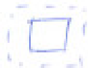
7. Thermal Stresses & Deformations

will look at

- free expansion of a ply
- constraint & thermal stress
- rotation of plies
- laminate & effective properties
- stresses & deformation

Consider a body changing temperature



$$\epsilon = \alpha \Delta T = \alpha (T - T_0)$$

coefficient of Thermal Expansion, CTE

Ref. Temperature

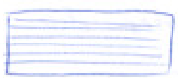
Fiber : Anisotropic CTE

$\alpha_L \simeq -.5 \ \mu\epsilon/°F$

α_T small positive $\simeq 2 \sim 3 \ \mu\epsilon/°F$

Matrix, isotropic

$\alpha \simeq 20$ to $30 \ \mu\epsilon/°F$



Ply? Did micromechanics

→ ply equivalent properties

Also microstresses between fiber & matrix some, will ignore these here

Ply Properties (Gr/Ep material)

$\alpha_L \simeq -1.0$ to $+5 \ \mu\epsilon/°F$

$\alpha_T \simeq 16 \ \mu\epsilon/°F$

Consider In-plane Thermal strains

$$\epsilon^T = \alpha\,\Delta T$$

$$\epsilon^T = \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} \qquad \alpha = \begin{Bmatrix} \alpha_L \\ \alpha_T \\ 0 \end{Bmatrix}$$

No stress!

Before had $\sigma = Q\,\epsilon$

Now need new constitutive law.

To modify, note $\epsilon = \epsilon^M + \epsilon^T$

total strain — "real" $\simeq \frac{\Delta l}{l}$ mechanical   thermal

$\epsilon^M = S\,\sigma$ mechanical stress-strain

$$\epsilon = S\,\sigma + \epsilon^T = S\,\sigma + \alpha\,\Delta T$$

or $$\sigma = Q\,\{\epsilon - \alpha\,\Delta T\}$$

[Thermoelastic stress-strain laminate]

What if constrained?

$$\epsilon = 0$$

$$\epsilon = 0 = S\,\sigma + \alpha\,\Delta T$$

$$S\,\sigma = -\alpha\,\Delta T$$

$$\sigma = -Q\,\alpha\,\Delta T$$

in 1-Dim, $\sigma_x = -E\,\alpha\,\Delta T$

The σ obtained is called "Thermal Stress"

Actually, this is a Misnomer

Thermal strain O.K.

Thermal stresses caused by mechanical forces due to constraints.

Also one defines "equivalent thermal stress"

$$\sigma^T = +Q\,\alpha\,\Delta T$$

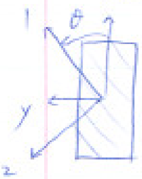
This is a fiction but is computationally useful.

$$\sigma = \sigma^m + \sigma^T$$

$$\epsilon = S\,\sigma = S\,\sigma^m + \overbrace{S\,Q}^{I}\,\alpha\,\Delta T$$

allows one to use old constitutive law with "false" thermal stress

- Ply at Arbitrary Angle



$\bar{\underset{\sim}{\epsilon}} = \underset{\sim}{T_\epsilon}^{-1} \underset{\sim}{\epsilon}$: strain transformation

(lamin.) (ply)

$\bar{\underset{\sim}{\alpha}} \Delta T = T_\epsilon^{-1} \alpha \Delta T$

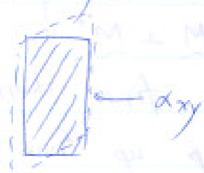
$\boxed{\bar{\underset{\sim}{\alpha}} = \underset{\sim}{T_\epsilon}^{-1} \underset{\sim}{\alpha}}$ ← CTE in laminate axes

where,

$$\underset{\sim}{T_\epsilon}^{-1} = \underset{\sim}{T_\sigma}^{T} = \begin{bmatrix} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2cs & -2cs & (c^2-s^2) \end{bmatrix}$$

In general, $\bar{\underset{\sim}{\alpha}} = \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}$

Can get shear



- Laminate Thermal Properties

Have for a single ply

$\underset{\sim}{\epsilon} = \epsilon^m + \underset{\sim}{\epsilon}^T = \underset{\sim}{S} \underset{\sim}{\sigma} + \underset{\sim}{\alpha} \Delta T$ (ply coords)

$\bar{\underset{\sim}{\epsilon}} = \underset{\sim}{\epsilon}^0 + \underset{\sim}{\kappa} z = \bar{\underset{\sim}{S}} \bar{\underset{\sim}{\sigma}} + \bar{\underset{\sim}{\alpha}} \Delta T$ (laminate coordinate)

For laminate, want Force & Moment Resultants,

$\underset{\sim}{N} = \int \bar{\underset{\sim}{\sigma}} \, dz , \quad \underset{\sim}{M} = \int \bar{\underset{\sim}{\sigma}} z \, dz$

Rewriting stress - strain, get

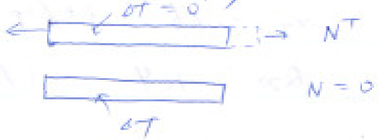
$\bar{\underset{\sim}{\sigma}} = \bar{\underset{\sim}{Q}} (\underset{\sim}{\epsilon}^0 + \underset{\sim}{\kappa} z - \bar{\underset{\sim}{\alpha}} \Delta T)$

