6. Controllability & observability

- ✓ Controllability
- ✓ Observability
- ✓ Canonical Decomposition
- ✓ Conditions in Jordan Form
- ✓ Discrete Time Case
- ✓ Time Varying Case

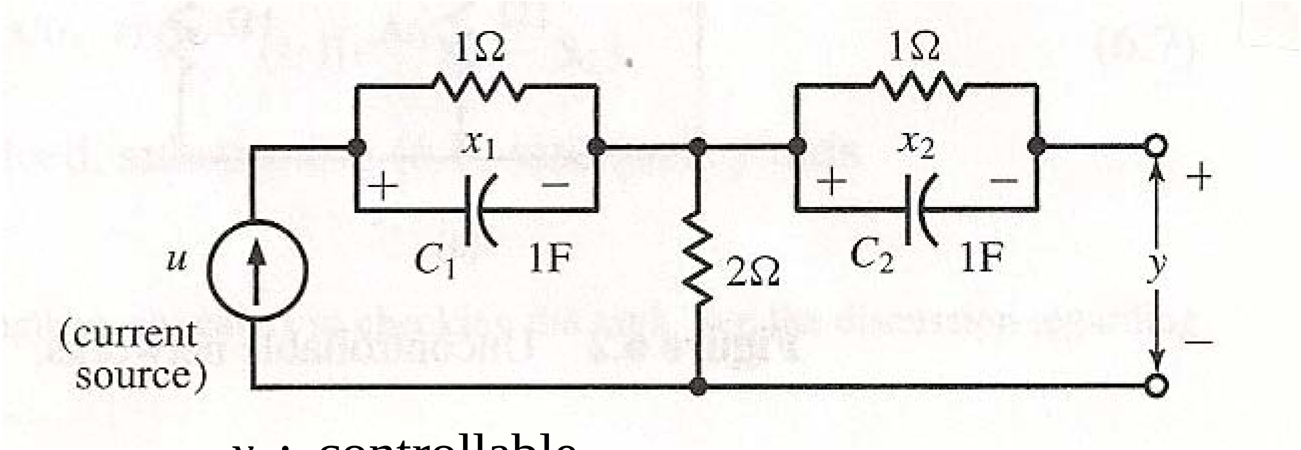
Controllability

Definition: Controllability

$\{A, B\}$ is said to be controllable if for any x_0, x_f,

$\exists$ an $u(t)$ that transfers x_0 to x_f in a finite time.

Otherwise, $\{A, B\}$ is said to be uncontrollable.

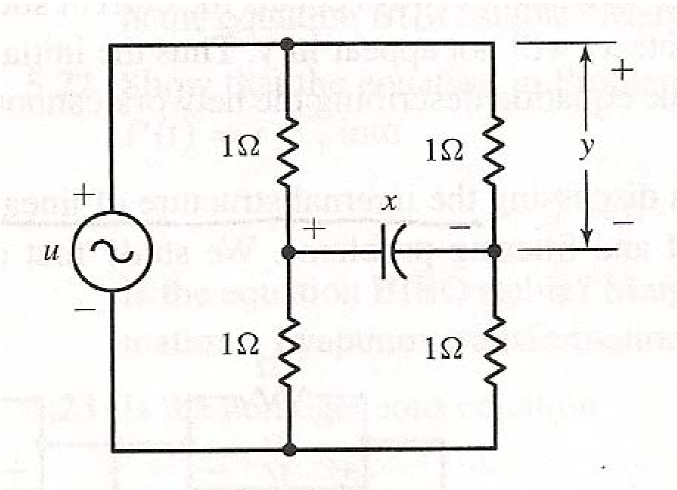


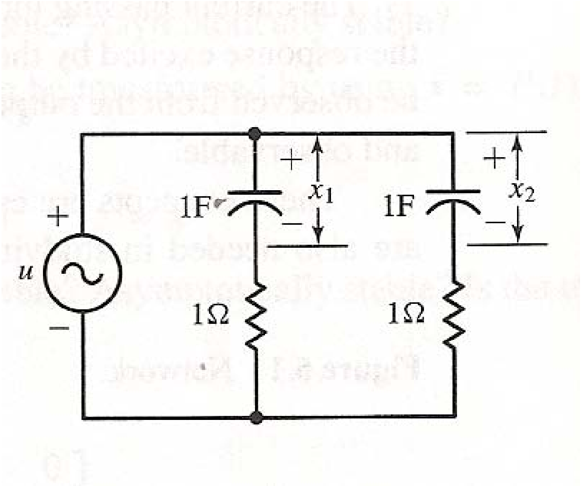
x_1 : controllable

x_2 : uncontrollable

Controllability

Example : Uncontrollable Case





Controllability

Theorem

The followings are equivalent

1. $\{A, B\}$ is controllable
2. $W_c(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau = \int_0^t e^{A(t-\tau)} BB' e^{A'(t-\tau)} d\tau$

   is nonsingular.
3. $C = \left[B \; AB \; A^2 B \cdots A^{n-1} B \right]$ has rank n.
4. $[A - \lambda \mathbf{I} \quad B]$ has full row rank for all λ
5. If $\mathrm{Re}\{\lambda_i\} < 0 \;\; \forall i$, then $AW_c + W_c A' = BB'$ has unique and

   positive definite. The solution is called

   Controllability Gramian expressed as

   $$W_c(\infty) = \int_0^\infty e^{A\tau} BB' e^{A'\tau} d\tau.$$

Controllability

(Pf. 1↔2)

(⇐) If $W_c(t) > 0$ (nonsingular) $\rightarrow$ controllable

$$x(t_1) = e^{At_1}x(0) + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau$$

$$x(0) = x_0,\ x(t_1) = x_1$$

Let $u(t) = -B'e^{A'(t_1-t)}W_c^{-1}(t_1)\left[e^{At_1}x_0 - x_1\right]$

$$x(t_1) = e^{At_1}x_0 - \int_0^{t_1} e^{A(t_1-\tau)}BB'e^{A'(t_1-\tau)}d\tau W_c^{-1}(t_1)\left[e^{At_1}x_0 - x_1\right]$$

$$= x_1$$

Controllability

(Pf. 1↔2)

($\Rightarrow$) by contradiction

Assume $\{A, B\}$ is controllable but $W_c(t_1)$ is singular.

$\exists\ \upsilon \neq 0 \ni W_c(t_1)\upsilon = 0$

$$\upsilon' W_c(t_1)\upsilon = \int_0^{t_1} \upsilon' e^{A(t_1-\tau)} BB' e^{A'(t_1-\tau)} \upsilon d\tau$$

$$= \int_0^{t_1} \left\| B' e^{A'(t_1-\tau)} \upsilon \right\|^2 d\tau = 0$$

$$B' e^{A'(t_1-\tau)} \upsilon = 0 \ \forall\ \tau \in [0, t_1]$$

$$x(0) = e^{-At_1}\upsilon,\ x(t_1) = 0$$

$$0 = \upsilon + \int_0^{t_1} e^{A(t_1-\tau)} Bu(\tau) d\tau$$

$$0 = \upsilon'\upsilon + \int_0^{t_1} \upsilon' e^{A(t_1-\tau)} Bu(\tau) d\tau = \upsilon'\upsilon$$

This is contradict.

Controllability

(Pf. 2↔3)

$(\Rightarrow)$ If $W_c(t)$ nonsingular $\rightarrow$ C has full rank

$$\upsilon' W_c(t)\upsilon = 0 \text{ means } \upsilon = 0 \quad .....(*)$$

$$\upsilon' e^{At} B = 0$$

$$e^{At} = \sum_{i=0}^{n-1} \alpha_i A^i \quad \text{(using minimal poly)}$$

$$\upsilon' e^{At} B = \upsilon' \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \underset{\neq 0}{\begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{n-1} \end{bmatrix}} = 0 \quad .....(**)$$

If C has not full rank,

$\exists\ \upsilon' \neq 0$ that satisfy (**), this contracts (*).

Hence C has full rank.

Controllability

(Pf. 2↔3)

$(\Leftarrow)$ If W_c singular $\rightarrow$ C does not have full rank

$$\exists\ \upsilon' \neq 0 \ \ni \upsilon' W_c \upsilon = 0$$

$$\upsilon' \left[B \ AB \ \cdots A^{n-1}B \right] = 0$$

$\left[B \ AB \ \cdots A^{n-1}B \right]$ has not full rank.

Controllability

(Pf. 3↔4)

$(\Rightarrow)$ If C has full rank $\rightarrow \left[A - \lambda I \;\; B \right]$ has full rank

If not, $\exists\ q \neq 0 \ \ni$

$$q\left[A - \lambda_1 I \;\; B \right] = 0$$

$$\Rightarrow qA = \lambda_1 q, \;\; qB = 0$$

$$qA^2 = qAA = \lambda_1{}^2 q, \cdots qA^k = \lambda_1{}^k q$$

$$q\left[B \; AB \cdots A^{n-1}B \right] = \left[qB \;\; \lambda_1 qB \cdots \lambda_1{}^{n-1} qB \right] = 0$$

C has not full rank (contradict).

