16.28 Adaptive Materials and Structures - N.W. Hagood

- Adaptive structure
  - -- characteristics can be beneficially changed in response to environment

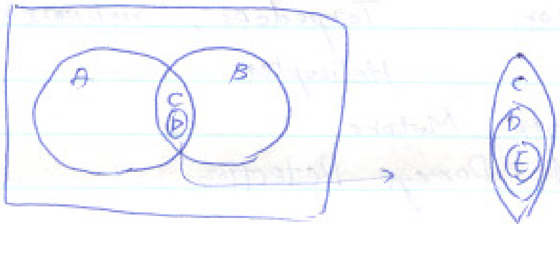
  Actuation
  Sensing
  Control

- Characteristics
  - shape, geometry
  - stiffness, damping
  - vibration characteristics
  - radar signature
  - acoustic reflectivity

- stimuli
  - external force
  - pressure field
  - applied voltage
  - " magnetic field

- Controllable Response
  - actuator, sensor control to increase performance to enhance functionality
  - complexity not pay for itself

- Hierarchy of Adaptive Structure



A : Actuated or Adaptive structures
B : Sensory Structure
... Monitoring of system stats.
C : Controlled
D : Active Materials and Structures
E : Intelligent Structure

- Active Structures
  - analysis tools for integration

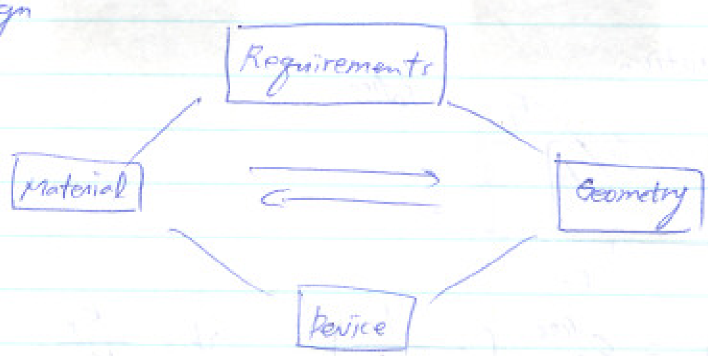
- Motivation
  - Increased Functionality
  - Complexity is genetic trend

- Applications
  - Vibration Suppression : multi-payload platform
  - Precision Optical [Pointing] System (Interferometry)
  - Precision Machining
  - Static shape Control
  - Helicopter vibration
    - Optical surface Correction
    - Active optics
    - Deformable Lifting surfaces
  - Active Noise Control
    - Interior ... Fuselage, Cockpit
      Elevator
      Rooms
    - Exterior ... Torpedoes, Submarines
      Helicopters
  - Solid State Motors
  - Structural Damage Detection

- Dynamic Flow Structure Interaction
- Gust Buffet Load Alleviation
- Flutter Suppression

- Adaptation Mechanisms
  - Material
  - Controllable Size
    - Electrical --- Piezoelectric Ceramics, Polymers
    - Thermal --- SMA
    - Magnetical --- Magnetostrictors
    - Optically
    - Chemically
  - Controllable Stiffners
  - " Viscosity
    - Electrorheological
    - Magnetorheological
- Transducer --- works both ways
  ⇒ Solid State Actuation Sensing

* Design

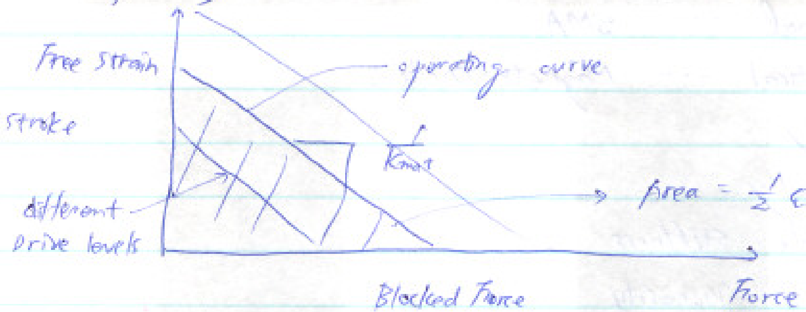


2 Problems
i) Elastic Actuation
ii) Solid state Physics
  - Couple of Fields
- Activation Figure of Merit
  - Stroke, ε^{free}

ε^{free} / Electric Field

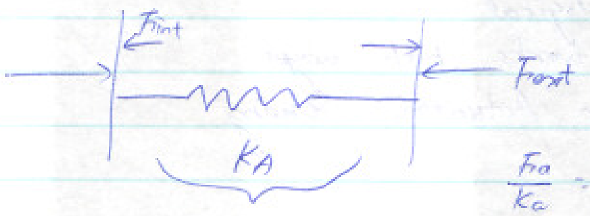
- max
- sensitivity
- force, stress
- max, clamped
- sensitivities

• Graphically



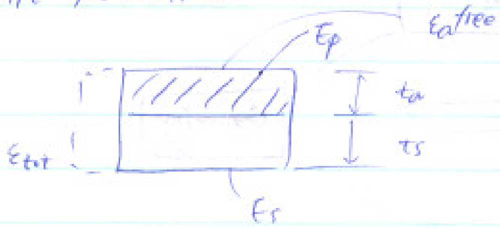
→ Area $= \frac{1}{2}\varepsilon_{max}\sigma_{max}$ = Actuation Energy/value

• 1-D (simple) model

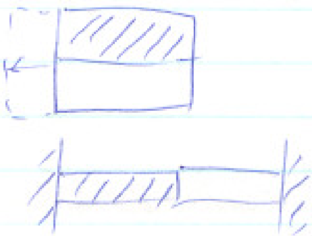


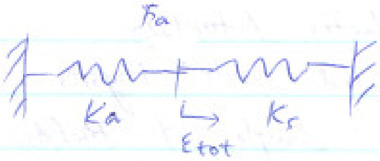
$\frac{F_a}{K_a} = x_a$

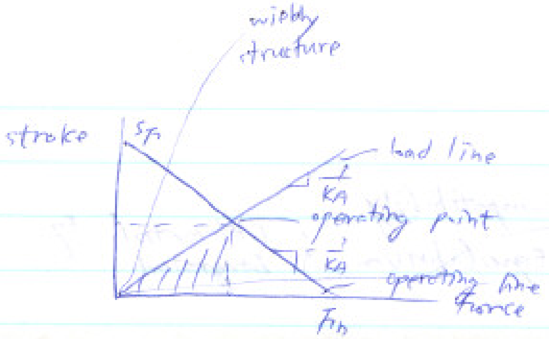
• Elastic Actuation



$$\varepsilon_{tot} = \varepsilon_a^{free}\left(\frac{1}{1+\psi}\right), \quad \psi = \frac{K_s}{K_a}$$





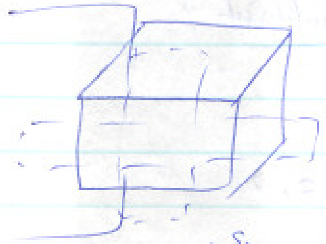


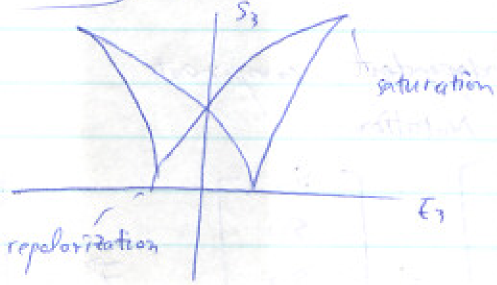
max. work developed : $\frac{1}{4} \cdot \frac{1}{2} \, \varepsilon_a^{free} \, \sigma_a^{blocked}$

- Actuation Energy Density $= \frac{\frac{1}{2} \varepsilon^f \sigma^b}{\rho}$

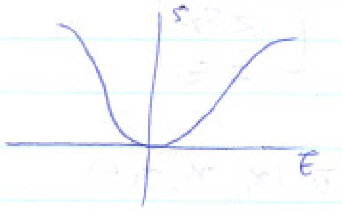
Introduction to Material Behavior

- Piezoelectric





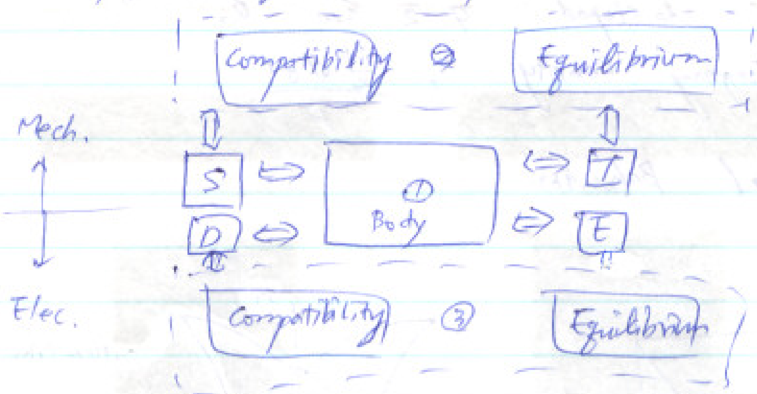
- Electrostrictive Ceramics



- Terfenol - D
  Iron - Terbium - Dysprosium
- Shape Memory Alloys

- Continuum Mechanics
  1) Kinematics ··· compatibility
  2) Kinetics ··· Equilibrium $\left\{ \begin{array}{l} \text{Differential Eq.} \\ \text{Integrated } '' \end{array} \right.$
  3) Constitutive Relation

Roadmap for Analysis of Coupled Continuum



- Mechanical field

  S : strain , S_{ij} $i=1,2,3$

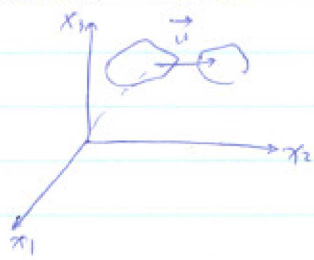
  T : stress , T_{ij}

  $T_{ij} = T_{ji}$ 6 independent components

Voight or Contracted Notation

$$\vec{S} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} S_x \\ S_y \\ S_z \\ 2S_{yz} \\ 2S_{zx} \\ 2S_{yx} \end{bmatrix} = \begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ 2S_{23} \\ 2S_{31} \\ 2S_{21} \end{bmatrix}, \quad \vec{T} = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{21} \end{bmatrix}$$

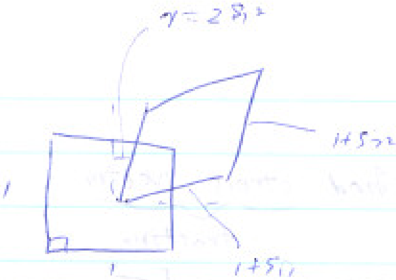
- Displacement Field



$\vec{u}\,(x_1, x_2, x_3, t)$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

- Strain : relative deformation

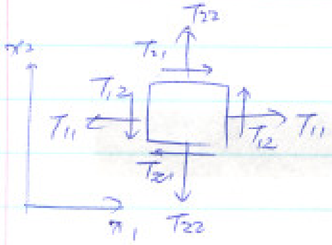


strain - Displacement Relation

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$\Rightarrow$ 6 independent eqn.

- Newton's Law $\quad \Sigma f_i = ma$



$T_{ij} = \frac{\text{Force}}{\text{Area}}$ = on ith-face, jth direction

- Equilibrium Eqn.

differential Form

$$\frac{\partial T_{ij}}{\partial x_j} + f_i = \rho a_i \quad \Rightarrow 3 \text{ Eqn.}$$

$$= \begin{cases} \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} + f_1 = \rho a_1 \\ \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} + f_2 = \rho a_2 \end{cases}$$

Unknowns

6 strains	3 Equilibrium
6 stresses	6 Stain - Displacement
3 displacement	6 constitutive Rel.
15	15 Eqn.

- Constitutive Relations

tensor $\quad$ stiffness tensor, elasticity

$$T_{ij} = E_{ijmn}\, S_{mn}$$

$$S_{ij} = C_{ijmn}\, T_{mn}$$

$$\underset{\sim}{T} = \underset{\sim}{c}\, \underset{\sim}{S} \quad , \quad \underset{\sim}{S} = \underset{\sim}{s}\, \underset{\sim}{T}$$

$\underset{\sim}{c}$: 6x6, stiffness $\qquad$ $\underset{\sim}{s}$: compliance.

- Boundary Conditions

on A_1

prescribed $\underbrace{\text{stress vector}}_{\text{traction}}$

$$\vec{T_s} = (T_{sn})\hat{i}_n = [T_{mn} \cos(N, m)]\,\hat{i}_n$$

$$\begin{bmatrix} T \\ T \\ T \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ & T_{22} & T_{23} \\ & & T_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

Differential Form

$$\frac{\partial T_{ij}}{\partial x_j} + f_i - \rho a_i = 0$$

Integral Form

$$\int_V \left\{ \left[\frac{\partial T_{ij}}{\partial x_j} + f_i \right] \cdot \delta \vec{v} \right\} dV = 0$$

$$\underbrace{\int_V T \delta S \, dV}_{\delta U:\ \text{Energy}} = \underbrace{\int_V (\vec{f} \cdot \delta u)\, dV + \int_S (\vec{f_s} \cdot \delta u)\, dS}_{\delta W:\ \text{virtual work}}$$

Principle of Minimum Potential Energy $\delta(U - W) = 0$

→ Principle of Total Min. Potential Energy

- Electric Fields

E, D (electrical displacement)

$$E = \lim_{q \to 0} \frac{F}{q} = \frac{\text{volts}}{m}$$

Coulomb's Law

$$F = \frac{q_1 q_2}{4\pi \varepsilon_0} \frac{\vec{r}}{|r|^3} = \frac{q_1 q_2}{4\pi \varepsilon_0} \nabla \frac{1}{r}$$

$$\varepsilon_0 = 8.2 \times 10^{-?} \text{ Farad/meter}$$

$$\vec{E} = \frac{-q}{4\pi\varepsilon_0} \nabla \frac{1}{r}$$

superposition gives field associated with other charge distribution

- Vector field facts
  - vector field is uniquely defined by its circulation density + source density

$$\nabla \cdot \vec{v} = s, \quad \nabla \times \vec{v} = \vec{c} \quad \Big\} \quad \vec{v} = -\nabla\varphi + \nabla \times \vec{A}$$

$\varphi(r)$: scalar potential $= \frac{1}{4\pi} \int \frac{s(r')}{|r-r'|} dr'$

$\vec{A}(r)$: vector potential $= \frac{1}{4\pi} \int \frac{\vec{c}(r')}{|r-r'|} dr'$

- Gauss's Flux Theorem

$$\int_S \vec{E} \cdot d\vec{s} = \frac{q}{\varepsilon_0}$$

Differential Form

$$\int_S \vec{E} \cdot d\vec{s} = \int_V \nabla \vec{E} \cdot dV = \frac{q}{\varepsilon_0} = \int_V \frac{\rho}{\varepsilon_0} dV$$

$$\boxed{\begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \\ \nabla \times \vec{E} &= 0 \end{aligned}}$$

$$\nabla \times \vec{E} = 0 \quad \Rightarrow \quad \oint \vec{E} \cdot d\vec{l} = 0$$

- Electric Potential

$$\vec{E} = -\nabla\varphi$$

gradient operator

$$E_1 = -\frac{\partial\varphi}{x_1}$$

$$\Rightarrow \nabla^2\varphi = -\frac{\rho}{\varepsilon_0} \quad \text{Poisson's Eq.}$$

$$\varphi(r) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(r')}{|r-r'|} dr'$$

point charge $\quad \varphi(r) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}$

- Polarization Field

$+q$, $-q$, dx

$$\vec{p} = \text{dipole moment} = q\, d\vec{x}$$

$$\varphi(r) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p} \cdot \vec{r}}{r^3}$$

- Volume distribution $\vec{P} = \vec{p}/\text{volume}$
- potential associated with volume distribution of dipoles

$$\varphi = \frac{1}{4\pi\varepsilon_0} \int \vec{P} \cdot \nabla\left(\frac{1}{r}\right) dr$$

$$\vec{\nabla} \cdot \left(\frac{\vec{P}}{r}\right) = \frac{1}{r} \vec{\nabla} \cdot \vec{P} + \vec{P} \cdot \nabla\left(\frac{1}{r}\right)$$

$$\varphi = \frac{1}{4\pi\varepsilon_0}\left[\int \nabla\cdot\frac{\vec{P}}{r}\,dV - \int \frac{1}{r}\nabla\cdot\vec{P}\,dV\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\underbrace{\int_S \frac{\vec{P}\cdot d\vec{s}}{r}}_{\text{potential equivalent to surface charge distribution } \sigma_P = P_n} - \underbrace{\int \frac{\nabla\vec{P}}{r}\,dV}_{\rho_R = -\nabla\vec{P}}\right]$$

free charge

bound charge

$\therefore$ ρ consists of $\rho_S + \rho_B$, $\rho_B = -\nabla\vec{P}$

- Electrical Displacement

$$\nabla^2\varphi = -\nabla\vec{E} = -\frac{\rho^{tot}}{\varepsilon_0} = -\frac{(\rho^f + \rho^B)}{\varepsilon_0}$$

let $\rho^B = -\nabla\vec{P}$

$$\nabla(\vec{E} + \frac{\vec{P}}{\varepsilon_0}) = \frac{\rho^f}{\varepsilon_0}$$

Describe a new vector field

$$\vec{D} = \text{charge/area} = \varepsilon_0\vec{E} + \vec{P}$$

$$\nabla\cdot\vec{D} = \rho^f = \int_S \vec{D}\cdot ds = q^f$$

- Electrical Constitutive Relations
  - Polarization is dependent on electrical field

$$\vec{P} = \varepsilon_0 \underset{\sim}{\chi}\vec{E}$$

$$\vec{P} = \varepsilon_0 \chi \vec{E} \text{ (3x3 matrix)}$$

$$P_i = \varepsilon_0 \chi_{ij} E_j$$

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0(1+\chi)E$$

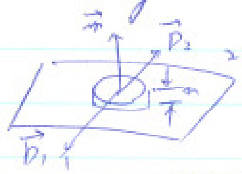
$$\vec{D} = \kappa\varepsilon_0\vec{E} = \varepsilon E$$

$$\to \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ & \text{etc.} & \\ & & \varepsilon_{33} \end{bmatrix}$$

- Eqns

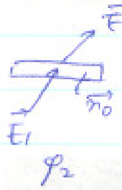
compatibility		Variable	
$E = -\nabla\varphi$	3	φ	1
Equilibrium		E	3
$\nabla D = \rho^f$	1	D	3
~~$\nabla\times\vec{D} = 0$~~	~~3~~		7
$D = \varepsilon E$	3		
	7		

- Boundary Condition



$$\int \vec{D} \cdot ds = q^f$$

$$\boxed{\vec{n} \ (\vec{D_2} - \vec{D_1}) = \sigma}$$



$$\boxed{\varphi_1 = \varphi_2}$$

- Differential Form

$$\nabla \vec{D} = \sigma \quad \text{applied free charge}$$

$$\int_V [\nabla \cdot \vec{D}] \delta\varphi \, dV = \int_V \sigma \delta\varphi \, dV$$

$$\int_V \vec{D} \cdot \delta\vec{E} = \Sigma q_i \delta\varphi_i$$

$$\delta U = \delta W$$

* Magnetism

- Current

  - current density = amps/area
  - conservation of charge $\quad \nabla \vec{j} = -\frac{\delta\rho}{\delta t}$
  - stationary current $\quad \nabla \vec{j} = 0$
  - constitutive relation

$$\hat{j} = \sigma \vec{E}$$

(A) Electromotive force

$$\hat{j} = \sigma(E + E')$$

where $\nabla \times E = 0$

$\nabla \times E' \neq 0$

$$\oint \frac{\hat{j}}{\sigma} \cdot d\vec{l} = \int (\cancel{E} + E') \, d\vec{l} = \int \vec{E'} \, dl = \mathcal{E},$$

$$\mathcal{E} = \oint \frac{\hat{j}}{\sigma} = I \int \frac{dl}{\sigma S} = IR \qquad R = \int \frac{dl}{\sigma S} \quad \Rightarrow \quad V = IR$$

$$\nabla \times E = 0$$

$$\nabla (\sigma E) = -\nabla (\sigma E')$$

$$\nabla \hat{j} = 0$$

$J = \sigma E$

$\nabla \times E = 0 \qquad\qquad \nabla \times E = 0$

$\nabla(\sigma E) = 0 \qquad\qquad \nabla \cdot D = 0$

E

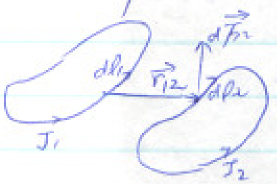
$\hat{J} \quad \vdots \quad \hat{D}$

$\hat{J} = \sigma E \quad \vdots \quad \vec{D} = \varepsilon \vec{E}$

B.C's $\quad n \cdot (\vec{D_2} - \vec{D_1}) \qquad n \cdot (\vec{J_2} - \vec{J_1})$

$n \times (\vec{E_2} - \vec{E_1}) = 0$

- Magnetic Field
  - Ampere's Law



$$F_2 = \frac{\mu_0}{4\pi} J_1 J_2 \oint_1 \oint_2 \frac{d\vec{l_2} \times (d\vec{l_1} \times \vec{r_{12}})}{(r_{12})^3}$$

$$F = \frac{1}{4\pi\varepsilon_0} q_1 q_2 \frac{|\vec{r}|}{|r|^3}$$

$$F_2 = J_2 \oint_2 d\vec{l_2} \times \vec{B_2} \quad , \quad B_2 = \frac{\mu_0}{4\pi} J_1 \oint_1 \frac{d\vec{l_1} \times \vec{r_{12}}}{|r|^3}$$

