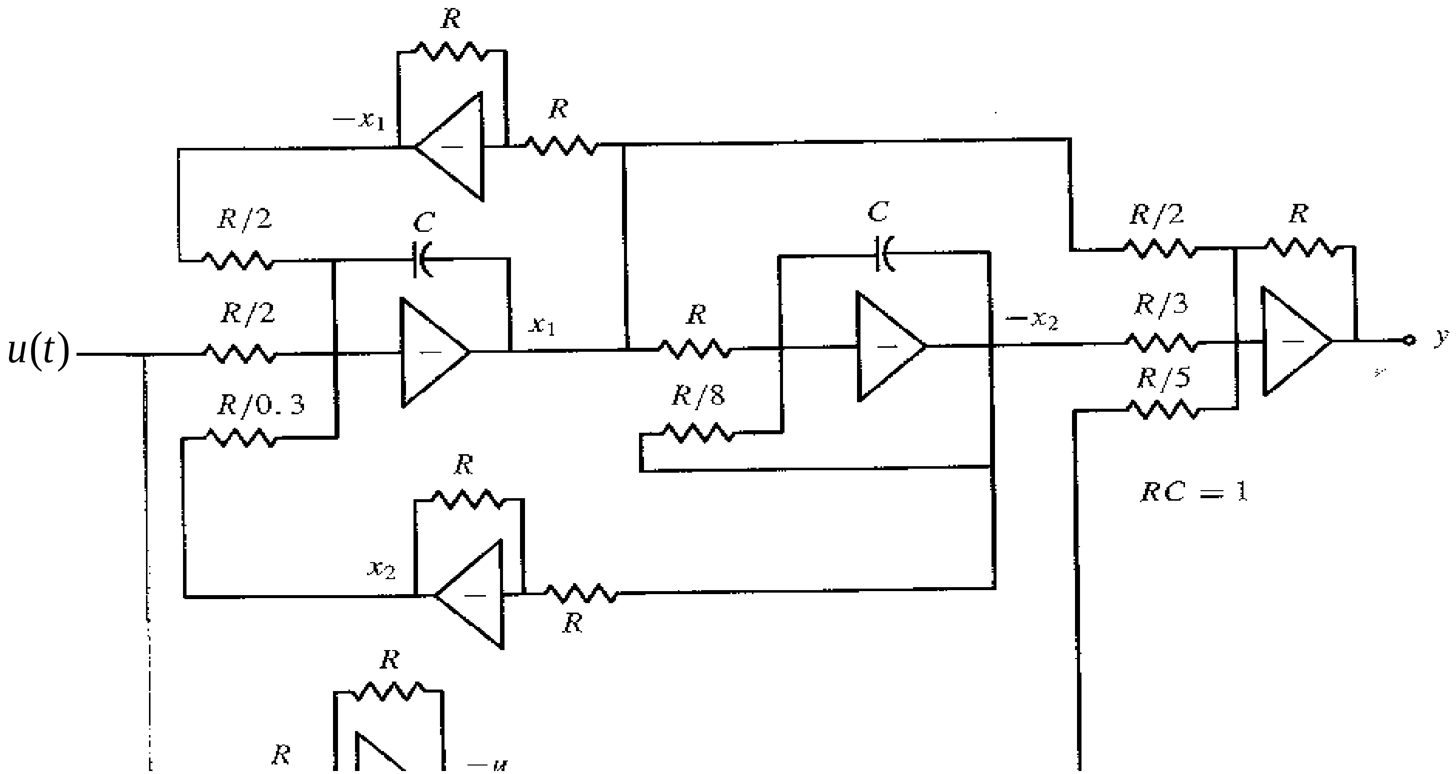
5. Stability

- ✓ Motivations
- ✓ Input Output Stability
- ✓ Internal Stability for LTI System
- ✓ General Definition of Internal Stability
- ✓ Lyapunov Stability Theory
- ✓ Stability of Linear Time Varying System

Motivations

Is $y(t)$ bounded for bounded $u(t)$? $\rightarrow$ IO stability

Is $x_1(t)(x_2(t),\cdots)$ bounded? $\rightarrow$ Internal stability



Input Output Stability

Input Output Stability of LTI System

$$y(t)=\int_0^t g(t-\tau)u(\tau)d\tau=\int_0^t g(\tau)u(t-\tau)d\tau........(*)$$

$$u(t):bounded$$

$$|u(t)|\le u_m<\infty \qquad \forall t>0$$

Theorem 5.1

(*) is BIBO stable iff

$$\int_0^\infty |g(t)|dt\le M<\infty.$$

Input Output Stability

Pf)

$$(\Leftarrow)\ \ |y(t)| = \left|\int_0^\infty g(\tau)u(t-\tau)d\tau\right|$$

$$\le \int_0^\infty |g(\tau)||u(t-\tau)|d\tau$$

$$\le u_m \int_0^\infty |g(\tau)|d\tau \le u_m M < \infty.$$

$(\Rightarrow)$ Assume $\exists t_1$ such that

$$\int_0^{t_1} |g(\tau)|d\tau = \infty.$$

Let us choose

$$u(t_1-\tau) = \begin{cases} 1 & \text{if } g(\tau) > 0 \\ -1 & \text{if } g(\tau) \le 0 \end{cases}$$

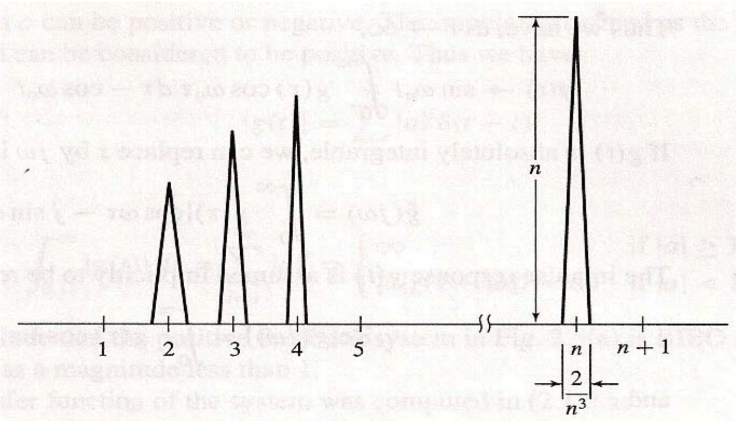
$$\mathrm{y}(t_1) = \int_0^{t_1} g(\tau)u(t_1-\tau)d\tau = \int_0^{t_1} |g(\tau)|d\tau = \infty$$

Input Output Stability

Note) Even if $\int |g(\tau)| d\tau < \infty$,

$g(\tau)$ may not be bounded or may not converge to zero.