$\underset{\sim}{N} = \int \bar{\underset{\sim}{\sigma}} \, dz = \underbrace{\left(\int \bar{\underset{\sim}{Q}} \, dz\right)}_{\text{"}A} \underset{\sim}{\epsilon}^0 + \underbrace{\left(\int \bar{\underset{\sim}{Q}} z \, dz\right)}_{\text{"}B} \underset{\sim}{\kappa} - \underbrace{\int \bar{\underset{\sim}{Q}} \bar{\underset{\sim}{\alpha}} \Delta T \, dz}_{\text{"}N^T}$

Last is what we call "Thermal Force"

$\underset{\sim}{N}^T = \int \bar{\underset{\sim}{Q}} \bar{\underset{\sim}{\alpha}} \Delta T \, dz$ (fake useful quantity)

$\underset{\sim}{N}^T$ is not a physical load, it is a convenience



N^T is mechanical load necessary to provide same deformation in laminate as ΔT with no N.

For "thermal stresses",

$$\bar{\sigma} = \bar{Q}\,(\epsilon^\circ + \kappa z - \bar{\alpha}\,\Delta T)$$

$$N = \int \bar{\sigma}\, dz = A\,\epsilon^\circ + B\,\kappa - N^T$$

Likewise,

$$M = \int \bar{\sigma}\, z\, dz = B\,\epsilon^\circ + D\,\kappa - \underbrace{\int \bar{Q}\,\bar{\alpha}\,\Delta T\, z\, dz}_{= M^T}$$

Combining,

$$\begin{Bmatrix} N + N^T \\ M + M^T \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \epsilon^\circ \\ \kappa \end{Bmatrix}$$

or

$$\begin{Bmatrix} \epsilon^\circ \\ \kappa \end{Bmatrix} = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{Bmatrix} N + N^T \\ M + M^T \end{Bmatrix}$$

If laminate is unloaded — free thermal deformation

One step up from single ply case

Plies may have stresses, but $N = M = 0$

$$\begin{Bmatrix} \epsilon^\circ \\ \kappa \end{Bmatrix} = \begin{bmatrix} a & b \\ b^T & d \end{bmatrix} \begin{Bmatrix} N^T \\ M^T \end{Bmatrix}$$

If ΔT constant with z

$$= \begin{bmatrix} a & b \\ b^T & d \end{bmatrix} \begin{Bmatrix} \int \bar{Q}\,\bar{\alpha}\, dz \\ \int \bar{Q}\,\bar{\alpha}\, z\, dz \end{Bmatrix} \Delta T$$

In symmetric case, $b = 0$, $M^T = 0$

$$\begin{Bmatrix} \epsilon_x^\circ \\ \epsilon_y^\circ \\ \gamma_{xy}^\circ \end{Bmatrix} = \underbrace{a \int \bar{Q}\,\bar{\alpha}\, dz}_{" \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}} \Delta T$$

$\leftarrow$ Engineering CTE of Laminate

Single ply case $\rightarrow$ $\epsilon^\circ = \bar{\alpha}\,\Delta T$

This is stiffness-weighted rotated average CTE of each ply.

— order doesn't matter. (like A)

NOTE: $\alpha_L = -.5$ $\quad \alpha_T = 16$ $\mu\epsilon/^\circ F$

$Q_{11} = 20$ $\quad Q_{22} = 1.4$ (AS4/3501-6)

Can play off α and ply angle θ to get zero CTE's
$E_L \gg E_T$ helps (scissor's effect with θ)

- Bending

If ΔT constant and laminate symmetric,

$$\underset{\sim}{M}^T = \int \bar{\underset{\sim}{Q}}\, \bar{\underset{\sim}{\alpha}}\, \Delta T\, z\, dz \rightarrow 0 \quad \text{no bending}$$

If ΔT gradient and laminate symmetric, $\underset{\sim}{M}^T \neq 0$
laminate bends / twists.

If laminate unsymmetric, $\underset{\sim}{B}$ and $\underset{\sim}{M}^T \neq 0$,
laminate bends / twists (some exceptions)

Note on ΔT

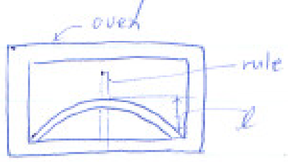
$$\Delta T = T - T_0, \quad \text{what is } T_0 ?$$

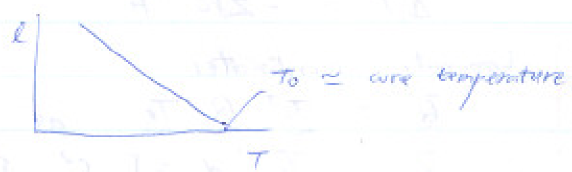
An experiment — $[0/90]_T$

$\leftarrow$ @ cure temperature

@ room temperature $\rightarrow$

$\Delta T \neq 0$





T_0 usually the cure temperature

Note: To calculate N^T & M^T for ΔT = const.

