

### 3. Flapping Dynamics of a Rotor Blade

#### 3.1 Forward Flight Case

① Consider the rotor blade shown on Figure 1 of the next page.

It consists of a hinge offset, denoted by  $e$ , and a torsional spring having spring  $K_{10} (= I_1 \omega_{10}^2)$ , which can be also represented by an appropriate fundamental frequency  $\omega_{10}$ , combined with flapwise moment of inertia of the blade about the hinge  $I_1$ . Such a model can be used as an approximate representation of hingeless blades, furthermore when  $K_{10} = 0$ , it represents an articulated blade. Assuming that the flapping motion is restricted to small angles ( $\beta < 6^\circ$ ) one can assume

$$\sin \beta \cong \beta ; \quad \cos \beta \cong 1.0.$$

The position vector of a masspoint on the blade, with the blade mass assumed to be concentrated at the cross sectional sectional center of mass.

$$\begin{aligned} \vec{r}_p &= \vec{i}x + \vec{k}z & (1) \\ x &= e + r \cos \beta \cong e + r \end{aligned}$$

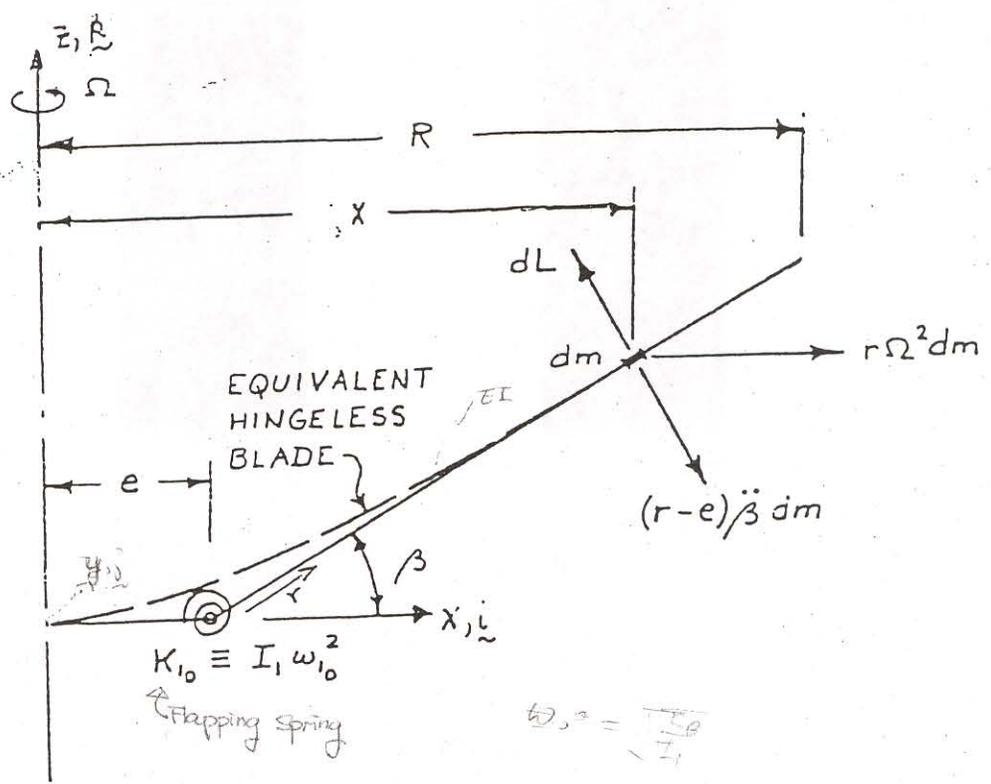


FIG. 1

STIFFNESS (EI)를 조정해서  
 EQUIVALENT MODEL을 만들수있다!

from the geometry

$$z = r \sin \beta \approx (x-e) \sin \beta \approx (x-e) \beta \quad (2)$$

before deflection  $\vec{r}_P = \vec{i}x + \vec{k}z = \vec{i}(e+r\cos\beta)$

$$\vec{r}_P = \vec{i}x + (x-e)\beta \vec{k} \quad (3)$$

from elementary mechanics

$$\vec{a} = \ddot{\vec{r}}_P + 2\vec{\omega} \times \dot{\vec{r}}_P + \vec{\omega} \times (\vec{\omega} \times \vec{r}_P) + \dot{\vec{\omega}} \times \vec{r}_P \quad (4)$$

where  $\vec{a}$  is the acceleration vector and  $\vec{\omega} = \vec{k}\Omega$ ,

and it is usually assumed that the speed of rotation  $\Omega$  is

a constant, i.e.  $\Omega = \text{const.}$  Denote by  $\vec{r}_P$  the position vectors of a mass point in the rotating reference frame, then

$$\dot{\vec{r}}_P = \vec{k} \dot{\beta} (x-e)$$

$$\ddot{\vec{r}}_P = \vec{k} (x-e) \ddot{\beta}$$

$$\vec{\omega} \times \dot{\vec{r}}_P = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \Omega \\ 0 & 0 & \dot{\beta}(x-e) \end{vmatrix} = 0$$

i.e. the small angle assumption leads to the neglect of the

Coriolis term. — Show!