Controllability

(Pf. 3↔4)

($\Leftarrow$) $[A-\lambda I \;\; B]$ has full rank $\rightarrow C$ has full rank

If $\rho[C]<n, \;\; \rho[A-\lambda I \;\; B]<n$ at some λ

By Theorem 6.6, if $\rho[C]=n-m, \; \exists \, P \neq 0 \;\; \ni$

$$\bar{A}=PAP^{-1}=\begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \quad \bar{B}=PB=\begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}$$

Let $q_1\bar{A}_{\bar{c}}=\lambda_1 q_1 \Rightarrow q_1(\bar{A}_{\bar{c}}-\lambda_1 I)=0$

Let $q=[0 \; q_1]$

$$q\left[\bar{A}-\lambda_1 I \;\; \bar{B}\right]=[0 \; q_1]\begin{bmatrix} \bar{A}_c-\lambda_1 I & \bar{A}_{12} & \bar{B}_c \\ 0 & \bar{A}_{\bar{c}}-\lambda_1 I & 0 \end{bmatrix}=0$$

$$\Rightarrow \rho\left(\left[\bar{A}-\lambda_1 I \;\; \bar{B}\right)<n\right.$$

By Theorem 6.2, controllability is invariant by equivalence transformation.

$$\Rightarrow \rho\left(\left[A-\lambda_1 I \;\; B\right)<n\right.$$

Controllability

(Pf. 1↔ 2↔ 5)

$$\{A, B\} \text{ controllable}$$

$$\Leftrightarrow \; p(C) = n$$

$\Leftrightarrow$ By Controllary 5.5, $\exists$ unique $W_c > 0 \;\ni$

$AW_c + W_c A = -BB'$ ...(*) for A with negative real part

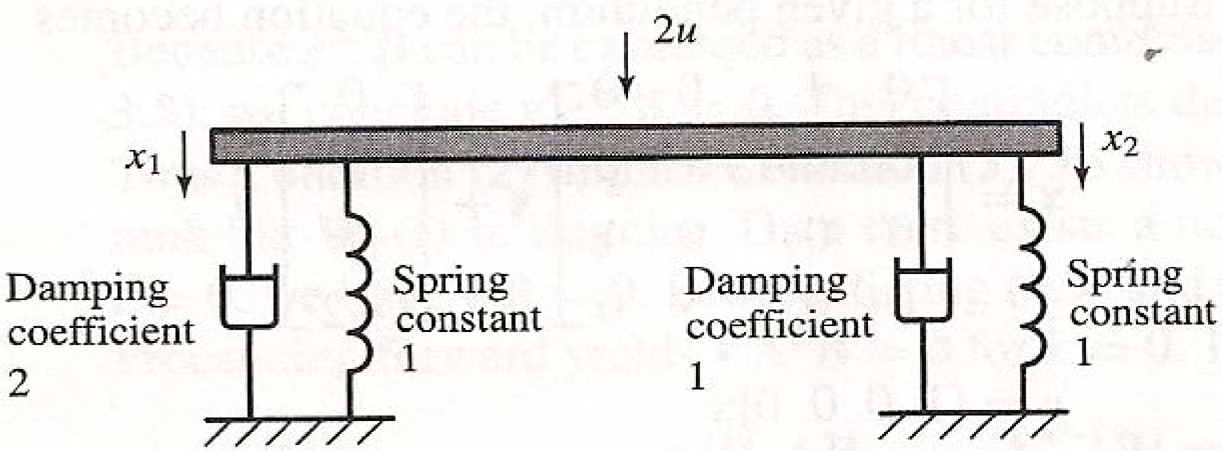
In addition, by Theorem 5.6,

$W_c(\infty) = \int_0^\infty e^{A\tau} BB' e^{A'\tau} d\tau$ is unique solution of (*).

Controllability

Example 6.3

Can we apply a force to bring the platform from $x_1(0)=10, x_2(0)= -1$ to equilibrium with 2 seconds?



$$x_1 + 2\dot{x}_1 = u, \quad x_2 + \dot{x}_2 = u$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}u$$

Controllability

Example 6.3 (cont)

$$\rho[B\ AB] = \rho\begin{bmatrix} 0.5 & -0.25 \\ 1 & -1 \end{bmatrix} = 2$$

$\rightarrow$ controllable

To find control $u(t)$ in $[0, 2]$,

$$W_c(2) = \int_0^2 \left(\begin{bmatrix} e^{-0.5\tau} & \\ & e^{-\tau} \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \end{bmatrix} \begin{bmatrix} e^{-0.5\tau} & \\ & e^{-\tau} \end{bmatrix} \right) d\tau$$

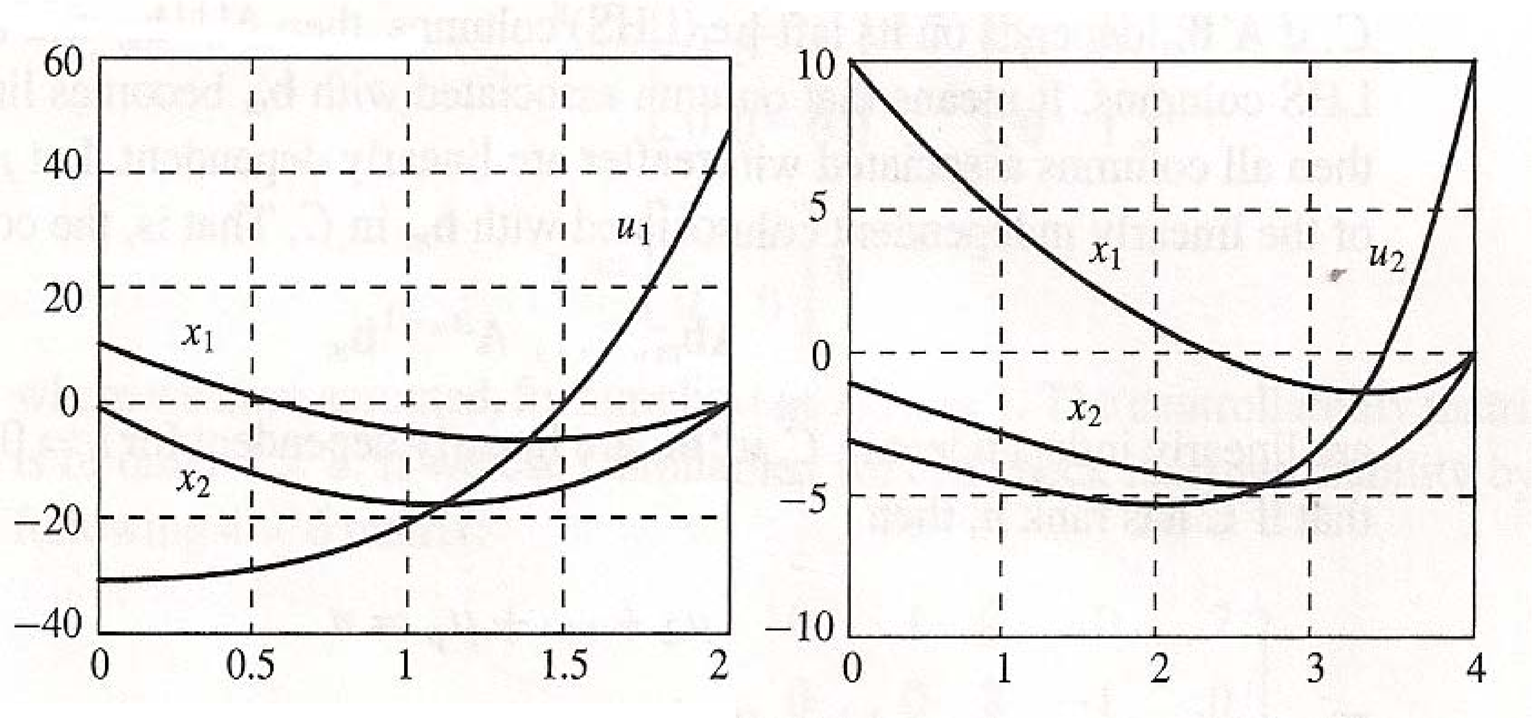
$$= \begin{bmatrix} 0.2162 & 0.3167 \\ 0.3167 & 0.4908 \end{bmatrix}$$

$$u(t) = -\begin{bmatrix} 0.5 & 1 \end{bmatrix} \begin{bmatrix} e^{-0.5(2-t)} & \\ & e^{-(2-t)} \end{bmatrix} W_c^{-1}(2) \begin{bmatrix} e^{-1} & \\ & e^{-2} \end{bmatrix} \begin{bmatrix} 10 \\ -1 \end{bmatrix}$$

$$= -58.82e^{0.5t} + 27.96e^{t}$$

Controllability

Example 6.3 (cont)



