2. Mathematical Description of Systems

- ✓ Linear System Representation
- ✓ Causality and Lumpedness
- ✓ Linear Systems
- ✓ Linear Time-Invariant (LTI) Systems
- ✓ Linearization
- ✓ Examples of Linear Systems
- ✓ Discrete-Time Systems

Linear System Representation

Differential Equation

$$y^{(n)}(t) = f(y^{(n)}(t), y^{(n-1)}(t), \ldots, y(t), u^{(n)}(t), u^{(n-1)}(t), \ldots, u(t))$$

Impulse Response

$$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$$

Transfer Function

$$\mathrm{y}(s) = \mathrm{G}(s)\,\mathrm{u}(s)$$

State Space Equation

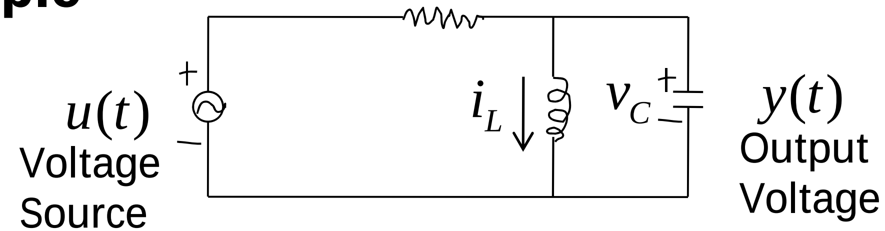
$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

Linear System Representation

Definition 2.1: The state $x(t_0)$ of a system at time t_0 is the information at t_0 that, together with the input $u(t)$, for $t \geq t_0$, determines uniquely the output $y(t)$ for all $t \geq t_0$.

Example



$y(t)$ can be uniquely determined for any input $u(t)$

if initial values of induction current and capacitor voltage at t_0

$\Longrightarrow$ State: $i_L(t_0), \quad v_C(t_0)$

$$y(t_0), \quad \dot{y}(t_0)$$

$$x_1(t_0), \quad x_2(t_0)$$

Causality and Lumpedness

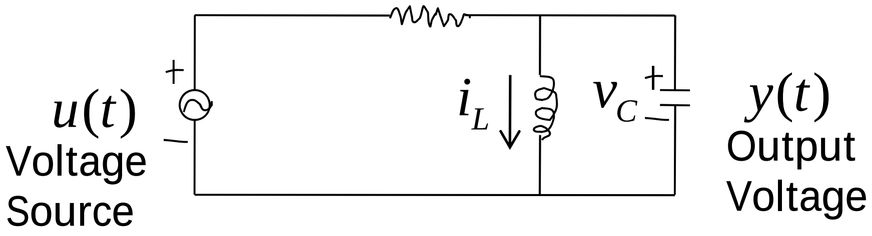
Causal System (Nonanticipatory System)

The current output depends on only the past and current inputs but not on the future inputs

Lumped System (Finite Dimensional System)

The number of state variables is finite

If infinite, Distributed System



Linear Systems

Linear System

Satisfying superposition principle: additivity + homogeneity

$$\text{For} \quad \left.\begin{matrix} x_i(t_0) \\ u_i(t), t \geq t_0 \end{matrix}\right\} \Rightarrow y_i(t), t \geq t_0, i = 1, 2$$

$$\text{then} \quad \left.\begin{matrix} \alpha_1 x_1(t_0) + \alpha_2 x_2(t_0) \\ \alpha_1 u_1(t_0) + \alpha_2 u_2(t_0), t \geq t_0 \end{matrix}\right\} \Rightarrow \alpha_1 y_1(t_0) + \alpha_2 y_2(t_0), t \geq t_0$$

Linear Systems

Zero-input Response

$$\left.\begin{array}{l} x(t_0) \\ u(t)=0, t \geq t_0 \end{array}\right\} \Rightarrow y_{zi}(t), t \geq t_0$$