- $F = \int_V \hat{J} \times \vec{B}\, dV$

$$B = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J} \times \vec{r}}{|r|^3}\, dV$$

$\nabla \cdot \vec{B} = 0 , \quad \nabla \times \vec{B} = \mu_0 J$

$\nabla \times \vec{E} = 0 , \quad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

- vector potential

$\vec{B} = \nabla \times \vec{A} \qquad\qquad E = -\nabla\varphi$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{|r|} dV \qquad \varphi = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho}{|r|} dV$$

- currents
  - free, bound
  - Polarization current $\frac{dP}{dt}$: neglect
  - magnetization current $\hat{J}_m$
  - displacement current $\frac{\partial \varepsilon_0 E}{\partial t}$

just as for polarization,

$$\vec{p} = \int \vec{P}\, dV = \int \rho \vec{\xi}\, dV$$

$$\vec{m} = \int \vec{M}\, dV = \frac{1}{2} \int (\xi \times \hat{J}_m)\, dV$$

$\rho_b = -\nabla \vec{P}$

$\hat{J}_m = \nabla \times \vec{M}$

• Magnetic Fields in matter

$$\nabla B = 0$$

$$\nabla \times B = \mu_0 \vec{j}_{tot} = \mu_0 (\vec{j}_{true} + \underbrace{\nabla \times \vec{M}}_{\hat{j}_m})$$

$$\nabla \times \vec{E} = 0$$

$$\nabla \cdot E = \frac{\rho_{tot}}{\varepsilon_0} = \frac{1}{\varepsilon_0}(\rho_{true} - \nabla \cdot \vec{P})$$

$$\nabla \times (B - \mu_0 M) = \mu_0 \hat{j}_{true}$$

or $H = \frac{B}{\mu_0} - \vec{M}$, $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$: coercive field

$\begin{cases} \nabla \times \vec{H} = \hat{j}_{true} \\ \vec{\nabla} \cdot \vec{D} = \rho_{true} \end{cases}$ $\begin{cases} \nabla \times B = J_{tot}\ \mu_0 \\ \nabla E = \rho_{tot}/\varepsilon_0 \end{cases}$

$$\boxed{\oint \vec{H} \cdot d\vec{l} = I_{true}}$$

• Permiable Material

$$\vec{M} = \chi \vec{H}$$

χ : magnetic susceptibility

$$B = \mu_0 (\chi_m + 1) H$$

$B = K_m \mu_0 H = \mu H$

μ : absolute permitivity

K_m : relative permitivity

$\mu_0 = 4\pi \times 10^{-7}$ Henry/m

Terfenol χ_m : 1.9

Iron χ_m 1,000

• B.C's $\nabla \cdot B = 0$ $n \cdot (B_2 - B_1) = 0$

$\nabla \times H = \hat{j}_{true}$ $n \cdot (\mu_2 H_2 - \mu_1 H_1) = 0$

$n \times (H_2 - H_1) = K$: surface current density

• Magnet Circuit Analysis

$\nabla \cdot B = 0$ $\nabla \cdot \hat{j} = 0$

$\oint \vec{H} \cdot dl = I_{true}$ $\oint E' \, dl = \varepsilon$

$B = \mu \vec{H}$ $\vec{j} = \sigma \vec{E}$

$$\vec{B} \Leftrightarrow \vec{j}$$

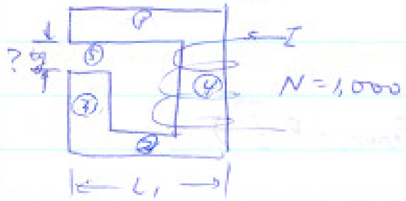
$$H \Leftrightarrow E'$$

$$\mu \Leftrightarrow \sigma \qquad \epsilon_i = I_i R_i$$

$\mathcal{E}_i = I_i R_i$

$I_1 = \int_A \vec{j}\, dA, \quad R_1 = \frac{l_1}{\sigma_1 S_1}$

$dH_1 = \varphi R, \quad \varphi = \int_A \vec{B}\, da, \quad R_1 = \frac{l_i}{\mu_i S_i}$



$\oint H\, dl = nI$

$R_1 = \frac{L_1}{A_1 \mu_1}, \quad R_2 = R_1,$

$R_4 = \frac{L_2}{A_2 \mu_2}, \quad R_5 = \frac{t_g}{A_5 \mu_0}$

$R_3 = \frac{L_3}{A_3 \mu_3}$

$V_{gap} = H_g t_g = \frac{R_5}{R_{tot}} V_{tot} = \frac{R_5}{R_{tot}} \underbrace{H_{tot} l_{tot}}_{NI}$

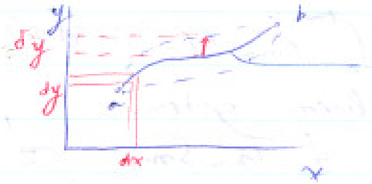
$H_{gap} = \frac{NI}{A_5 \mu_0 R_{tot}} \rightarrow H_{gap} = \frac{NI}{t_g}$

$B_{gap} = \mu_0 H_{gap}$

- Variational Principles
  1. Vectorial Approach
     $\vec{F} = m\vec{a}$
  2. Energy or Variational Approach
     - scalars
     - governing principle
     - correct condition is defined by
       $\delta(\text{scalar}) = 0$, stationary
     - integral
- Calculus of Variations
  - Chap. 2 Lanczos "Variational Principles of Mechanics"
  - Chap. 2 Hildebrand "Methods of Applied Math."
  - Chap. 3 Langhaar "Energy Methods in Applied Mech."

Behavior described by finding condition when some meaningful quantity is stationary

$I = \int_a^b F(y, y', x)\, dx$



$\delta I = 0$ correct path.

δy is called "variation in y". y is dependent variable.

$$\delta I = \int_a^b \delta F(y, y', x)\, dx$$

$$\delta F(y, y', x) = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \cancel{\frac{\partial F}{\partial x}\delta x} = 0$$

in operations

$$\delta g(y) = \frac{dg(x)}{dy}\delta y = g'(y)\,\delta y$$

$$\delta (y^2) = 2y\,\delta y$$

$$\delta \sin y = \cos y\,\delta y$$

$$\delta\left(\frac{dw}{dx}\right) = \frac{d\,\delta w}{dx}, \quad \delta\left(\frac{d^n w}{dx^n}\right) = \frac{d^n(\delta w)}{dx^n}$$

$$\delta\left(\int_a^b F\,dx\right) = \int_a^b \delta F\,dx$$

Consider

$$I = \int_a^b F(y, y', x)\,dx$$

$$\delta I = \int_a^b \delta F\,dx = \int_a^b \left[\frac{\partial F}{\partial y}\delta y + \underbrace{\frac{\partial F}{\partial y'}\delta y'}_{\text{integrating by parts}}\right] dx$$

$$= \int_a^b \left[\frac{\partial F}{\partial y}\delta y\right] dx + \frac{\partial F}{\partial y'}\overset{0}{\delta y}\Big|_a^b - \int_a^b \delta y \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

$$\delta I = -\int_a^b \left[\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}\right]\delta y\,dx = 0$$

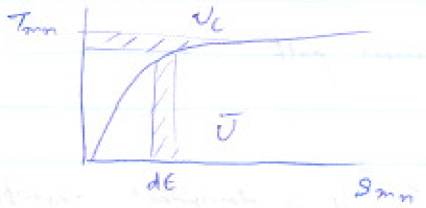
$\delta I = 0$, if and only if

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$: Euler's Equations

if $F = T - U + W$

$x = t$ (time) $\Rightarrow$ Lagrange's Eqn.

- structural Mechanics
- scalar's of interest
  - Strain Energy Density $\bar{U} = \bar{U}(S_{mn})$
  - Complementary Strain Energy Density $\bar{U}_c = \bar{U}(T_{mn})$



$\bar{U} = \int_0^{S_{mn}} T_{mn}(S'_{mn})\, dS'_{mn}$

- for linear system

$\bar{U} = \frac{1}{2} T_{mn} S_{mn} = \frac{1}{2} T^T S$

Tensor Voight

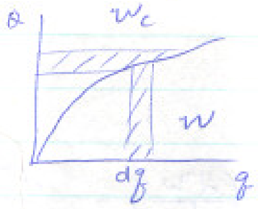
$U = \iiint \bar{U}(S_{mn})\, dV = U$

$U_c = \int_0^{T_{mn}} S_{mn}(T'_{mn})\, dT'_{mn}$

- Work = Force times distance

$\int_a^b \vec{F} \cdot d\vec{r}$

Work, Complementary Work



$W = \int_0^q Q(q')\, dq'$

$W_c = \int_0^Q q(Q')\, dQ'$

- Variations of strain Energy

$$U(S_{mn}) = \iiint \left\{ \int_0^{S_{mn}} T_{mn}(S'_{mn})\, dS'_{mn} \right\}$$

perturbation in strain field $S_{mn} + \delta S_{mn}$

consider $U(S_{mn} + \delta S_{mn}) - U(S_{mn})$

$$= \iiint \left\{ \int_{S_{mn}}^{S_{mn}+\delta S_{mn}} T_{mn}(S'_{mn})\, dS'_{mn} \right. = \Delta U(S_{mn})$$

$$= \underbrace{\iiint \left\{ T_{mn}(S_{mn})\, \delta S_{mn} \right\} dV}_{\delta U}$$

- Variation in work

$$\delta W = \iint_A \hat{T}_{sn}\, \delta U_n\, dA + \iiint_V f_n\, \delta U_n\, dV$$

- Derivation of Principle of Minimum Total Potential Energy

given loaded body in equilibrium

displacement

$$\iiint_V \left\{ \frac{\partial T_{mn}}{\partial x_m} + f_n \right\} \delta U_n\, dV = 0$$

1st term : $\iiint \frac{\partial T_{mn}}{\partial x_m} \delta U_n\, dV = \iiint \left\{ \frac{\partial}{\partial x_m} (T_{mn}\, \delta U_n) \right.$

$$- T_{mn} \frac{\partial}{\partial x_m} (\delta U_n) \} dx,$$

$$\vec{B} = \hat{i}_n (T_{mn} \delta U_n)$$

S_{mn} (normal strain)

$$= \iiint \{ \nabla \vec{B} - T_{mn} \delta \left(\frac{\partial U_n}{\partial x_m} \right) \} dV$$

$$\iiint_V \nabla \vec{B} \, dV = \iint_A \vec{B} \cdot \vec{N} \, dA$$

$$\iint_A \overbrace{T_{mn} \cos(N x_m)}^{\hat{T}_{mn}} \delta U_n \, dA - \iiint_V T_{mn} \delta S_{mn} \, dV + \iiint_V f_n \delta U \, dV = 0$$

$$\underbrace{\iiint_V T_{mn} \delta S_{mn} \, dV}_{\delta U} = \underbrace{\iint_{A_T} \hat{T}_{mn} \delta U_n \, dA + \iiint f_n \delta U_n \, dV}_{\delta W}$$

$$\delta(\underbrace{U - W}_{I}) = 0$$

I : variational indicator

- Principle of Stationary Total Potential Energy

of all displacements of a loaded structure satisfying geometric B.C's – the right ones (equilibrium) are those that minimize π.

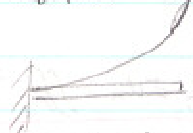
$$\pi = U - W$$

$$\delta \pi = \delta (U - W) = 0$$

$$U = \iiint \{ \int_0^{S_{mn}} T_{mn} (S_{mn}) \, dS_{mn} \} dV$$

$$W = \iint_A \hat{T}_{sn} u_n \, dA + \iiint_V f_n u_n \, dV$$

Solution by minimization

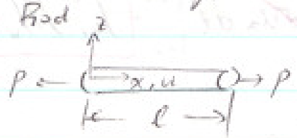


usually consider some subset.

→ Approximate solution

- U of typical structure

i) Rod



$$T_{11} = E S_{11} - E \alpha \Delta T$$

$$S_{11} = \frac{du}{dx}$$

$$\delta U = \iiint T_{11} \delta S_{11} \, dV$$

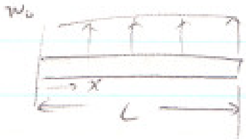
$$U = \iiint_V \{ \tfrac{1}{2} E S_{11}^2 - E \alpha \Delta T S_{11} \} \, dA \, dx$$

$$= \frac{1}{2} \int_0^l EA \left(\frac{du}{dx} \right)^2 dx - \int_0^l \left(\iint E \alpha \Delta T \frac{du}{dx} dA \right) dx$$

ii) Beam in Bending

$$T_{11} = E S_{11} - E\alpha\Delta T$$

$$S_{11} = -z\frac{d^2w}{dx^2}$$



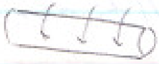
$$U = \iiint\left\{\frac{1}{2}E S_{11}^2 - E\alpha\Delta T S_{11}\right\}dA\,dx$$

$$U = \frac{1}{2}\int_0^l E I\left(\frac{d^2w}{dx^2}\right)^2dx + \int_0^l\left(\iint E\alpha\Delta T z\,dA\right)\frac{d^2w}{dx^2}dx$$

$I = \int z^2 dA$

$$W = \int_0^l f_w(x)\,w(x)\,dx$$

iii) Torsion of Bar



$$\frac{d\alpha}{dx} = \frac{T}{GJ}, \quad T = GJ\frac{d\alpha}{dx}$$

$$U = \frac{1}{2}\int_0^l T\frac{d\alpha}{dx}dx = \frac{1}{2}\int_0^l GJ\left(\frac{d\alpha}{dx}\right)^2dx$$

- Rayleigh - Ritz Approach

$$\vec{U}(x,y,z) = \sum_{i=1}^{N}\vec{a_i}(t)\varphi_i(x,y,z)$$

→ φ ⇒ satisfy B.C.'s

φ must be twice differentiable

$$\Pi_p = (U - W) = \Pi_p(a_1, a_2, \cdots, a_n)$$

$$\delta\Pi_p = \frac{\partial\Pi_p}{\partial a_1}\delta a_1 + \frac{\partial\Pi_p}{\partial a_2}\delta a_2 + \cdots$$

$$\frac{\partial\Pi_p}{\partial a_1} = 0, \quad \frac{\partial\Pi_p}{\partial a_2} = 0, \quad \cdots \text{ etc.}$$

For linear elastic problems,

$$[K]\{q\} = \{f\}$$

- Note on dynamics

- D'Alembert's Force $\quad \vec{f_I} = -\rho\ddot{\vec{u}}_n$

PSTPE

$$\int_{t_1}^{t_2}\iiint_V T_{mn}\,\delta S_{mn}\,dV + \iiint_V\rho\ddot{u}_n\,\delta u_n - \underbrace{\iiint_V f_{nB}\,\delta u_n - \iint_A\hat{T}_{sn}\,\delta u_n\,dA}_{\delta W} = 0 \quad dt$$

$\dot{u} \quad \delta U$

$$\int_{t_1}^{t_2}\iiint_V\rho\ddot{u}_n\,\delta u_n\,dV\,dt = \iiint_V\rho\cancel{\dot{u}_n\delta u_n}^{\,0}\,dt\Big|_{t_1}^{t_2} - \boxed{\int_{t_1}^{t_2}\iiint_V\rho\dot{u}\,\delta\dot{u}_n\,dV\,dt}$$

$\delta\dot{u}_n \qquad\qquad -\delta T$

$$T = \iiint_V\frac{1}{2}\rho\dot{u}_n\dot{u}_n\,dV$$

$$\int_{t_1}^{t_2}(\delta U - \delta T - \delta W)\,dt = 0$$

$$\boxed{\delta\int_{t_1}^{t_2}(T-U)\,dt + \int_{t_1}^{t_2}\delta W\,dt = 0}$$

Hamilton's principle

- Principle of Minimum Complementary Energy
  - increments in stress field δT_{mn}

$$\frac{\partial \delta T_{mn}}{\partial x_m} + \cancel{\delta f_n} = 0$$

$$\iiint_V \left\{ S_{mn} - \frac{1}{2}\left(\frac{\partial u_n}{\partial x_m} + \frac{\partial u_m}{\partial x_n}\right) \right\} \delta T_{mn}\, dV = 0$$

$$\iiint_V S_{mn}\, \delta T_{mn}\, dV - \iint_A u_n\, \delta \vec{T}_{sn}\, dA = 0$$

$$\delta U^c \quad - \quad \delta W^c \quad = 0$$

$$\delta \hat{T}_{sn} = \delta T_{mn} \cos(N x_n)\, dA$$

- Electrical Variational Systems

$$\nabla \vec{D} = \sigma$$

premultiply by allowable variation of φ, $d\varphi$

$$\int_V (\nabla \vec{D})\, \delta\varphi\, dV = \int_V \sigma\, \delta\varphi\, dV$$

$\delta\varphi = 0$ on fixed conditions

$\delta\varphi = $ const. along conductors

$\delta E = -\nabla \delta\varphi$

divergence theorem
$\iiint \operatorname{div} \vec{F}\, dV = \iint_S \vec{F}\cdot\vec{n}\, dA$

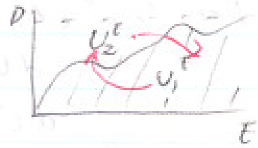
$$\underbrace{\int_V \nabla\cdot(\delta\varphi \vec{D})}_{\int_S \cancel{\delta\varphi \vec{D}\, d\vec{s}}} - \int_V \vec{D}\cdot\nabla\delta\varphi = \int_V \sigma\,\delta\varphi\, dV$$

$$\int_V \vec{D}\cdot\delta\vec{E}\, dV = \int_V \sigma\, \delta\varphi\, dV \rightarrow \Sigma q_i\, \delta\varphi_i \text{ (discrete)}$$

$$\delta U_1^E = \int_V \vec{D}\cdot\delta\vec{E}\, dV, \quad \delta W_1^E = \Sigma q_i\, \delta\varphi_i$$

$$\Downarrow$$

$$U_1^E = \int_V \int_0^E \vec{D}(\tilde{E})\, d\tilde{E}\, dV$$



$$\delta(U_1^E - W_1^E) = 0$$

- Non-Complementary Principle
  - allow variation of free charge distribution
  - constant with E-equal.

$$\nabla\cdot\delta\vec{D} = \delta\sigma$$

premultiply by φ

$$\int_V \nabla\cdot\delta\vec{D}\,\varphi\, dV = \int_V \delta\sigma\,\varphi\, dV$$

$$\int_V \nabla\cdot(\delta\vec{D}\varphi)\, dV - \int_V \delta\vec{D}\cdot\nabla\varphi\, dV = \int_V \delta\sigma\,\varphi\, dV$$

$$\int_V \delta \vec{D} \cdot \vec{E} = \int_V \delta\sigma \varphi \, dV$$

$$\int_V \delta \vec{D} \cdot \vec{E} = \sum_i \delta q_i \varphi_i$$

$$\delta (U_2^E - W_2^E) = 0$$

$$U_2^E = \int_V \int_0^D \vec{E}(\tilde{D}) \cdot d\tilde{D} \, dV$$

linear relationship $\vec{D} = \varepsilon \vec{E}$

$$\Rightarrow U_1 = \frac{1}{2} \int_V \vec{E}^T \varepsilon \vec{E} \, dV$$

$$U_2 = \frac{1}{2} \int_V \vec{D}^T \{\varepsilon^{-1}\} \vec{D} \, dV$$

$$U_1^E = D \cdot E - U_2^E$$

$\delta (U_1^M - W_1^M) = 0$	PMTPE	Mechanical	δu, δS
$\delta (U_2^M - W_2^M) = 0$	PMTCPE		δT, δf
$\delta (U_1^E - W_1^E) = 0$	complementary		$\delta \varphi$, δE
$\delta (U_2^E - W_2^E) = 0$	non-complementary		$\delta \sigma$, δD

- Combined Electrical - Mechanical

$$U^{TOT} = U_1^M + U_2^E$$

$$\delta U^{TOT} = \delta U_1^M + \delta U_2^E = \delta W_1^M + \delta W_2^E = \delta W$$

$$\delta U_1^M = \int_V \vec{T} \cdot \delta \vec{S} \, dV, \quad \vec{T} \to T(\vec{S}, \vec{D})$$

$$\delta U_2^E = \int_V \vec{E} \cdot \delta \vec{D} \, dV, \quad \vec{E} \to \vec{E}(S, D)$$

From Thermodynamics;

1st law of Thermodynamics,

$$dU^{TOT} = dQ + dW$$

$$dU^{TOT} = \underset{\text{reversible}}{\theta ds} + \vec{T} \cdot d\vec{S} + \vec{E} \cdot d\vec{D}$$

$$dU = \frac{\delta U}{\delta s} ds + \frac{\delta U}{\delta \vec{S}} d\vec{S} + \frac{\delta U}{\delta \vec{D}} \cdot d\vec{D}$$

entropy

$$\vec{T} = \left[\frac{\partial U}{\partial \vec{S}}\right]_{s,D}, \quad \theta = \left[\frac{\partial U}{\partial s}\right]_{S,D}, \quad \vec{E} = \left[\frac{\partial U}{\partial \vec{D}}\right]_{S,s}$$

temperature

$$A = U - \theta s, \quad dA = -s \, d\theta + \vec{T} \cdot d\vec{S} + \vec{E} \cdot d\vec{D}$$: Helmholtz Free Energy

$$\vec{T} = \left[\frac{\partial A}{\partial S}\right]_{\theta, \vec{D}}, \quad s = \left[\frac{\partial A}{\partial \theta}\right]_{\vec{S}, \vec{D}}, \quad \vec{E} = \left[\frac{\partial A}{\partial \vec{D}}\right]_{\vec{S}, \theta}$$

$G_1 = U - \theta s - \vec{T}\cdot\vec{S}$, $dG_1 = -s\,d\theta - \vec{S}\cdot d\vec{T} + \vec{E}\cdot d\vec{D}$: Gibbs F.E.