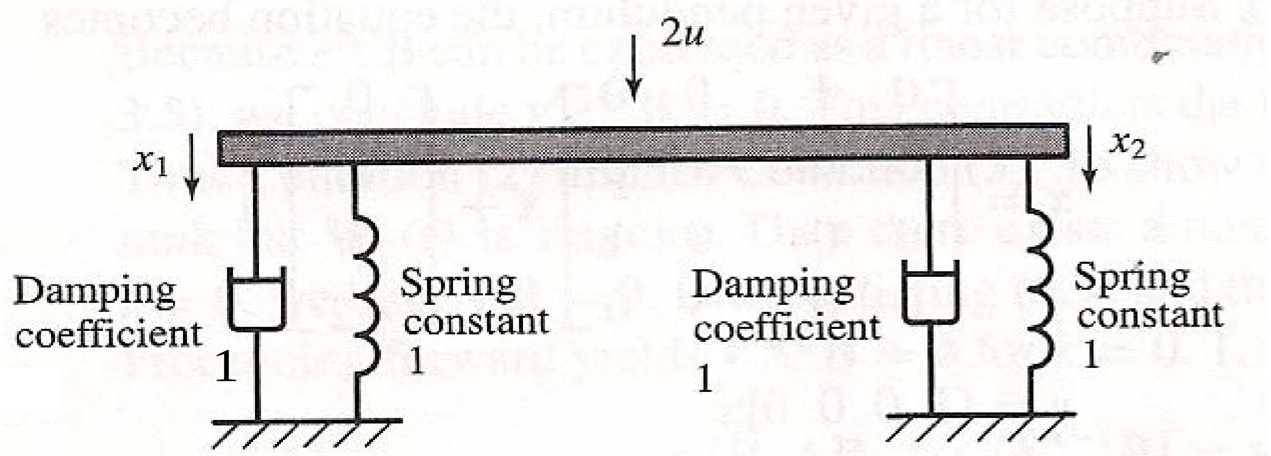
Controllability

Example 6.4

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$\rho[B\ AB] = \rho \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = 1$$

$$\rightarrow uncontrollable$$



Controllability

Controllability indices

Define $U_k = \left[B \vdots AB \vdots \cdots \vdots A^k B \right] \quad k = 0,1,2\cdots$

$(U = U_{n-1}$: controllability matrix)

If $\{A, B\}$ is controllable,

$\Leftrightarrow \ \rho U_{n-1} = n \Leftrightarrow \exists \ n$ LI columns among np columns

$U_k = \left[b_1 \ b_2 \cdots b_p \vdots Ab_1 \cdots Ab_p \vdots \cdots \vdots A^k b_1 \cdots A^k b_p \right]$

Note) If $A^j b_i$ is LD to its left-hand-side(LHS) vectors,

$A^k b_i, \ k > j,$ is LD to its LHS vectors

$$\begin{cases} \because \ Ab_2 = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_{p+1} Ab_1 \\ A^2 b_2 = \alpha_1 Ab_1 + \alpha_2 Ab_2 + \cdots + \alpha_{p+1} A^2 b_1 \end{cases}$$

Note) column search algorithm (Appendix A in 2nd Ed.)

Controllability

Define $r_i :=$ number of LD columns in $\{A^i b_1, \cdots, A^i b_p\}$

$$\Rightarrow 0 \le r_0 \le r_1 \le r_2 \cdots \le p$$

$$\Rightarrow \exists\, \mu \ni$$

$$0 \le r_0 \le r_1 \qquad \le r_{\mu-1} < p, r_\mu = r_{\mu+1} = \cdots = p$$

$$\Leftrightarrow \exists\, \mu \ni$$

$$\rho U_0 < \rho U_1 < \cdots < \rho U_{\mu-1} = \rho U_\mu = \rho U_{\mu+1} \cdots$$

$\Rightarrow$ If $\rho U_{\mu-1} = n$, $\{A \quad B\}$ is controllable.

$\Rightarrow \mu$: controllability index

Controllability

Rearrange of U

$$\left\{b_1, Ab_1, A^2b_1 \cdots A^{\mu_1 -1}b_1 \vdots \cdots \vdots b_p, Ab_p, \cdots, A^{\mu_p -1}b_p\right\}$$

$$\mu = \max\left\{\mu_1, \cdots, \mu_p\right\}, \ \left\{\mu_1, \cdots, \mu_p\right\} \text{ controllability indices}$$

$$\Rightarrow \text{If } \sum \mu_i = n, \ \left\{A \quad B\right\} \text{ is controllable}$$

Claim :

$$\frac{n}{p} \le \mu \le \min(\bar{n}, \ n - \bar{p} + 1)$$

where $\bar{n}$ is degree of minimal polynomial.

$\bar{p}$ is rank of B.

Controllability

Pf)

i) $n \le p\mu \Rightarrow \dfrac{n}{p} \le \mu$

ii) $A^{\bar{n}} = \alpha_1 A^{\bar{n}-1} + \cdots + \alpha_{\bar{n}} \mathrm{I}$

$A^{\bar{n}} B = \alpha_1 A^{\bar{n}-1} B + \quad + \alpha_{\bar{n}} B$ is LD to its LHS vectors

$\Rightarrow \mu \le \bar{n}$

iii) The rank of $\begin{bmatrix} B & AB & \cdots & A^{\mu-1}B \end{bmatrix}$ increases at least one

whenever μ increases by one, for example,

$\rho\begin{bmatrix} B & AB & A^2B \end{bmatrix} - \rho\begin{bmatrix} B & AB \end{bmatrix} \ge 1.$

The largest μ is achieved when the rank increases just by one

in every increase of μ. i.e.,

$\bar{p} + \mu \; - 1 \le n \Rightarrow \mu \; \le n - \bar{p} + 1.$

Controllability

Corollary 6.1

$\{A, B\}$ is controllable iff

$$C_{n-\bar{p}+1} = \begin{bmatrix} B & AB & \cdots & A^{n-\bar{p}}B \end{bmatrix} \text{ has rank } n.$$

Theorem 6.2

Controllability is invariant by any equivalence transformation.

Pf)

$$C = \begin{bmatrix} B & AB & & A^{n-1}B \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & & \bar{A}^{n-1}\bar{B} \end{bmatrix}$$

$$= \begin{bmatrix} PB & PAP^{-1}PB & \cdots & PA^{n-1}P^{-1}PB \end{bmatrix}$$

$$= P\begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix} = PC, \; P : \text{nonsingular}$$

$$\Rightarrow \rho(C) = \rho(\bar{C}).$$

Controllability

Example 6.5

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x$$

$$C_{n-\bar{p}+1} = \begin{bmatrix} B & AB & A^2 B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 & 0 & -4 \end{bmatrix}$$

$$\mu = 2, \;\; C_n = \begin{bmatrix} B & AB & A^2 B & A^3 B \end{bmatrix}$$

Controllability

Theorem 6.3

The set of controllability indices is invariant

by any equivalence transformation and any ordering of columns in *B*.

Pf)

$$\text{i)}\rho(C_k) = \rho(\bar{C}_k) \text{ by theorem 6.2}$$

$$\text{ii) } \tilde{B} = BM \ \ (M = p \times p \text{ pumutation matrix, } M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$$

$$\tilde{U}_k = \begin{bmatrix} \tilde{B} \cdots & A^k \tilde{B} \end{bmatrix}$$

$$= U_k \underbrace{\text{diag}\{M, M \cdots M\}}_{\text{nonsingular}}$$

$$\Rightarrow \mu_k = \rho(U_k) = \rho(\tilde{U}_k)$$

HW 6-1

Problem 6-2, Text, p. 180

What do you have to get ?

Ideal candidates should have excellent mathematical and programming skills, outstanding research potential in machine learning (e.g, recurrent networks, reinforcement learning, evolution, statistical methods, unsupervised learning, the recent theoretically optimal universal problem solvers, adaptive robotics), and good ability to communicate results.

General intellectual ability:
Analytical / theoretical skills:
Programming skills:
Experimental skills:
Motivation:
Written communication skills:
Verbal communication skills:
Ability to organise workload:
Originality / creativity:
Social skills:

Observability

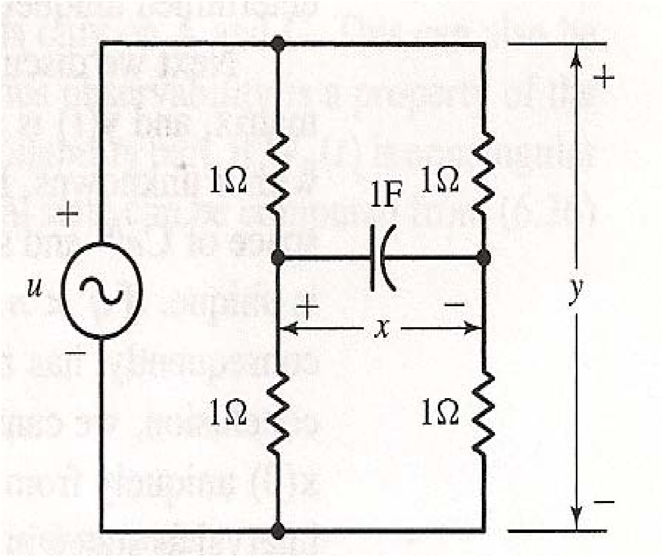
Definition 6.O1

$\{A, C\}$ is said to be observable if

for any unknown $x(0)$, $\exists$ finite $t_1 > 0$ such that

u & $y \in [0, t_1]$ suffices to find $x(0)$.