$$\underset{\sim}{N}^T \equiv \int \bar{\underset{\sim}{Q}}\, \bar{\underset{\sim}{\alpha}}\, \Delta T\, dz = \Delta T \sum^{K} \bar{\underset{\sim}{Q}}^k\, \bar{\underset{\sim}{\alpha}}^k\, (z_{uk} - z_{lk})$$

$$\underset{\sim}{M}^T \equiv \int \bar{\underset{\sim}{Q}}\, \bar{\underset{\sim}{\alpha}}\, \Delta T\, z\, dz = \Delta T\, \frac{1}{2} \sum^{K} \bar{\underset{\sim}{Q}}^k\, \bar{\underset{\sim}{\alpha}}^k\, (z_{uk}^2 - z_{lk}^2)$$

Same as $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{D}$ matrices

Always use z_{uk} and z_{lk} for each ply, rather than z_K and z_{K-1}

(confusing, z direction signs, Jones book)

- Thermal Stresses in Plies

Have
$$\begin{Bmatrix} \underset{\sim}{\epsilon}^0 \\ \underset{\sim}{\kappa} \end{Bmatrix} = \begin{bmatrix} a & b \\ b^T & d \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N} + \underset{\sim}{N}^T \\ \underset{\sim}{M} + \underset{\sim}{M}^T \end{Bmatrix}$$

What happens at ply level ?

Total strains are just

$$\bar{\epsilon} = \epsilon^{\circ} + z\,\kappa\,, \quad \text{(laminate coordinates)}$$

Just transform to get ply coordinate

$$\epsilon = T_\epsilon\, \bar{\epsilon} \qquad \text{(ply coordinates)}$$

Mechanical strain (these cause stress in material)

$$\bar{\epsilon}^M = \bar{\epsilon} - \bar{\alpha}\,\Delta T$$

↳ total ↳ thermal

What are stresses ?

Recall, $\bar{\sigma} = \bar{Q}\,\bar{\epsilon}^M = \bar{Q}\,(\bar{\epsilon} - \bar{\alpha}\,\Delta T)$ (laminate coordinate)

Also $\sigma = T_\sigma\, \bar{\sigma}$ (ply coordinate)

Example → $[0/90]_s$ T300/934 material

$$Q = \begin{bmatrix} 20.1 & .4 & 0 \\ .4 & 1.4 & 0 \\ 0 & 0 & .7 \end{bmatrix} \text{Msi}, \quad \alpha = \begin{Bmatrix} +.05 \\ 16.0 \\ 0 \end{Bmatrix} \mu\epsilon/^{\circ}F$$

$T_0 = 350\,^{\circ}F$ (stress - free)

$T = 70\,^{\circ}F$

$\Delta T = -280\,^{\circ}F$

Laminate coordinates

$$\bar{Q} = T_\epsilon^T\, Q\, T_\epsilon \quad \text{as before}$$

$$\bar{\alpha} = T_\epsilon^{-1}\,\alpha = \begin{bmatrix} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2sc & -2sc & (c^2-s^2) \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{Bmatrix}$$

$$\bar{n} = \bar{Q}\,\alpha = \begin{Bmatrix} n_x \\ n_y \\ n_{xy} \end{Bmatrix}$$

$$N^T = \int \bar{Q}\,\bar{\alpha}\,\Delta T\,dz = \Delta T \sum \bar{n}^k (z_{uk} - z_{\ell k})$$

$$M^T = \int \bar{Q}\,\bar{\alpha}\,\Delta T\,z\,dz = \Delta T\,\tfrac{1}{2} \sum \bar{n}^k (z_{uk}^2 - z_{\ell k}^2)$$

(for symmetric laminate, $M^T = 0$)

For 0° ply,

$$\bar{\alpha} = \begin{Bmatrix} c^2\alpha_1 + s^2\alpha_2 \\ s^2\alpha_1 + c^2\alpha_2 \\ 2cs(\alpha_1 - \alpha_2) \end{Bmatrix} = \begin{Bmatrix} +.05 \\ 16.0 \\ 0 \end{Bmatrix} \mu\epsilon/^{\circ}F$$

$$\bar{n} = \begin{bmatrix} 20.1 & .4 & 0 \\ .4 & 1.4 & 0 \\ 0 & 0 & .7 \end{bmatrix} \begin{Bmatrix} .05 \\ 16 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 7.41 \\ 22.4 \\ 0 \end{Bmatrix} \text{ lbs/in}^2\,{}^\circ F$$

$\times 10^6$ $\times 10^{-6}$

For 90° ply

$$\underset{\sim}{\bar{\alpha}} = \begin{Bmatrix} \\ \\ \\ \end{Bmatrix} = \begin{Bmatrix} 16.0 \\ .05 \\ 0 \end{Bmatrix} \mu\epsilon/{}^\circ F$$

$$\underset{\sim}{\bar{n}} = \begin{bmatrix} 1.4 & .4 & 0 \\ .4 & 20.1 & 0 \\ 0 & 0 & .7 \end{bmatrix} \begin{Bmatrix} 16.0 \\ .05 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 22.4 \\ 7.41 \\ 0 \end{Bmatrix}$$

ply	z_{ku}	z_{kl}	$z_{uk} - z_{lk}$	$\bar{n}_x$	$\bar{n}_y$	$\bar{n}_{xy}$
0°	.010	.005	.005	7.41	22.4	0
90°	.005	0	.005	22.4	7.41	0