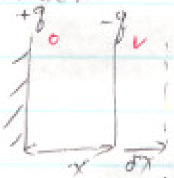
$\delta G_1 = \delta W \Rightarrow -\delta U_c^M + \delta U_2^E + \delta W_2^M - \delta W_2^E = 0$

$G_2 = U - \theta s - \vec{D}\cdot\vec{E}$, $dG_2 = -s\,d\theta + T\cdot d\vec{S} - \vec{D}\cdot d\vec{E}$

$$\boxed{\delta U_1^M - \delta U_1^E - \delta W_1^M + \delta W_1^E = 0}$$

Displacement, Electrical Potential

- Consider



$$q = CV = \frac{\varepsilon_0 A}{x} V = \varepsilon_0 A\, E$$

$$E = \frac{q}{\varepsilon_0 A}$$

$$U = \frac{1}{2}\varepsilon_0 \int_V E^2\, dV = \frac{1}{2}\varepsilon_0 \int_0^x \frac{q^2}{\varepsilon_0^2 A^2} A\, dx$$

$$U = \frac{1}{2}\frac{q^2}{\varepsilon_0 A}$$

$$\frac{\partial U}{\partial x} = \frac{1}{2}\frac{q^2}{\varepsilon_0 A}$$

$$F = -\left(\frac{\partial U}{\partial x}\right)^{q=\text{const}} = -\frac{1}{2}Eq$$

- Consider Constant Voltage

$$U = \frac{1}{2}\varepsilon_0 \int_{Vol} E^2\, dVol = \frac{1}{2}\varepsilon_0 \int \left(\frac{V}{x}\right)^2 dVol = \frac{1}{2}\varepsilon_0 \int_0^x \left(\frac{V}{x}\right)^2 dx \cdot A$$

distance

$$U = \frac{1}{2}\varepsilon_0 A \frac{V^2}{x} = \frac{1}{2}CV^2$$

$$\frac{\partial U}{\partial x}\Big|^{V=\text{const}} = -\frac{1}{2}\varepsilon_0 A\frac{V^2}{x^2}$$

$$F = -\left(\frac{\partial U}{\partial x}\right)^{V=\text{const}} = \frac{1}{2}\varepsilon_0 A \frac{V^2}{x^2}$$

What charge is required for V to be constant?

$$V = Ex = \frac{q}{\varepsilon_0 A}x$$

$$dV = 0 = \frac{q}{\varepsilon_0 A}dx + \frac{x}{\varepsilon_0 A}dq$$

$$dq = -\frac{q}{x}dx$$

Total energy $d\tilde{U} = dU - V\,dq$

$$dU = \frac{dU}{dx}dx$$

$$dU = \left(-\frac{1}{2}\frac{\varepsilon_0 A}{x^2}V^2\right)dx$$

$$-V\,dq = \frac{q}{x}V\,dx = \frac{\varepsilon_0 A}{x^2}V^2 dx$$

$$d\tilde{U} = \frac{1}{2}\frac{\varepsilon_0 A}{x^2}V^2 \quad , \quad F = -\frac{d\tilde{U}}{dx} = \frac{1}{2}\frac{\varepsilon_0 A}{x^2}V^2$$

$\tilde{U} = U - \Sigma q \varphi$

$= U - \vec{E} \cdot \vec{D}$: electrical enthalpy

The force

$F_i = -\frac{\partial \tilde{U}}{\partial x}\Big|^{\varphi} = \frac{\partial U}{\partial x}\Big|^{\varphi} = -\frac{\partial U}{\partial x}\Big|^{q}$

More formally $\partial U = \left(\frac{\partial U}{\partial q}\right)^{x} dq + \left(\frac{\partial U}{\partial x}\right)^{q} dx$

$= \varphi \, dq + F_i \, dx$: total energy

$\tilde{U} = U - \varphi q$ $\qquad dU = E dD - T dS$: energy per volume

$\delta \tilde{U} = \delta U - \delta \varphi q - \varphi \delta q$

$= -q \delta \varphi - F_i \delta x$

$dU = \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial q} dq$

$T = \left(\frac{\partial U}{\partial S}\right)^{D},$

$E = \left(\frac{\partial U}{\partial D}\right)^{S}.$

temperature enthalpy

Gibbs free energy : $G = U - T_{ij} S_{ij} - E_m D_m - \theta \sigma$

Elastic " $\quad G_1 = U - T_{ij} S_{ij} - \theta \sigma$

Electric " $\quad G_2 = U - E_m D_m - \theta \sigma$

Helmhol free energy : $A = U - \theta \sigma$

enthalpy : $H = U - T \cdot S - E_m D_m$

Elastic enthalpy $\quad H_1 = U - T \cdot S$

Electric enthalpy : $H_2 = U - E_m D_m$

$dG = -\sigma d\theta - S_{ij} \, dT_{ij} - D_m \, dE_m$

$dG_1 = -\sigma d\theta - S_{ij} \, dT_{ij} + E_m \, dD_m$

$dG_2 = -\sigma d\theta + T_{ij} \, dS_{ij} - D_m \, dE_m$

$dA = -\sigma d\theta + T_{ij} \, dS_{ij} + E_m \, dD_m$

$dH = \theta d\sigma - S_{ij} \, dT_{ij} - D_m \, dE_m$

$dH_1 = \theta d\sigma - S_{ij} \, dT_{ij} + E_m \, dD_m$

$dH_2 = \theta d\sigma + T_{ij} \, dS_{ij} - D_m \, dE_m$

$dG = dU - \cancel{TdS} - S dT - \cancel{E dD} - D dE - \cancel{\theta d\sigma} - \sigma d\theta$

$\cancel{\theta d\sigma} + \cancel{E dD} + \cancel{T dS}$

$dG(\theta, T_{ij}, E_m) = \frac{\partial G}{\partial \theta} d\theta + \frac{\partial G}{\partial T_{ij}} dT_{ij} + \frac{\partial G}{\partial E_m} dE_m$

$\sigma = -\left(\frac{\partial G}{\partial \theta}\right)^{T,E}$

$S_{ij} = -\left(\frac{\partial G}{\partial T_{ij}}\right)^{E,\theta}$

$D_m = -\left(\frac{\partial G}{\partial E_m}\right)^{T,\theta}$

$$S_{ij} = s^{E}_{ijkl} T_{kl} + f_1(T, E)$$
$$D_m = \varepsilon^{T}_{mn} E_m + f_2(T, E)$$

Ref : 1. "Dynamics and Mechanics of Electrical Systems"
by Crandall - Chap. 6

2. "Principle & Applications of Ferroelectrics & Related Material" by M.E. Lines & A.M. Glass - Chap. 3

$$dG = -S_{ij}\, dT_{ij} - D_m\, dE_m$$
$$dG = \frac{\partial G}{\partial T_{ij}} dT_{ij} + \frac{\partial G}{\partial E_m} dE_m$$
$$S_{ij} = -\left(\frac{\partial G}{\partial T_{ij}}\right)^E , \qquad D_m = -\left(\frac{\partial G}{\partial E_m}\right)$$

Expand S_{ij}

$$dS_{ij} = \left(\frac{\partial S_{ij}}{\partial T_{kl}}\right)^E dT_{kl} \quad \left(\frac{\partial S_{ij}}{\partial E_m}\right)^T dE_m$$
$$s^E_{ijkl} = \left(\frac{\partial S_{ij}}{\partial T_{kl}}\right)^E = -\left(\frac{\partial^2 G}{\partial T_{ij}\partial T_{kl}}\right)^{E,\theta}$$
$$d^T_{ijk} = \left(\frac{\partial S_{ij}}{\partial E_k}\right)^T = -\left(\frac{\partial^2 G}{\partial T_{ij}\partial E_k}\right)^{T,\theta}$$: Piezo free Strain
$$\alpha^{T,E}_{ij} = \left(\frac{\partial S_{ij}}{\partial \theta}\right)^{T,E} = -\left(\frac{\partial^2 G}{\partial T_{ij}\partial \theta}\right)^{T,E}$$: Coefficient of Thermal Expansion

Linearize
$$dD_m \to D_m$$
$$dS_{ij} \to S_{ij}$$
$$dT_{kl} \to T_{kl}$$
$$dE_m \to E_m$$

- Constitutive Equation

$$S_{ij} = s^E_{ijkl} T_{kl} + d_{mij} E_m + \alpha^E_{ij}\theta$$
$$D_n = d_{nkl} T_{kl} + \varepsilon^T_{nm} E_m + p^T_n \theta$$
$$\sigma = \alpha_{ij} T_{ij} + p^T_m E_m + \left(\frac{\partial \sigma}{\partial \theta}\right)^{T,E}\theta$$
→ pyroelectric effect

Simplify
- pyroelectric $\to 0$
- ignore entropy
- assume constant θ, H

Linear Piezoelectric Constitutive Eq.
$$S_{ij} = s^E_{ijkl} T_{kl} + d_{mij} E_m$$
$$D_n = d_{nkl} T_{kl} + \varepsilon^T_{nm} E_m$$

- Gibbs free Energy
$$G = -\frac{1}{2}\varepsilon_{mn} E_m E_n - \frac{1}{3}\varepsilon_{mno} E_m E_n E_o - \frac{1}{4}\varepsilon_{mnop} E_m E_n E_o E_p - \dots$$
$$- \frac{1}{2} s_{ijkl} T_{ij} T_{kl} - \frac{1}{3} s_{ijklmn} T_{ij} T_{kl} T_{mn} - \dots$$

$$- u_{mnijkl} E_m T_{ij} T_{kl} - r_{mnijkl} E_m E_n T_{ij} T_{kl} - \cdots$$
$$- d_{mij} E_m T_{ij} - m_{mnij} E_m E_n T_{ij} - \cdots$$
$$D_m = -\left(\frac{\partial G}{\partial E_m}\right)^T$$
$$S_{ij} = \left(\frac{\partial G}{\partial T_{ij}}\right)^E$$
$$D_m = \varepsilon_{mn} E_n + \varepsilon_{mno} E_m E_n + \cdots$$
$$+ u_{mijkl} T_{ij} T_{kl} + 2 r_{mnijkl} E_n T_{ij} T_{kl} + \cdots$$
$$+ d_{mij} T_{ij} + 2 m_{mnij} E_n T_{ij} \cdots$$
$$S_{ij} = s_{ijkl} T_{kl} + s_{ijklmn} T_{kl} T_{mn} + \cdots$$
$$+ \cancel{2 u_{mijkl} E_m T_{kl}} + 2 r_{mnijkl} E_m E_n T_{kl} + \cdots$$
$$+ \cancel{d_{mij} E_m} + m_{mnij} E_m E_n + \cdots$$

- Quadratic Electrostrictor Equations

Simplifications … Electrostrictors are symm.

→ odd rank permittivity → 0

→ $m_{ijmn} = m_{mnij}$ $d, g, u \to 0$

→ Drop higher order terms

$$D_m = \varepsilon^T_{mn} E_n + 2 m_{mnij} E_n T_{ij}$$
$$S_{ij} = s^E_{ijkl} T_{kl} + 2 r_{mnijkl} E_m E_n T_{kl} + m_{mnij} E_m E_n$$

m : electrostrictive coupling

r : elastostriction

$$s^*_{ijkl} = s^E_{ijkl} + 2 r_{mnijkl} E_m E_n$$: electric field varing component

- Piezoelectric Constitutive Behavior

$$S_{ij} = s^E_{ijkl} T_{kl} + d_{mij} E_m$$
$$D_n = d_{nkl} T_{kl} + \varepsilon^T_{mn} E_m$$

$$\{S\} = \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ 2S_{23} \\ 2S_{13} \\ 2S_{12} \end{Bmatrix} = \begin{Bmatrix} S_1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{Bmatrix},$$

$$\{D\} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix}$$: charge

electric field and strees are similar

$$\begin{Bmatrix} D \\ S \end{Bmatrix} = \begin{bmatrix} \epsilon^T & d \\ d_t & s^E \end{bmatrix} \begin{Bmatrix} E \\ T \end{Bmatrix}$$

transpose

$$\epsilon^T = \begin{bmatrix} \epsilon_{11}^T & 0 & 0 \\ 0 & \epsilon_{22}^T & 0 \\ 0 & 0 & \epsilon_{33}^T \end{bmatrix}$$

$$s^E = \begin{bmatrix} s_{11}^E & s_{12}^E & s_{13}^E & & 0 & \\ s_{12}^E & s_{11}^E & s_{13}^E & & & \\ s_{13}^E & s_{13}^E & s_{33}^E & & & \\ & & & s_{55}^E & 0 & 0 \\ & 0 & & 0 & s_{55}^E & 0 \\ & & & 0 & 0 & s_{66}^E \end{bmatrix}$$

$$d = \begin{bmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{bmatrix}$$

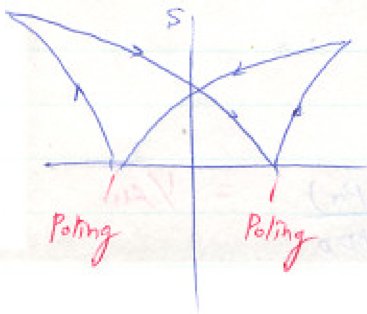
IEEE Standard STD - 176 - 1978

ferroelectric : able to be poled by E-field
transversely istropic in 1-2 directions,
poling in 3-direction

Poling

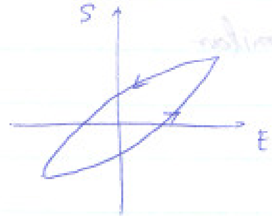
in the beginning

Apply Large electric field



"Butterfly Curve"

used



- Relation of Coupling terms

from form 2

$$T = (s^E)^{-1} S - (s^E)^{-1} d_t E$$

1st eqn. of form 2

$$D = d(s^E)^{-1} S - d(s^E)^{-1} d_t E + \epsilon^T E$$

Compare to form #1

$$C^E = (s^E)^{-1}$$

$$e = d C^E$$

$$\varepsilon^S = \varepsilon^T - d C^E d_t$$

clamped dielectric < free dielectric

- Example 2

From form 2 $\quad E = (\varepsilon^T)^{-1} D - (\varepsilon^T)^{-1} d T$

substitute into equation for S

$$S = s^E T + d_t (\varepsilon^T)^{-1} D - d_t (\varepsilon^T)^{-1} d T$$

Compare to 3,

$$s^D = s^E - d_t (\varepsilon^T)^{-1} d$$

- Material Constants

Electrical $\quad \varepsilon_{mn} = K \cdot \epsilon_0$

where $\epsilon_0 = 8.85 \times 10^{-12}$ F/m

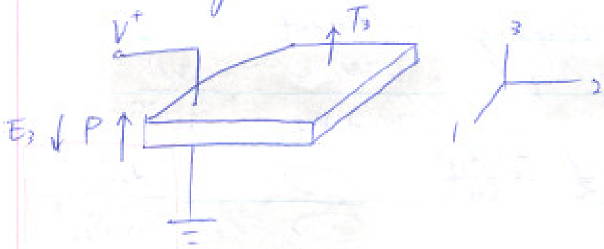
Material	K $= \varepsilon_{33}/\varepsilon_0$
Air (vacuum)	1
rubber	6
epoxy	3-6
water	80
PZT	3,400

Mechanical

Material	E (MPa) $= 1/s_{11}$
epoxy	~ 2,800

steel	200,000
Al	70,000
PZT-5H	60,600

- Modes of Operation
  - Uni-axial stress cases
    - Longitudinal mode

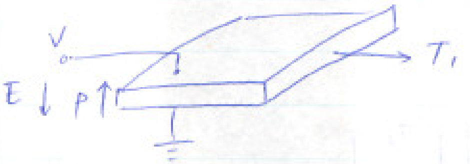


characterized E_3, T_3, All the other $T \to 0$

$$\begin{Bmatrix} S_3 \\ D_3 \end{Bmatrix} = \begin{bmatrix} s_{33}^E & d_{33} \\ d_{33} & \varepsilon_{33}^T \end{bmatrix} \begin{Bmatrix} T_3 \\ E_3 \end{Bmatrix}$$

- Transverse mode

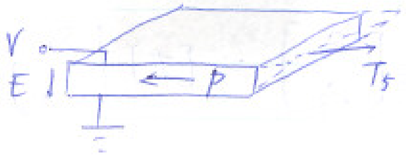
characterized E_3, T_1



$$\begin{Bmatrix} S_1 \\ D_3 \end{Bmatrix} = \begin{bmatrix} s_{11}^E & d_{31} \\ d_{31} & \varepsilon_{33}^T \end{bmatrix} \begin{Bmatrix} T_1 \\ E_3 \end{Bmatrix}$$

- Shear mode

characterized by $E_1 (\perp P)$, T_5



$$\begin{Bmatrix} S_5 \\ D_1 \end{Bmatrix} = \begin{bmatrix} s_{55}^E & d_{15} \\ d_{15} & \varepsilon_{11}^T \end{bmatrix} \begin{Bmatrix} T_5 \\ E_1 \end{Bmatrix}$$

- Lecture 3

3/9 Form Relationships

$$\varepsilon^S = \varepsilon^T - d\,c^E d_t$$

$$\varepsilon^S = \varepsilon^T \left(1 - \frac{d\,c^E d_t}{\varepsilon^T}\right)$$

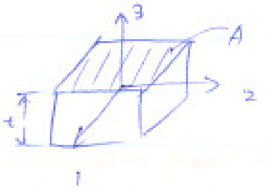
$$= \varepsilon^T \left(1 - \underbrace{\left(\frac{d\,d_t}{s^E \varepsilon^T}\right)}_{\text{coupling coefficient}}\right)$$

$$s^D = s^E - d_t\, {\varepsilon^T}^{-1} d$$

$$= s^E \left(1 - \frac{d_t d}{s^E \varepsilon^T}\right)$$

Coupling Coefficient

definition : Ratio of Electrical / Mechanical

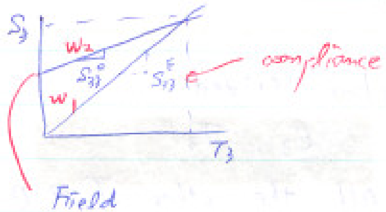


i) Load up mechanically, T_3, $E_3 = 0$

$$U = \int T_3 \, \delta S_3 = \boxed{\frac{1}{2} s_{33}^E T_3^2}$$

ii) Field

$$E_3 = -\frac{T_3 d_{33}}{\varepsilon_{33}^T}$$



Coupling coefficient

$$k^2 = \frac{W_1}{W_1 + W_2} = \frac{W^E}{W^m}$$

Field

$$W^E = \frac{1}{2} \varepsilon_3^T E_{33}^2 = \frac{1}{2} \frac{T_3^2 d_{33}^2}{\varepsilon_{33}^T}$$

$$W^m = \frac{1}{2} T_3^2 s_{33}^E$$

$$\frac{\bigcirc}{\bigcirc} = \boxed{\frac{d_{33}^2}{\varepsilon_{33}^T s_{33}^E}} = k_{33}^2$$

Transverse mode $\quad k_{31}^2 = \dfrac{d_{31}^2}{\varepsilon_{33}^T s_{11}^E}$

shear mode $\quad k_{15}^2 = \dfrac{d_{15}^2}{\varepsilon_{33}^T s_{55}^E}$

· Change of coordinates

longitudinal

$$\begin{bmatrix} S_3 \\ D_3 \end{bmatrix} = \begin{bmatrix} s_{33}^E & d_{33} \\ d_{33} & \varepsilon_{33}^T \end{bmatrix} \begin{bmatrix} T_3 \\ E_3 \end{bmatrix}$$

Define a transformation

$$\begin{bmatrix} \tilde{S}_3 \\ \tilde{D}_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{s_{33}^E} & 0 \\ 0 & 1/\sqrt{\varepsilon_{33}^T} \end{bmatrix} \begin{bmatrix} S_3 \\ D_3 \end{bmatrix} = [T] \begin{bmatrix} S_3 \\ D_3 \end{bmatrix}$$

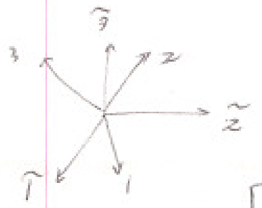
$$\begin{bmatrix} \tilde{T}_3 \\ \tilde{E}_3 \end{bmatrix} = [T^{-1}] \begin{bmatrix} T_3 \\ E_3 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{S}_{33} \\ \tilde{D}_{33} \end{bmatrix} = \underbrace{\left\{ T \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} T \right\}} \begin{bmatrix} \tilde{T}_{33} \\ \tilde{E}_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{s_{33}^E} & 0 \\ 0 & 1/\sqrt{\varepsilon_{33}^T} \end{bmatrix} \begin{bmatrix} s_{33}^E & d_{33} \\ d_{33} & \varepsilon_{33}^T \end{bmatrix} \begin{bmatrix} 1/\sqrt{s_{33}^E} & 0 \\ 0 & 1/\sqrt{\varepsilon_{33}^T} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & k_{33} \\ k_{33} & 1 \end{bmatrix} \qquad k_{33} = \sqrt{\frac{d_{33}^2}{\varepsilon_{33}^T s_{33}^E}}$$