Ex)



$$\sum_{n=2}^{\infty} \frac{1}{n^2} < \infty, \text{ but } \lim_{t\to\infty} g(t) = \infty$$

Lemma: The uniformly continuous function

satisfying $\int |g(\tau)| d\tau < \infty$, converge to zero.

i.e. $\lim_{t\to\infty} g(t) = 0$

Input Output Stability

Theorem 5.2

If $g(t)$ is BIBO stable,

for $u(t) = a, \quad \lim_{t\to\infty} y(t) = \mathbf{g}(0)a$

for $u(t) = \sin w_0 t, \; \lim_{t\to\infty} y(t) = |\mathbf{g}(jw_0)| \sin(w_0 t + \angle\mathbf{g}(jw_0)),$

where $\mathbf{g}(s)$ is Lapace transform of $g(t)$.

Pf)

$$\mathbf{g}(s) = \int_0^\infty g(\tau)e^{-s\tau}d\tau$$

$$u(t) = a, \; y(t) = \int_0^t g(\tau)u(t-\tau)d\tau = a\int_0^t g(\tau)d\tau$$

$$\lim_{t\to\infty} y(t) = a\mathbf{g}(0)$$

Input Output Stability

Pf_cont)

$$u(t) = \sin w_0 t$$

$$y(t) = \int_0^t g(\tau)\sin w_0(t-\tau)d\tau$$

$$= \int_0^t g(\tau)\left[\sin w_0 t \cos w_0\tau - \cos w_0 t \sin w_0\tau)\right]d\tau$$

$$= \sin w_0 t \int_0^t g(\tau)\cos w_0\tau d\tau - \cos w_0 t \int_0^t g(\tau)\sin w_0\tau d\tau$$

$$\mathbf{g}(jw) = \int_0^\infty g(\tau)\left[\cos w\tau - j\sin w\tau)\right]d\tau$$

$$\mathrm{Re}\left[\mathbf{g}(jw)\right] = \int_0^\infty g(\tau)\cos w\tau d\tau$$

$$\mathrm{Im}\left[\mathbf{g}(jw)\right] = -\int_0^\infty g(\tau)\sin w\tau d\tau$$

$$\lim_{t\to\infty} y(t) = \sin w_0 t \,\mathrm{Re}\left[\mathbf{g}(jw_0)\right] + \cos w_0 t \,\mathrm{Im}\left[\mathbf{g}(jw_0)\right]$$

$$= \left|\mathbf{g}(jw)\right| \sin(w_0 t + \angle \mathbf{g}(jw))$$

Input Output Stability

Theorem 5.3

A SISO system with proper rational transfer function **g**(s) is BIBO stable iff every pole of **g**(s) has a negative real part.

Note) MIMO is BIBO.

iff $g_{ij}(t)$ is BIBO stable (Theorem 5.M1, Theorem 5.M3).

Note)

Time varying system is BIBO stable iff

$$\int_{t_0}^{t} \left| g(t,\tau) \right| d\tau \le M < \infty \text{ for all } t, t_0 > 0$$

Input Output Stability

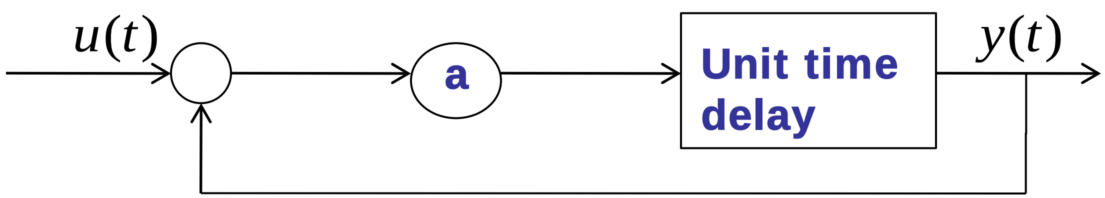
Example) unity-feedback system

$$g(t) = \sum_{i=1}^{\infty} a^i \delta(t-i) \rightarrow |g(t)| = \sum_{i=1}^{\infty} |a|^i \delta(t-i)$$

$$\int_0^{\infty} |g(t)|\, dt = \sum_{i=1}^{\infty} |a|^i = \begin{cases} \infty & \text{if } |a| \geq 1 \\ |a|/(1-|a|) < \infty & \text{if } |a| < 1 \end{cases}$$

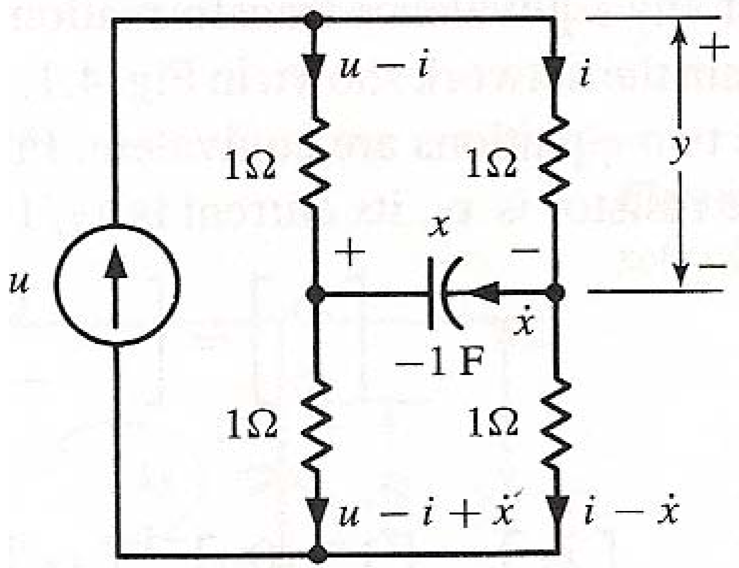
This system is BIBO stable iff the gain a has a magnitude less than 1.