Example 6.6



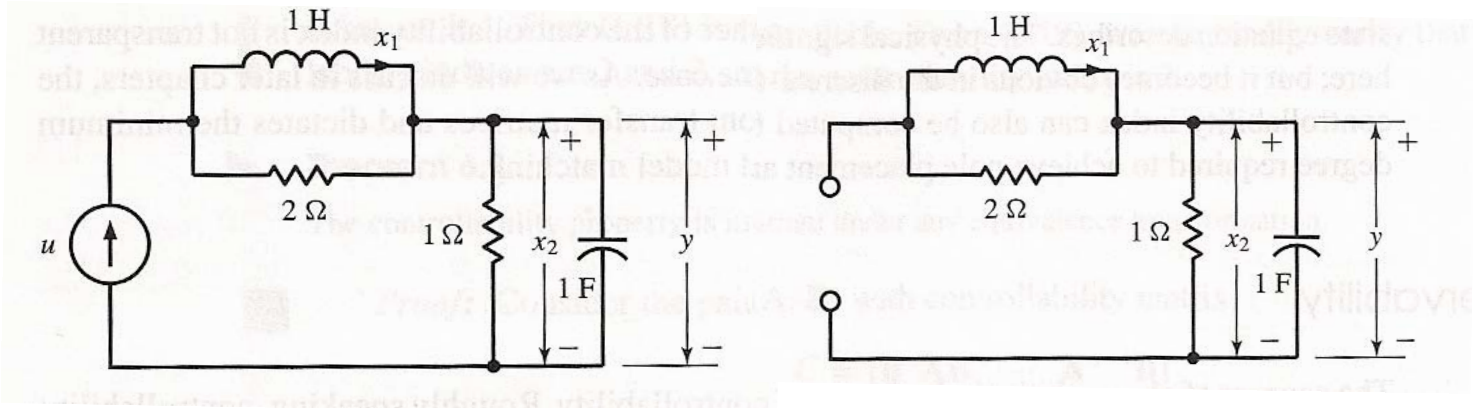
When $u = 0$, $y = 0$ always

regardless of initial state $x(0)$.

$\Rightarrow$ unobservable

Observability

Example 6.7



When $u = 0,\ x_1(0) = a \neq 0,\ \ x_2(0) = 0,$ the output $y(t) = 0.$

There is no way to determine the initial state $[a,\ 0]$

form $y(t)$ and $u(t)$.

$\Rightarrow$ unobservable

Observability

Observability Matrix

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

$\bar{y} = Ce^{At}x(0)$ where $\bar{y} = y(t) - u_0$,

and $u_0 = C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$.

$$\begin{bmatrix} \bar{y} \\ \bar{y}' \\ \vdots \\ \bar{y}^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} e^{At}x(0) = \mathbf{O}e^{At}x(0).$$

If $\rho(\mathbf{O}) = n$, $\rho(\mathbf{O}e^{At}) = n$ ($\because e^{At}$ is nonsingular).

Hence the solution $x(0)$ is uniquely determined. $\Rightarrow$ Observable.

O is called **Observability Matrix**.

Observability

Theorem 6.4

The state is observable iff the n × n matrix

$$W_0(t) = \int_0^t e^{A'\tau} C' C e^{A\tau} d\tau$$

is nonsingular $\forall\ t > 0$.

Pf)

$$(\Leftarrow)\ Ce^{At} x_0 = \bar{y}(t)$$

$$\int_0^{t_1} e^{A't} C' C e^{At} dt x_0 = \int_0^{t_1} e^{A't} C' \bar{y}(t) dt$$

$$x_0 = [W_0^{t_1}(t_1)]^{-1} \int_0^{t_1} e^{A't} C' \bar{y}(t) dt, \text{ for any fixed } t_1.$$

Observability

Pf_continued)

$(\Rightarrow)$ $W_0(t_1)$ is singular $\rightarrow$ x_0 is not observable.

$\rightarrow \exists\ \upsilon \neq 0$ such that $W_0(t_1)\upsilon = 0.$

$$0 = \upsilon' W_0(t_1)\upsilon = \int_0^{t_1} \upsilon' e^{A'\tau} C' C e^{A\tau} \upsilon d\tau$$

$$= \int_0^{t_1} \left\| C e^{A\tau} \upsilon \right\|^2 d\tau$$

$$\rightarrow C e^{A\tau} \upsilon = 0 \ \forall\ t \in [0, t_1]$$

$$\rightarrow \bar{y}(t) = C e^{A\tau} \upsilon = 0 \text{ for } x^{(1)}(0) = \upsilon \neq 0$$

$$\bar{y}(t) = C e^{A\tau} x_2(0) = 0 \text{ for } x^{(2)}(0) = 0$$

$\rightarrow \exists$ two different initial states for $\bar{y}(t) = 0.$

$\rightarrow$ not observable.

Observability

Theorem 6.5 (Theorem of duality)

$\{A, B\}$ is controllable iff (A', B') is observable.

Pf)

$\{A, B\}$ is controllable iff

$$W_c(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau \text{ is nonsingular for all } t > 0.$$

$\{A', B'\}$ is observable iff

$$W_0(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau \text{ is nonsingular for all } t > 0.$$

$$W_c(t) = W_0(t).$$

Observability

Theorem 6.O1 : The following statements are equivalent.

1. $\{A, C\}$ is observable.

2. $W_0(t) = \int_0^t e^{A'\tau} C'C e^{A\tau} d\tau$ is nonsingular $\forall\ t > 0$.

3. The $nq \times n$ observability matrix $\mathbf{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has rank n.

4. The $(n+q) \times n$ matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full rank at every eigenvalue λ of A.

5. If $\mathrm{Re}\{\lambda_i(A)\} < 0$, $\exists\ W_0 > 0$ such that

   $A'W_0 + W_0 A = -C'C,\ W_0 = \lim_{t \to \infty} W_0(t)$: Observability Gramian.

Observability

Observability index ν

$$\mathbf{O}_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \rightarrow \rho \mathbf{O}_n = \rho \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_q \\ \cdots \\ \vdots \\ \vdots \\ \cdots \\ C_1 A^{n-1} \\ \vdots \\ C_q A^{n-1} \end{bmatrix} = \rho \mathbf{O}_\nu = \rho \begin{bmatrix} C \\ \cdots \\ \vdots \\ \cdots \\ CA^{\nu-1} \\ \cdots \end{bmatrix} = n$$

Observability

Linearly Independent Vectors in $\mathbf{O}_\nu$

$$\begin{bmatrix} c_1 \\ \vdots \\ c_1 A^{\nu_1 - 1} \\ \vdots \\ \vdots \\ c_q \\ \vdots \\ c_q A^{\nu_q - 1} \end{bmatrix}$$

observability indices:

$$\{\nu_1 \cdots, \nu_q\}.$$

observability index:

$$\nu = \max(\nu_1 \cdots \nu_q).$$

Observability

Claim

$$n/q \le \nu \le \min(\bar{n}, n-\bar{q}+1)$$

where $\rho(C)=\bar{q}$.

Corollary 6.O1

$\{A, C\}$ is observable iff

$$O_{n-\bar{q}+1} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-\bar{q}} \end{bmatrix} = n$$

where $\rho(C)=\bar{q}$.

Observability

Theorem 6.O2

Observability property is invariant by equivalence transformstion.

Theorem 6.O3

The set of observability indices of $\{A, C\}$ is invariant
under equivalence transformation and any reordering of the rows of C.

HW 6-2

Problem 6.11, in Text, p.181

Canonical Decomposition

Equivalence Transformation (Remind)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Let $\bar{x} = Px$, where P is a nonsingular matrix. Then

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$
$$y = \bar{C}\bar{x} + \bar{D}u$$

with $\bar{A} = PAP^{-1}, \bar{B} = PB, \bar{C} = CP^{-1}, \bar{D} = D.$

They are equivalent. i.e.,

$$\{A, B, C, D\} \leftrightarrow \{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}.$$

And $\bar{\mathbf{C}} = P\mathbf{C}$, $\bar{\mathbf{O}} = \mathbf{O}P^{-1}$, i.e.,

Stability, Controllability, Observability are preserved.