- Rotation of Material Properties



$(\tilde{\ })$: material coordinate system
$(\ )$: structure coordinate system

$$\begin{bmatrix} \tilde{D} \\ \tilde{T} \end{bmatrix} = \begin{bmatrix} \tilde{\varepsilon}^S & \tilde{e} \\ \tilde{e} & \tilde{c}^E \end{bmatrix} \begin{Bmatrix} \tilde{E} \\ \tilde{S} \end{Bmatrix}$$

In general, use tensor transformations

$\tilde{D}_m = a_{mp} D_p$ First order tensor transformation (D, E)

$\tilde{T}_{mn} = a_{\tilde{m}p}\, a_{\tilde{n}q}\, T_{pq}$

$\boxed{T_{mn} = a_{m\tilde{p}}\, a_{n\tilde{q}}\, \tilde{T}_{pq}}$ Inverse transformation

- good for any tensor matrial property rotations

- Matrix

$\tilde{D} = F\, D$ --- (1)

$\tilde{E} = F\, E$ --- (2)

	1	2	3
$\tilde{1}$	$a_{\tilde{1}1}$	$a_{\tilde{1}2}$	$a_{\tilde{1}3}$
$\tilde{2}$	$a_{\tilde{2}1}$	$a_{\tilde{2}2}$	$a_{\tilde{2}3}$
$\tilde{3}$	$a_{\tilde{3}1}$	$a_{\tilde{3}2}$	$a_{\tilde{3}3}$

$a_{\tilde{i}j}$: direction cosine $\tilde{i}j$

$$F = \begin{bmatrix} a_{\tilde{1}1} & a_{\tilde{1}2} & a_{\tilde{1}3} \\ & \text{etc.} & \end{bmatrix}$$

$\tilde{T} = A\, T$ --- (3)

$\hat{T}_{11} = (a_{\tilde{i}j},\ T_{ij})$

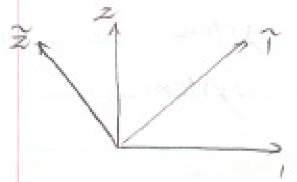
$\tilde{S} = B\, S$ --- (4)

$B = A\ (\beta = 2,\ \alpha = 1)$

$$\begin{bmatrix} F & 0 \\ 0 & A \end{bmatrix} \begin{Bmatrix} D \\ T \end{Bmatrix} = \begin{bmatrix} \tilde{\varepsilon}^S & \tilde{e} \\ -\tilde{e}_t & \tilde{c}^E \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & B \end{bmatrix} \begin{Bmatrix} E \\ S \end{Bmatrix}$$

$$\begin{bmatrix} \varepsilon^S & e \\ -e & C^E \end{bmatrix} = \begin{bmatrix} F^{-1} & 0 \\ 0 & A^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\varepsilon}^S & \tilde{e} \\ -\tilde{e}_t & \tilde{c}^E \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & B \end{bmatrix}$$

- 2D specialization



$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline \tilde{1} & c & s & 0 \\ \tilde{2} & -s & c & 0 \\ \tilde{3} & 0 & 0 & 1 \end{array}$$

$\tilde{D} = R_E D = F D$

$\tilde{E} = R_E E = F E$

$$R_E = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\tilde{S} = \beta S = R_S S$

$$R_S = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & cs \\ s^2 & c^2 & 0 & 0 & 0 & -cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -cs & cs & 0 & 0 & 0 & c^2-s^2 \end{bmatrix}$$

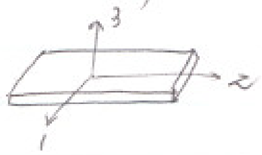
$R_T = (R_{S_T})^{-1}$

$\tilde{T} = (R_{S_T})^{-1} T$

$R_E^T = (R_E)^{-1}$

$$\begin{Bmatrix} D \\ T \end{Bmatrix} = \begin{bmatrix} R_{E_T} \tilde{\varepsilon}^S R_E & R_{E_T} \tilde{e} R_S \\ -R_{S_T} \tilde{e}_T R_E & R_{S_T} \tilde{c}^E R_S \end{bmatrix} \begin{Bmatrix} E \\ S \end{Bmatrix}$$

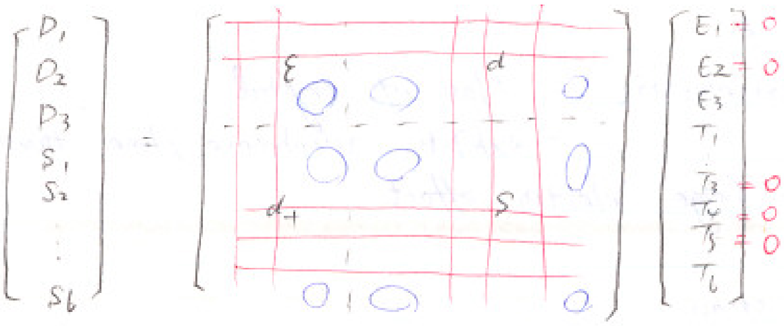
- Plane Stress + strain
  - Plane stress
    - reduction of material properties



i) $T_3 \ll T_1, T_2$

ii) Ignore shear $T_4, T_5 \rightarrow S_4, S_5 = 0$

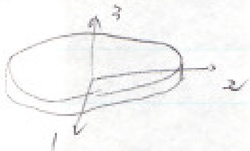
iii) $\varepsilon_3 \gg \varepsilon_1, \varepsilon_2$



$$\Rightarrow \begin{bmatrix} D_3 \\ S_1 \\ S_2 \\ S_6 \end{bmatrix} = \begin{bmatrix} \varepsilon_{33}^T & d_{31} & d_{31} & 0 \\ d_{31} & s_{11}^E & s_{12}^E & 0 \\ d_{31} & s_{12}^E & s_{22}^E & 0 \\ 0 & 0 & 0 & s_{66}^E \end{bmatrix} \begin{bmatrix} E_3 \\ T_1 \\ T_2 \\ T_6 \end{bmatrix}$$

$$\begin{Bmatrix} D_3 \\ T_1 \\ T_2 \\ T_6 \end{Bmatrix} = \begin{bmatrix} & \\ & C^E \end{bmatrix} \begin{Bmatrix} E_3 \\ S_1 \\ S_2 \\ S_6 \end{Bmatrix}$$

- Plane Strain
  - $E_3 \gg E_1, E_2$



  - S_3, S_4, S_5 are zero

$$\begin{Bmatrix} D \\ T \end{Bmatrix} = \begin{bmatrix} \varepsilon^S & e \\ -e^t & C^E \end{bmatrix} \begin{Bmatrix} E \\ S \end{Bmatrix}$$

transpose

- Electrostrictors and Relaxor Ferroelectrics

  Ref. : - The one given out.
  - Blackwood & Ealey "Electrostrictive Bias in PMN Actuators" Smart Material & Structures 2 (1993) pg. 124-133

Electrostriction Effect

strain $\propto$ (Polarization)^2

What is Polarization ?

D : electrical displacement

$D = \varepsilon_0 E + P$

$= \varepsilon_0 E + \varepsilon_r \varepsilon_0 E$

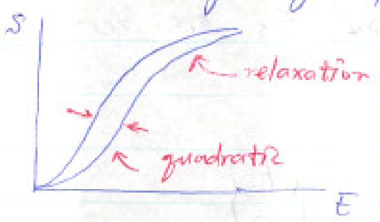
$D \simeq P$

Relaxor Ferroelectric - Class of crystal

- exhibits dispersive phase transition

- Exhibits large electro effect

- Behavior
  - Large strain
  - High stiffness
  - require no poling (sym.)
    - little hysteresis
    - long stability
    - low thermal expansion
    - high sensitivity of coupling ← to temperature
    - " of dielectric permittivity
    - nonlinear
    - brittle
    - very high permittivity



- Low temperature
  - ferroelectric (like piezo)
  - hysteresis ↗
- High temperature
  - hysteresis ↘
  - strain ↘
- Applications
  - Little hysteresis → high frequency
    → micro positioning
  - Long term stability → remote application

• Constitutive Eqn.

$$D_m = \varepsilon^T_{mn} E_n + 2\, m_{mnij} E_n T_{ij}$$
$$S_{ij} = m_{pqij} E_p E_q + s^E_{ijkl} T_{kl}$$

$$\begin{Bmatrix} \vec{D} \\ \vec{S} \end{Bmatrix} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ S_1 \\ S_2 \\ \vdots \\ S_6 \end{Bmatrix} = \begin{Bmatrix} \\ \\ \\ S_{11} \\ S_{22} \\ S_{33} \\ 2S_{23} \\ 2S_{31} \\ 2S_{12} \end{Bmatrix} = \begin{bmatrix} \varepsilon^T & 2m^* \\ m^*_t & s^E \end{bmatrix} \begin{Bmatrix} E \\ T \end{Bmatrix}$$

$$m^* = \begin{bmatrix} m_{11}E_1 & m_{12}E_1 & m_{12}E_1 & 0 & m_{44}E_3 & m_{44}E_2 \\ m_{12}E_2 & m_{11}E_2 & m_{12}E_2 & m_{44}E_3 & 0 & m_{44}E_1 \\ m_{12}E_3 & m_{12}E_3 & m_{11}E_3 & m_{44}E_2 & m_{44}E_1 & 0 \end{bmatrix}$$

If $\bar{E} = \begin{Bmatrix} 0 \\ 0 \\ E_3 \end{Bmatrix}$ then $\underline{m}$ looks like $\underline{d}$.

$$\varepsilon^T = \begin{bmatrix} \varepsilon^T_{11} & 0 & 0 \\ 0 & \varepsilon^T_{11} & 0 \\ 0 & 0 & \varepsilon^T_{11} \end{bmatrix},$$

$$s^E = \begin{bmatrix} s^E_{11} & s^E_{12} & s_{12} & 0 & 0 & 0 \\ s^E_{12} & s^E_{11} & s_{12} & 0 & 0 & 0 \\ s^E_{12} & s^E_{12} & s_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{44} \end{bmatrix}$$

quadratic $S \propto E^2$ is good low E

$$S \propto \tanh^2(kE)$$

$$S_{ij} = \frac{1}{k^2} m_{mnij} \tanh^2(k|E|) \frac{E_m E_n}{|E|^2}$$

relaxation parameter

as $E \to$ small

$$\tanh^2(k|E|) \to k^2 E^2$$

$$S \rightarrow m E^2$$

Hyperbolic Constitutive Relation

$$D_m = \epsilon^T_{mn} E_n + \frac{Z}{K} m_{mnij} T_{ij} \sinh(k|E|)/\cosh^3(k|E|) \cdot \frac{E_n}{|E|}$$

$$S_{ij} = s^E_{ijkl} T_{kl} + \frac{1}{K^2} m_{mnij} \tanh^2(k|E|) \frac{E_m E_n}{|E|^2}$$

- Polarization

$$S \propto P^2 , \quad P(E) = \epsilon_r \epsilon_0 E$$

$$P_n = P^s \tanh(k E_n)$$

↳ saturation polarization = const.

Material Properties @ room temperature for 0.9 PMN − 0.1 PT

$C^E_{1111} = 120$ GPa

$\nu^{1111} = 0.38$

$\epsilon_{33} = 17000\ \epsilon_0$ ← max

$m_{3311} = 6.6 e^{-16}\ m^2/V^2$

$K = 1.6 e^{-6}\ V/m$

$r_{33} = 3.25 e^{-24}\ m^2/V^2 \cdot Pa$

$\alpha < 1.0 e^{-6}$ °/C

$K_{IC} = 0.9\ MPa\sqrt{m}$

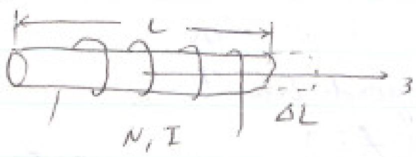
- Magnetostrictors
  - James Joule, 1840
    Nickel 50 ppm
  - 1970's : Giant Magnetostriction
    1,500 ~ 2,000 ppm

Terfenol-D ← Dysprosium

Terbium Iron Naval Ord. Lab.

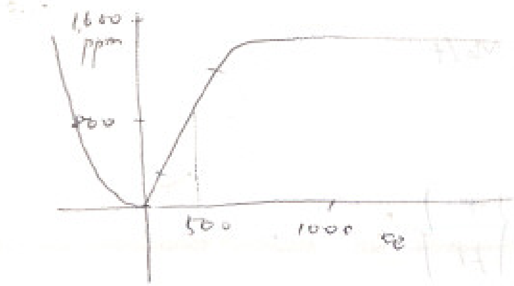
Arthur Clarke

- Phenomena



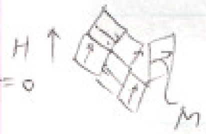
H : coercive field (A/m or Oersted)

$$H = n I$$



- Material Behavior
  - Alignment of Magnetic Domains

$\bar{M} = 0$ random alignments



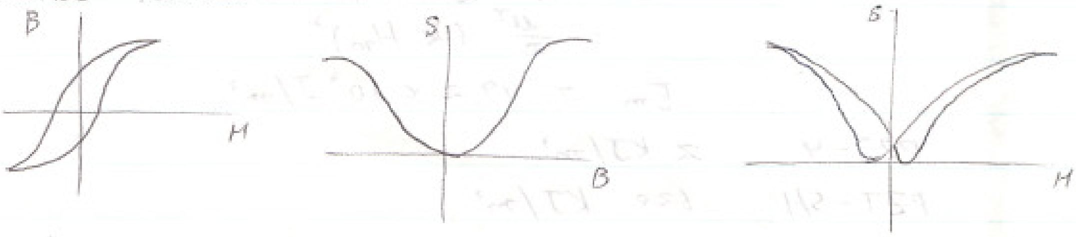
$H = 0$

$M \neq 0$, alignment & strain

Why is this material important?

- High force ~ piezo's
- Monolithic ⇒ Reliable
- Low Voltage of Operation
- Low Hysteresis
- Moderate to Bandwidth (5 ~ 10 KHz)
- Moderate Temperature stability
- No poling-aging Fatigue

- Application
  - Machine tools
  - Isolation → Machinary
  - Sonar → High power, low frequency
- Material Behavior



- Manufacturing Process
  - Bridgmen

- Free Standing Zone Melt
  Compressive Prestress

◦ Linearization

$$\begin{Bmatrix} S \\ B \end{Bmatrix} = \begin{bmatrix} s^H & d^t \\ d & \mu^T \end{bmatrix} \begin{Bmatrix} T \\ H \end{Bmatrix}$$

$y^H = 2.5 \sim 3.5 \times 10^{10}$ Pa : soft

$y^B = 5.0 \sim 7.0 \times 10^{10}$ Pa : aluminum

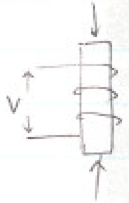
$y^H = y^B(1-k^2) \Rightarrow k = 0.7$
0.75

$d : 1.5 \to 2 \times 10^{-8}$ m/A

$\mu^T : 9.2 \times \underbrace{4\pi \times 10^{-7}}_{\mu_0}$ Tesla m/A

$\mu^s = 4.5 \times 4\pi \times 10^{-7}$

$\mu^s = \mu^T (1-k^2)$



Force - Voltage Sensors

$\Downarrow$

$B \neq 0$
$H = 0$

$B = dT$

$\varphi = BA$, $V = -N \frac{d\varphi}{dt}$

$V = -NAd \frac{dT}{dt}$

$= -Nd \frac{df}{dt}$

B=0  current introduced!

◦ Energy Density

$$E = k^2 \left(\frac{1}{2} B_{max} H_{max}\right)$$

$$= \frac{\mu^T}{2} (k H_m)^2$$

$E_m = 19.2 \times 10^3$ J/m³

PZT-4 2 KJ/m³

PZT-5H 620 KJ/m³

$$\int_{t_1}^{t_2} [\delta T - \delta U_1^M + \delta U_1^E + \delta W_1^M - \delta W_1^E]\, dt = 0$$

Generalized Hamilton's Principle
for elastoelectric Bodies

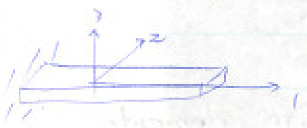
$$\int_V \delta E^T D\, dV$$

$$S = L_U \vec{U}(x,y,z) \Rightarrow \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \end{bmatrix}}_{L_u} \begin{Bmatrix} U_1(x,y,z) \\ U_2(x,y,z) \\ U_3(x,y,z) \end{Bmatrix}$$

$$\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial z} \end{bmatrix}}_{L_p} \phi(x)$$

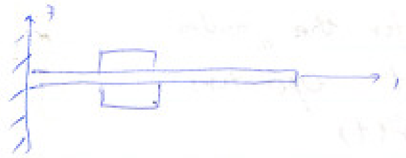
For Bernoulli - Euler Beam

$$\begin{bmatrix} U_1(x,y,z) \\ U_2(x,y,z) \\ U_3(x,y,z) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -y\frac{\partial}{\partial x} & -z\frac{\partial}{\partial x} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L_{uz}} \begin{bmatrix} \bar{U}_1(x) \\ \bar{U}_2(x) \\ \bar{U}_3(x) \end{bmatrix}$$



$$\Rightarrow L_U L_{uz}$$

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & -y\frac{\partial^2}{\partial x^2} & -z\frac{\partial^2}{\partial x^2} \\ & \underset{\sim}{0} & \end{bmatrix} \begin{bmatrix} U_1(x) \\ U_2(x) \\ U_3(x) \end{bmatrix}$$



- Classical Plate

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & -z\frac{\partial^2}{\partial x^2} \\ 0 & \frac{\partial}{\partial y} & -z\frac{\partial^2}{\partial y^2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & -2z\frac{\partial^2}{\partial x \partial y} \end{bmatrix} \begin{bmatrix} u_1(x,y) \\ u_2(x,y) \\ u_3(x,y) \end{bmatrix}$$

Generalized Coordinate

$$\vec{u}(\vec{x},t) = \psi_r(\vec{x})\,\vec{r}(t) = [\psi_{r_1}(\vec{x}) \cdots \psi_{r_n}(\vec{x})] \begin{bmatrix} r_1(t) \\ \vdots \\ r_n(t) \end{bmatrix}$$

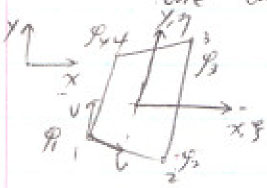
voltage

$$\varphi(\vec{x},t) = \psi_v(\vec{x})\,\vec{V}(t) = [\psi_{v_1}(\vec{x}) \cdots \psi_{v_m}(\vec{x})] \begin{bmatrix} V_1(t) \\ \vdots \\ V_m(t) \end{bmatrix}$$

$$\vec{u}(\vec{x},t) = \psi_r(\vec{x})\,\vec{r}(t)$$
$$\varphi(\vec{x},t) = \psi_v(\vec{x})\,\vec{v}(t)$$

Rayleigh - Ritz ⇒ over whole domain

- Finite Elements
  - displacement - voltage based finite elements are directly analogous to Rayleigh - Ritz procedure



Example - 2-D quadrilateral
- 3 DOF per node: 2 mechanical, 1 electrical

$$[\,u(x,y) \;\; v(x,y) \;\; \varphi(x,y)\,] = \frac{1}{4}\sum_{i=1}^{4}(1-\xi_i\xi)(1-\eta_i\eta)[\,u_i \;\; v_i \;\; \varphi_i\,]$$

ξ_i, η_i are $\xi-\eta$ coordinate for the nodes

Combining shapes with Differential Operators

$$S(x,t) = \underbrace{L_u\,\psi_r(x)}_{N_r(x)}\,\vec{r}(t)$$

$$E(x,t) = \underbrace{L_\varphi\,\psi_v(x)}_{N_v(x)}\,V(t)$$

- Linear Piezoelectric

linear piezoelectric material

$$(-M_s + M_p)\ddot{r} + (K_s + K_p^E) r - \Theta v = \underbrace{B_f \vec{f}}_{Q_p} + Q_B + Q_s$$

$$(\Theta^T) r + (C_s + C_p^S) v = B_q q$$ ← vector of applied charges

$$M_{s,p} = \int_{V_s, V_p} \psi_r^T \rho_{s,p}(x) \psi_r \, dV$$

$$K_{s,p} = \int_{V_s, V_p} N_r^T C_{s,p}^E N_r \, dV$$

$$C_{s,p} = \int_{V_s, V_p} N_v^T \epsilon_{s,p}^S N_v \, dV$$

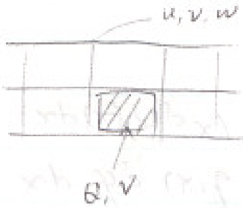
$$\Theta = \int_{V_p} N_R^T e_t N_v \, dV$$

$$B_f = \begin{bmatrix} \psi_{r_1}^T(x_{f_1}) & \cdots & \psi_{r_1}^T(x_{f_\ell}) \\ \vdots & & \vdots \\ \psi_{r_n}^T(x_{f_1}) & \cdots & \psi_{r_n}^T(x_{f_\ell}) \end{bmatrix}$$

$$B_q = \begin{bmatrix} \psi_{v_1}(x_{q_1}) & \cdots & \psi_{v_1}(x_{q_k}) \\ \vdots & & \vdots \\ \psi_{v_m}(x_{q_1}) & \cdots & \psi_{v_m}(x_{q_k}) \end{bmatrix}$$

$$Q_s = \int_S \psi_r^T(x) f^S(x) \, dS$$

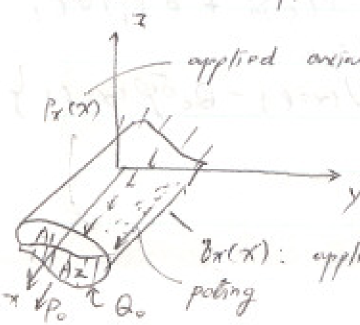
$$Q_v = \int_V \psi_r^T(x) f^B(x) \, dV$$



$$\begin{Bmatrix} F \\ Q \end{Bmatrix} = \begin{bmatrix} K_{uu} & K_{uv} \\ -K_{uv}^T & K_{vv} \end{bmatrix} \begin{Bmatrix} U \\ V \end{Bmatrix}$$

capacitance $Q = CV$

$U = 0 \;:\; F = K_{uv} V$



Prismatic

piezo poled in x-direction

- Kinematic Assumptions

$$\varphi = \varphi(x,t) \Rightarrow E_1 = -\varphi'(x,t)$$
$$E_2 = E_3 = 0$$
$$U_1 = U_1(x,t) \Rightarrow S_1 = U_1'(x,t)$$
$$U_2 = U_3 = 0 \qquad S_2 = S_6 = 0$$

$$\int_{t_1}^{t_2} [\delta T - \delta U_1^M + \delta U_1^E + \delta W_1^M - \delta W_1^E] dt = 0$$

$$\delta T = \delta \int_V^{\dot{\xi}} \{ \int_0^{\xi} \rho \dot{U} \dot{U}' d\xi \} dV = \int_\ell [\bar{m} \dot{U} \delta \dot{U}] dx$$