$\mathbf{g}(\mathrm{s}) = \dfrac{se^{-s}}{1-ae^{-s}}$ is not rational function and Theorem 5.3 is not applicable.



Input Output Stability

Example



(b)

$$\dot{x} = x(t), \quad y = 0.5x(t) + 0.5u(t),$$

$$\bar{A} = 1, \ \bar{B} = 0, \ \bar{C} = 0.5, \ \bar{D} = 0.5$$

$$\mathbf{g}(s) = C^T(sI - A)^{-1}B + D$$

$$= 0.5(s-1)^{-1}0 + 0.5 = 0.5$$

$\rightarrow$ BIBO stable even if it has positive real part eigenvalue.

$\rightarrow$ Internal stability (state stability is needed).

Input Output Stability

Discrete-Time Case

$$y[k]=\sum_{m=0}^{k} g[k-m]u[m]$$

Theorem 5.D.1

Discrete-time SISO system is BIBO

iff $g[k]$ is absolutely summable, i.e.,

$$\sum_{k=0}^{\infty} \left|g[k]\right| \leq M < \infty$$

Input Output Stability

Theorem 5.D.2

If a discrete-time system is BIBO stable, then

$$1.\ u[k]=a$$

$$\lim_{k\to\infty} y[k]=\mathbf{g}(1)a$$

$$2.\ u[k]=\sin w_0 k$$

$$\lim_{k\to\infty} y[k]\to\left|\mathbf{g}(e^{jw_0})\right|\sin(w_0 k+\angle\mathbf{g}(e^{jw_0}))$$

$$\text{where}\quad \mathbf{g}(z)=\sum_{m=0}^{\infty} g[m]z^{-m}$$

Theorem 5.D.3

Discrete-time SISO system with proper rational transfer function $\mathbf{g}(z)$ is BIBO stable iff every pole has a magnitude less than 1.

Internal Stability for LTI System

Internal Stability (State Stability)

$$\dot{x}(t) = Ax(t)$$
$$x(t) = e^{At}x_0$$

Definition : Zero-input Response of $\dot{x} = Ax$ is
marginally stable or stable in the sense of Lyapunov
if $|x(t)| < \infty$ for all $t > 0$ & all x_0,
and is asymptotically stable if $x(t) \to 0$ as $t \to \infty$

Linear System for LTI System

Theorem 5.4

1. The state equation is marginally stable iff all eigenvalues of A have zero or negative real part and those with zero real part are simple roots of minimal polynomial of A.
2. The state equation is asymptotically stable iff all eigenvalues of A have negative real part.

① $\begin{cases} \mathrm{Re}(\lambda_i) \le 0 \text{ and} \\ \mathrm{Re}(\lambda_i) = 0 \,\&\, \overline{n}_i = 1 \end{cases}$

$\Rightarrow$ marginally stable

② All $\mathrm{Re}\,\lambda_i < 0 \rightarrow \mathrm{A.S}$

$$e^{At} = Qe^{\hat{A}t}Q^{-1}$$

$$= Q\begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{1}{2}t^2e^{\lambda_1 t} & & \\ & e^{\lambda_1 t} & te^{\lambda_1 t} & & \\ & & e^{\lambda_1 t} & & \\ & & & e^{\lambda_2 t} & \\ & & & & \ddots \end{bmatrix}Q^{-1}$$

Internal Stability for LTI System

Example 5.4

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x \rightarrow x(t) = c_1 + c_2 e^{-t} \leq c_1$$

Characteristic polynomial is $\lambda^2(\lambda + 1)$

Minimal polynomial is $\lambda(\lambda + 1)$

$\lambda = 0$ is simple root $\rightarrow$ marginally stable

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x \rightarrow x(t) = c_1 + c_2 t + c_3 e^{-t} \rightarrow \infty$$

Minimal polynomial is $\lambda^2(\lambda + 1)$

$\rightarrow$ not marginally stable

Internal Stability for LTI System

Discrete-Time Case

$$x[k+1] = Ax[k]$$
$$x[k] = A^k x_0$$

Theorem 5.D.4

1. $|\lambda_{\mathrm{i}}| \leq 1$ and $|\lambda_{\mathrm{i}}| = 1$ with $\bar{n}_i = 1 \rightarrow$ marginally stable
2. $|\lambda_{\mathrm{i}}| < 1 \rightarrow$ asymtotically stable

General Definition of Internal Stability

Definition (Equilibrium Point)

x_e is said to be equibrium point at t_0 iff

$$x(t) = \Phi(t, t_0)x_e = x_e \quad \forall t \geq t_0.$$

Note)

$$\dot{x} = A(t)x(t)$$

$$x(t) = \Phi(t,t_0)x(t_0)$$

$$\Phi(t,t_0) = e^{\int_{t_0}^{t} A(\tau)d\tau} \text{ or } X(t)X^{-1}(t_0)$$

$$x_e = \Phi(t,t_0)x_e$$

$$\left[\mathbf{I} - \Phi(t,t_0)\right]x_e = 0 \quad \forall t \geq t_0$$

If all columns of $\left[\mathbf{I} - \Phi(t,t_0)\right]$ are linearly independent

$$x_e = 0,$$

otherwise it may be

$$x_e \neq 0.$$

General Definition of Internal Stability

Definition

x_e is stable i.s.L at t_0 iff

for every $\varepsilon > 0$, $\exists\ \delta(\varepsilon, t_0) > 0$ such that

$$\left\| x_0 - x_e \right\| \le \delta(\varepsilon, t_0) \rightarrow \left\| x(t) - x_e \right\| \le \varepsilon \ \ \forall t \ge t_0.$$

x_e is uniformly stable i.s.L $[t_0, \infty)$ iff

for every $\varepsilon > 0$, $\exists\ \delta(\varepsilon) > 0$ such that

$$\left\| x_0 - x_e \right\| \le \delta(\varepsilon) \rightarrow \left\| x(t) - x_e \right\| \le \varepsilon \ \ \forall t \ge t_0.$$