$$\vec{\omega} \times \vec{r}_P = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \Omega \\ x & 0 & (x-e)\beta \end{vmatrix} = +\vec{j} \times \Omega$$

(3)

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}_P) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \Omega \\ 0 & x\Omega & 0 \end{vmatrix} = -\hat{i} x \Omega^2$$

and  $\dot{\underline{\omega}} = 0$ , thus

$$\underline{a} = \underline{k}(x-e)\ddot{\beta} - \hat{i} x \Omega^2 = -\hat{i} x \Omega^2 + \underline{k}(x-e)\ddot{\beta} \quad (5)$$

The distributed inertia force along the blade is given by

D'Alembert principle, using  $m$  as mass per unit length of the bl.

$$\underline{P} = P_x \hat{i} + P_y \hat{j} + P_z \hat{k} = -m \underline{a} = -m [-\hat{i} x \Omega^2 + \underline{k}(x-e)\ddot{\beta}]$$

$$\underline{P} = m x \Omega^2 \hat{i} - (x-e)\ddot{\beta} m \hat{k} \quad (6)$$

Thus  $P_x$  and  $P_z$  are given by Eq (6) and  $P_y = 0$

Consider next moment equilibrium of the blade about

its flapping hinge, taking clockwise moments as positive

$$dM_I = \underline{r}_0 \times \underline{P} \quad (6)$$

where  $\underline{r}_0$  = position vector from hinge line, and

$$M_I = \int_e^R (\underline{r}_0 \times \underline{P}) dx \quad (7)$$

$M_I$  is the moment due to inertia loads, and

$$\underline{r}_0 = (x-e) \hat{i} + (x-e)\beta \hat{k} \quad (8)$$

$$\underline{r}_0 \times \underline{p} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ (x-e) & 0 & (x-e)\beta \\ m x \Omega^2 & 0 & -m(x-e)\beta'' \end{vmatrix} =$$

$$= -\underline{j} \left[ -m(x-e)^2 \beta'' - m x \Omega^2 (x-e) \beta \right]$$

$$\underline{M}_I = \underline{j} \int_e^R m \left[ (x-e)^2 \beta'' + x(x-e) \Omega^2 \beta \right] dx \quad (9)$$

The restoring moment of the spring is

$$\underline{M}_E = \underline{j} k_{10} \beta = \underline{j} I \omega_{10}^2 \beta \quad (10)$$

Finally consider the moment due to the aerodynamic forces acting on the blade. Assume that the lift per unit span on the blade is denoted by  $L$  which acts perpendicular to the blade as shown, due to the small angle assumption the aerodynamic moment is simply given by

$$\underline{M}_A = -\underline{j} \int_e^R (x-e) L dx \quad (11)$$

For moment equilibrium about the hinge one has

$$\underline{M}_I + \underline{M}_E + \underline{M}_A = 0 \quad (12)$$

Combining Eqs (9), (10), (11) and (12) one

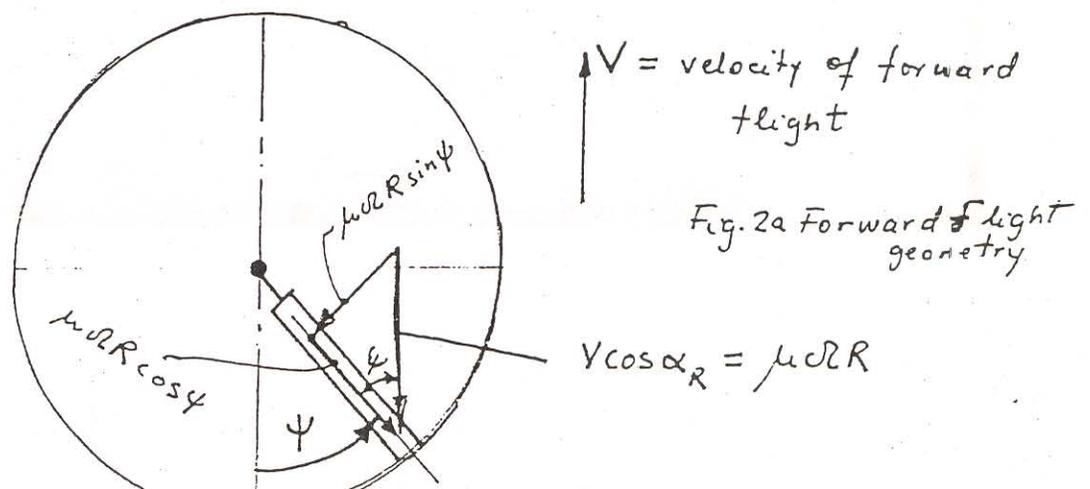
finally obtains

$$\int_e^R m[(x-e)^2 \ddot{\beta} + x(x-e) \Omega^2 \beta] dx + I_1 \omega_{10}^2 \beta = \int_e^R L(x-e) dx \quad (13)$$