Canonical Decomposition

Canonical Decomposition

Theorem 6.6

If $\rho(\mathbf{C}) = \rho\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n_1 < n$

Let $\mathrm{Q} = P^{-1} = \begin{bmatrix} q_1 \cdots q_{n_1} & q_{n_1+1} \cdots q_n \end{bmatrix}$

$q_i, \ i = 1, \ldots, n_1$ LI column vectors in $\mathbf{C}$

$q_i, \ i = n_1 + 1, \ldots, n$ LI vectors to $q_i, \ i = 1, \ldots, n_1$.

Then $\overline{\mathrm{x}} = \mathrm{Px}$ leads to

$$\begin{bmatrix} \dot{\overline{\mathrm{x}}}_c \\ \dot{\overline{\mathrm{x}}}_{\overline{c}} \end{bmatrix} = \begin{bmatrix} \overline{\mathrm{A}}_c & \overline{\mathrm{A}}_{12} \\ 0 & \overline{\mathrm{A}}_{\overline{c}} \end{bmatrix} \begin{bmatrix} \overline{\mathrm{x}}_c \\ \overline{\mathrm{x}}_{\overline{c}} \end{bmatrix} + \begin{bmatrix} \overline{\mathrm{B}}_c \\ 0 \end{bmatrix} \mathrm{u}$$

$$\mathrm{y} = \begin{bmatrix} \overline{\mathrm{C}}_c & \overline{\mathrm{C}}_{\overline{c}} \end{bmatrix} \begin{bmatrix} \overline{\mathrm{x}}_c \\ \overline{\mathrm{x}}_{\overline{c}} \end{bmatrix} + \mathrm{Du}$$

Canonical Decomposition

Theorem 6.6 (continued)

where

$$\bar{A}_c : n_1 \times n_1$$

$$\bar{A}_{\bar{c}} : (n - n_1) \times (n - n_1)$$

And the n_1 dimensional subequation

$$\dot{\bar{x}}_c = \bar{A}_c \bar{x}_c + \bar{B}_c u$$

$$y = \bar{C}_c \bar{x}_c + Du$$

is controllable and has the same transfer function matrix as the original state equation.

Canonical Decomposition

Pf) $\mathrm{Q} = \mathrm{P}^{-1} = \left[\mathrm{q}_1 \cdots \mathrm{q}_{n_1}, \cdots \mathrm{q}_n \right]$

$$\{n_i\} \quad \mathrm{x} \xrightarrow{\mathrm{A}} \dot{\mathrm{x}}$$

$$\mathrm{Q} \uparrow \quad \mathrm{x} = \mathrm{Q}\overline{\mathrm{x}},$$

$$\{q_i\} \quad \overline{\mathrm{x}} \xrightarrow{\overline{\mathrm{A}}} \dot{\overline{\mathrm{x}}}, \quad \overline{a}_i : \text{rep. of } \mathrm{Aq}_i \ w.r.t.\{q_i\}$$

$$\mathrm{Aq}_i = [\mathrm{q}_1 \quad \cdots \quad \mathrm{q}_n]\overline{a}_i$$

$\mathrm{Aq}_i, i = 1, \ldots n_1,$ is linearly dependent on its LHS vectors, i.e., $\{\mathrm{q}_i, i = 1, \ldots n_1\}$ (see 6.2.1) and they are linearly independent on $\{\mathrm{q}_i, i = n_1 + 1, \ldots, n\}$. Hence $\overline{a}_i^{\,T} = [\overline{a}_{i1} \ \ldots \ \overline{a}_{in_1} \ \ 0 \ldots 0], i = 1, \ldots, n_1$.

$$\mathrm{A}[\mathrm{q}_1 \quad \cdots \quad \mathrm{q}_n] = [\mathrm{q}_1 \quad \cdots \quad \mathrm{q}_n][\overline{a}_1 \quad \cdots \overline{a}_{n_1} \quad \cdots \quad \overline{a}_n]$$

$$\mathrm{AQ} = \mathrm{Q}\overline{\mathrm{A}}$$

$$= \left[\mathrm{q}_1, \cdots, \mathrm{q}_{n_1} \quad \cdots \mathrm{q}_n \right] \begin{bmatrix} \overline{\mathrm{A}}_c & \overline{\mathrm{A}}_{12} \\ 0 & \overline{\mathrm{A}}_{\overline{c}} \end{bmatrix}.$$

Canonical Decomposition

Pf_continued)

$$\bar{B} = PB$$

$$B = P^{-1}\bar{B} = Q\bar{B} = \begin{bmatrix} q_1, \cdots, q_{n_1} & \cdots q_n \end{bmatrix} \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}$$

All columns in $\begin{bmatrix} B & AB & \cdots \end{bmatrix}$ are spaned by $\{q_1 \quad \cdots \quad q_{n_1}\}$,
as a results, B is spaned by $\{q_1 \quad \cdots \quad q_{n_1}\}$.

$$\bar{\mathbf{C}} = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \bar{B}_c & \bar{A}_c\bar{B}_c \cdots \bar{A}_c^{n_1-1}\bar{B}_c & \cdots & \bar{A}_c^{n-1}\bar{B}_c \\ 0 & 0 \quad \ldots \quad 0 & \cdots & 0 \end{bmatrix} \begin{matrix} n_1 \\ n-n_1 \end{matrix}$$

$$= \begin{bmatrix} \bar{\mathbf{C}}_c & \bar{A}_c^{n_1}\bar{B}_c & \cdots \\ 0 & 0 & 0 \end{bmatrix} \cdots \rho(\bar{\mathbf{C}}) = \rho(\bar{\mathbf{C}}_c) = n_1$$

$$\bar{C}(sI\text{-}\bar{A})^{-1}\bar{B} + \bar{D} = \bar{C}_c(sI\text{-}\bar{A}_c)^{-1}\bar{B}_c + D \text{ (see p.160)}$$

Canonical Decomposition

Example 6.8

$$\dot{x}=\begin{bmatrix}1&1&0\\0&1&0\\0&1&1\end{bmatrix}x+\begin{bmatrix}0&1\\1&0\\0&1\end{bmatrix}u \qquad y=\begin{bmatrix}1&1&1\end{bmatrix}x$$

Since ρ(B)=2, [B AB] is used instead of [B AB A^2B].

$$\rho[B\ AB]=\rho\begin{bmatrix}0&1&1&1\\1&0&1&0\\0&1&1&1\end{bmatrix}=2<3\rightarrow \textit{uncontrollable.}$$

$$Q=P^{-1}=\begin{bmatrix}0&1&1\\1&0&0\\0&1&0\end{bmatrix},\ \text{and let } \bar{x}=Px.$$

$$\bar{A}=PAP^{-1}=\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix},\ \bar{B}=PB=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix},\ \bar{C}=CP^{-1}=\begin{bmatrix}1&2&1\end{bmatrix}$$

Canonical Decomposition

Theorem 6.O6

$$\text{If } \rho(\mathbf{O}) = \rho\begin{bmatrix} \mathrm{C} \\ \mathrm{CA} \\ \cdots \\ \mathrm{CA}^{\mathrm{n\text{-}1}} \end{bmatrix} = n_2 < n, \ \text{let } \mathrm{Q}^{-1} = \mathrm{P} = \begin{bmatrix} p_1 \\ \cdots \\ p_{n_2} \\ \cdots \\ p_n \end{bmatrix},$$

where

$p_i, \ i = 1, \ldots, n_2$ LI column vectors in $\mathbf{O}$

$p_i, \ i = n_2 + 1, \ldots, n$ LI vectors to $p_i, \ i = 1, \ldots, n_2$.

Then $\bar{\mathrm{x}} = \mathrm{Px}$ leads to

$$\begin{bmatrix} \dot{\bar{\mathrm{x}}}_o \\ \dot{\bar{\mathrm{x}}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathrm{A}}_o & \\ \bar{\mathrm{A}}_{21} & \bar{\mathrm{A}}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathrm{x}}_o \\ \bar{\mathrm{x}}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathrm{B}}_o \\ \bar{\mathrm{B}}_{\bar{o}} \end{bmatrix} \mathrm{u}$$

$$\mathrm{y} = \begin{bmatrix} \bar{\mathrm{C}}_o & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathrm{x}}_o \\ \bar{\mathrm{x}}_{\bar{o}} \end{bmatrix} + \mathrm{Du}$$

Canonical Decomposition

Theorem 6.6 (continued)

where

$$\bar{A}_o : n_2 \times n_2$$

$$\bar{A}_{\bar{o}} : (n - n_2) \times (n - n_2)$$

And the n_2 dimensional subequation

$$\dot{\bar{x}}_o = \bar{A}_o \bar{x}_o + \bar{B}_0 u$$

$$y = \bar{C}_o \bar{x}_o + Du$$

is obsevable and has the same transfer function matrix as the original state equation.