General Definition of Internal Stability

Example

$$\dot{x} = (6t\sin t - 2t)x(t)$$

$$x(t) = x(t_0)\exp^{\int_{t_0}^{t}(6\tau\sin\tau - 2\tau)d\tau}$$

$$= x(t_0)\exp(6\sin t - 6t\cos t - t^2 - 6\sin t_0 + 6t_0\cos t_0 + t_0^2)$$

$$\text{Define } c(t_0) = \sup_{t\geq t_0}\exp(6\sin t - 6t\cos t - t^2 - 6\sin t_0 + 6t_0\cos t_0 + t_0^2)$$

$$< \sup_{t\geq t_0}\exp(12 + 6(t+t_0) - (t^2 - t_0^2),$$

$$< \sup_{t\geq t_0}\exp(12 + 6T + 12t_0 - T^2),\ \ T := t - t_0$$

$$< \infty$$

$$|x(t)| < |x(t_0)|c(t_0)$$

General Definition of Internal Stability

Example (cont)

For any given $\varepsilon > 0$, if we choose

$$\delta(\varepsilon, t_0) = \varepsilon / c(t_0),$$

then

$$\left| x(t_0) - x_e \right| \le \delta(\varepsilon, t_0), \quad x_e = 0$$

$$\rightarrow \left| x(t) \right| < \left| x(t_0) \right| c(t_0) \le \varepsilon \ \forall t \ge t_0.$$

This implies the system is stable i.s.L.

General Definition of Internal Stability

Example (cont)

On the other hand, if we choose

$$t_0 = 2n\pi, \;\; t = (2n+1)\pi$$

$$x\left[(2n+1)\pi\right] = x(2n\pi)\exp^{(4n+1)(6-\pi)\pi}$$

for

$$x\left[(2n+1)\pi\right] < \varepsilon$$

$$x(t_0) = x(2n\pi) < \varepsilon \cdot \exp^{-((4n+1)(6-\pi)\pi)} = \delta(\varepsilon, t_0 = 2n\pi) \;.$$

It is not possible to choose a single $\delta(\varepsilon)$

independent of t_0=$2n\pi$.

That is $\left|x(t_0)\right| \leq \delta(\varepsilon, t_0) \rightarrow 0$ as $t_0 \rightarrow \infty$.

This implies the system is not uniformly stable.

General Definition of Internal Stability

Example: Pendulum

$$\text{Let } x_1 = \theta$$

$$x_2 = \dot{\theta}$$

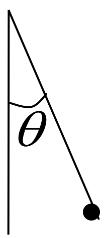
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \left(-\frac{g}{l}\right)\sin x_1 + \frac{\cos x_1}{ml}u(t)$$

$$\dot{x}_1 = x_2 = 0 \text{ with } u(t) = 0$$

$$x_e = \begin{bmatrix} k\pi \\ 0 \end{bmatrix},\ k = 0, \pm1, \pm2 \cdots$$

$$\begin{cases} x_e = \left[k\pi \ \ 0\right]^T,\ \ k = 0, \pm2, \pm4 \text{ uniformly stable eq. pt.} \\ x_e = \left[k\pi \ \ 0\right]^T,\ \ k = \pm1, \pm3, \qquad \text{unstable eq. pt.} \end{cases}$$



General Definition of Internal Stability

Definition

(1) x_e is asymptotically stable at t_0 if

– x_e is stable i.s.L at t_0, and

– $\|x(t) - x_e\| \to 0$ as $t \to \infty$

i.e.) for any $\bar{\varepsilon}$, $\exists\ \gamma > 0$ *and* $T(\bar{\varepsilon}, \gamma, t_0) > 0$ such that

$\|x(t_0) - x_e\| \le \gamma$ yields

$\|x(t) - x_e\| \le \bar{\varepsilon} \quad \forall t \ge t_0 + T$

(2) x_e is uniformly asymptotically stable if

– x_e is uniformly stable i.s.L. over $[t_0, \infty)$

– T is independent to t_0

General Definition of Internal Stability

Example

$$\dot{x} = -\frac{1}{1+t}x(t) \quad \to x(t) = \frac{x_0(1+t_0)}{1+t}$$

1) $\lim_{t\to\infty} x(t) = 0 \to$ asymptotically stable

2) $\|x_o\| < \varepsilon \to \|x(t)\| < \dfrac{\varepsilon(1+t_0)}{1+t} \leq \varepsilon \to$ uniformly stable