Zero-state Response

$$\left.\begin{array}{l} x(t_0)=0 \\ u(t), t \geq t_0 \end{array}\right\} \Rightarrow y_{zs}(t), t \geq t_0$$

By additivity

$$\left.\begin{array}{l} x(t_0) \\ u(t), t \geq t_0 \end{array}\right\} \Rightarrow y_{zi}(t)+y_{zs}(t), t \geq t_0$$

Response = Zero-input Response+Zero-state Response

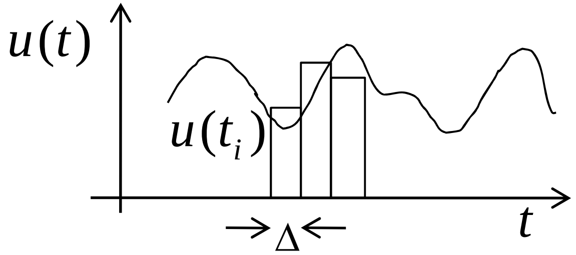
Linear Systems

Input-Output Description

Assume initial state is zero.

Define piecewise continuous function of input:

$$u(t) \approx \sum_i u(t_i)\delta_\Delta(t - t_i)\Delta$$



where

$$\delta_\Delta(t - t_i) = \begin{cases} 0, & t < t_i \\ 1/\Delta, & t_i \le t < t_i + \Delta \\ 0, & t \ge t_i + \Delta \end{cases}$$

$$u(t_i)\delta_\Delta(t - t_i)\Delta = u(t_i)\frac{1}{\Delta}\Delta = u(t_i)$$

Linear Systems

Input-Output Description

Let $g_\Delta(t,t_i)$ be the output for the input $\delta_\Delta(t-t_i)$, i.e.

$$\delta_\Delta(t-t_i) \to g_\Delta(t,t_i)$$

By homogeneity

$$\delta_\Delta(t-t_i)u(t_i)\Delta \to g_\Delta(t,t_i)u(t_i)\Delta$$

By additivity

$$\underbrace{\sum_i \delta_\Delta(t-t_i)u(t_i)\Delta}_{\approx u(t)} \to \underbrace{\sum_i g_\Delta(t,t_i)u(t_i)\Delta}_{\approx y(t)}$$

Linear Systems

Input-Output Description

The output $y(t)$ for the input $u(t)$

$$y(t) \approx \sum_i g_\Delta(t, t_i) u(t_i) \Delta$$

$$y(t) = \lim_{\Delta \to 0} \sum_i g_\Delta(t, t_i) u(t_i) \Delta$$

$$y(t) = \int_{-\infty}^{\infty} g(t, \tau) u(\tau) d\tau$$

where

$$\delta(t - \tau) \to g(t, \tau): \text{ Impulse Response}$$

Linear Systems

Input-Output Description

Causal $g(t,\tau)=0, \quad \text{for } t<\tau$

Relaxed at t_0 :initial state at t_0 is 0

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} g(t,\tau)u(\tau)d\tau \\
&= \int_{-\infty}^{t} g(t,\tau)u(\tau)d\tau \Leftarrow causal \\
&= \int_{-\infty}^{t_0} g(t,\tau)u(\tau)d\tau + \int_{t_0}^{t} g(t,\tau)u(\tau)d\tau \\
&= \int_{t_0}^{t} g(t,\tau)u(\tau)d\tau \Leftarrow relaxed
\end{aligned}
$$

Linear Systems

Input-Output Description

MIMO System

$$\mathbf{y}(t) = \int_{t_0}^{t} \mathbf{G}(t,\tau)\mathbf{u}(\tau)d\tau$$

where

$$\mathbf{G}(t,\tau) = \begin{bmatrix} g_{11}(t,\tau) & \dots & & g_{1p}(t,\tau) \\ g_{21}(t,\tau) & g_{22}(t,\tau) & \dots & \dots \\ \dots & & & \dots \\ g_{q1}(t,\tau) & \dots & & g_{qp}(t,\tau) \end{bmatrix}$$

