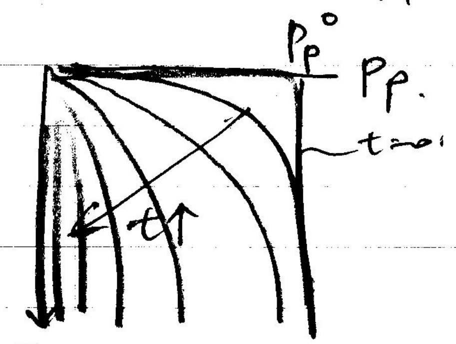
Solve this solution.

$t=0$, $P_p(z,0) = P_p^{\circ} = \frac{\alpha M}{\lambda+2G+\alpha^2 M} p$.

$P_p(z=0,t) = 0$, $\frac{\partial P_p}{\partial z}(z=h,t) = 0$.

The solution of this can be obtained from heat conduction Equation. (Carslaw & Jaeger (1959))

$$P_p(z,t) = \frac{\alpha M p}{\lambda+2G+\alpha^2 M}\sum_{n=1,3\ldots}^{\infty}\frac{4}{n\pi}\sin\left(\frac{n\pi z}{2h}\right)\exp\left(\frac{-n^2\pi^2 kt}{4\mu S h^2}\right)$$



$D = 1\times10^{-12}\ m^2$

critical time, $5 \cdot \exp(-5) < 0$.

$$t_{eq} = \frac{20\mu S h^2}{\pi^2 k} \doteq 2\mu S h^2 / k.$$

Berea sandstone

t a few second ~ several years.

$C_f = 5\times10^{-4}$/MPa.

$k = 190$ mD $= 190\times10^{-15}\ m^2$. Table 7.2.

$\mu = 10^{-3}$ Pa·s.

< Berea Sandstone
Westerly granite.

$$S = \phi C_f = 0.2 \times 5 \times 10^{-4} \times 10^{-6}$$

$$\frac{2 \times 10^{-3} \times 0.2 \times 5 \times 10^{10} \times 1}{\pi^2 \times 180 \times 10^{-15}} = \frac{100}{\pi^2 \times 180} =$$

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$$w(z,t) = \frac{P}{\lambda+2G}\left[(z-h) + \frac{\alpha^2 M h}{\lambda+2G+\alpha^2 M}\sum_{n=1,3}^{\infty}\frac{8}{n^2\pi^2}\cos\left(\frac{n\pi z}{2h}\right) \times \exp\left(\frac{-n^2\pi^2 kt}{4\mu S h^2}\right)\right]$$

w at $z=0$,

$$w(0,t) = \frac{-Ph}{\lambda+2G}\left[1 - \frac{\alpha^2 M}{\lambda+2G+\alpha^2 M}\sum_{n=1,3}^{\infty}\frac{8}{n^2\pi^2}\exp\left(\frac{-n^2\pi^2 kt}{4\mu S h^2}\right)\right]$$

$t=0$ $\quad \frac{1}{1}+\frac{1}{3^2}+\frac{1}{5^2}+\dots = \frac{\pi^2}{8}$. $\quad w = \frac{-Ph}{(\lambda+2G+\alpha^2 M)}$.

undrained uniaxial strain.

$t=\infty$. $\quad w = \frac{-Ph}{\lambda+2G}$.