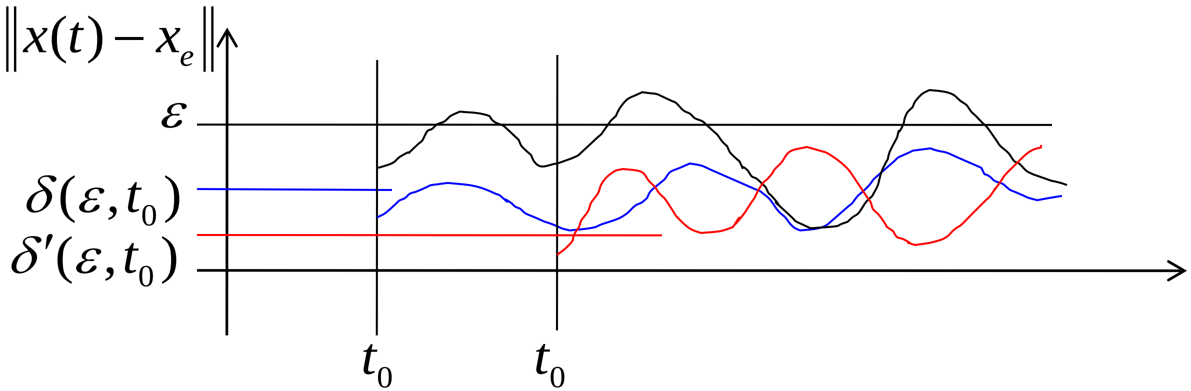
3) for $\|x(t_0+T)\| = \dfrac{x_0(1+t_0)}{1+t_0+T} < \varepsilon$

$$T > \frac{x_0(1+t_0)}{\varepsilon} - 1 - t_0 = \left(\left(\frac{x_0}{\varepsilon} - 1\right)(1+t_0)\right)$$

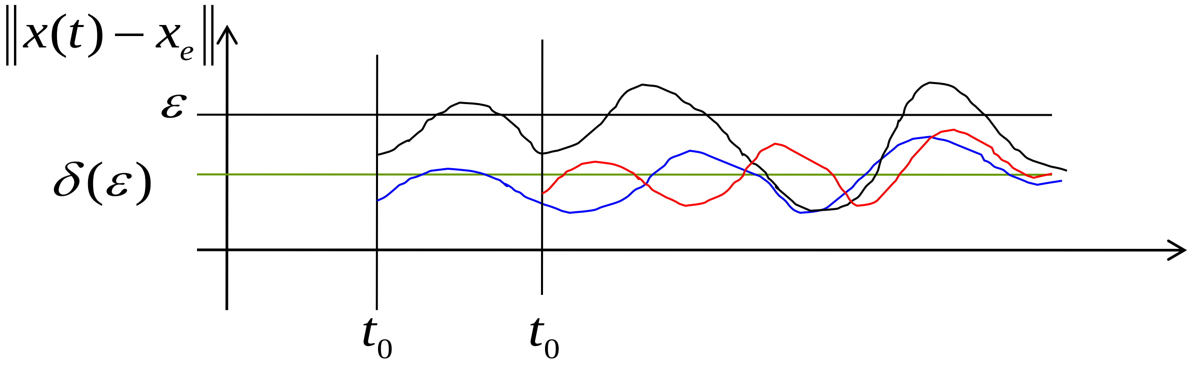
$\Rightarrow$ not uniformly asymptotically stable.

Review

Stable i.s.L.

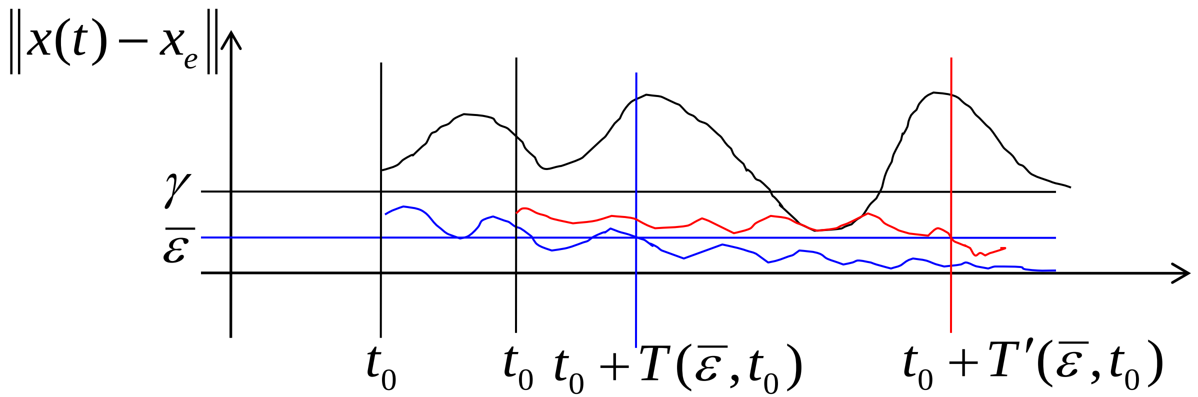


Uniformly Stable i.s.L.

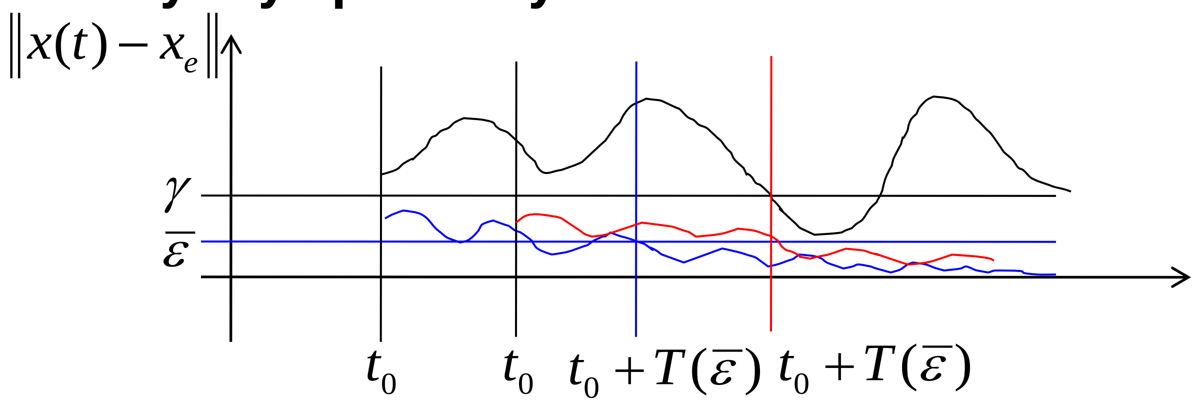


Review

Asymptotically Stable i.s.L.