Linear Time Invariant Systems

Linear Time Invariant (LTI)

A system is said to be time invariant if

$$\left.\begin{array}{l} x(t_0) \\ u(t), t \geq t_0 \end{array}\right\} \Rightarrow y(t), t \geq t_0$$

and any T, we have

$$\left.\begin{array}{l} x(t_0 + T) \\ u(t - T), t \geq t_0 + T \end{array}\right\} \Rightarrow y(t - T), t \geq t_0 + T$$

Linear Time Invariant Systems

If the system is LTI,

$$\begin{aligned} g(t,\tau) &= g(t+T,\tau+T) \\ &= g(t-\tau,0) \quad (\text{let } T=-\tau) \\ &\equiv g(t-\tau) \end{aligned}$$

Output of LTI system

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = \int_0^t g(\tau)u(t-\tau)d\tau$$

Linear Time Invariant Systems

Example) unity-feedback system

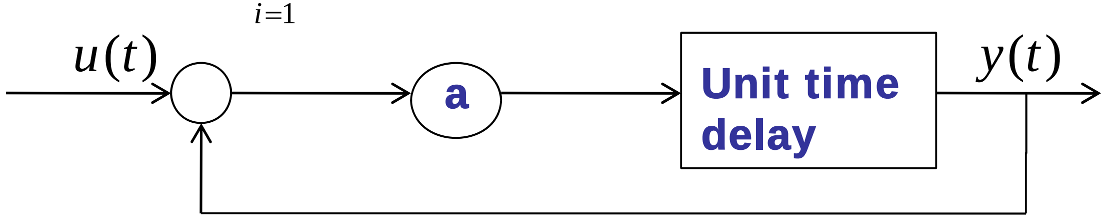
$$g(t) = a\delta(t-1) + a^2\delta(t-2) + a^3\delta(t-3) + \ldots$$

$$= \sum_{i=1}^{\infty} a^3\delta(t-3)$$

Output

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = \sum_{i=1}^{\infty} a^i \int_0^t \delta(t-\tau-i)u(\tau)d\tau$$

$$= \sum_{i=1}^{\infty} a^i u(t-i)$$



Linear Systems

Transfer Function

g(s) : Laplace transform of $g(t)$

$$\mathrm{g(s)} := \mathcal{L}(g) = \int_0^{\infty} g(t)e^{-st}dt$$

$$\mathrm{y(s)} = \int_0^{\infty} \left(\int_0^{t} g(t-\tau)u(\tau)d\tau\right)e^{-st}dt \Leftarrow relaxed$$

$$= \int_0^{\infty} \left(\int_0^{\infty} g(t-\tau)u(\tau)d\tau\right)e^{-st}dt \Leftarrow causality$$

$$= \int_0^{\infty} g(v)e^{-sv}dv\int_0^{\infty} u(\tau)e^{-s\tau}d\tau \Leftarrow v = t-\tau, t = v+\tau$$

$$\mathrm{y(s)} = \mathrm{g(s)\,u(s)}$$

Transfer Function Matrix: MIMO case

$$\mathbf{y}(\mathrm{s}) = \mathbf{G}(\mathrm{s})\mathbf{u}(\mathrm{s})$$

Linear Systems

Properness of Transfer Function

$$\mathrm{g(s)} = N(s)/D(s)$$

$$-\,\mathrm{g(s)}\ \text{ proper} \Leftrightarrow \deg D(s) \geq \deg N(s)$$
$$\Leftrightarrow \mathrm{g}(\infty) = \text{zero or constant}$$

$$-\,\mathrm{g(s)}\ \text{ strictly proper} \Leftrightarrow \deg D(s) > \deg N(s)$$
$$\Leftrightarrow \mathrm{g}(\infty) = \text{zero}$$

$$-\,\mathrm{g(s)}\ \text{ biproper} \Leftrightarrow \deg D(s) = \deg N(s)$$
$$\Leftrightarrow \mathrm{g}(\infty) = \text{non-zero constant}$$

$$-\,\mathrm{g(s)}\ \text{ improper} \Leftrightarrow \deg D(s) < \deg N(s)$$
$$\Leftrightarrow |\mathrm{g}(\infty)| = \infty$$