$\frac{\partial \dot{U}''}{\partial \xi}$

$$\bar{m} = A_1 \rho_1 + A_2 \rho_2$$

$$\delta U_1^M = \delta \int_V \{ \int_0^{\bar{\xi}} T S' dS \} dV = \int_V [T_1 \delta S_1] dV$$

Introduce Constitutive Relationship

structure ...
$$\begin{bmatrix} T_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} C_{11} & 0 \\ 0 & \varepsilon_{11} \end{bmatrix} \begin{bmatrix} S_1 \\ E_1 \end{bmatrix}$$

piezo ...
$$\begin{bmatrix} T_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} C_{11}^E & -e_{11} \\ -e_{11} & \varepsilon_{11}^S \end{bmatrix} \begin{bmatrix} S_1 \\ E_1 \end{bmatrix}$$

$$\delta U_1^M = \int_\ell [\bar{c} S_1 - \bar{e} E_1] \delta S_1 dx$$
$$\bar{c} = A_1 C_{11} + A_2 C_{11}^E$$
$$\bar{e} = A_2 e_{11}$$

likewise

$$\delta U_1^E = \int_V D \delta E dV = \int_\ell [\bar{e} S_1 + \bar{\varepsilon} E_1] \delta E_1 dx$$
$$\bar{\varepsilon} = A_2 \varepsilon_{11}^S + A_1 \varepsilon_{11}$$

Works

$$\delta W_1^M = P_0 \delta U_1 (\ell_{x=\ell}) + \int_\ell P_x \delta U(x) dx$$
$$\delta W_1^E = Q_0 \delta \varphi (\ell_{x=\ell}) + \int_\ell q(x) \delta \varphi dx$$

stuffing variational principle

$$\int_{t_1}^{t_2} \{ \int_0^\ell [\bar{m} \dot{U} \delta \dot{U} - (\bar{c} S_1 - \bar{e} E_1) \delta S_1 + (\bar{e} S_1 + \bar{\varepsilon} E_1) \delta E_1$$
$$+ P_x \delta U - q_x \delta \varphi] dx + P_0 \delta U(x=\ell) - Q_0 \delta \varphi(x=\ell) \} dt = 0$$

Decisions :

1) go continuous

$$S_1 = \frac{dU}{dx}, \quad E_1 = -\frac{d\varphi}{dx}$$

$$\int_{t_1}^{t_2}\int_0^{\ell}\{[\qquad]\delta u + [\qquad]\delta\varphi\}\,dx\,dt \quad + \text{ B.C's terms}$$

ii) Discrete Representation

Rayleigh - Ritz

$$u(x) = U_0\left(\frac{x}{\ell}\right) \Rightarrow \delta S_1 = \frac{\delta U_0}{\ell}$$

$$\delta u = \delta U_0\left(\frac{x}{\ell}\right)$$

$$\varphi(x) = V_0\left(\frac{x}{\ell}\right) \Rightarrow \delta E_1 = -\frac{\delta V_0}{\ell} \qquad \delta\varphi = \delta V_0\left(\frac{x}{\ell}\right)$$

can assume more complex distribution if you want better accuracy

$$\int_{t_1}^{t_2}\left\{\int_0^{\ell}\left[m\dot{U}_0\,\delta\dot{U}_0\left(\frac{x}{\ell}\right)^2 - \left(\bar{c}\frac{U_0}{\ell}+\bar{e}\frac{V_0}{\ell}\right)\frac{\delta U_0}{\ell} + \left(\bar{e}\frac{U_0}{\ell} - \frac{\bar{\varepsilon}V_0}{\ell}\right)\left(-\frac{\delta V_0}{\ell}\right) + p_x\left(\frac{x}{\ell}\right)\delta U_0 - q_x\left(\frac{x}{\ell}\right)\delta V_0\right]dx + P_0\,\delta U_0 - Q_0\,\delta V_0\right\}dt = 0$$

$$\int_{t_1}^{t_2}\left\{\frac{\bar{m}\ell}{3}\dot{U}_0\,\delta\dot{U}_0 - \left(\frac{\bar{c}}{\ell}U_0 + \frac{\bar{e}}{\ell}V_0\right)\delta U_0 + \left(\bar{e}\frac{U_0}{\ell} - \bar{\varepsilon}\frac{V_0}{\ell}\right)(-\delta V_0) + p_x\frac{\ell}{2}\delta U_0 - q_x\frac{\ell}{2}\delta V_0 + P_0\,\delta U_0 - Q\,\delta V_0\right\}dt = 0$$

Integrating by parts

$$\int_{t_1}^{t_2}\frac{\bar{m}\ell}{3}\dot{U}_0\,\delta\dot{U}_0\,dt = \cancel{\frac{\bar{m}\ell}{3}\dot{U}_0\,\delta U_0\Big|_{t_1}^{t_2}} - \int_{t_1}^{t_2}\frac{\bar{m}\ell}{3}\ddot{U}_0\,\delta U_0\,dt$$

- for arbitrary variations in δU_0 δV_0

$$\delta U_0:\ \underbrace{\frac{\bar{m}\ell}{3}}_{M}\ddot{U}_0 + \overbrace{\frac{\bar{c}}{\ell}}^{K}U_0 + \overbrace{\frac{\bar{e}}{\ell}}^{-\theta}V_0 = p_x\frac{\ell}{2} + P_0 \quad : \text{Actuator}$$

$$\delta V_0:\ -\frac{\bar{e}}{\ell}U_0 + \frac{\bar{\varepsilon}}{\ell}V_0 = q_x\frac{\ell}{2} + Q_0 \quad : \text{Sensor}$$

$$M = \frac{\bar{m}\ell}{3}$$

$$K = \frac{\bar{c}}{\ell}$$

$$\theta = -\frac{\bar{e}}{\ell}$$

$$C_p = \frac{\bar{\varepsilon}}{\ell}$$

- simplifying case

Free response to applied voltage, V_0 at end electrode

→ quantitative

$$U_0 = -\frac{\bar{e}}{\bar{c}}\frac{V_0}{\ell}$$

$$\bar{\varepsilon} = A_1\varepsilon_{11} + A_2\varepsilon_{11}^S$$

$$\bar{c} = A_1 c_{11} + A_2 c_{11}^E$$

$$\bar{e} = A_2 e_{11}$$

$$\frac{U_0}{\ell} = S_1 = -\left(\frac{A_2}{A_1c_{11} + A_2c_{11}^S}\right)e_{11}E_0$$

if you say $e_{11}/c_{11}^S \simeq d_{11}$, $d_{11}E_0 = \Lambda_0$

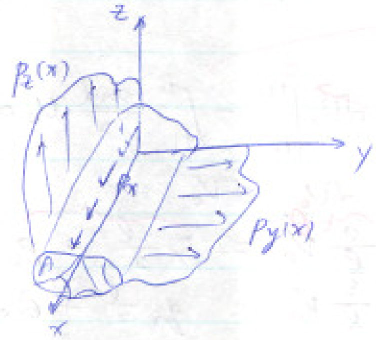
$$S_1 = -\left(\frac{1}{1+\psi}\right)\Lambda_0$$

$$\psi = \left(\frac{A_1 \; C_{11}}{A_2 \; C_{11}^{s}}\right)$$

- Beams

  - z Dimensions $\ll 1$
  - Could have
    - out of plane bending
    - extensions
    - twisting
    - warping
    - transverse shear

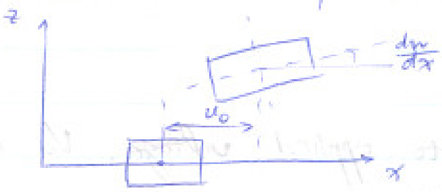
  $\Rightarrow$ all kinematic assumptions



Kinematics

$\rightarrow$ Bernoulli - Euler Beam

- plane section remain plane $\perp$ to the centerline



$$u(x,y,z) = u_0 - y\frac{dv}{dx} - z\frac{dw}{dx}$$

$$v(x,y,z) = v_0(x)$$

$$w(x,y,z) = w_0(x)$$

- Strain - Displacement Relation.

$$S_1 = \frac{du}{dx} = \frac{du_0}{dx} + y\left(-\frac{d^2 v_0}{dx^2}\right) + z\left(-\frac{d^2 w_0}{dx^2}\right)$$

$$= \varepsilon_0 + y\,K_z + z\,K_y$$

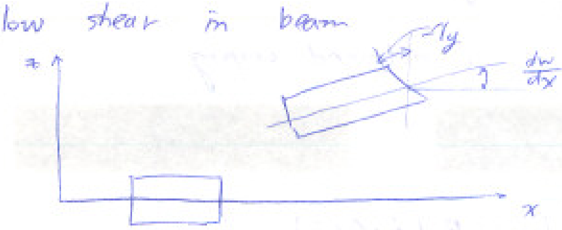
$$S_2 = \frac{dv}{dy} = 0$$

$$S_3 = \frac{dw}{dz} = 0$$
$$S_6 = 2 S_{12} = \frac{du}{dy} + \frac{dv}{dx} = 0$$
$$S_4 = S_5 = 0$$

unknowns ε_0, K_y, K_z at each section

- Transverse shear Kinematics – Timoshenko beam theory

Allow shear in beam



$$u(x, y, z) = u_0 + y\gamma_z + z\gamma_y$$
$$v = v_0$$
$$w = w_0$$

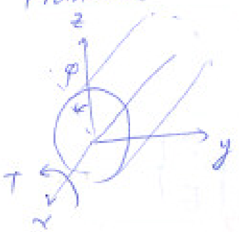
Strains

$$S_1 = \varepsilon_0 + y\overbrace{\frac{d\gamma_z}{dx}}^{K_z} + z\overbrace{\frac{d\gamma_y}{dx}}^{K_y}$$
$$S_5 = \frac{dw_0}{dx} + \gamma_y$$
$$S_6 = \frac{dv_0}{dx} + \gamma_z$$

5 parameters

$$\underset{\text{extension}}{\varepsilon_0}, \quad \underbrace{\frac{d\gamma_z}{dx}, \frac{d\gamma_y}{dx}}_{\text{2 curvatures}}, \quad \underbrace{\frac{dw_0}{dx} + \gamma_y, \frac{dv_0}{dx} + \gamma_z}_{\text{2 shear}}$$

- Torsion Kinematics
  - simplest is St. Verants torsion
  - inherently a 2-D problem
  - shear distribution on cross section
  - * Prandtl ... stress distribution



any cross section

→ $\varphi' = \frac{d\varphi}{dx}$ is constant

→ each section rotates as a rigid body.

$u = f(y, z)\varphi'$: warping

$v = -z\varphi$, $w = y\varphi$

→ cross section is free to warp.

$$S_1 = \underset{\hookrightarrow 0}{\varphi''} f(y, z) = 0$$

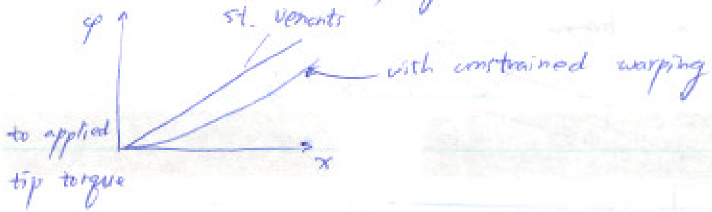
$S_2 = S_3 = S_4 = 0$

$S_5 = \varphi'\left(\frac{\partial f}{\partial z} + y\right)$

$S_6 = \varphi'\left(\frac{\partial f}{\partial y} - z\right)$

can allow φ' to vary as a function of x

$\Rightarrow$ constrained warping



- General Beam Model ("Rehfield")

$u = u_0 + y\,\gamma_z + z\,\gamma_y + \varphi'' f(y,z)$

$v = v_0 - z\varphi$

$w = w_0 + y\varphi$

- 7 Descriptive variables per cross section

$\varepsilon_0, \frac{d\gamma_z}{dx}, \frac{d\gamma_y}{dx}, \varphi', \varphi''$

$\frac{dw_0}{dx} + \gamma_y, \quad \frac{dv_0}{dx} + \gamma_z$

- Bending of a Beam
  - BE Kinematics

$u = u_0 - y\frac{dv_0}{dx} - z\frac{dw_0}{dx}$

$v = v_0(x)$

$w = w_0(x)$

$S_1 = \frac{du_0}{dx} + y\left(-\frac{d^2v_0}{dx^2}\right) + z\left(-\frac{d^2w_0}{dx^2}\right)$

$= \varepsilon_0 + y\,K_z + z\,K_y$

$S_2 \to S_6 = 0$

- Constitutive Properties

Options

i) reduce $\begin{bmatrix} T \\ D \end{bmatrix} = \begin{bmatrix} c^E & -e_t \\ e & \epsilon^S \end{bmatrix}_{9\times 9} \begin{Bmatrix} S \\ E \end{Bmatrix}$ . . . (1)

to $\begin{bmatrix} T_1 \\ D \end{bmatrix} = \begin{bmatrix} c_{11}^E & -e_t \\ e_t & \epsilon^S \end{bmatrix}_{4\times 4} \begin{Bmatrix} S_1 \\ E \end{Bmatrix}$ . . . (2)

Option ii) reduce (1) to include only normal strains

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ D \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ E \end{bmatrix}$$

then say $S_2 = S_3 = -\nu S_1$

reduce to

$$\begin{bmatrix} T_1 \\ D \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} S_1 \\ E \end{bmatrix}$$

Option (iii) stress form (2)

$$\begin{bmatrix} S \\ D \end{bmatrix} = \begin{bmatrix} s^E & d_t \\ d & \epsilon^T \end{bmatrix} \begin{bmatrix} T \\ E \end{bmatrix}$$

reduce this assuming only T_1

$$\begin{bmatrix} S_1 \\ D \end{bmatrix} = \begin{bmatrix} s_{11}^E & d_t \\ d & \epsilon^T \end{bmatrix} \begin{bmatrix} T_1 \\ E \end{bmatrix}$$

$S_1 = s_{11} T_1 + dE$

$T_1 = \underbrace{s_{11}^{-1}}_{c_{11}^E} S_1 - \underbrace{s_{11}^{-1} d}_{-e} E$

invert this to obtain

$$\begin{bmatrix} T_1 \\ D \end{bmatrix} = \underbrace{\begin{bmatrix} c_{11}^E & -e_t \\ e & \epsilon^S \end{bmatrix}}_{\text{effective}} \begin{bmatrix} S_1 \\ E \end{bmatrix}$$

no piezoelectric coupling in structure

$$\begin{bmatrix} T_1 \\ D_1 \\ D_2 \\ D_3 \end{bmatrix} = \underbrace{\left[\begin{array}{c|ccc} c_{11}^E & -e_{11} & -e_{12} & -e_{13} \\ \hline e_{11} & \epsilon_1^S & 0 & 0 \\ e_{12} & 0 & \epsilon_2^S & 0 \\ e_{13} & 0 & 0 & \epsilon_3^S \end{array}\right]}_{\text{effective properties}} \begin{bmatrix} S_1 \\ E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

- Variational Principle

$$\int_{t_1}^{t_2} [\delta T - \delta U_1^m + \delta U_1^E + \delta W_1^m - \delta W_1^E] \, dt = 0$$

– First Kinetic Energy

$$\delta T = \int_V \rho \vec{V} \cdot \delta \vec{V} \, dV$$

$$= \int_x \bar{m} \begin{bmatrix} \dot{u}_0 & \dot{v}_0 & \dot{w}_0 \end{bmatrix} \begin{bmatrix} \delta \dot{u}_0 \\ \delta \dot{v}_0 \\ \delta \dot{w}_0 \end{bmatrix} dx$$

$\bar{m} = \int_A \rho \, dA$ = effective mass per unit length

$\ddot{u} \cong \ddot{u}_0$: Ignore rotary inertia

- Mechanical energies

$$\delta U_1^m = \int_V T \, \delta S \, dV$$

$$= \int_x \Big\{ \int_{A_1} \Big\{ C_{11}^E (\varepsilon_0 + y K_z + z K_y) - \overbrace{\vec{e}_1 \cdot \vec{E}}^{T_1^E} \Big\} \cdot (\delta\varepsilon_0 + y \delta K_z + z \delta K_y) \, dA + \int_{A_0} C_{11} (\varepsilon_0 + y K_z + z K_y)(\delta \varepsilon_0 + y \delta K_z + z \delta K_y) \, dA \Big\} dx$$

$$\delta U_1^m = \int_x \left\{ [\varepsilon_0 \;\; K_z \;\; K_y] \underset{3\times3}{\bar{K}} \begin{bmatrix} \delta\varepsilon_0 \\ \delta K_z \\ \delta K_y \end{bmatrix} - \underset{1\times3}{\bar{O}} \begin{bmatrix} \delta\varepsilon_0 \\ \delta K_z \\ \delta K_y \end{bmatrix} \right\} dx$$

where,

$$\bar{K} = \int_A \begin{bmatrix} C_{11} & y C_{11} & z C_{11} \\ y C_{11} & y^2 C_{11} & yz C_{11} \\ z C_{11} & yz C_{11} & z^2 C_{11} \end{bmatrix} dA$$

$$\bar{O} = [\, P^E \;\; -M_z^E \;\; -M_y^E \,]$$

where, $P^E = \int_A \vec{e} \cdot \vec{E} \, dA = \int_A T_1^E \, dA$

$$-M_z^E = \int_A y \, T_1^E \, dA$$

$$-M_y^E = \int_A z \, T_1^E \, dA$$

- Work Terms

$$\delta W_1^m = \int_x [P_x \delta\varepsilon_0 - M_z \delta K_z - M_y \delta K_y] \, dx$$

$$\begin{cases} P_x = \int_A T_1 \, \delta x \\ -M_z = \int_A y T_1 \, \delta x \\ -M_y = \int_A z T_1 \, \delta x \end{cases}$$

$$\delta W_1^m = \int \Big\{ \overset{\text{small}}{p_x} \delta u_0 + \overset{\text{small}}{p_y} \delta v_0 + \overset{\text{small}}{p_z} \delta w_0 + m_x \, 0 + m_y \left(-\frac{dw_0}{dx}\right) + m_z \cdot \left(\frac{dv_0}{dx}\right) \Big\} dx$$

Integrating by parts,

$$= \Big(\overset{\text{small}}{\textstyle\int p_x}\Big) \delta u_0 \Big|_0^l - \int_x \Big(\overset{\text{large}}{\textstyle\int p_x}\Big)\left(\frac{d\delta u_0}{dx}\right) dx + \Big(\overset{\text{small}}{\textstyle\int p_y}\Big) \delta v_0 \Big|_0^l + \int_x \Big[\Big(-\overset{\text{large}}{\textstyle\int p_y}\Big) + m_z\Big]\left(\frac{d\delta v_0}{dx}\right) dx$$

$$+ \Big(\overset{\text{small}}{\textstyle\int p_z}\Big) \delta w_0 \Big|_0^l + \int_x \Big[\Big(-\overset{\text{large}}{\textstyle\int p_z}\Big) - m_y\Big]\left(\frac{d\delta w_0}{dx}\right) dx$$

$$= \underline{\Big(\textstyle\int p_x\Big) \delta u_0} \Big|_0^l - \int_x \overbrace{\Big[\textstyle\int p_x\Big]}^{P_x} \overbrace{\left(\frac{d\delta u_0}{dx}\right)}^{\delta\varepsilon_0} dx + \underline{\Big(\textstyle\int p_y\Big) \delta v_0 \Big|_0^l}$$

$$+ \underline{\Big[\int_x \Big\{\Big(-\textstyle\int p_y\Big) + m_z\Big\}\Big] \frac{d\delta v_0}{dx}} \Big|_0^l$$

$$+ \int_x \Big[\underbrace{\int_x \Big\{\Big(-\textstyle\int p_y\Big) + m_z\Big\}}_{-M_z} \underbrace{\left(-\frac{d^2 \delta v_0}{dx^2}\right)}_{\delta K_z} \Big] dx$$

$$+ \underline{(\int P_x)\,\delta W_0\Big|_0^{\ell}} + \underline{\left[\int_x \{(-\int P_z) - m_x\}\right] \frac{d\,\delta W_0}{dx}\Big|_0^{\ell}}$$

$$+ \int_x \Big[\underbrace{\int_x \{(-\int P_z) - m_y\}}_{-M_y} \underbrace{\left(-\frac{d^2 \delta W_0}{dx^2}\right)}_{\delta K_y}\Big] dx$$

$$M_z'' + m_z' = P_y$$

$$M_y' - m_y'' = P_z$$

$$P_x' = -P_x$$

$$\delta W_1^m = \int_x [P_x \,\delta \epsilon_0 - M_z K_z - M_y K_y] dx \quad \text{stress resultants}$$