Sym.

$$\underset{\sim}{N}^T = \begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix} = \Delta T \sum^K \underset{\sim}{\bar{n}}^k (z_{uk} - z_{lk}) = (-280) \begin{Bmatrix} .298 \\ .298 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} -84 \\ -84 \\ 0 \end{Bmatrix} \text{ lbs/in}$$

$$\underset{\sim}{M}^T = \begin{Bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix} = \Delta T \tfrac{1}{2} \sum \underset{\sim}{\bar{n}}^k (z_{uk}^2 - z_{lk}^2) = 0 \quad \text{(Symmetric)}$$

$$\underset{\sim}{A} = \begin{bmatrix} .215 & .0082 & 0 \\ .0082 & .215 & 0 \\ 0 & 0 & .014 \end{bmatrix} \times 10^6$$

$$\underset{\sim}{\epsilon}^\circ = \underset{\sim}{a}\, \underset{\sim}{N}^T = \begin{bmatrix} 4.65 & -.177 & 0 \\ -1.77 & 4.65 & 0 \\ 0 & 0 & 71.4 \end{bmatrix} \begin{Bmatrix} -84 \\ -84 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -377 \\ -377 \\ 0 \end{Bmatrix} \mu\epsilon$$

$\times 10^{-6}$

$$\underset{\sim}{\bar{\alpha}}_{average} = a \int \bar{Q} \bar{\alpha} \, dz = \epsilon^\circ / \Delta T = \begin{Bmatrix} 1.3 \\ 1.3 \\ 0 \end{Bmatrix} \mu\epsilon/^\circ F$$

0° Ply

$$\underset{\sim}{\bar{\epsilon}}^m = \underset{\sim}{\epsilon}^\circ - \bar{\alpha} \Delta T = \begin{Bmatrix} -377 \\ -377 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0.5 \\ 16 \\ 0 \end{Bmatrix} (-280) = \begin{Bmatrix} -363 \\ 4103 \\ 0 \end{Bmatrix} \mu\epsilon$$

$$\underset{\sim}{\bar{\sigma}} = \underset{\sim}{\bar{Q}} \underset{\sim}{\epsilon}^m = \underbrace{\begin{bmatrix} 20.1 & .4 & 0 \\ .4 & 1.4 & 0 \\ 0 & 0 & .7 \end{bmatrix}}_{\times 10^6} \underbrace{\begin{Bmatrix} -363 \\ 4103 \\ 0 \end{Bmatrix}}_{\times 10^{-6}} = \begin{Bmatrix} -5600 \\ +5600 \\ 0 \end{Bmatrix} lbs/in^2$$

Similarly obtain 90° ply

In ply coordinate, $\underset{\sim}{\sigma}^\circ = \underset{\sim}{T_\sigma} \underset{\sim}{\bar{\sigma}}^\circ = \underset{\sim}{\bar{\sigma}}^\circ$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} -5.6 \\ 5.6 \\ 0 \end{Bmatrix} ksi$$

Recall allowables

σ_1	σ_2	σ_6
+190	+6	+10
-160	-25	

Residual stresses close to allowable Y_t here.

Can be quite significant even if flat.

In progressive failure analysis, should include this.

$$\underset{\sim}{N}^{TOT} = \lambda \underset{\sim}{N}^0 + \underset{\sim}{N}^T$$

include this

A little complicates

See Jones Sec. 4 failure with ΔT

◦ Summary

Thermal strains $\underset{\sim}{\alpha} \Delta T$ cause residual stresses due to cool down,

$\Delta T = -280\ ^\circ F$

For symmetric laminates, $\underset{\sim}{\kappa} = 0 \rightarrow$ no accompanying warping

For unsymmetric laminate, $\underline{\kappa} \neq 0$ →

warping $\begin{cases} \kappa_x, \kappa_y & : \text{bending} \\ \kappa_{xy} & : \text{twisting} \\ = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \alpha & \end{cases}$

One unsymmetric laminate that doesn't warp,
$[\theta/(\theta-90)_2/\theta]_A$
i.e., $[\theta/(\theta-90)_2/\theta/-\theta/-(\theta-90)_2/-\theta]_t$
(Also give extension - twist coupling)

◦ Moisture

See Tsai & Hahn, Chap. 8
Matrix absorbs water, and swells
By micromechanics, can calculate ply swelling
(also have microstresses, ignore here)

Hydro $\epsilon^h = \beta \, \Delta M$ — Moisture change = weight of moisture/dry weight
↳ CME : Coefficient of Moisture Expansion

$\Delta M = M - M_0$
$M_0 = 0$: Dry condition

Careful : ΔM sometimes expressed as percent (factor of 100)
$\Delta M \cong .5$ to 2% typical.