Canonical Decomposition

Theorem 6.7

Every state equation can be transformed into

$$\begin{bmatrix} \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\bar{c}o} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_{co} \\ \bar{B}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_{co} & 0 & \bar{C}_{\bar{c}o} & 0 \end{bmatrix} \bar{x} + Du.$$

$$\dot{\bar{x}}_{co} = \bar{A}_{co}\bar{x}_{co} + \bar{B}_{co}u$$

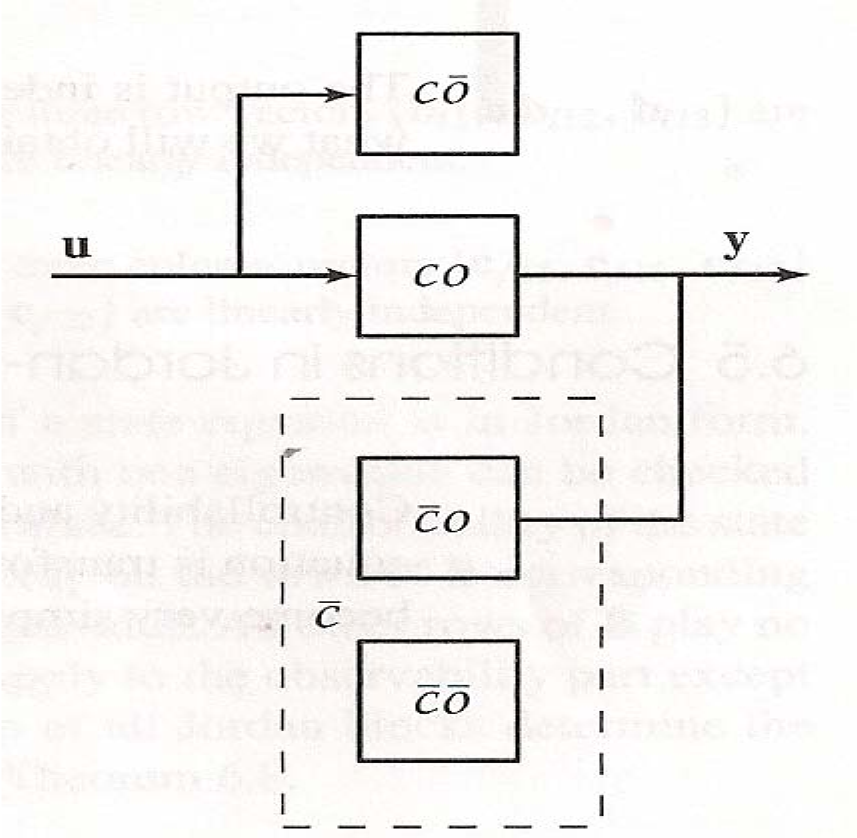
$$y = \bar{C}_{co}\bar{x}_{co} + Du.$$

$\Rightarrow$ controllable and observable.

$$G(s) = \bar{C}_{co}(sI - \bar{A}_{co})^{-1}\bar{B}_{co} + D.$$

Canonical Decomposition

Kalman Decomposition



Canonical Decomposition

Example 6.9

$$\dot{\mathrm{x}} = \begin{bmatrix} 0 & -0.5 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \mathrm{x} + \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathrm{u}$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathrm{x} + \mathrm{u}.$$

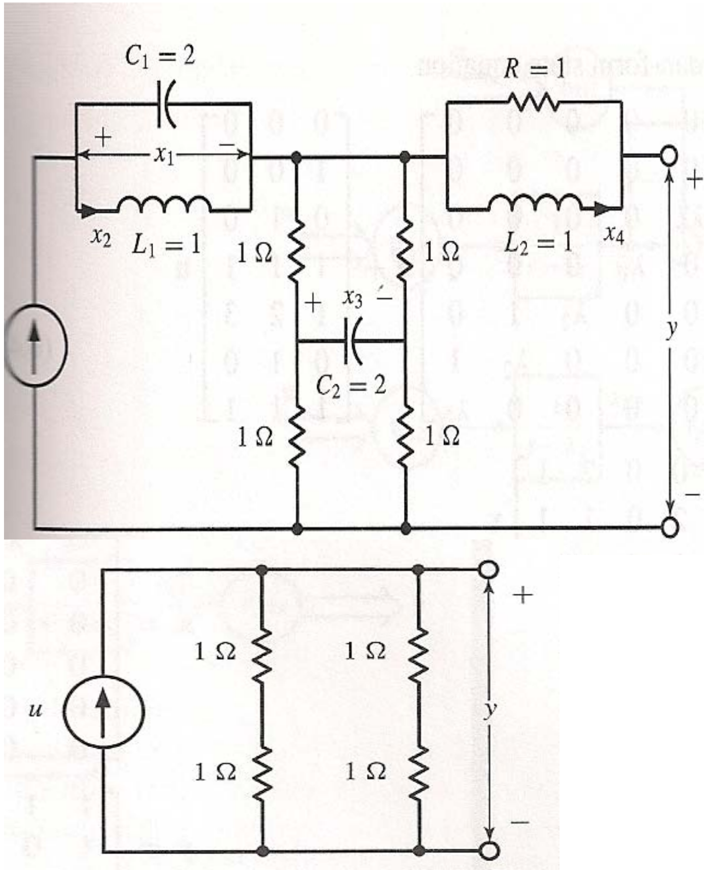
Controllable part is

$$\dot{\mathrm{x}}_c = \begin{bmatrix} 0 & -0.5 \\ 1 & 0 \end{bmatrix} \mathrm{x}_c + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \mathrm{u}$$

$$y = \begin{bmatrix} 0 & 0 \end{bmatrix} \mathrm{x}_c + \mathrm{u}.$$

Controllable and observable part is

$$y = \mathrm{u}.$$



Conditions in Jordan Form

Jordan-form Dynamical Equations.

$$\dot{x} = Jx + Bu$$

$$y = Cx + Du$$

$$J = \mathrm{diag}(J_1,\ J_2) = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

$$J_1 = \mathrm{diag}(J_{11},\ J_{12},\ J_{13}),\ J_2 = \mathrm{diag}(J_{21},\ J_{22})$$

b_{lij}: the row of B corresponding to the *last* row of J_{ij}.

c_{fij}: the column of C corresponding to the *first* column of J_{ij}.

Conditions in Jordan Form

Example 6.10

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u \qquad \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \end{matrix} \begin{bmatrix} b_{l11} \\ b_{l12} \\ b_{l13} \end{bmatrix} := B_1^l, \qquad \leftarrow \begin{bmatrix} b_{l21} \end{bmatrix} := B_2^l$$

If the rows of B_i^l are LI, $\{J, B\}$ is controllable.

Conditions in Jordan Form

$$y = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 3 & 1 & 2 & 2 \end{bmatrix} x$$

$$\uparrow \qquad \uparrow \quad \uparrow \quad \uparrow$$

$$\begin{bmatrix} c_{f11} & c_{f12} & c_{f13} \end{bmatrix} \begin{bmatrix} c_{f21} \end{bmatrix}$$

$$:= C_1^f \qquad\qquad := C_2^f$$

$r(i)$: number of Jordan block for λ_i , for example.

$r(1) = 3,\ r(2) = 1.$

If the columns of C_i^f are LI, {J, C} is observable.

Conditions in Jordan Form

Theorem 6.8

1) JFE is controllable iff for each i,

the rows of $r(i)\times p$ matrix

$$\mathrm{B}_i^l := \begin{bmatrix} \mathrm{b}_{li1} \\ \mathrm{b}_{li2} \\ \vdots \\ \mathrm{b}_{lir(i)} \end{bmatrix}$$ are linearly independent to each other.

2) JFE is observable iff for each i,

the columns of $q\times r(i)$ matrix

$$\mathrm{C}_i^f := \begin{bmatrix} \mathrm{c}_{fi1} & \mathrm{c}_{fi2} & \cdots & \mathrm{c}_{fir(i)} \end{bmatrix}$$ are LI to each other.