Few options

i) Add Electrical terms δW_1^E, δV_1^E

ii) Just look at Actuation

→ Assume Prescribed $\vec{E}$

⇒ $\delta E = 0$ ignore electrical terms

→ Actuation Equations only

Assume for now quasistatic $\delta t = 0$

Simplified

$$\int_{t_1}^{t_2} [\delta V_1^m - \delta W_1^m] dt = 0$$

substituting

$$\int_{t_1}^{t_2}\int_x \left\{ [\epsilon_0 \;\; K_z \;\; K_y]\, \bar{C} \begin{bmatrix} \delta\epsilon_0 \\ \delta K_z \\ \delta K_y \end{bmatrix} - [P^E \;\; -M_z^E \;\; -M_y^E] \begin{bmatrix} \delta\epsilon_0 \\ \delta K_z \\ \delta K_y \end{bmatrix} \right.$$

$$\left. - [P^M \;\; -M_z^m \;\; -M_y^m] \begin{bmatrix} \delta\epsilon_0 \\ \delta K_z \\ \delta K_y \end{bmatrix} \right\} dx\,dt = 0$$

$$\int_{t_1}^{t_2}\int_x \left\{ [\;\;?\;\;] \begin{bmatrix} \delta\epsilon_0 \\ \delta K_z \\ \delta K_y \end{bmatrix} \right\} dx\,dt$$

i) Assume shape functions

$$\begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = \Psi(\vec{x})\, r(t)$$

ii) Get Station Equations

$\delta\epsilon_0$, δK_z, δK_y are arbitrary

$[\ ?\ ] = 0 \quad @ \text{ all } x$

$$\bar{c}_{3\times3} \begin{bmatrix} \epsilon_0 \\ \kappa_z \\ \kappa_y \end{bmatrix} = \begin{bmatrix} P^M + P^E \\ -M_z^m - M_z^E \\ -M_y^m - M_y^E \end{bmatrix}$$

$\bar{c}$ has form

$$\bar{c} = \int_A \begin{bmatrix} c_{11} & y c_{11} & z c_{11} \\ y c_{11} & y^2 c_{11} & yz c_{11} \\ z c_{11} & yz c_{11} & z^2 c_{11} \end{bmatrix} dA$$

↳ these are dependent on position

Sometimes

$$\bar{c} = E_R \begin{bmatrix} \bar{A} & \bar{y}\bar{A} & \bar{z}\bar{A} \\ \bar{y}\bar{A} & \bar{I}_{zz} & \bar{I}_{yz} \\ \bar{z}\bar{A} & \bar{I}_{yz} & \bar{I}_{yy} \end{bmatrix}$$

Modulus weighted properties

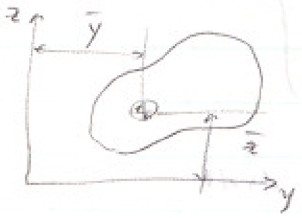
$\bar{A}$: modulus weighted area

$$= \int \frac{c_{11}}{E_r}\, dA$$

$\bar{y}$: location of modulus weighted centroid

$$\bar{y} = \frac{1}{\bar{A}} \int y \frac{c_{11}}{E_r}\, dA$$

$$\bar{z} = \frac{1}{\bar{A}} \int z \frac{c_{11}}{E_r}\, dA$$

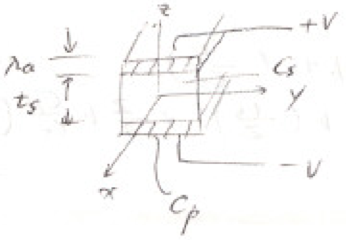


if you set axis such that $\bar{y} = \bar{z} = 0$

$$\epsilon_0 = \frac{1}{E_r \bar{A}} (P^M + P^E)$$

$$\begin{bmatrix} \kappa_z \\ \kappa_y \end{bmatrix} = \frac{1/E_r}{\bar{I}_{zz}\bar{I}_{yy} - \bar{I}_{yz}^2} \begin{bmatrix} \bar{I}_{yy} & -\bar{I}_{yz} \\ -\bar{I}_{yz} & \bar{I}_{zz} \end{bmatrix} \begin{bmatrix} -(M_z^M + M_z^E) \\ -(M_y^M + M_y^E) \end{bmatrix}$$

- Simple Example



Notes :

- we are already at modulus weighted centroid
- $\bar{I}_{yz} = 0$
- $M_z = 0$

• Material Properties

Structure $C_{11} = C_s = C_p$

Piezo $\begin{bmatrix} T_1 \\ D \end{bmatrix} = \begin{bmatrix} C_{11}^E & -e_{31} \\ e_{31} & \epsilon_{33}^S \end{bmatrix} \begin{bmatrix} S_1 \\ E_3 \end{bmatrix}$

Piezo in z direction

transverse act

Electric field Assumption

Case A : $E_3 = +E_0 \ (z > 0)$

$E_3 = -E_0 \ (z < 0)$

Case B : $E_3 = E_0$ all z

Due to symmetry z equations (no mechanical loading)

$$\epsilon_0 = \frac{1}{C_s A} P^E$$

$$\kappa_y = -M_y^E / C_s \bar{I}_{yy}$$

$$P^E = \int_A \vec{e}_r \cdot \vec{E} \, dA$$

$$-M_y^E = \int_A z \, \vec{e}_r \cdot \vec{E} \, dA$$

$$\vec{e}_r \cdot \vec{E} = e_{31} E_0$$

Case A : $P^E = 0$

$$M_y^E = 2b \int_{\frac{t_s}{2}}^{\frac{t_s}{2}+t_a} z \, e_{31} E_0 \, dz$$

$$= b \, e_{31} E_0 \left[\left(\frac{t_s}{2} + t_a \right)^2 - \left(\frac{t_s}{2} \right)^2 \right]$$

Case B : $P^E = 2b \, e_{31} E_0 \, t_a$

$-M_y^E = 0$

Furthermore

$$\bar{A} = \int_A \frac{C_{11}(z)}{C_s}\, dA = b t_s + 2 b t_a \left(\frac{C_p}{C_s} \right)$$

$$\bar{I}_{yy} = b \int_z z^2 \frac{C_{11}(z)}{C_s}\, dA = \frac{2}{3} b \left(\frac{t_s}{2} \right)^3 + \frac{2}{3} b \frac{C_p}{C_s} \left[\left(\frac{t_s}{2} + t_a \right)^3 - \left(\frac{t_s}{2} \right)^3 \right]$$

Case B :

$$\epsilon_0 = \frac{2 b e_{31} E_0 t_a}{C_s b t_s + 2 b t_a C_p} = \underbrace{\left(\frac{e_{31} E_0}{C_p} \right)}_{\Lambda} \frac{1}{1+\psi}, \quad \psi = \frac{C_s t_s}{C_p t_a}$$

$$\kappa_y = 0$$

Case A :

$$\epsilon_0 = 0$$

$$\kappa_y = -\frac{M_y^E}{C_s \bar{I}_{yy}} = \frac{3 \overbrace{\left(\frac{e_{31} E_0}{C_p} \right)}^{\Lambda} \left[\left(\frac{t_s}{2} + t_a \right)^2 - \left(\frac{t_s}{2} \right)^2 \right]}{2 \left[\frac{C_s}{C_p} \left(\frac{t_s}{2} \right)^3 + \left(\frac{t_s}{2} + t_a \right)^3 - \left(\frac{t_s}{2} \right)^3 \right]}$$

limiting case

$$t_s = 0 \qquad \frac{3 \left(\frac{e_{31} E_0}{C_p} \right) t_a^2}{2 t_a^3} = \frac{3}{2} \frac{1}{t_a} \underbrace{\left(\frac{e_{31} E_0}{C_p} \right)}_{\Lambda}$$

Some comments on the Electrical side

$\Rightarrow$ Known E

$$E = \psi(x, y, z)\, v(t)$$

$$\Rightarrow \begin{bmatrix} P^E \\ -M_z^E \\ -M_y^E \end{bmatrix} = v(t) \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

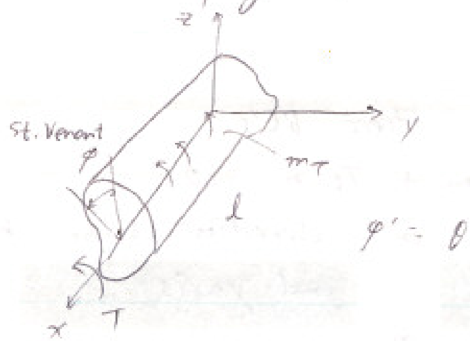
- Torsion

- very different from bending
- 2D distribution of stresses on cross section
- 2 way to approach
  - a) assumed stress function approach
    - membrane analogy

$$\tau = \frac{d\varphi}{dy}, \quad \tau_b = \frac{d\varphi}{dz}$$

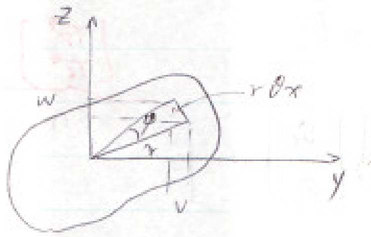
$$\nabla^2 \varphi = 2 G \theta, \quad \theta : \text{rate of twist}$$

b) assumed displacement function } st. Venant's Torsion
"warping" function



$\varphi' = \theta$: constant

- Assumption
  - each section rotates as a rigid body
  - rate of twist θ = constant
  - cross section is free to warp but, same in all cross sections



$v = -xz\theta$

$w = xy\theta$

$u = \theta f(y,z)$

$v = -r\theta x \sin\alpha$

$= -r\theta x \frac{z}{r}$

$= -xz\theta$

Strains

$S_1, S_2, S_3, S_4 = 0$

$S_5 = \frac{du}{dz} + \frac{dw}{dx} = \theta\left(\frac{\partial f}{\partial z} + y\right)$

$S_6 = \frac{du}{dy} + \frac{dv}{dx} = \theta\left(\frac{\partial f}{\partial y} - z\right)$

consider the warping function

equilibrium → only shear present

$$\boxed{\frac{\partial T_6}{\partial y} + \frac{\partial T_5}{\partial z} = 0}$$

$\frac{\partial T_6}{\partial x} = 0, \quad \frac{\partial T_5}{\partial x} = 0$

assume

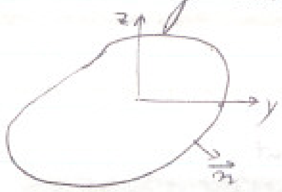
$\begin{bmatrix} T_5 \\ T_6 \end{bmatrix} = G \begin{bmatrix} S_5 \\ S_6 \end{bmatrix} \Rightarrow \nabla^2 f = 0$ Laplace's eqn.

$$\nabla^2 \varphi = -2G\theta$$

↳ stress function

BC's :

- along contour



→ From stress BC's

$$T_6\, m + T_5\, n = 0$$

where, m : direction cosine between $\vec{n}$ and y axis

n : " z axis

This gives $\left(\frac{\partial f}{\partial y} - z\right)m + \left(\frac{\partial f}{\partial z} + y\right)n = 0$

on stress-free contour

f must be continuous and differentiable

∘ Constitutive Relations

$$T = c^E S - e_t E$$

only read shear

$$\begin{bmatrix} T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} C_{55} & \cancel{C_{56}} \\ \cancel{C_{65}} & C_{66} \end{bmatrix} \begin{bmatrix} S_5 \\ S_6 \end{bmatrix} + \overbrace{\begin{bmatrix} e_t^{(5,6)} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}}^{\begin{bmatrix} T_5^E \\ T_6^E \end{bmatrix}}$$

$$T_5 = C_5 S_5 - T_5^E$$

$$T_6 = C_6 S_6 - T_6^E$$

∘ Variational principle

$$\int_{t_1}^{t_2} \int_V \left[\delta \overset{=0}{T} - \delta U_1^m - \delta W_1^m \right] dt = 0$$

$$\delta U_1^m = \int_V T_5\, \delta S_5 + T_6\, \delta S_6 \, dV$$

$$= \int_x \int_A \left\{ \left[C_{55}\, \theta \left(\frac{\partial f}{\partial z} + y\right) - T_5^E \right] \delta\theta \left(\frac{\partial f}{\partial z} + y\right) + \left[C_{66}\, \theta \left(\frac{\partial f}{\partial y} - z\right) - T_6^E \right] \delta\theta \left(\frac{\partial f}{\partial y} - z\right) \right\} dA\, dx$$

$$= \int_x \left[\theta \bar{K} \delta\theta - M_t^E\, \delta\theta \right] dx$$

$$\bar{K} = \int_A \left[C_{55} \left(\frac{\partial f}{\partial z} + y\right)^2 - C_{66} \left(\frac{\partial f}{\partial y} - z\right)^2 \right] dx$$

$$M_t^E = \int_A \left\{ T_5^E \left(\frac{\partial f}{\partial z} + y\right) + T_6^E \left(\frac{\partial f}{\partial y} - z\right) \right\} dA$$

- Work terms
  - assume there is a distributed torque
  - end tip torque

$$\delta W_1^m = T\delta\varphi_{x=L} + \int_x \underset{\hookrightarrow \delta\theta}{m_t \delta\varphi}\, dx$$

$$= T\delta\varphi_{x=L} + \left(\int m_t dx\right)\delta\varphi\Big|_0^L - \int_x \left(\int m_t dx\right)\delta\theta\, dx$$

BC's

$$\delta W_1^m = \left(T + \int_0^L m_t dx\right)\delta\varphi_{x=L} - \cancel{\left(\int m_t dx\right)\delta\varphi_{x=0}} \; {}^{\prime\prime}0$$

$$- \int_x \underbrace{\left(\int m_t dx\right)}_{M_t^m}\delta\theta\, dx$$

$$M_t^{m\,\prime} = -m_t$$

$$M_t = -\int_0^L m_t\, dx = -T.$$

$$\Rightarrow \int_{t_1}^{t_2}\int_A \left[\theta\bar{K}\delta\theta - M_t^E\delta\theta - M_t^m\delta\theta\right] dx\, dt = 0$$

$\delta\theta$: arbitrary @ each section

$$\theta\bar{K} = M_t^E + M_t^m$$

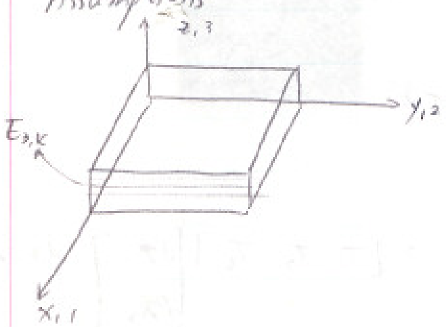
$$\bar{K} = G_r \bar{J}$$

$$= G_r \int_A \left[\frac{C_{55}}{G_r}\left(\frac{\partial f}{\partial z} + y\right)^2 + \frac{C_{66}}{G_r}\left(\frac{\partial f}{\partial y} - z\right)^2\right] dA$$

$$M_t^E = \int_A \left[T_5^E\left(\frac{\partial f}{\partial z} + y\right) + T_6^E\left(\frac{\partial f}{\partial y} - z\right)\right] dA$$

· Plate

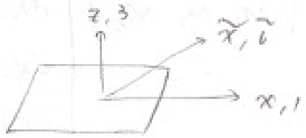
Assumptions



$T_z \ll T_x, T_y$ (plane stress)

- Kirchoff Plates
  - plane sections remain plane $\perp$ to midline
  - $\rightarrow$ ignore T_4, T_5, S_4, S_5
- Piezoelectrics polarized in z direction

· Constitutive Relations for Single Lamina



plane stress property

$$\begin{bmatrix}\tilde{T}_1\\ \tilde{T}_2\\ \tilde{T}_6\\ \tilde{D}_3\end{bmatrix} = \begin{bmatrix}\tilde{C}_{11}^E & \tilde{C}_{12}^E & 0 & -\tilde{e}_{13}\\ \tilde{C}_{12}^E & \tilde{C}_{22}^E & 0 & -\tilde{e}_{23}\\ 0 & 0 & \tilde{C}_{66}^E & 0\\ \tilde{e}_{31} & \tilde{e}_{32} & 0 & \tilde{\varepsilon}_{33}\end{bmatrix}\begin{bmatrix}\tilde{S}_1\\ \tilde{S}_2\\ \tilde{S}_6\\ \tilde{E}_3\end{bmatrix}$$

- rotations lead to fully coupled 4×4 material properties.

$$\begin{bmatrix} \tilde{C}_{11}^E & \tilde{C}_{12}^E & C_{16} & -\tilde{e}_{13} \\ & \tilde{C}_{22}^E & C_{26} & -\tilde{e}_{23} \\ & & \tilde{C}_{66}^E & e_{63} \\ & \text{etc.} & & \varepsilon_{33} \end{bmatrix}$$

2. Strain - Displacement for plate

$$u = u_0 - z\frac{\partial w}{\partial x} \qquad\qquad S_1 = \frac{\partial u}{\partial x}$$
$$v = v_0 - z\frac{\partial w}{\partial y} \quad \Rightarrow \quad S_2 = \frac{\partial v}{\partial y}$$
$$w = w_0 \qquad\qquad S_6 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} = \begin{bmatrix} S_1^\circ \\ S_2^\circ \\ S_6^\circ \end{bmatrix} + z\begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} = \begin{bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} \end{bmatrix} + z\begin{bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix}$$

plugging into constitutive relation

$$\begin{bmatrix} T_1 \\ T_2 \\ T_6 \end{bmatrix} = [C]\begin{bmatrix} S_1^\circ \\ S_2^\circ \\ S_6^\circ \end{bmatrix} + z[C]\begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} - \underbrace{\begin{bmatrix} e_{13} \\ e_{23} \\ e_{63} \end{bmatrix}}_{T_E} E_3$$

$$D_3 = [e]^T\begin{bmatrix} S_1^\circ \\ S_2^\circ \\ S_6^\circ \end{bmatrix} + z[e]^T\begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} + \varepsilon_{33}^S E_3$$

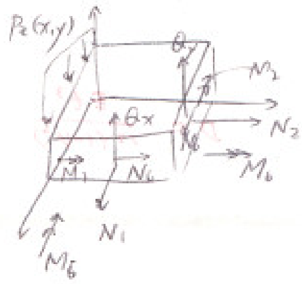
$$\delta U_1^m = \int_V \vec{T}\,\delta S\, dV$$
$$= \int_V [T_1 \; T_2 \; T_6]\, \delta\begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} dV$$

$$\delta U_1^m = \int_A \int_T [T_1 \; T_2 \; T_6]\begin{bmatrix} \delta S_1^\circ \\ \delta S_2^\circ \\ \delta S_6^\circ \end{bmatrix} + z[T_1 \; T_2 \; T_6]\begin{bmatrix} \delta\kappa_1 \\ \delta\kappa_2 \\ \delta\kappa_6 \end{bmatrix} dz\, dA$$

$$= \int_A [N_1 \; N_2 \; N_6]\begin{bmatrix} \delta S_1^\circ \\ \delta S_2^\circ \\ \delta S_6^\circ \end{bmatrix} + [M_1 \; M_2 \; M_6]\begin{bmatrix} \delta\kappa_1 \\ \delta\kappa_2 \\ \delta\kappa_6 \end{bmatrix} dA$$

$$N_1 = \int_t T_1\, dz \qquad M_1 = \int z T_1\, dz$$
$$N_2 = \int_t T_2\, dz \qquad M_2 = \int z T_2\, dz$$
$$N_6 = \int_t T_6\, dz \qquad M_6 = \int z T_6\, dz$$

$p_x, p_y(x,y)$, $m(x,y)$



$$q = \int_{A_{elec.}} D_3 \, dx\, dy$$

plugging in material properties

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} S_1^\circ \\ S_2^\circ \\ S_6^\circ \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \varkappa_1 \\ \varkappa_2 \\ \varkappa_6 \end{bmatrix}$$

$$- \sum_{k=1}^{n} \begin{bmatrix} e_{13} \\ e_{23} \\ e_{63} \end{bmatrix} E_{3k} h_k$$

$$A_{ij} = \sum_{k=1}^{n} (c_{ij}^E)_k (z_k - z_{k-1})$$
$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} (c_{ij}^E)_k (z_k^2 - z_{k-1}^2)$$

$$\begin{bmatrix} M_1 \\ M_2 \\ M_6 \end{bmatrix} = [B] \begin{bmatrix} S_1^\circ \\ S_2^\circ \\ S_6^\circ \end{bmatrix} + [D]_{3\times3} \begin{bmatrix} \varkappa_1 \\ \varkappa_2 \\ \varkappa_6 \end{bmatrix} - \sum_{k=1}^{n} \frac{1}{2}(z_k + z_{k-1}) \begin{bmatrix} e_{13} \\ e_{23} \\ e_{63} \end{bmatrix} E_{3k} h_k$$

$$h_k = z_k - z_{k-1}$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} (c_{ij}^E)_k (z_k^3 - z_{k-1}^3)$$

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}_{6\times6} \begin{bmatrix} S^\circ \\ \varkappa \end{bmatrix} - \begin{bmatrix} N^E \\ M^E \end{bmatrix}$$

- Plates (continued)