$\beta = \begin{Bmatrix} 45 \\ 5500 \\ 0 \end{Bmatrix} \mu\epsilon/\%$ for T300/934, $\alpha = \begin{Bmatrix} .05 \\ 16 \\ 0 \end{Bmatrix} \mu\epsilon/°F$

Note : typically, $\Delta T \simeq -280$, $\Delta M = 1\%$
$\epsilon^T + \epsilon^h = \alpha \Delta T + \beta \Delta M$

$= \begin{Bmatrix} -14 \\ -4480 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 45 \\ 5500 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 31 \\ 1{,}020 \\ 0 \end{Bmatrix} \mu\epsilon$

Moisture partly cancels some of strains.
Fortunate Relaxes Stresses.
CLPT works exactly same as before.
$N^h = \int \bar{Q} \, \bar{\beta} \, \Delta M \, dz$
$M^h = \int \bar{Q} \, \bar{\beta} \, \Delta M \, z \, dz$

where, $\bar{\beta} = T_\epsilon^{-1}\,\beta$

rotated β

$$\begin{Bmatrix}\epsilon^0\\ \chi\end{Bmatrix} = \begin{bmatrix} a & b \\ b^T & d \end{bmatrix}\begin{Bmatrix} N + N^T + N^h \\ M + M^T + M^h\end{Bmatrix}$$

$$\bar{\sigma} = \bar{Q}\,(\epsilon^0 + z\chi - \bar{\alpha}\,\Delta T - \bar{\beta}\,\Delta M)$$

Do CLPT as before

- Moisture Absorption

Define $m = \dfrac{\text{mass of water}}{\text{mass of dry material}}$

$M = m \times 100\ (\%)$

$\bar{m}$ = average through specimen

can measure

- Fick's Diffusion

$q^H = -D\dfrac{\partial m}{\partial x}$ (or generally, $q_i^H = -D_{ij}\dfrac{\partial M}{\partial x_j}$)

$\dfrac{\partial m}{\partial t} = -\dfrac{\partial}{\partial x}q^H = D_{ij}\dfrac{\partial^2 m}{\partial x_j^2}$ (3-Dim.)

$\dfrac{\partial m}{\partial t} = D\dfrac{\partial^2 m}{\partial z^2}$ (1-Dim.)

(like heat conduction)

D : diffusion constant $= K^H$

generally, $D = D_0\,e^{-C/T}$

T 300/1034 $\rightarrow$ $D_0 = 2.28\ \text{mm}^2/\text{sec}$

$C = 5554\ ^\circ K$

Moisture can effect cracks, cyclic effects, edge effects.

Equilibrium moisture content is, M_∞

Typically $m_\infty = m_{\infty,0}\,\phi$ in air

property of material

Ref. Humidity

$m_\infty = m_{\infty w}$ in water $\neq$ 100 °RH air

The m_∞ is usually the B.C.,

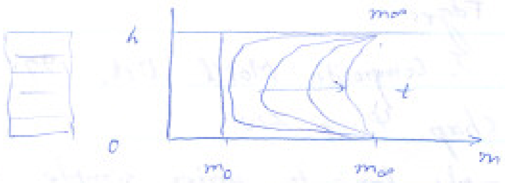
Differential Equation $\dfrac{\partial m}{\partial t} = D\dfrac{\partial^2 m}{\partial z^2}$

B.C. : @ $z = 0,\ h \rightarrow m = m_\infty$

Initial Condition : @ $t = 0 \rightarrow m = m_0$

Solution

$$m^* = \frac{m-m_0}{m_\infty - m_0} = 1 - \frac{4}{\pi}\sum_{j=0}^{\infty} \frac{1}{2j+1}\sin\frac{(2j+1)\pi z}{h}\, e^{-\frac{(2j+1)^2\pi^2}{h^2}Dt}$$



similar to heat conduction but very long times ($\times 10^5$)

Also interested in average moisture in specimen

$$\bar{m} = \frac{1}{h}\int_0^h m\,dz \longleftarrow \text{can measure}$$

Can then show

$$G = \frac{\bar{m} - m_0}{m_\infty - m_0} = 1 - \frac{8}{\pi^2}\sum \frac{1}{(2j+1)^2}\, e^{-\frac{(2j+1)^2\pi^2}{h^2}Dt}$$

A simple approximation to above is

$$G \simeq 1 - e^{-7.3\left(\frac{Dt}{h^2}\right)^{.75}}$$

Time t_p to reach 95 % final value

$$e^{-7.3(Dt/h^2)^{.75}} = .05$$

or

$$7.3\left(\frac{Dt}{h^2}\right)^{.75} = 3$$

$$t_p = \left(\frac{3}{7.3}\right)^{1.33}\frac{h^2}{D} \simeq .3\,\frac{h^2}{D}$$

Same formulas apply to heat conduction with appropriate constants

In addition to swelling, moisture causes deterioration of material properties.