Conditions in Jordan Form

Pf)

$$\rho\left[\lambda I - A \vdots B\right] = \text{n}, \ \text{ for all } \lambda_i .$$

For λ_1

$$\left[\lambda_1 I - A \vdots B\right] = \begin{bmatrix} 0 & -1 & & & & & & b_{111} \\ & 0 & -1 & & & & & b_{211} \\ & & 0 & & & & & b_{l11} \\ & & & 0 & -1 & & & b_{112} \\ & & & & 0 & & & b_{l12} \\ & & & & & \lambda_1 - \lambda_2 & -1 & b_{121} \\ & & & & & & \lambda_1 - \lambda_2 & b_{l21} \end{bmatrix}$$

$$\rho\left[\lambda_1 I - A \vdots B\right] = n \leftrightarrow \text{b}_{l11} \ \text{ and } \text{b}_{l12} \ \text{ is LI.}$$

Discrete Time Case

Discrete-Time State Equation

Theorem 6.D1

The followings are equivalent to each other;

1. $\{A, B\}$ is controllable
2. $W_{dc}[n-1] = \sum_{m=0}^{n-1} (A)^m BB'(A')^m : n \times n$ matrix is nonsingular
3. $\mathbf{C}_d = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ has rank n
4. $\rho[A - \lambda\mathbf{I} \ B] = n \ \forall \lambda$
5. If $|\lambda_i(A)| < 1$, $\exists \ W_{dc} > 0$ such that

$$W_{dc} - AW_{dc}A' = BB'$$

$$W_{dc} = W_{dc}[\infty].$$

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Discrete Time Case

Note)

$$x[n] = A^n x[0] + \sum_{m=0}^{n-1} A^{n-1-m} Bu[m]$$

$$x[n] - A^n x[0] = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} u[n-1] \\ \vdots \\ u[0] \end{bmatrix}$$

$$\bar{x} = \mathbf{C}_d \mathbf{u}$$

$$\rho(\mathbf{C}_d) = n \leftrightarrow \mathbf{u} \text{ is unique}$$

By Theorem 3.8

$$\rho(\mathbf{C}_d) = n \leftrightarrow \rho W_{dc}[n-1] = \rho \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} B' \\ B'A' \\ \vdots \\ B'(A')^{n-1} \end{bmatrix} = n$$

Discrete Time Case

Theorem 6.DO1

The followings are equivalent to each other;

1. $\{A, C\}$ is observable

2. $W_{do}[n-1] = \sum_{m=0}^{n-1} (A')^m C'C(A)^m : n \times n$ matrix

   is nonsingular

3. $\mathbf{O}_d = \begin{bmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{bmatrix}$ has rank n, 4. $\rho \begin{bmatrix} A & -\lambda \mathbf{I} \\ & C \end{bmatrix} = n \ \forall \lambda(A)$

5. If $|\lambda_i(A)| < 1$, $\exists\, W_{dc} > 0$ such that

   $W_{do} - A'W_{do}A = C'C$, $W_{do} = W_{do}[\infty]$.

Discrete Time Case

Controllability to the origin & reachability

- Controllability from any x_0 to any x_f
- Controllability from any $\mathrm{x}_0 \neq 0$ to $\mathrm{x}_f = 0$
- Controllability from any $\mathrm{x}_0 = 0$ to any $\mathrm{x}_f \neq 0$

$= \text{reachability}$

$$\mathrm{x}[k+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{x}[k] + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \mathrm{u}[k]$$

$\rho(\mathbf{C}_d) = 0:$ not controllable

$\mathrm{x}[3] = A^3 \mathrm{x}[0] = 0$ controllable to origin

Discrete Time Case

$$\mathrm{x}[k+1]=\begin{bmatrix}2 & 1\\ 0 & 0\end{bmatrix}\mathrm{x}[k]+\begin{bmatrix}-1\\ 0\end{bmatrix}\mathrm{u}[k]$$

$$\mathrm{x}_1[0]=\alpha,\ \mathrm{x}_2[0]=\beta$$

$$\mathrm{u}[0]=2\alpha+\beta \rightarrow \mathrm{x}[1]=0$$

controllable to origin

not reachable

$$\rho(\mathbf{C}_d)=1$$

Controllability after sampling

$$\dot{x}=\mathrm{Ax}(t)+\mathrm{Bu}(t)$$

$$\mathrm{u}[k]=\mathrm{u}(kT)=\mathrm{u}(t)\ \text{for}\ kT\le t<(k+1)T$$

$$\bar{\mathrm{x}}[k+1]=\bar{\mathrm{A}}\bar{\mathrm{x}}[k]+\bar{\mathrm{B}}\mathrm{u}[k]$$

$$\bar{\mathrm{A}}=e^{AT},\bar{\mathrm{B}}=\int_0^T e^{At}dt\mathrm{B}=\mathrm{MB}$$

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Discrete Time Case

Theorem 6.9

Suppose $\{A, B\}$ is controllable.

Sufficient condition for $\{\bar{A}, \bar{B}\}$ to be controllable is that

$$\left|\operatorname{Im}\left[\lambda_i - \lambda_j\right]\right| \neq 2\pi\, m/T \quad \text{for } m = 1, 2, \cdots$$

whenever $\operatorname{Re}\left[\lambda_i - \lambda_j\right] = 0.$

For single input case, the condition is necessary as well.

Note)

$$\text{Let} \begin{cases} \lambda_1 = \alpha + j\beta & \bar{\lambda}_1 = e^{(\alpha + j\beta)T} \\ \lambda_2 = \alpha - j\beta & \bar{\lambda}_2 = e^{(\alpha - j\beta)T} \end{cases}$$

If $\operatorname{Im}[\lambda_1 - \lambda_2] = 2\beta = 2\, m\pi/T$, then $T = m\pi/\beta$

$$\bar{\lambda}_1 = e^{\lambda_1 T} = e^{\alpha T}, \ \bar{\lambda}_2 = e^{\lambda_2 T} = e^{\alpha T}$$

$$\rightarrow \bar{\lambda}_1 = \bar{\lambda}_2$$

Discrete Time Case

$$\overbrace{\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \ddots & \\ & & & & \lambda_n \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}}^{B} \rightarrow \begin{array}{c} \text{controllable} \\ \rho\begin{bmatrix} A - \lambda_1 I & B \end{bmatrix} = n \end{array}$$

$$\overbrace{\begin{bmatrix} \bar{\lambda}_1 & & & 0 \\ & \bar{\lambda}_2 & & \\ & & \ddots & \\ 0 & & & \ddots & \\ & & & & \bar{\lambda}_n \end{bmatrix}}^{\bar{A}} \overbrace{\begin{bmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}}^{\bar{B}} \rightarrow \begin{array}{c} \text{Since } \bar{\lambda}_1 = \bar{\lambda}_2 \\ \rho\begin{bmatrix} \bar{A} - \bar{\lambda}_1 I & \bar{B} \end{bmatrix} = n - 1 \end{array}$$

Discrete Time Case

Pf. of Theorem 6.9

Controllability is invariant by ET

→ can be proved by Jordan form

$$A = \begin{bmatrix} \lambda_1 & 1 & & & & \\ & \lambda_1 & 1 & & 0 & \\ & & \lambda_1 & & & \\ & & & \lambda_1 & & \\ & & & & \lambda_2 & 1 \\ & & 0 & & & \lambda_2 \end{bmatrix} \quad B = \begin{bmatrix} * & * \\ * & * \\ 0 & 1 \\ 1 & 1 \\ * & * \\ 1 & 0 \end{bmatrix}$$

controllable

Discrete Time Case

Pf. of Theorem 6.9 (cont.)

$$
\bar{\mathrm{A}} = \begin{bmatrix} e^{\lambda_1 T} & Te^{\lambda_1 T} & T^2 e^{\lambda_1 T}/2 & & & \\ & e^{\lambda_1 T} & Te^{\lambda_1 T} & & 0 & \\ & & e^{\lambda_1 T} & & & \\ & & 0 & e^{\lambda_1 T} & & \\ & & & & e^{\lambda_2 T} & Te^{\lambda_2 T} \\ & & 0 & & & e^{\lambda_1 T} \end{bmatrix}
$$

$$
\bar{\mathrm{B}} = \mathrm{MB} = \mathrm{M} \begin{bmatrix} * \\ 01 \\ 11 \\ * \\ 10 \end{bmatrix}
$$

Discrete Time Case

Pf. of Theorem 6.9 (cont.)

If $I_m\left[\lambda_i-\lambda_j\right]\neq 2\pi\, m/T$ for $\mathrm{Re}\left[\lambda_i-\lambda_j\right]=0$,

$$e^{\lambda_1 T}\neq e^{\lambda_2 T}.$$

If M is nonsingular, $\{\bar{A},\bar{B}\}$ is controllable.

To show M is nonsingular,

$$m_{ii}=\int_0^T e^{\lambda_i\tau}d\tau=\begin{cases}(e^{\lambda_i T}-1)/\lambda_i & \text{for } \lambda_i\neq 0\\ T & \text{for } \lambda_i=0\end{cases}$$

$$\neq 0,$$

if $2\beta_i T\neq 2\pi m$ ($\because m_{ii}=0$ only for $\alpha_i=0$ & $\beta_i T=\pi m$).

Discrete Time Case

Example 6.12

Consider

$$g(s) = \frac{s+2}{s^3+3s^2+7s+5} = \frac{s+2}{(s+1)(s+1+j2)(s+1-j2)}$$

Using (4.41), the state equation is

$$\dot{x} = \begin{bmatrix} -3 & -7 & -5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} x$$

$$|\lambda_i - \lambda_j| = 2, 4 \rightarrow T \neq 2\pi m/2 = \pi m \text{ and } T \neq 2\pi m/4 = 0.5\pi m.$$

The second condition includes the first one.

The discretized equation is controllable iff $T \neq 0.5\pi m$.