  - 6 strain-displacement

$$S_1^\circ = \frac{\partial u_0}{\partial x} \qquad \varkappa_1 = -\frac{\partial^2 w}{\partial x^2}$$
$$S_2^\circ = \frac{\partial v_0}{\partial y} \qquad \varkappa_2 = -\frac{\partial^2 w}{\partial y^2}$$
$$S_6^\circ = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \qquad \varkappa_6 = -2\frac{\partial^2 w}{\partial x \partial y}$$

  - Actual strain

$$S = S_0 + z\varkappa$$

  - 6 Stress - Strain

$$N_1 = \int_{-t/2}^{t/2} T_1 \, dz = A(S_1^\circ + \nu S_2^\circ)$$

(constant thickness , isotropic

where, $A = \frac{Et}{1-\nu^2}$

$$N_2 = \int T_2\, dz = A(S_2^{\circ} + \nu S_1^{\circ})$$
$$N_6 = \int T_6\, dz = A \frac{(1-\nu)}{2} S_6^{\circ}$$
$$M_1 = \int T_1 z\, dz = D(\kappa_1 + \nu \kappa_2) \qquad D = \frac{Et^3}{12(1-\nu^2)}$$
$$M_2 = \int T_2 z\, dz = D(\kappa_2 + \nu \kappa_1)$$
$$M_6 = \int T_6 z\, dz = D \frac{(1-\nu)}{2} \kappa_6$$

- Add piezo
$$N_E = \int_{-t/2}^{t/2} T_E\, dz$$
$$M_E = \int z T_E\, dz$$
$$\{N\} = [A]\{S_0\} + [B]\{\kappa\} - \{N_E\}$$
B = 0 if symm. laminate
$$\{M\} = [B]\{S_0\} + [D]\{\kappa\} - \{M_E\}$$

- Mechanical strain - Displacement } → Mechanical Eqn. of Motion
  " Stress - Strain
Piezo Constitutive Relationship → Hamilton's Equation
↳ Actuator, Sensor

- Piezo are sensor
  - sense charge
  - use D_3 equations.
  - assume $E_3 = 0$

$$q(t) = \int_A D_3\, dA$$
$$= \int_A (e_{31} S_1 + e_{32} S_2 + e_{36} S_6 + \varepsilon_3^S \cancel{E_3})\, dA$$
$$q = \int_A [e_{31}(S_1^{\circ} + z\kappa_1) + e_{32}(S_2^{\circ} + z\kappa_2) + e_{36}(S_6 + z\kappa_6)]\, dA$$
$z_k = z$ midplane of k-th active layer

- Mechanical Equations of Motion

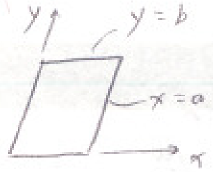
the "equilibrium" equations ($F = ma$)
$$\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} = m \frac{\partial^2 u_0}{\partial t^2} - p_x(t)$$
$$\frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} = m \frac{\partial^2 v_0}{\partial t^2} - p_y(t)$$
$$\frac{\partial^2 M_1}{\partial x^2} + 2\frac{\partial^2 M_6}{\partial x \partial y} + \frac{\partial^2 M}{\partial y^2} = m \frac{\partial^2 w}{\partial t^2} - p_z(x,y,t)$$
$$m = \frac{\text{mass}}{\text{area}} = \rho t$$

- No in-plane stress
  - stress / bending coupling $= 0 \Rightarrow B = 0$
  - Isotropic · Quasi-static

$$D \nabla^2 \nabla^2 w = -\nabla^2 M^E - p_z$$

$$D = \frac{Et^3}{12(1-\nu^2)}$$

- B.C. options



on $x = a$

Clamped : $w = 0$, $u^\circ = 0$, $v^\circ = 0$

$\frac{\partial w}{\partial x} = 0$

Simply supported : $w = 0$, $M_1 = 0$

$u^\circ = 0$, $v^\circ = 0$

Free : $N_1 = 0$, $N_6 = 0$,

$M_1 = 0$, $M_6 = 0$

$V_1 = 0$ (shear)

$\longrightarrow \frac{\partial M_6}{\partial y} + V_1 = 0$

$\Rightarrow \frac{\partial M_1}{\partial x} + 2\frac{\partial M_6}{\partial y} = 0$

- Principle of Virtual Work

$$-\int_V D \cdot \delta E \, dV + \int_V T \cdot \delta S \, dV = \int_V F \cdot dU \, dV + \int_S t_n \cdot \delta U \, dS - q\delta\varphi$$

$$T = c^E S - eE$$

$$= c^E S - T^E$$

$$D = eS + \epsilon E$$

$$\int_V -(eS + \epsilon E)\,\delta E \, dV + \int_V \left[(c^E S - T^E)\,\delta S - F\delta U\right] dV$$

$$- \int_S t_n \cdot \delta U \, dS - q\, d\varphi = 0$$

coefficients of δE et $\delta S \rightarrow 0$

$p_z(x,y)$

actuator : $\int_V (\underbrace{S_t\, c^E \delta S}_{K} - F_t \delta U - \underbrace{T^E \delta S}_{\Theta})\, dV - \int_S t_n \cdot \delta U \, dS = 0$

sensor : $\int_V (\underbrace{-T^E \delta E}_{\Theta^T} - \underbrace{E_t\, \epsilon_t\, \delta E}_{C})\, dV - \Sigma\, q_n \delta\varphi = 0$

Integrating by parts,

$$\Pi_p = \int_V (\overbrace{\tfrac{1}{2} S_t\, c^E S}^{K} - \overbrace{T_t^E S}^{\Theta})\, dV - \int_S p_z(x,y)\, w\, dS = 0$$

$$\int_V (\underbrace{-T^E E}_{\Theta^T} - \tfrac{1}{2}\underbrace{E_t\, \epsilon_t\, E}_{C})\, dV - \Sigma\, q\varphi = 0$$

- Substitute plate terms

looking at actuator equation

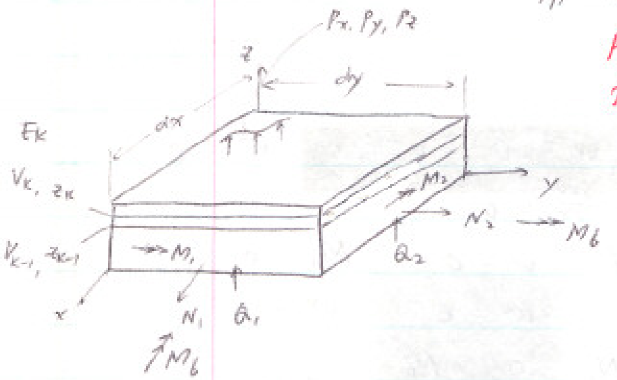
$$S = S^o + z\kappa$$

Internal energy

$$\Pi_p = \frac{1}{2}\int_A \left(S^o_t A S^o + S^o_t B \kappa + \kappa_t B S^o + \kappa_t D \kappa\right) dA - \int_A \left(N_t^E S^o_1 + M_t^E \kappa\right) dA - \int_A p_z(x,y)\, w\, dA = 0$$

Piezo or thermal force & moment

mechanical forcing



- Kirchhoff Plate

$$u = u_0 - z\frac{dw}{dx} \quad \cdots (1a)$$

$$v = v_0 - z\frac{dw}{dx} \quad \cdots (1b)$$

$$w = w_0 - z\frac{dw}{dx} \quad \cdots (1c)$$

- Kinematics

strain displacement

$$\begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} = \begin{bmatrix} S_1^o \\ S_2^o \\ S_6^o \end{bmatrix} + z\begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial x} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{bmatrix} + z\begin{bmatrix} -\frac{\partial^2 w_0}{\partial x^2} \\ -\frac{\partial^2 w_0}{\partial y^2} \\ -2\frac{\partial^2 w_0}{\partial x \partial y} \end{bmatrix}$$

$$\begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} = L_o \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} \quad \cdots (3), \qquad L_o = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & -z\frac{\partial^2}{\partial x^2} \\ 0 & \frac{\partial}{\partial y} & -z\frac{\partial^2}{\partial y^2} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & -2z\frac{\partial^2}{\partial x \partial y} \end{bmatrix}$$

Also,

$$E_K = -\frac{1}{h_K}\left(V_K - V_{K-1}\right)$$

$$\begin{bmatrix} E_n \\ \vdots \\ E_1 \end{bmatrix} = L_V \begin{bmatrix} V_n \\ \vdots \\ V_1 \end{bmatrix} \quad \cdots (4)$$

- Energy Principle

$$\int_{t_1}^{t_2} \left[\delta T - \delta U_1^M + \delta U_1^E + \delta W_1^M - \delta W_1^E\right] dt = 0$$

- Kinetic Energy

$$\delta T = \int_{\rho} [\dot u \ \dot v \ \dot w] \begin{bmatrix} \dot u \\ \dot v \\ \dot w \end{bmatrix} dV \Rightarrow \int_A \underset{\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}}{\underset{\parallel}{m(x,y)}} [\dot u_0 \ \dot v_0 \ \dot w_0] \begin{bmatrix} \delta \dot u_0 \\ \delta \dot v_0 \\ \delta \dot w_0 \end{bmatrix} dA$$

$$\dot u = \dot u_0 - \cancel{z \frac{\partial \dot w_0}{\partial x}}$$

$$\dot v = \dot v_0 - \cancel{z \frac{\partial \dot w_0}{\partial y}}$$

$$\dot w = \dot w_0$$

- Mechanical energy

$$\delta U_1^M = \int_V T \, \delta S \, dV = \int_A [N \ M]_{1\times 6} \begin{bmatrix} \delta S_0 \\ \delta \kappa \end{bmatrix} dV$$

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \int_t \begin{bmatrix} T_1 \\ T_2 \\ T_6 \\ zT_1 \\ zT_2 \\ zT_6 \end{bmatrix} dt$$

- Stress - strain

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} S_0 \\ \kappa \end{bmatrix} - \begin{bmatrix} N^E \\ M^E \end{bmatrix}$$

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} S_0 \\ \kappa \end{bmatrix} - \begin{bmatrix} C \\ F \end{bmatrix} \begin{bmatrix} E_{2n} \\ \vdots \\ E_1 \end{bmatrix}$$

$$C = [\, C_n \ \ C_{n-1} \ \cdots \ C_1 \,]$$

$$F = [\, F_n \ \ F_{n-1} \ \cdots \ F_1 \,]$$

$$C_K = \begin{bmatrix} e_{13} \\ e_{23} \\ e_{63} \end{bmatrix}_K h_K, \qquad F_K = \begin{bmatrix} e_{13} \\ e_{23} \\ e_{63} \end{bmatrix}_K h_K \frac{1}{2}(z_K + z_{K-1})$$

Plugging

$$\delta U_1^M = \int_A [N \ M] \begin{bmatrix} \delta S_0 \\ \delta \kappa \end{bmatrix} dA$$

$$= \int_A \left\{ [S_0 \ \kappa_0] \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \delta S_0 \\ \delta \kappa \end{bmatrix} - [E_n \cdots E_1][C^T F^T] \begin{bmatrix} \delta S_0 \\ \delta \kappa \end{bmatrix} \right\} dA$$

- Work

$$\delta W_1^M = \int_A [p_x \; p_y \; p_z] \begin{bmatrix} \delta u_0 \\ \delta v_0 \\ \delta w_0 \end{bmatrix} dA$$

- Derivation of Equilibrium equation.

$$\int_{t_1}^{t_2} [\delta T - \delta U_1^M + \delta W_1^m] dt +$$

$$\int_{t_1}^{t_2} \int_A \left\{ [\dot{u}_0 \; \dot{v}_0 \; \dot{w}_0] m(x,y) \begin{bmatrix} \delta u_0 \\ \delta v_0 \\ \delta w_0 \end{bmatrix} - [N \; M][L_u] \begin{bmatrix} \delta u_0 \\ \delta v_0 \\ \delta w_0 \end{bmatrix} \right.$$

$$\left. + [p_x \; p_y \; p_z] \begin{bmatrix} \delta u_0 \\ \delta v_0 \\ \delta w_0 \end{bmatrix} \right\} dA \, dt$$

$$\begin{cases} \frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} = m\ddot{u}_0 - p_x & \cdots \delta u_0 \\ \frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} = m\ddot{v}_0 - p_y & \cdots \delta v_0 \\ \frac{\partial^2 M_1}{\partial x^2} + 2\frac{\partial^2 M_6}{\partial x \partial y} + \frac{\partial^2 M_2}{\partial y^2} = m\ddot{w}_0 - p_z & \cdots \delta w_0 \end{cases}$$

- Electrical terms

$$\delta U_1^E = \int_V D \, \delta E \, dV$$

$$= \int_A \int_t \left\{ [e] \begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} + \varepsilon_{33} E \right\} \delta E \, dt \, dA$$

$[e_{13} \; e_{23} \; e_{63}]$

$$\delta U_1^E = \int_A [S_0 \; \kappa] \begin{bmatrix} C \\ F \end{bmatrix} \begin{bmatrix} \delta E_n \\ \vdots \\ \delta E_1 \end{bmatrix} + [E_n \cdots E_1][\varepsilon] \begin{bmatrix} \delta E_n \\ \vdots \\ \delta E_1 \end{bmatrix} dA$$

$$[\varepsilon] = \begin{bmatrix} \varepsilon^S_{33n} h_n & & \\ & \ddots & \\ & & \varepsilon^S_{33,1} h_1 \end{bmatrix}$$

- Electrical Work

$$\delta W_1^E = \int_V q \, \delta\varphi \, dV = \int_A [q_n \cdots q_1] \begin{bmatrix} \delta V_n \\ \vdots \\ \delta V_1 \end{bmatrix} dA$$

$$\int_{t_1}^{t_2} (\delta U_1^E - \delta W_1^E) \, dt = 0$$

$$[E] = L_v \begin{bmatrix} V_n \\ \vdots \\ V_1 \end{bmatrix}$$

$$\int_A \left\{ [S_0 \ \varkappa] \begin{bmatrix} C \\ F \end{bmatrix} L_V + [E_n \cdots E_1][\varepsilon] L_V - [q_n \cdots q_0] \right\} \begin{bmatrix} \delta V_n \\ \vdots \\ \delta V_0 \end{bmatrix} dA = 0$$

- Electrical equation of motion

$$L_V^T [C^T \ F^T] \begin{bmatrix} S_0 \\ \varkappa \end{bmatrix} + L_V^T [\varepsilon] L_V [V] = [q]$$

Everything so far

$\int_b \Rightarrow$ section equations

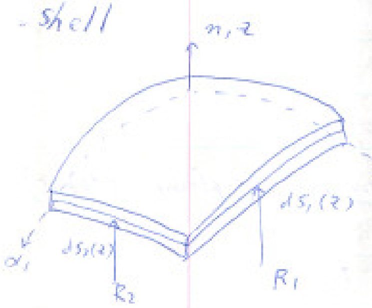
- Rita Analysis

→ just consider bending in general,

$$u_0 = \sum_{i=1}^{n_u} \varphi_{u_i}(x,y)\, q_{u_i}(t)$$
$$v_0 = \sum \varphi_{v_i}(x,y)\, q_{v_i}(t)$$
$$w_0 = \sum \varphi_{w_i}(x,y)\, q_{w_i}(t)$$

$[B] = 0$

- Shell



Assume the following

$$u = u_0(\alpha_1, \alpha_2) + z\,\beta_1(\alpha_1, \alpha_2)$$
$$v = v_0(\alpha_1, \alpha_2) + z\,\beta_2(\alpha_1, \alpha_2)$$
$$w = w_0(\alpha_1, \alpha_2)$$

plug into expression for

S_5, S_4 etc $= 0$

to solve for β_1, β_2

$$\beta_1 = \frac{u_0}{R_1} - \frac{1}{A_1}\frac{\partial w_0}{\partial \alpha_1}$$
$$\beta_2 = \frac{v_0}{R_2} - \frac{1}{A_2}\frac{\partial w_0}{\partial \alpha_2}$$

$$\begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} = \Delta_1 \begin{bmatrix} S_1^0 \\ S_2^0 \\ S_{12}^0 \\ S_{21}^0 \end{bmatrix} + z\,\Delta_1 \begin{bmatrix} \varkappa_1 \\ \varkappa_2 \\ \varkappa_{12} \\ \varkappa_{21} \end{bmatrix}, \qquad \Delta_1 = \begin{bmatrix} \frac{1}{\varphi_1} & 0 & 0 & 0 \\ 0 & \frac{1}{\varphi_2} & 0 & 0 \\ 0 & 0 & \frac{1}{\varphi_1} & \frac{1}{\varphi_2} \end{bmatrix}$$

$$S_3 \approx 0$$
$$S_4 = S_5 = 0$$
$$S_1^0 = \frac{1}{A_1}\frac{\partial u_0}{\partial \alpha_1} + \frac{v_0}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{w_0}{R_1}$$
$$S_2^0 = \frac{1}{A_2}\frac{\partial v_0}{\partial \alpha_2} + \frac{u_0}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{w_0}{R_2}$$

$$S_{12}^\circ = \frac{1}{A_1}\frac{\partial v_0}{\partial \alpha_1} - \frac{u}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}$$

$$S_{21}^\circ = \frac{1}{A_2}\frac{\partial u_0}{\partial \alpha_1} - \frac{v_0}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}$$

$$S_6^\circ = S_{12}^\circ + S_{21}^\circ$$

$$S_6 = \frac{1}{\varphi_1 \varphi_2}\left[\left(1 - \frac{z^2}{R_1 R_2}\right) S_6^\circ + z\left(1 + \frac{z}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right) \varkappa_6^\circ\right]$$

$$\varkappa_6^\circ = z\left(\varkappa_{12}^\circ + \frac{S_{21}^\circ}{R_1}\right) = z\left(\varkappa_{21}^\circ + \frac{S_{12}^\circ}{R_2}\right)$$

$$\varkappa_1^\circ = \frac{1}{A_1}\frac{\partial \beta_1}{\partial \alpha_1} + \frac{\beta_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}$$

$$\varkappa_2^\circ = \frac{1}{A_2}\frac{\partial \beta_2}{\partial \alpha_2} + \frac{\beta_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}$$

$$\varkappa_{12}^\circ = \frac{1}{A_1}\frac{\partial \beta_2}{\partial \alpha_1} - \frac{\beta_1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}$$

$$\varkappa_{21}^\circ = \frac{1}{A_2}\frac{\partial \beta_1}{\partial \alpha_2} - \frac{\beta_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_1}$$

$$\begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} = \Delta_1' \begin{bmatrix} S_1^\circ \\ S_2^\circ \\ S_6^\circ \end{bmatrix} + z\Delta_1'' \begin{bmatrix} \varkappa_1^\circ \\ \varkappa_2^\circ \\ \varkappa_6^\circ \end{bmatrix}$$

$$\Delta_1' = \begin{bmatrix} \frac{1}{\varphi_1} & 0 & 0 \\ 0 & \frac{1}{\varphi_2} & 0 \\ 0 & 0 & \frac{1}{\varphi_1\varphi_2}\left(1 - \frac{z^2}{R_1R_2}\right) \end{bmatrix}$$

$$\Delta_1'' = \begin{bmatrix} \frac{1}{\varphi_1} & 0 & 0 \\ 0 & \frac{1}{\varphi_2} & 0 \\ 0 & 0 & \frac{1}{\varphi_1\varphi_2}\left(1 + \frac{z}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right) \end{bmatrix}$$

Introducing the stress-strain relations for a lamina (plane stress)

$$\begin{bmatrix} T_1 \\ T_2 \\ T_6 \end{bmatrix} = \begin{bmatrix} C_{11}^E & C_{12}^E & C_{16}^E \\ C_{12}^E & C_{22}^E & C_{26}^E \\ C_{16}^E & C_{26}^E & C_{66}^E \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} - \underbrace{\begin{bmatrix} e_{13} \\ e_{23} \\ e_{33} \end{bmatrix} E_3}_{\begin{bmatrix} T_1^E \\ T_2^E \\ T_6^E \end{bmatrix}}$$

→ in force aligned with $(\vec{\alpha}_1, \vec{\alpha}_2, \vec{n})$

⇒ plane stress properties

$$\begin{bmatrix} T_1 \\ T_2 \\ T_6 \end{bmatrix} = [C][\Delta_1] \begin{bmatrix} S_1^\circ \\ S_2^\circ \\ S_{12}^\circ \\ S_{21}^\circ \end{bmatrix} + z[C]\Delta_1 \begin{bmatrix} \varkappa_1 \\ \varkappa_2 \\ \varkappa_{12} \\ \varkappa_{21} \end{bmatrix} - \begin{bmatrix} T_1^E \\ T_2^E \\ T_6^E \end{bmatrix}$$

$$[T] = [C]\,[\Delta_1 \;\; z\Delta_1] \begin{bmatrix} S \\ \varkappa \end{bmatrix} - [T^E]$$

Note you also have

$$\begin{bmatrix} \varepsilon^0 \\ \kappa \end{bmatrix} = \underbrace{[\Omega][\theta]}_{\text{Differential operator for shell theory}} \begin{bmatrix} u^0 \\ v^0 \\ w^0 \end{bmatrix}$$

Refer to "Jia - Rogers"