See Tsai, "Composite Design" 4th Ed. 1988
Chap. 16, 17

Summary

1. Moisture tends to relieve residual thermal stresses obtained from cure (some moisture better than dry)
2. Similarly can do other strains, e.g. → Piezoelectric

$$\rightarrow \epsilon^P = d_T\,\Delta V$$

← voltage

coefficient of piezo expansion

$$\text{Then}\quad \underset{\text{mechanical strain}}{\epsilon^M} = \underbrace{\epsilon^\circ + \kappa z}_{\text{Total strain}} - (\underset{\epsilon^T}{\alpha\Delta T} + \underset{\epsilon^m}{\beta\Delta M} + \underset{\epsilon^P}{d_T\,\Delta V})$$

For computing convenience, can sometimes combine

$$\alpha\Delta T + \beta\Delta M + d_T\,\Delta V \longrightarrow \alpha_{\text{equivalent}}\,\Delta T_{Eq}$$

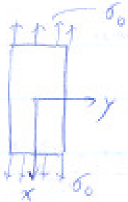
and do analysis with Equivalent $\alpha_{Eq}\,\Delta T_{Eq}$.

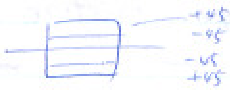
8. Advanced Topics

- Interlaminar Stresses due to Free Edges

Ref. { Pipes & Pagano, J. Composite Mat'l, Oct. 1970, p. 538
Jones book, Chap. 4 p. 210 }

Consider $[\pm 45]_s$ angle-ply laminate under simple tension σ_0





For each ply, have

$$\underset{\sim}{\sigma} = \underset{\sim}{\bar{Q}}\,\underset{\sim}{\epsilon}$$

Material Properties: $E_L = 20.0$ Msi, $E_T = 2.1$ Msi,
$G_{LT} = .85$ Msi, $\nu_{LT} = .21$, $t_p = 0.005''$

$$\underset{\sim}{Q} = \begin{bmatrix} 20.1 & .44 & 0 \\ .44 & 2.11 & 0 \\ 0 & 0 & .855 \end{bmatrix} \times 10^6 \ (\text{lbs/in}^2)$$

Using transformation $\bar{Q} = T_\epsilon^T\, Q\, T_\epsilon$, get

$$\underset{\sim}{\bar{Q}}^{45} = \begin{bmatrix} 6.63 & 4.92 & 4.50 \\ 4.92 & 6.63 & 4.50 \\ 4.50 & 4.50 & 5.33 \end{bmatrix} \times 10^6$$

$\underset{\sim}{\bar{Q}}^{-45} \rightarrow$ same with $\bar{Q}_{16} = \bar{Q}_{26} = -4.50 \times 10^6$

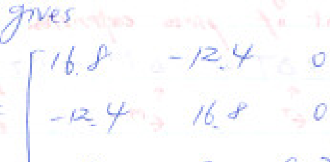
Laminate Stiffness $\underset{\sim}{A} = \Sigma\, \bar{Q}^k (z_{uk} - z_{\ell k})$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} .1325 & .0983 & 0 \\ .0983 & .1325 & 0 \\ 0 & 0 & .1066 \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} \times 10^6$$

For problem, $N_x = \sigma_0 h$, $N_y = N_{xy} = 0$,

Inverting gives

$$\begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{bmatrix} 16.8 & -12.4 & 0 \\ -12.4 & 16.8 & 0 \\ 0 & 0 & 9.38 \end{bmatrix} \begin{Bmatrix} \sigma_0 h \\ 0 \\ 0 \end{Bmatrix} \times 10^{-6} = \begin{Bmatrix} 16.8 \\ -12.4 \\ 0 \end{Bmatrix} \sigma_0 h \times 10^{-6}$$



$$\nu_{laminate} = \frac{-\epsilon_y^0}{\epsilon_x^0} = .74$$

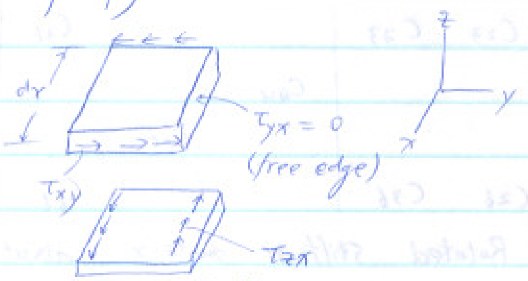
Stresses in Top ply $+45^\circ$ are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} 6.63 & 4.92 & 4.50 \\ 4.92 & 6.63 & 4.50 \\ 4.50 & 4.50 & 5.33 \end{bmatrix} \begin{Bmatrix} 16.8 \\ -12.4 \\ 0 \end{Bmatrix} \sigma_0 h = \begin{Bmatrix} 50.3 \\ 0 \\ 19.8 \end{Bmatrix} \sigma_0 h$$

since $h = 4(.005) = .020$

$$= \begin{Bmatrix} 1.00\,\sigma_0 \\ 0 \\ .396\,\sigma_0 \end{Bmatrix}$$ Note big shear stress

Look @ top ply element dx

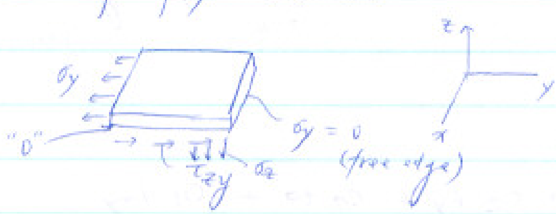


To balance τ_{xy}, $\Sigma M_z = 0$, must have τ_{zx} develop on the interface.