Properness of Transfer Function Matrix

$-\mathbf{G}(s)$ is (strictly) proper

if all entries are (strictly) proper

$-\mathbf{G}(s)$ is biproper

if $\mathbf{G}(s)$ is square and

both $\mathbf{G}(s)$ and $\mathbf{G}^{-1}(s)$ are proper

Linear System

State Space Equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Laplace Transform

$$\mathrm{sx(s)} - \mathrm{x(0)} = A\,\mathrm{x(s)} + B\,\mathrm{u(s)}$$
$$\mathrm{y(s)} = C\,\mathrm{x(s)} + D\,\mathrm{u(s)}$$

Which implies

$$\mathrm{x(s)} = (\mathrm{sI} - A)^{-1}\,\mathrm{x(0)} + (\mathrm{sI} - A)^{-1} B\,\mathrm{u(s)}$$
$$\mathrm{y(s)} = C(\mathrm{sI} - A)^{-1}\,\mathrm{x(0)} + C(\mathrm{sI} - A)^{-1} B\,\mathrm{u(s)} + D\,\mathrm{u(s)}$$

Transfer Function Matrix

$$\mathbf{G}(\mathrm{s}) = C(\mathrm{sI} - A)^{-1} B + D$$

Linearization

Nonlinear System

$$\dot{x}(t) = f(x(t), u(t), t)$$
$$y(t) = h(x(t), u(t), t)$$

Linearization at an operating point $x_0(t), u_0(t)$

$$x(t) = x_0(t) + \overline{x}(t), \quad u(t) = u_0(t) + \overline{u}(t)$$

$$\dot{x}(t) = \dot{x}_0(t) + \dot{\overline{x}}(t) = f(x_0 + \overline{x}, u_0 + \overline{u}, t)$$
$$= f(x_0, u_0, t) + \frac{\partial f}{\partial x_0}\overline{x} + \frac{\partial f}{\partial u_0}\overline{u} + O(.)$$

$$\Rightarrow$$

$$\dot{\overline{x}}(t) = A\overline{x} + B\overline{u}$$

$$\text{where} \quad A = \frac{\partial f}{\partial x_0}\overline{x}, \quad B = \frac{\partial f}{\partial u_0}\overline{u} + O(.)$$

Implementation and Examples

Op-Amp Circuit Implementation: P. 17

Figure 2.7,

Examples: (p. 18 -29)

Cart with inverted pendulum,

Satellite in orbit,

Hydraulic tanks,

RLC circuits

Discrete-Time Systems

Sampling Period: T

$$u[k] = u(kT), \quad y[k] = y(kT)$$

$$x[k] = x(kT)$$

Linear System: homogeneity, additivity

Response= Zero-state response + Zero-input response

Impulse Sequence

$$\delta[k] = \begin{cases} 1 & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k \neq 0 \end{cases}$$

Discrete-Time Systems

Input Sequence

$$u[k] = \sum_{m=-\infty}^{\infty} u[m]\delta[k-m]$$

Impulse Response Sequence

$$\delta[k-m] \to g[k,m]$$

By homogeneity

$$\delta[k-m]u[m] \to g[k,m]u[m]$$

By additivity

$$\sum_{m} \delta[k-m]u[m] \to \sum_{m} g[k,m]u[m]$$

Input-Output Description

$$y[k] = \sum_{m} g[k,m]u[m] \qquad \Rightarrow y[k] = \sum_{m=0}^{k} g[k-m]u[m]$$