Uniformly Asymptotically Stable i.s.L.



HW 5-1

Problem: Determine the eigenvalues and stability of the eq. and discuss the relation between eigenvalues and stability of the time varying system.

$$\dot{x}(t) = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} x(t)$$

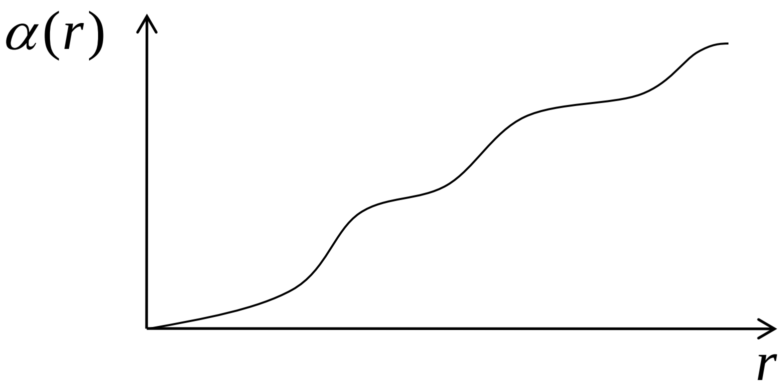
Lyapunov Stability Theory

Lyapunov Theory

Definition: class K functions [Hahn, 1967]

Function $\alpha(r): R^+ \to R^+$ belongs to class K if

- $\alpha(0) = 0$
- continuous
- strictly increasing.



Lyapunov Stability Theory

Definition: locally positive definite functions

Continuous function $V(x,t): B_h \subset R^n \times R_+ \to R_+$ is

$l.p.d.f.$ if for some $h, \alpha \in K$

$$\begin{cases} V(x,t) \geq \alpha(\|x\|) \ \forall \ \mathrm{x} \in \ B_h \subset R^n, \ \ B_h := \left\{ x \middle| x \in R^n, \|x\| < h \right\} \\ V(o,t) = 0. \end{cases}$$

In case of $B_h = R^n$, $V(x,t)$ is *globally* $p.d.f.$

Lyapunov Stability Theory

Definition: Descrescent Function

$V(x,t)$ is descrescent if

$\exists\ \alpha(\cdot) \in K$ such that

$V(x,t) \le \alpha(\|x\|), \forall x \in R^n, t \ge 0$

Lyapunov Stability Theory

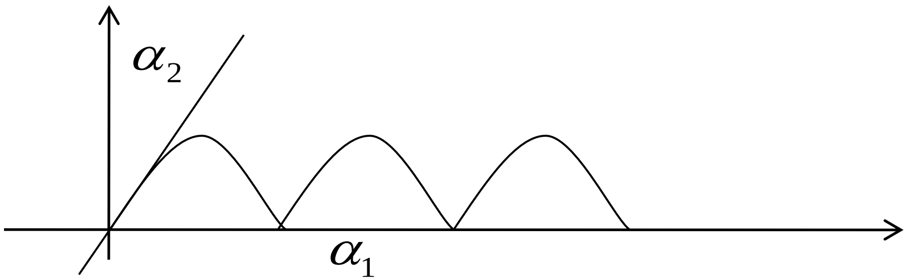
Example

$$\left.\begin{aligned} &V(x,t) = \|x\|^2 = x^T x \\ &V(x,t) = x^T M x, \quad M > 0 \end{aligned}\right\} \text{g.p.d.f. descrescent}$$

$$V(x,t) = (t+1)\|x\|^2 \qquad : \text{g.p.d.f.}$$

$$V(x,t) = e^{-t}\|x\|^2 \qquad : \text{descrescent}$$

$$V(x,t) = \sin^2(\|x\|^2) \;:\; \text{l.p.d.f. , descrescent in } B_h = \{x \,|\, \|x\| < \pi\}$$



(not K-class ftn bound because not strictly increasing)

Lyapunov Stability Theory

Theorem (Lyapunov Stability Theorem)

If $\exists$ a continuously differentiable function $V(x,t)$ such that

$$\dot{x} = f(x,t), \quad f(0,t) = 0 \ \ (x(t) = 0 \text{ is equilibrium point})$$

$$\alpha_1(\|x\|) \le V(x,t) \le \alpha_2(\|x\|) \quad \forall x \in B_\gamma \,(\text{or } x \in R^n)$$

$$\dot{V} = \frac{dV}{dt} + \frac{dV}{dx} f(x,t) \le -\alpha_3(\|x\|)$$

$$(\equiv \alpha_3(\|x\|) \le -\dot{V})$$

then $x(t) = 0$ is (globally) uniformly asymptotically stable.

Note :

$$\alpha_1(\|x\|) \le V(x,t), \ \alpha_3(\|x\|) \le -\dot{V} \Rightarrow \text{asymptotically stable}$$

$$\alpha_1(\|x\|) \le V(x,t), \ 0 \le -\dot{V} \Rightarrow \text{marginally stable (i.s.L)}$$

Lyapunov Stability Theory

$V(x,t)$	$-\dot{V}(X,t)$	stability
p.d.f., descrescent	p.d.f.	G. U. A. S
l.p.d.f., descrescent	l.p.d.f.	U. A. S
l.p.d.f.,	l.p.d.f	A. S
l.p.d.f., descrescent	≥0 locally	U. . S i.s.L.
l.p.d.f.,	≥0 locally	S i.s.L.
p.d.f.,	p.d.f.	G. A. S
p.d.f.,	≥0	G. S i.s.L.
p.d.f., descrescent	≥0	G. U. . S i.s.L.