Similarly consider $[0/90]_s$ cross-ply laminate under tension σ_0

would obtain
$$\begin{cases} \sigma_x = 1.811\,\sigma_0 \\ \sigma_y = .032\,\sigma_0 \\ \tau_{xy} = 0 \end{cases}$$

Look @ top ply element



For $\Sigma F_y = 0$, must have τ_{zy} develop

For $\Sigma M_{x_0} = 0$, must have σ_z develop, but $\int \sigma_z\, dy = 0$

τ_{zx}, τ_{zy}, σ_z : "Interlaminar stresses", Develop on z face

σ_x, σ_y, τ_{xy} : "In-plane stresses"

Note : For $[\pm\theta]_s \Rightarrow$ only τ_{zx} develops

9-468 T. Pagano(?) office

For $[0/90]_s$ $\rightarrow$ only τ_{zy}, σ_z

For general combinations, all 3 interlaminar stresses present.

Must use 3-D Elasticity to solve complete problem.

- 3-Dim. Solution

Ref. Pipes & Pagano, J. Composite Mat'l Oct. 1970 p. 583 ~

Consider $[\pm 45°]_s$ laminate under tension σ_0

For each ply,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{zy} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & & & C_{16} \\ C_{12} & C_{22} & C_{23} & & & C_{26} \\ C_{13} & C_{23} & C_{33} & & & C_{36} \\ & & & C_{44} & & \\ & & & & C_{55} & \\ C_{16} & C_{26} & C_{36} & & & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{zy} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix}$$

↑ Rotated stiffness matrix about z-axis

(depends on 9 constants only → orthotropic)

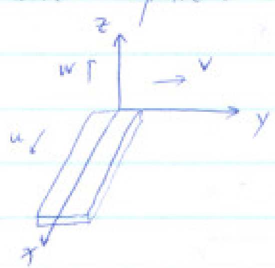
Strain-Displacement

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\epsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$\epsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Displacement pattern is



$$u = Kx + U(y, z)$$

$$v = V(y, z)$$

$$w = W(y, z)$$

↑ From symmetry, no x dependence of stresses

Stresses found as,

$$\sigma_x = C_{11}\epsilon_x + C_{12}\epsilon_y + C_{13}\epsilon_z + C_{16}\gamma_{xy}$$

$$= C_{11}\frac{\partial u}{\partial x} + C_{12}\frac{\partial v}{\partial y} + C_{13}\frac{\partial w}{\partial z} + C_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$= C_{11}K + C_{12}V_{,y} + C_{13}W_{,z} + C_{16}U_{,y}$$

Similarly,

$$\sigma_y = C_{12}K + C_{22}V_{,y} + C_{23}W_{,z} + C_{26}U_{,y}$$

$$\tau_{yz} = C_{44}(V_{,z} + W_{,y}) + C_{45}(W_{,x} + U_{,z})$$

Placing into Equilibrium Eqn s

$$\cancel{\frac{\partial \sigma_x}{\partial x}} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \cancel{F_x} = 0$$

$$\cancel{\frac{\partial \tau_{xy}}{\partial x}} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \cancel{F_y} = 0$$

no body force

Equations reduce to

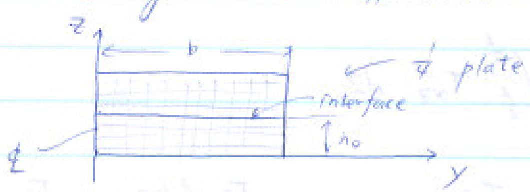
$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12} & L_{22} & L_{23} \\ L_{13} & L_{23} & L_{33} \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = 0$$

symmetric

where,
$$L_{11} = C_{66}\frac{\partial^2}{\partial y^2} + C_{55}\frac{\partial^2}{\partial z^2}$$
$$L_{12} = C_{26}\frac{\partial^2}{\partial y^2} + C_{45}\frac{\partial^2}{\partial z^2}$$
$$L_{13} = (C_{36} + C_{45})\frac{\partial^2}{\partial x \partial y}$$

6th order set of Differential Equations

Solve by Finite Difference (like CFD)



used up to 400 points

Boundary Conditions :

on Top face → $\tau_{zx} = 0$, $\tau_{zy} = 0$, $\sigma_z = 0$

on interface → τ_{zx}, τ_{zy}, σ_z, u, v, w continuous

at side face → $\tau_{yx} = 0$, $\sigma_y = 0$, $\tau_{yz} = 0$

@ $z = 0$ → Symmetry, $\frac{\partial u}{\partial z} = 0$, $\frac{\partial v}{\partial z} = 0$, $w = 0$