Time Varying Case

LTV State Equation

$$\dot{x} = A(t)x(t) + B(t)u(t)$$
$$y = C(t)x(t)$$

Theorem 6.11

$\{A(t), B(t)\}$ is controllable at t_o iff

$\exists$ a finite $t_1 > t_0$ such that

$$W_c(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B'(\tau)\Phi'(t_1, \tau)d\tau$$

is nonsingular,

where $\Phi(t, \tau)$ is the state transition matrix.

Time Varying Case

Pf. of Theorem 6.11

($\Leftarrow$)

$W_c(t_0,t_1)$ is nonsingular $\rightarrow \{A(t), B(t)\}$ is controllable at t_o

$$\mathrm{x}(t_1) = \Phi(t_1,t_0)\mathrm{x}_0 + \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau$$

We claim that the input

$$\mathrm{u}(t) = -B'(t)\Phi'(t_1,t)W^{-1}{}_c(t_0,t_1)[\Phi(t_1,t_0)\mathrm{x}_0 - \mathrm{x}_1]$$

will transfer x_0 to x_1. Then

$$\mathrm{x}(t_1) = \Phi(t_1,t_0)\mathrm{x}_0 - \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B'(\tau)\Phi'(t_1,\tau)d\tau \cdot W^{-1}{}_c(t_0,t_1)[\Phi(t_1,t_0)\mathrm{x}_0 - \mathrm{x}_1]$$

$$= \mathrm{x}_1.$$

Time Varying Case

Pf. of Theorem 6.11 (cont.)

($\Rightarrow$) (By contraction)

$W_c(t_0,t_1)$ is nonsingular $\leftarrow \{A(t),B(t)\}$ is controllable at t_o

Assume $W_c(t_0,t_1)$ be singular even if controllable,

$\exists\ v \neq 0$ such that $W_c(t_0,t_1)v = 0,$ so

$$v'W_c(t_0,t_1)v = \int_{t_0}^{t_1} v'\Phi(t_1,\tau)B(\tau)B'(\tau)\Phi'(t_1,\tau)v d\tau$$

$$= \int_{t_0}^{t_1} \left\|B'(\tau)\Phi'(t_1,\tau)v\right\|^2 d\tau = 0,\ \ \forall \tau \text{ in } [t_0,t_1].$$

This implies $B'(\tau)\Phi'(t_1,\tau)v = 0,\ \ \forall \tau$ in $[t_0,t_1]$.

If controllable, $\exists\ u(t)$ that transfer $\mathrm{x}_0 = \Phi(t_0,t_1)v$ to $\mathrm{x}_1 = 0$. i.e.,

$$0 = \Phi(t_1,t_0)\Phi(t_0,t_1)v + \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau.$$

Its premultiplication by v' yields

$$0 = v'v + \int_{t_0}^{t_1} v'\Phi(t_1,\tau)B(\tau)u(\tau)d\tau = v'v.$$

This contradicts $v \neq 0$.

Time Varying Case

Controllability condition without $\Phi(t,\tau)$

Define $M_0(t) = B(t)$

$$M_{m+1}(t) = -A(t)M_m(t) + \frac{d}{dt}M_m(t)$$

Theorem 6.12

Let $A(t), B(t)$ be $(n-1)$ times continuously differentiable.

$\{A(t), B(t)\}$ is controllable at t_o if

there exists a finite $t_1 > t_0$ such that

$$\rho\begin{bmatrix} M_0(t_1) & M_1(t_1) & \cdots & M_{n-1}(t_1) \end{bmatrix} = n.$$

Time Varying Case

Claim $$\frac{\partial^m}{\partial t^m}\Phi(t_2,t)B(t)=\Phi(t_2,t)M_m(t)$$

Pf) $$\frac{\partial}{\partial t}\left[\Phi(t_1,t)B(t)\right]=\frac{\partial}{\partial t}\left[\Phi(t_1,t)\right]B(t)+\Phi(t_1,t)\frac{d}{dt}B(t)$$

$$=\Phi(t_1,t)\left[-A(t)M_0(t)+\frac{d}{dt}M_0(t)\right]$$

$$=\Phi(t_1,t)M_1(t)$$

$$\vdots$$

$$\frac{\partial^m}{\partial t^m}\Phi(t_1,t)B(t)=\Phi(t_1,t)M_m(t)$$

$$\frac{\partial}{\partial t}\Phi(t_2,t)=-\Phi(t_2,t)A(t)$$

$$\begin{cases}\dfrac{\partial}{\partial t}\Phi(t,t_2)=A(t)\Phi(t,t_2)\\ \Phi(t_2,t)=\Phi(t,t_2)^{-1}\end{cases}$$

Time Varying Case

Pf) (By contraction)

$$(\text{not controllable} \rightarrow \rho\left[M_0,\ldots,M_{n-1}\right] < n)$$

Assume $W_c(t_0,t_1)$ be singular $\forall\ t_1 \geq t_0$.

$\exists\ \upsilon \neq 0$ such that

$$W_c(t_0,t_1)\upsilon = 0$$

$$\upsilon' W_c(t_0,t_1)\upsilon = \int_{t_0}^{t_1} \upsilon'\Phi(t_1,\tau)BB'\Phi'(t_1,\tau)\upsilon d\tau$$

$$= \int_{t_0}^{t_1} \left\| B'(\tau)\Phi'(t_1,\tau)\upsilon \right\|^2 d\tau = 0.$$

Time Varying Case

Pf) (cont.)

This implies

$$B'(\tau)\Phi'(t_1,\tau)\upsilon = 0 \ \forall \tau \in [t_0 \quad t_1]$$

$$\upsilon'\Phi(t_1,\tau)B(\tau) = 0.$$

By $m-$times derivatives,

$$\upsilon'\Phi(t_1,\tau)M_m(\tau) = 0$$

$$\Rightarrow \upsilon'\Phi(t_1,\tau)\begin{bmatrix} M_0(\tau) & \cdots & M_{n-1}(\tau)\end{bmatrix} = 0.$$

Since $\upsilon'\Phi(t_1,\tau) \neq 0,$

$$\rho\begin{bmatrix} M_0(\tau) & \cdots & M_{n-1}(\tau)\end{bmatrix} < n \quad \text{for all } \tau > t_0.$$

Time Varying Case

Example 6.13

Consider

$$\dot{\mathrm{x}} = \begin{bmatrix} t & -1 & 0 \\ 0 & -1 & t \\ 0 & 0 & t \end{bmatrix} \mathrm{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u.$$

We have $\mathrm{M}_0 = [0\ 1\ 1]'$ and compute

$$\mathrm{M}_1 = -\mathrm{A}(t)\mathrm{M}_0 + \frac{d}{dt}\mathrm{M}_0 = \begin{bmatrix} 1 \\ 0 \\ -t \end{bmatrix}, \quad \mathrm{M}_2 = -\mathrm{A}(t)\mathrm{M}_1 + \frac{d}{dt}\mathrm{M}_1 = \begin{bmatrix} -t \\ t^2 \\ t^2 - 1 \end{bmatrix}.$$

The determinant of

$$\begin{bmatrix} \mathrm{M}_0 & \mathrm{M}_1 & \mathrm{M}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -t \\ 1 & 0 & t^2 \\ 1 & -t & t^2 - 1 \end{bmatrix}$$

is $t^2 + 1$. This implies the system is controllable at every t.

Time Varying Case

Example 6.14

Consider

$$\dot{\mathrm{x}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \mathrm{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \rightarrow \text{controllable by Corollary 6.8.}$$

How about the following time varying case:

$$\dot{\mathrm{x}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \mathrm{x} + \begin{bmatrix} e^{t} \\ e^{2t} \end{bmatrix} u$$

Controllability Grammian is

$$\mathrm{W_c}(t_0, t) = \begin{bmatrix} e^{2t}(t - t_0) & e^{3t}(t - t_0) \\ e^{3t}(t - t_0) & e^{4t}(t - t_0) \end{bmatrix}.$$

Its determinant is zero for all t_0, t, hence uncontrollable.

Time Varying Case

Theorem 6.O11

$\{A(t), C(t)\}$ is controllable at t_o iff

$\exists$ a finite $t_1 > t_0$ such that

$$W_o(t_0, t_1) = \int_{t_0}^{t_1} \Phi'(t_1, \tau) C'(\tau) C(\tau) \Phi(t_1, \tau) d\tau$$

is nonsingular.

Theorem 6.O12

Let $A(t), C(t)$ be $(n-1)$ times continuously differentiable.

$\{A(t), C(t)\}$ is observable at t_o if

there exists a finite $t_1 > t_0$ such that

$$\rho \begin{bmatrix} \mathrm{N}_0(t_1) \\ \mathrm{N}_1(t_1) \\ \cdots \\ \mathrm{N}_{n-1}(t_1) \end{bmatrix} = n, \quad \text{where } \mathrm{N}_0(t) = C(t)$$

$$\mathrm{N}_{m+1}(t) = \mathrm{N}_m(t) A(t) + \frac{d}{dt} \mathrm{N}_m(t).$$

HW 6-3

Problem 6.21 in Text P. 183