Lyapunov Stability Theory

Lemma : Barbalat's Lemma: (convergence)

If a real valued function $g(t)$ is uniformly continuous $\forall t \geq 0$ &

$$\lim_{t \to \infty} \int_0^t g(\tau) d\tau < \infty,$$

then $\lim_{t \to \infty} g(t) = 0.$

Lyapunov Stability Theory

Theorem : boundedness & convergence set

Suppose $f(x,t)$ is locally Lipschitz on $B_r \times R_+$

Let $V(x,t)$ be continuously differentiable function such that

$$\alpha_1(\|x\|) \le V(x,t) \le \alpha_2(\|x\|) \text{ (l.p.d.f. \& descrescent)}$$

and $\dot{V} \le -W(x) \le 0$

Assume $\dot{V}$ is uniformly continuous ($\equiv \ddot{V}$ is bounded)

Then solutions of

$$\dot{x} = f(x,t), \ \|x(t_0)\| \le \alpha_2^{-1}(\alpha_1(r))$$

are bounded, i.e.

$$\|x\| \le r,$$

and $W(x(t)) \to 0$ as $t \to \infty$.

Lyapunov Stability Theory

Brief bf)

$$l = \min_{\|x\|=r}(V(x)) = \max_{\|x\|=\delta}(V(x))$$

$$\Rightarrow \|x(t_0)\| \le \delta \rightarrow \|x\| \le r$$

$$\alpha_1(r) = \min_{\|x\|=r}(V(x,t)) = \max_{\|x\|=\delta}(V(x,t)) = \alpha_2(\delta)$$

$$\Rightarrow \delta = \alpha_2^{-1}(\alpha_1(r))$$

$$\therefore \|x(t_0)\| \le \delta = \alpha_2^{-1}(\alpha_1(r))$$

$$\Rightarrow \quad \|x\| \le r$$

Lyapunov Stability Theory

Brief bf(continued)

Since $V(x,t)$ is decrescent,

$V(x,t)$ is bounded for bounded x, hence

$$\lim_{t\to\infty} V(x,t) = \lim_{t\to\infty} \int_0^t \dot{V}(\tau)d\tau < \infty.$$

Since $\dot{V}$ is uniformly continuous

$$\lim_{t\to\infty} \dot{V}(t) = 0 \le -\lim_{t\to\infty} W(x) \le 0$$

$$\lim_{t\to\infty} W(x) = 0$$

If $W(x) \in K, \quad W(x) = 0 \to x = 0.$

Hence $\lim_{t\to\infty} x(t) = 0 \ \ (x(t) \to 0, \ \ A.\ S.)$

Lyapunov Stability Theory

Example

$$\dot{x}_1 = x_2 + cx_1(x_1{}^2 + x_2{}^2)$$
$$\dot{x}_2 = -x_1 + cx_2(x_1{}^2 + x_2{}^2)$$

Is this system stable ? How can we determine the stability?

The candidate for Lyapunov function

$$V(x) = x_1{}^2 + x_2{}^2, \leftarrow \text{p.d.f and decrescent.}$$
$$\dot{V} = 2c(x_1{}^2 + x_2{}^2)^2.$$

If $c = 0, \dot{V} = 0,$ and therefore $x_e = 0$ u.s..

If $c < 0, \dot{V} \le -\alpha(\|x\|),$ and therefore $x_e = 0$ g.u.a.s..

If $c > 0, \dot{V} \ge \alpha(\|x\|),$ and therefore $x_e = 0$ unstable.

Lyapunov Stability Theory

Example

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 - e^{-t} x_1$$

Is this system stable ? How can we determine the stability?

The candidate for Lyapunov function

$$V(x) = x_1^{\,2} + x_2^{\,2},$$
$$\dot{V} = -2x_2^{\,2} + 2x_1 x_2 (1 - e^{-t}).$$

We can not say anything.

$$V(x) = x_1^{\,2} + e^t x_2^{\,2},$$
$$\dot{V} = -e^t x_2^{\,2}.$$

If $c = 0, \dot{V} \le 0,$ and therefore $x_e = 0$ stable i.s.L.

HW5-2

Problem

$$\dot{x}_1 = -2x_1 + x_1 x_2 + x_2$$
$$\dot{x}_2 = -x_1^{\,2} - x_1$$

Determine the stability and discuss the convergence property.

Lyapunov Stability Theory

Theorem 5.5

$$\mathrm{Re}\{\lambda_i(A)\} < 0 \text{ iff}$$

$$\text{for any } N' = N > 0, \exists \text{ unique } M = M' > 0$$

$$\text{such that } A'M + MA = -N.$$

Lyapunov Stability Theory

Pf)

$$(\Leftarrow) \quad M > 0 \rightarrow \mathrm{Re}\{\lambda_i\} < 0$$

$$V(x) = x'Mx > 0, \text{ decrescent}$$

$$\begin{aligned}\dot{V}(x) &= \dot{x}'MX + x'M\dot{x} \\ &= x'A'Mx + x'MAx \\ &= -x'Nx \le -\lambda_m(N)\|x\|^2\end{aligned}$$

By Lyapunov Theorem,

$$\lim_{t\to\infty} x = 0.$$

By Theorem 5.4, $\mathrm{Re}\{\lambda_i(A)\} < 0.$

$$(\Rightarrow) \quad \mathrm{Re}\{\lambda_i\} < 0 \rightarrow M > 0$$

$$\mathrm{Re}\{\lambda_i(A)\} + \mathrm{Re}\{\lambda_j(A)\} \neq 0 < 0$$

Lyapunov Stability Theory

Pf_continued)

$$\Rightarrow A'M + MA = -N \text{ has unique solution (see Section 3.7)}$$

$$M = \int_0^\infty e^{A't} N e^{At} dt \text{ is solution.}$$

$$(\because A'M + MA = \int_0^\infty A'e^{A't} N e^{At} dt + \int_0^\infty e^{A't} N e^{At} A dt$$

$$= \int_0^\infty \frac{d}{dt}(e^{A't} N e^{At}) dt = e^{A't} N e^{At}\Big|_{t=0}^{\infty} = -N)$$