@ $y = 0$ → Symmetry, $u = 0$, $v = 0$, $\frac{\partial w}{\partial y} = 0$

Numerical solution by computer

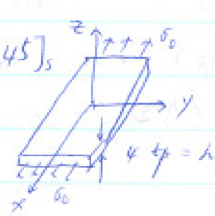
Input = $K = \frac{\partial u}{\partial x} = \epsilon_{x_0}$

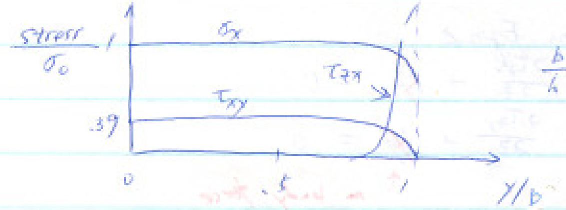
Enter from B.C.'s

$\sigma_y = 0$ @ side face

$\sigma_z = 0$ @ top and interface

Results — $[\pm 45]_s$



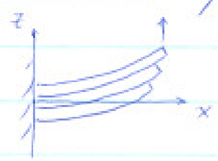


$\frac{b}{h} = 2$

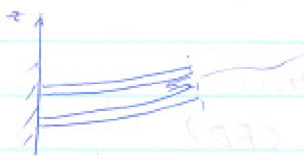
$\left|\frac{y/b}{\Delta y}\right|$ — boundary layer develops ($\Delta y < h$)

· Interlaminar stresses due to Bending

Consider a symmetric laminate in Bending ($\underset{\sim}{B} = 0$, $\underset{\sim}{\epsilon}^\circ = 0$)

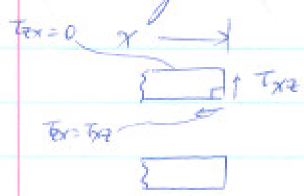


layers would slide if not glued together



τ_{zx} develops (same as beam theory)

@ any section x



$$Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz}\, dz$$

To obtain τ_{xz}, go to Equilibrium Equation

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + p_x = 0$$

$$\frac{\partial \tau_{zx}}{\partial z} \simeq -\underbrace{\frac{\partial \sigma_x}{\partial x}}_{\text{Main contribution}} - \underbrace{\cancel{\frac{\partial \tau_{xy}}{\partial y}}}_{\text{small}} - \underbrace{\cancel{p_x}}_{0}$$

Now, $\underset{\sim}{\bar{\sigma}} = \underset{\sim}{\bar{Q}}\, \underset{\sim}{\kappa}\, z$

$$\sigma_x \simeq z\, \bar{Q}_{11}\, \kappa_x \simeq z\, \bar{Q}_{11}\, d_{11}\, M_x$$

Placing into Equilibrium Equation & Integrating,

$$\int_{z_{lk}}^{z_{uk}} \frac{\partial \tau_{zx}}{\partial z}\, dz \simeq -\int_{z_{lk}}^{z_{uk}} \frac{\partial}{\partial x}\left(z\, \bar{Q}_{11}\, d_{11}\, M_x\right) dz$$

$$\simeq -\int z\, \bar{Q}_{11}\, d_{11}\, \underbrace{\frac{\partial M_x}{\partial x}}_{= Q_x}\, dz$$

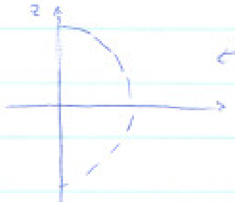
Integrating gives

$$\tau_{zx}(z_{uk}) - \tau_{zx}(z_{lk}) = -\, d_{11}\, Q_x\, \bar{Q}_{11}^{\,k}\, \frac{z^2}{2}\Big|_{z_{lk}}^{z_{uk}}$$

$$\tau_{zx}(z_{lk}) = \tau_{zx}(z_{uk}) + d_{11}\, Q_x\, \bar{Q}_{11}^{\,k}\left(z_{uk}^2 - z_{lk}^2\right)/2$$

start at tip where $\tau_{zk}(h/2) = 0$

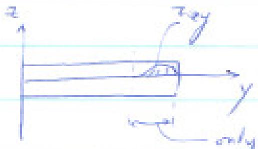
and work down.



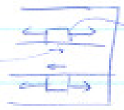
Similarly do for τ_{zy} distribution

Interlaminar stresses due to bending $\rightarrow$ everywhere in plate

" free edges $\rightarrow$ Boundary layer near edges



For free edges



σ_y can be different

τ_{zy} must be the same

τ_{zy} must be the same

- Outline
  - Introduction
  - Micromechanics
  - Ply Elasticity (orthotropic)
  - Laminate Theory
  - Failure
  - Bending of Plate
  - Thermal Stresses
  - Advanced $\rightarrow$ Interlaminar stresses
  - Composites generally
  - Deal in design
  - Physical parameters
  - Organization $\rightarrow$ Computer program
- Many current problems
  - Failure, Fracture, holes
  - Cracking, Delamination, Fiber Breaking
  - Impact, Thermal stresses
  - Environmental Degradation
  - Buckling of the plates, Large deflections

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16.251 Longevity (Lagace)
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16.29 Seminar (current work)
16.230 Plates & shells