Causal
Relaxed
LTI

Discrete-Time Systems

Z-Transform

$$y(z) = \mathcal{Z}(y[k]) = \sum_{k=0}^{\infty} y[k] z^{-k}$$

Discrete Transfer Function

$$y(z) = \sum_{k=0}^{\infty} (\sum_{m=0}^{\infty} g[k-m] u[m]) z^{-(k-m)} z^{-m}$$

$$= \sum_{k=0}^{\infty} g[k-m] z^{-(k-m)} \sum_{m=0}^{\infty} u[m] z^{-m}$$

$$= \sum_{l=0}^{\infty} g[l] z^{-l} \sum_{m=0}^{\infty} u[m] z^{-m}$$

$$= g(z)\, u(z)$$

Discrete-Time Systems

State-Space Equations

$$x[k+1] = A[k]x[k] + B[k]u[k]$$

$$y[k] = C[k]x[k] + D[k]u[k]$$

Z-Transform of $x[k+1]$

$$\mathcal{Z}(x[k+1]) = \sum_{k=0}^{\infty} x[k+1]z^{-k} = z\sum_{k=0}^{\infty} x[k+1]z^{-(k+1)}$$

$$= z(\sum_{l=1}^{\infty} x[l]z^{-l} + x[0] - x[0])$$

$$= z(\mathrm{x}(\mathrm{z}) - x[0])$$

Discrete-Time Systems

Z-Transform of State-Space Equations

$$z\,\mathrm{x}(z) - zx[0] = A\,\mathrm{x}(z) + B\,\mathrm{u}(z)$$

$$\mathrm{y}(z) = C\,\mathrm{x}(z) + D\,\mathrm{u}(z)$$

$$\mathrm{x}(z) = (z\,\mathrm{I} - A)^{-1} zx[0] + (z\,\mathrm{I} - A)^{-1} B\,\mathrm{u}(z)$$

$$y(z) = C(z\,\mathrm{I} - A)^{-1} zx[0] + (C(z\,\mathrm{I} - A)^{-1} B + D)\,\mathrm{u}(z)$$

If zero initial state

$$y(z) = (C(z\,\mathrm{I} - A)^{-1} B + D)\,\mathrm{u}(z)$$

Transfer Function

$$\mathbf{G}(z) = C(z\,\mathrm{I} - A)^{-1} B + D$$

Discrete-Time Systems

Example: compound interest calculation

Impulse Response

Interest: 0.015%

$$u[0]=1,\quad u[i]=0,\quad i=1,2,$$

$$g[k]=(1.00015)^k$$

Output

$$y[k]=\sum_{m=0}^{k}(1.00015)^{k-m}u[m]$$

Transfer Function

$$g(z)=\sum_{k=0}^{\infty}(1.00015)^k z^{-k}=\sum_{k=0}^{\infty}(1.00015z^{-1})^k$$

$$=\frac{1}{1-1.00015z^{-1}}=\frac{z}{z-1.00015}$$

Concluding Remarks

Example: compound interest calculation

System Type	Internal Description	External Description
Distributed, Linear (Causal, Relaxed)		$y(t)=\int_{-t_0}^{t} g(t,\tau)u(\tau)d\tau$
Lumped, Linear (Causal, Relaxed)	$\dot{x}=A(t)x+B(t)u$ $y=C(t)x+D(t)u$	$y(t)=\int_{-t_0}^{t} g(t,\tau)u(\tau)d\tau$
Distributed, Linear, Time-invariant		$y(t)=\int_{0}^{t} g(t-\tau)u(\tau)d\tau$ $\mathrm{y(s)}=\mathrm{G(s)\,u(s)}$
Rumped, Linear, Time-invariant	$\dot{x}=Ax+Bu$ $y=Cx+Du$	$y(t)=\int_{0}^{t} g(t-\tau)u(\tau)d\tau$ $\mathrm{y(s)}=\mathrm{G(s)\,u(s)}$