- stress Resultants

concisely

$$[N_1 \quad M_1] = \int_{-h/2}^{h/2} T_1 \varphi_2 \, [1 \quad z] \, dz$$
$$[N_2 \quad M_2] = \int T_2 \varphi_1 \, [1 \quad z] \, dz$$
$$[N_{12} \quad M_{12}] = \int T_6 \, [1 \quad z] \, \varphi_2 \, dz$$
$$[N_{21} \quad M_{21}] = \int T_6 \, [1 \quad z] \, \varphi_1 \, dz$$

$$\begin{bmatrix} N \\ M \end{bmatrix} = \int \begin{bmatrix} \Delta_2 \\ z\Delta_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_6 \end{bmatrix} dz$$

$$\Delta_2 = \begin{bmatrix} \varphi_2 & 0 & 0 \\ 0 & \varphi_1 & 0 \\ 0 & 0 & \varphi_2 \\ 0 & 0 & \varphi_1 \end{bmatrix} \qquad \Delta_2 \neq \Delta_1^T$$
$$\Delta_2 = \varphi_1 \varphi_2 \Delta_1^T$$

in contracted notation

$$N_6 = \frac{1}{2}\left(N_{12} - \frac{M_{21}}{R_2} + N_{21} - \frac{M_{12}}{R_1}\right)$$
$$M_6 = \frac{1}{2}\left(M_{12} + M_{21}\right)$$

$$\underset{6\times1}{\begin{bmatrix} N' \\ M' \end{bmatrix}} = \int \begin{bmatrix} \Delta_2' \\ z\Delta_2'' \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_6 \end{bmatrix} dz$$

$$\Delta_2' = \begin{bmatrix} \varphi_2 & 0 & 0 \\ 0 & \varphi_1 & 0 \\ 0 & 0 & \frac{\varphi_2+\varphi_1}{2} - \frac{1}{2}\left(\frac{\varphi_1}{R_1} + \frac{\varphi_2}{R_2}\right) z \end{bmatrix}$$

$$\Delta_2'' = \begin{bmatrix} \varphi_2 & 0 & 0 \\ 0 & \varphi_1 & 0 \\ 0 & 0 & \frac{\varphi_2+\varphi_1}{2} \end{bmatrix}$$

Equivalent Electrically induced stress resultants

$$\underset{6\times1}{\begin{bmatrix} N^E \\ M^E \end{bmatrix}} = \int \begin{bmatrix} \Delta_2 \\ z\Delta_2 \end{bmatrix} \begin{bmatrix} T_1^E \\ T_2^E \\ T_6^E \end{bmatrix} dz$$

$$\begin{bmatrix} N^{\varepsilon'} \\ M^{\varepsilon'} \end{bmatrix} = \int \begin{bmatrix} \Delta_2' \\ z\Delta_2'' \end{bmatrix} \begin{bmatrix} T_1^{\varepsilon} \\ T_2^{\varepsilon} \\ T_6^{\varepsilon} \end{bmatrix} dz$$

plugging in expression for stresses

$$\underset{6\times1}{\begin{bmatrix} N \\ M \end{bmatrix}} = \underset{6\times8}{\left\{ \int \begin{bmatrix} \Delta_2 \\ z\Delta_2 \end{bmatrix} [C] \begin{bmatrix} \Delta_1 & z\Delta_1 \end{bmatrix} dz \right\}} \begin{bmatrix} S^{\circ} \\ \chi \end{bmatrix} - \begin{bmatrix} N^{\varepsilon} \\ M^{\varepsilon} \end{bmatrix}$$

$$\underset{6\times1}{\begin{bmatrix} N' \\ M' \end{bmatrix}} = \underset{6\times6}{\left\{ \int \begin{bmatrix} \Delta_2' \\ z\Delta_2'' \end{bmatrix} [C] \begin{bmatrix} \Delta_1' & z\Delta_1'' \end{bmatrix} dz \right\}} \begin{bmatrix} S^{\circ\prime} \\ \chi^{\circ\prime} \end{bmatrix} - \begin{bmatrix} N^{\varepsilon'} \\ M^{\varepsilon'} \end{bmatrix}$$

- shells (continued)

$$\begin{bmatrix} N \\ M \end{bmatrix} = \underset{8\times8}{\left\{ \int \begin{bmatrix} \Delta_2 \\ z\Delta_2 \end{bmatrix} [C] \begin{bmatrix} \Delta_1 & z\Delta_1 \end{bmatrix} dz \right\}} \underset{8\times1}{\begin{bmatrix} S^{\circ} \\ \chi \end{bmatrix}} - \begin{bmatrix} N^{\varepsilon} \\ M^{\varepsilon} \end{bmatrix}$$

section properties

we can write

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} S^{\circ} \\ \chi \end{bmatrix} - \begin{bmatrix} N^{\varepsilon} \\ M^{\varepsilon} \end{bmatrix}$$

$A = \int \Delta_2 C \Delta_1 = A^T$

$B = \int z \Delta_2 C \Delta_1$

$D = \int z^2 \Delta_2 C \Delta_1$

$\Delta_2 = \varphi_1 \varphi_2 \Delta_1^T$

$A = \int \Delta_1^T C \Delta_1 \varphi_1 \varphi_2 \, dz$

$B = \int z \Delta_1^T C \Delta_1 \varphi_1 \varphi_2 \, dz$

$D = \int z^2 \cdots \, dz$

- Total Potential Energy

$$\delta U = \iint_{\alpha_1 \alpha_2} \left\{ [N \ M] \begin{bmatrix} \delta S^{\circ} \\ \delta \chi \end{bmatrix} \right\} A_1 A_2 \, d\alpha_1 \, d\alpha_2$$

$$\delta U = \int_{\alpha_1} \int_{\alpha_2} \left\{ [S^{\circ} \ \chi] \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \delta S^{\circ} \\ \delta \chi \end{bmatrix} - [N^{\varepsilon} \ M^{\varepsilon}] \begin{bmatrix} \delta S^{\circ} \\ \delta \chi \end{bmatrix} \right\} A_1 A_2 \, d\alpha_1 \, d\alpha_2$$

- Kinetic Energy

$$\delta T = \int_V [\dot{u} \ \dot{v} \ \dot{w}] \, \rho \begin{bmatrix} \delta \dot{u} \\ \delta \dot{v} \\ \delta \dot{w} \end{bmatrix} dV = \int_V [\dot{u}^{\circ} \ \dot{v}^{\circ} \ \dot{w}^{\circ}] \, \rho \begin{bmatrix} \delta \dot{u}^{\circ} \\ \delta \dot{v}^{\circ} \\ \delta \dot{w}^{\circ} \end{bmatrix} dV$$

$$\int_V = \int \int_{\alpha_1} \int_{\alpha_2} (\quad) A_1 A_2 \underbrace{\varphi_1 \varphi_2}_{\simeq 1} \, d\alpha_1 \, d\alpha_2 \, dz$$

if assume $\varphi_1 \varphi_2 \simeq 1$

$$\delta T = \int_{\alpha_1} \int_{\alpha_2} \rho h \, [\ddot{u}^\circ \;\; \ddot{v}^\circ \;\; \ddot{w}^\circ] \begin{bmatrix} \delta u^\circ \\ \delta v^\circ \\ \delta w^\circ \end{bmatrix} A_1 A_2 \, d\alpha_1 \, d\alpha_2$$

if $\varphi_1 \varphi_2 \neq 1$,

let $h = h' = \int_{-h/2}^{h/2} \rho/\rho_0 \, \varphi_1 \varphi_2 \, dz$

- Work term

equivalent forces / area in α_1, α_2, z direction called $\bar{F}_1, \bar{F}_2, \bar{F}_3$

Then,

$$\delta W = \int_{\alpha_1} \int_{\alpha_2} (\bar{F}_1 \delta u^\circ + \bar{F}_2 \delta v^\circ + \bar{F}_3 \delta w^\circ) A_1 A_2 \, d\alpha_1 \, d\alpha_2$$

- Approximate Solution by Rayleigh - Ritz

$$\begin{bmatrix} \varepsilon^\circ \\ \kappa \end{bmatrix} = \underbrace{[\Omega][\theta]}_{\Lambda_u} \begin{bmatrix} u^\circ \\ v^\circ \\ w^\circ \end{bmatrix}$$

(29), (42) in "Jia & Rogers"

plugging into Energy,

$$\delta U = \int_{\alpha_1} \int_{\alpha_2} \Big\{ [u^\circ \;\; v^\circ \;\; w^\circ] \, [\theta]^T [\Omega]^T \begin{bmatrix} A & B \\ B & D \end{bmatrix} [\Omega][\theta] \begin{bmatrix} \delta u^\circ \\ \delta v^\circ \\ \delta w^\circ \end{bmatrix}$$

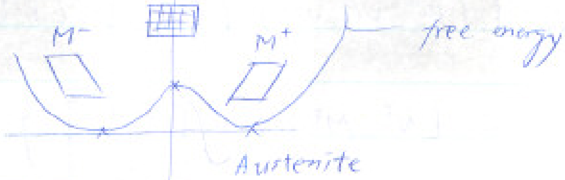
$$- [N^E \;\; M^E] \, \Omega \, \theta \begin{bmatrix} \delta u_0 \\ \delta v_0 \\ \delta w_0 \end{bmatrix} \Big\} A_1 A_2 \, d\alpha_1 \, d\alpha_2$$

· Ritz solution

$$\begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = \psi_U(\alpha_1, \alpha_2) \vec{r} = \sum_{i=1}^{n} \psi_{u_i}(\alpha_1, \alpha_2) \, r_i$$

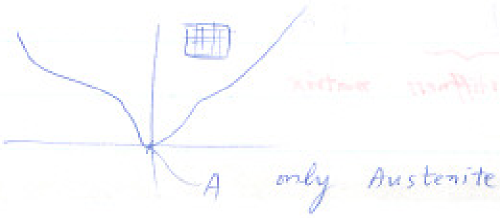
$$\delta U = r^T \underbrace{[\qquad]}_{\text{stiffness matrix}} \delta r$$

- Shape Memory Alloys
  - has an internal solid state phase transformation mechanisms which allows 2 stable states depending on of applied stress and temperatures.
  - Nickel - Titanium ("Nitinol")
    - utilized in robot applications -
      hose clamps
      large space structure vibration control
      adaptive acoustics
    - current :
      - slow adaptive structures
        - twist control of rotors & propellers.
        - adaptive fixed-wing lifting surfaces
        - air foil twist control
- The phenomena
  - stress + temperature induced martensitic phase transformation
  - depends on comp temperature, stress, history temperature
- Heuristic phenomology
  - room temperature



  - apply load

  - High temperature



- Temperature + stress induced phase transformation
- Constitutive Relation

$$\sigma - \sigma_0 = \underbrace{D(\varepsilon - \varepsilon_0)}_{\text{Mechanical}} + \underbrace{\theta(T - T_0)}_{\text{Thermal}} + \underbrace{\Omega(\xi - \xi_0)}_{\text{Phase change}}$$

ξ = % of martensite

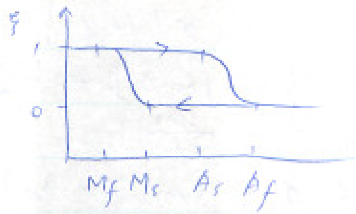
- martensite function

$$\begin{cases} \xi = 1 & \text{all martensite} \\ \xi = 0 & \text{all austenite} \end{cases}$$

- look at phase transformation

$$\xi = f(T, \sigma)$$

First, $\xi = f(T)$



Transformation is characterized by 4 temperatures.

M_f : martensite finish

M_s : " start

A_s : Austenite start

A_f : " finish

two types of material

1) $A_s > M_s$

2) $A_s < M_s$

where, in room temperature

$M \to A$: $\xi = \frac{1}{2}\{\cos[a_A(T - A_s)] + 1\}$

$A \to M$: $\xi = \frac{1}{2}\{\cos[a_M(T - M_f)] + 1\}$

$A_s < T < A_f$

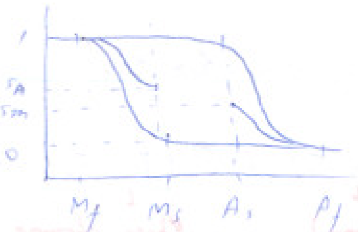
$M_f < T < M_s$

say $M - A$ $\quad \xi_0 = \xi_m$

$\xi = \xi/2 \{\cos[a_A(T - A_s)] + 1\}$

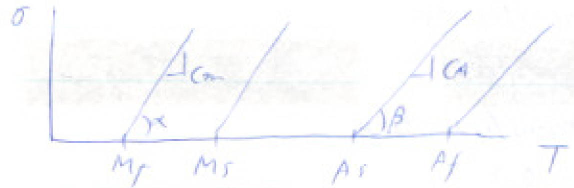
$A - M$ $\quad \xi_0 = \xi_A$

$\xi = \frac{1 - \xi_A}{2}\{\cos[a_M(T - M_f)]\} + \frac{1 + \xi_A}{2}$



∘ stress dependence of ξ

- Transformation temperatures increased with applied stress



$\Rightarrow$
$$\begin{cases} C_m = \tan(\alpha) \\ C_A = \tan(\beta) \end{cases}$$
$$\begin{cases} M_f' = M_f + \dfrac{\sigma}{C_M} \\ A_s' = A_s + \dfrac{\sigma}{C_A} \end{cases}$$

plugging in stress effects

M-A $\quad \xi = \dfrac{\xi_m}{2}\left\{\cos\left[a_A(T-A_s) - b_A\sigma\right] + 1\right\}$

A-M $\quad \xi = \dfrac{1-\xi_A}{2}\cos\left[a_M(T-M_f) + b_M\sigma\right] + \dfrac{1+\xi_A}{2}$

$$b_A = -\frac{a_A}{C_A}, \quad b_m = -\frac{a_M}{C_A}$$

M-A $\quad C_A(T-A_s) - \dfrac{\pi}{|b_A|} \le \sigma \le C_A(T-A_s)$

A-M $\quad \underbrace{C_M(T-M_f) - \dfrac{\pi}{|b_m|}}_{C_M(T-M_s)} \le \sigma \le C_M(T-M_f)$

Constitutive Modelling

1-D

Assume $M_f < M_s < T_R < A_s < A_f$ (room temperature)

∘ Case A.

- Isothermal loading
  - all austenite ($\xi = 0$)
- initial conditions

$$\sigma_0 = 0, \quad \xi_0 = 0, \quad \xi = 0, \quad T = T_0 \text{ isothermal}$$
$$\sigma - \sigma_0 = D(\varepsilon - \varepsilon_0) + \theta(T - T_0) + \Omega(\xi - \xi_0)$$

to start

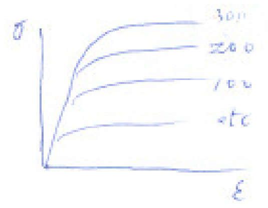
$\sigma = D\varepsilon$: linear elastic Austenite

→ fine until stresses reaches range where martensite start to form

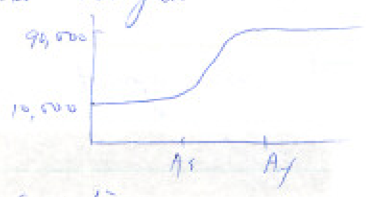
$$\sigma_{1,M} = C_M(T_0 - M_s) \rightarrow \varepsilon_{1,M} = \sigma_{1,m}/D$$

stress - strain

3.



- Yield strength



Transformation

once it begins, $\sigma - \sigma_0 = D(\varepsilon - \varepsilon_0) + \Omega(\xi - \xi_0)$

$$\begin{cases} \sigma_0 = \sigma_{lim} \\ \varepsilon = \varepsilon_{lim} \\ \xi_0 = 0 \end{cases} \Rightarrow \sigma = D\varepsilon + \Omega(\xi)$$

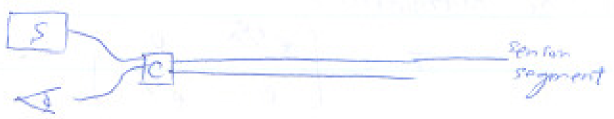
where,

$$\xi = \frac{1-\xi_A}{2} \cos\left[a_M\left(T - \left(M_f + \frac{\sigma}{C_M}\right)\right)\right] + \frac{1+\xi_A}{2}$$

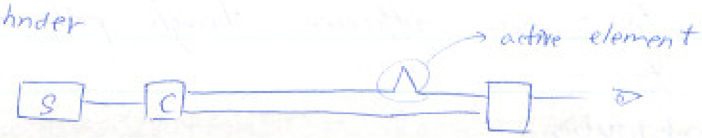
this progresses until $\xi = 1$ or wherever

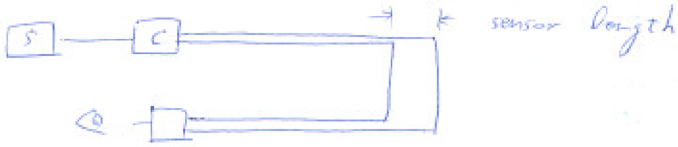
$$\xi = 1 \Rightarrow \sigma = C_m(T_0 - M_f)$$

- Fiber Optic Sensor
  - Intensometric
  - Interferometric
  - Polarmetric
  - Modalmetric
  - Spectral
  - OTDR
- Interferometric
  3 types
  i) Michelson



ii) Mach - Zehnder



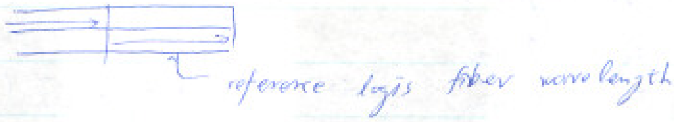


(iii) Fabrey - Perot

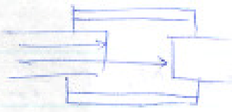
a) intrinsic

b) extrinsic

- intrinsic

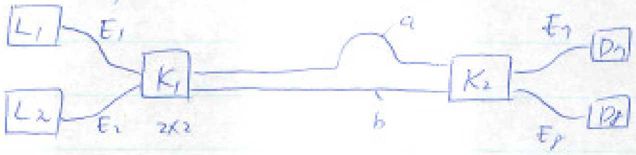


- extrinsic



· Simple model

Model of Interferometric Sensor



Transmission matrices

$$E_{in} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \quad E_{out} = \begin{bmatrix} E_7 \\ E_8 \end{bmatrix} = \underbrace{[K_2]}_{2\times 2}[T]\underbrace{[K_1]}_{2\times 2} E_{in}$$

For standard 3 db coupler,

- coupler $K_1 = K_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$

- Path (no attenuation)

$$T = \begin{bmatrix} e^{i\varphi_A} & 0 \\ 0 & e^{i\varphi_B} \end{bmatrix}$$

φ_A = phase difference through path a

φ_B = " b

substitution

- $E_2 = 0$

$$\begin{bmatrix} E_A \\ E_B \end{bmatrix} = \begin{bmatrix} E_1 \{e^{i\varphi_A} - e^{i\varphi_B}\}/2 \\ iE_1 \{e^{i\varphi_A} - e^{i\varphi_B}\}/2 \end{bmatrix}$$

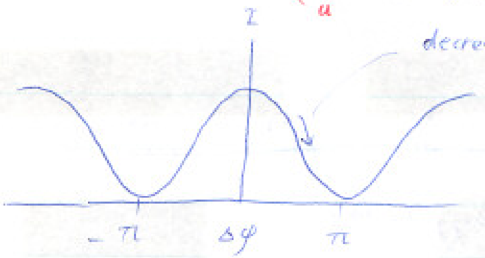
intensities

u (for attenuation)

$$I_A = I_1 [1 - \cos(\varphi_A - \varphi_B)]/2 = I_1 \sin^2\left(\frac{\Delta\varphi}{2}\right)$$

$$I_B = I_1 [1 + \cos(\varphi_A - \varphi_B)]/2 = I_1 \cos^2\left(\frac{\Delta\varphi}{2}\right)$$

u



decreasing intensities

- ambiguity between $+\varphi$ and $-\varphi$
  cos is even function.
- Solution - dither.

slope + sensitivity as well as mean

$\Delta\varphi$

- how changes in environment effect $\Delta\varphi$

$$\Delta\varphi = K[n\Delta L + L\Delta n]$$

$= \frac{\lambda}{2\pi}$

- Effects

$\Delta L = \varepsilon_{11} L$: elongation in change in path length

$\Delta\varphi = Kn\Delta L$

2) $\Delta n = f(\varepsilon)$: photoelastic effect

$$\Delta n = -n^3[P_{11}\varepsilon_{33} + P_{12}\varepsilon_{22} + P_{12}\varepsilon_{11}]/2$$

photoelastic constants

Putting it together,

$$\Delta\varphi = KnL\{\underbrace{\varepsilon_{11}^f}_{\text{long strain}} \underbrace{- \tfrac{1}{2}n^2[P_{11}\varepsilon_{33}^f + P_{12}\varepsilon_{22}^f + P_{12}\varepsilon_{11}^f]}_{\text{photoelastic}} + \underbrace{\alpha\Delta T}_{\text{thermal}}\}$$

- Typical : silica core fibers

$P_{11} = 0.113$, $P_{12} = 0.252$

$\varepsilon_{22} = \varepsilon_{33} = \nu\varepsilon_{11}$

$$\Delta\varphi = KnL\,\varepsilon_{11}\left[1 - \frac{n^2}{2}\{(1-\nu)P_{12} - \nu P_{11}\}\right]$$

$$\Delta\varphi = S\,L\,\varepsilon_{11}, \quad S = 1.13 \times 10^7 \text{ rad/strain-m}$$

scale factor

1 cm given length

1 at $\frac{I_2}{I_1} = 0.9975$

other application - polimetry + Multi-mode fibers

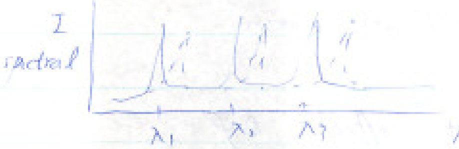
- 2 orthogonal polarization fiber

n_1, n_2

- 2 propagating modes

beating

- Bragg Grating Based Sensors



- changes in grating wavelengths due to strain

- Multiplexing