$$\text{and } N = \bar{N}'\bar{N} (\because N \text{ is symetric})$$

$$\rightarrow x'Mx = \int_0^\infty x'e^{A't}\bar{N}'\bar{N}e^{At} x dt = \int_0^\infty \left\| \bar{N}e^{At}x \right\|_2^2 dt > 0.$$

$$\Rightarrow M > 0.$$

Lyapunov Stability Theory

Corollary 5.5

$\mathrm{Re}\{\lambda_i(A)\} < 0$ iff

for any given $m \times n$ matrix $\bar{N}$ with $m < n$, together with the property

$$\text{rank(O):=rank}\begin{bmatrix} \bar{N} \\ \bar{N}A \\ \cdots \\ \bar{N}A^{n-1} \end{bmatrix} = n \text{ (full column rank)}$$

where O is an $nm \times n$ matrix, the Lyapunov equation

$$A'M + MA = -\bar{N}'\bar{N} = -N.$$

has an unique $M = M' > 0.$

Lyapunov Stability Theory

Pf) Since $N = \bar{N}'\bar{N}$ is positive semidefinite,

$\exists\ x \neq 0$ such that $Ne^{At}x = 0$ in $[0, \infty)$. Hence

$$x'Mx = \int_0^\infty x'e^{A't}\bar{N}'\bar{N}e^{At}xdt = \int_0^\infty \left\| Ne^{At}x \right\|_2^2 dt \geq 0.$$

$\Rightarrow M \geq 0.$

By derivative of $Ne^{At}x = 0$,

$$\begin{bmatrix} \bar{N} \\ \bar{N}A \\ \vdots \\ \bar{N}A^{n-1} \end{bmatrix} e^{At}x = \mathrm{O}e^{At}x = 0.$$

If O has full rank, $x = 0$. This implies

there is no $x \neq 0$ such that $Ne^{At}x = 0$ in $[0, \infty)$.

$\Rightarrow M > 0$ if O has full rank.

HW5-3

Problem 5.21 in the Text P. 142

Lyapunov Stability Theory

Discrete-Time Case

Theorem 5.D5

$$\left|\lambda_i(A)\right| < 1 \text{ iff } \text{ for any } N > 0 \text{ or } N = \bar{N}'\bar{N} \geq 0 \ \& \ \bar{N} = m \times n$$

with full rank O, the discrete Lyapunov equation

$$M - A'MA = N$$

has unique symmetric solution $M > 0$.

Lyapunov Stability Theory

b.pf) $\Rightarrow$ If $|\lambda_i| < 1,$ the solution

$$M = \sum_{m=0}^{\infty} (A')^m NA^m \text{ is well defined.}$$

$$(\because \sum_{m=0}^{\infty} (A')^m NA^m - A' \sum_{m=0}^{\infty} (A')^m NA^{m+1}$$

$$= N + \sum_{m=1}^{\infty} (A')^m NA^m - A' \sum_{m=0}^{\infty} (A')^m NA^{m+1} = N)$$

Since $N > 0,$ for $A\upsilon = \lambda\upsilon$

$$\upsilon^* N\upsilon = \upsilon^* M\upsilon - \upsilon^* A'MA\upsilon$$

$$= \upsilon^* M\upsilon - \lambda^* \upsilon^* M\upsilon\lambda$$

$$= (1-\lambda^2)\upsilon^* M\upsilon > 0$$

$$\upsilon^* M\upsilon > 0 \rightarrow M > 0.$$

Stability of Linear Time Varying System

Theorem

x_e of $\dot{x} = A(t)x$ is stable i.s.L at t_0 iff

$\exists\, K(t_0) \ni$

$$\left\|\Phi(t,t_0)\right\| \le K(t_0) < \infty \quad \forall t \ge t_0,$$

and x_e is uniformly stable i.s.L if K is independent of t_0.

Pf) $(\Leftarrow)$ $x_e = \Phi(t,t_0)x_e,\ \forall t \ge t_0$

$$x(t) - x_e = \Phi(t,t_0)(x_0 - x_e)$$

$$\left\|x(t) - x_e\right\| \le \left\|\Phi(t,t_0)\right\|\left\|x_0 - x_e\right\| \le K(t_0)\left\|x_0 - x_e\right\|$$

for any ε, $\exists\ \delta(t_0) > 0$ such that

$$\left\|x_0 - x_e\right\| \le \frac{\varepsilon}{K(t_0)} = \delta(t_0) \rightarrow \left\|x(t) - x_e\right\| \le \varepsilon.$$

$(\Rightarrow)$ by contradiction, it can be easily shown.

Stability of Linear Time Varying System

Theorem

Zero state of $\dot{x} = A(t)x$ is asymptotically stable at t_0

iff $\|\Phi(t,t_0)\| \le K(t_0) < \infty \ \& \ \|\Phi(t,t_0)\| \to 0$ as $t \to \infty$.

And it is uniformly asymptotically stable over $[t_0, \infty]$

iff $\exists\, K_1 > K_2 > 0$ such that

$$\|\Phi(t,t_0)\| \le K_1 e^{-K_2(t-t_0)} \quad \forall t \ge t_0.$$

Note

$$AS \not\Rightarrow BIBO \leftarrow y(t) = \int_{t_0}^{t} C(\tau)\Phi(\tau,t_0)B(\tau)u(\tau)d\tau$$

Stability of Linear Time Varying System

Theorem 5.7

Marginal and asymptotical stabilities are invariant under any Lyapunov transformation.

Pf)

By Lyapunov transformation,

$$\bar{X}(t) = P(t)X(t)$$

$$\begin{aligned}\bar{\Phi}(t,\tau) = \bar{X}(t)\bar{X}^{-1}(\tau) &= P(t)X(t)X^{-1}(\tau)P^{-1}(\tau) \\ &= P(t)\Phi(t,\tau)P^{-1}(\tau)\end{aligned}$$

Since $P(t)$ and $P^{-1}(t)$ is bounded, $\bar{\Phi}(t,\tau)$ is bounded.