In order to proceed with the analysis, certain assumptions have to be introduced. First let us assume that the helicopter rotor is in forward flight. From the general, introductory comments which have been made, it is clear that for forward flight a collective pitch setting  $\theta_0$  on the blade is required, furthermore in forward flight it has been mentioned that cyclic pitch is required to trim the pitching and rolling moments acting on the rotor, thus

$$\theta(\psi) = \theta_0 + \theta_{1c} \cos \psi + \theta_{1s} \sin \psi \quad (14)$$

where  $\psi = \Omega t$  is the azimuth angle measured from the straight aft position as shown in Fig. 2a below.



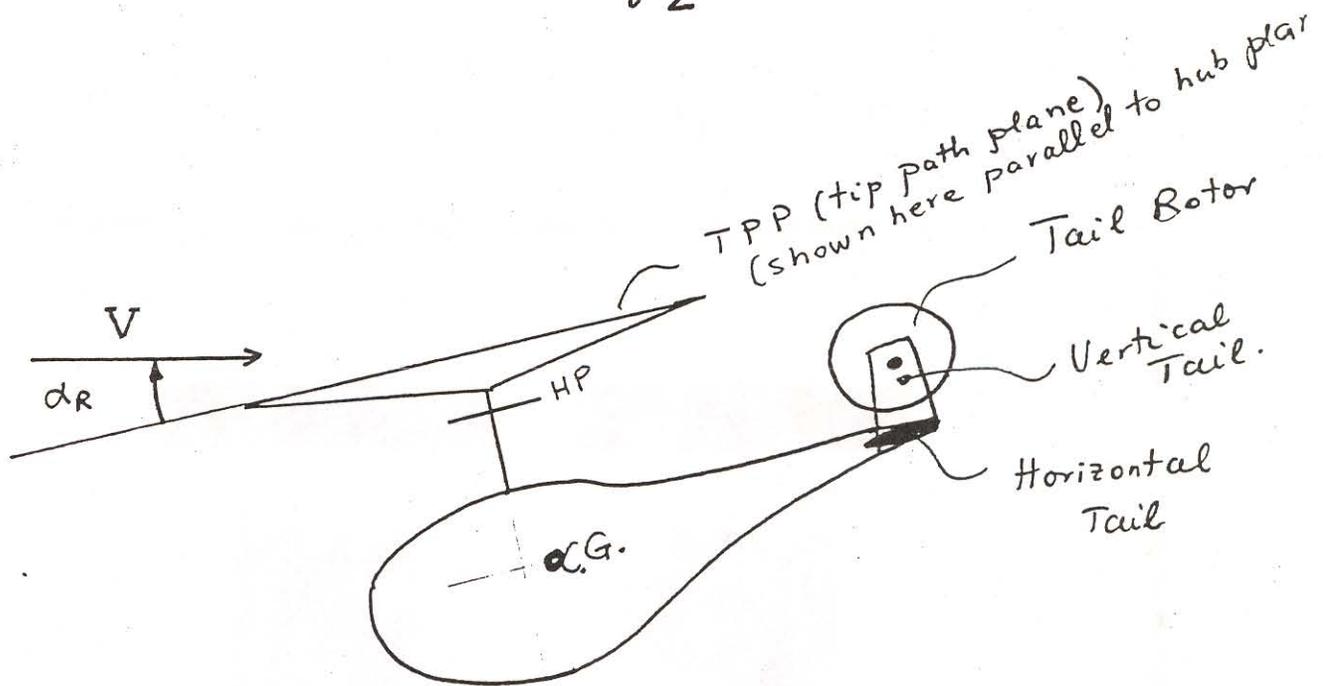


Fig 2b. Geometry for forward flight continued.

The advance ratio is defined as

$$\mu = \frac{V \cos \alpha_R}{\Omega R} \quad (15)$$

where  $V$  is the velocity of forward flight,  $R$  is the rotor radius and  $\alpha_R$  is the rotor angle of attack shown in Fig 2b.

As a consequence of the pitch program represented by Eq (14), the time history of the flapping angle can be assumed to have a similar mathematical form thus

$$\beta(\psi) = \beta_0 + \beta_{1c} \cos \psi + \beta_{1s} \sin \psi \quad (16)$$

$$\dot{\beta} = -\beta_{1c} \Omega \sin \psi + \beta_{1s} \Omega \cos \psi \quad (17)$$

$$\ddot{\beta} = -\beta_{1c} \Omega^2 \cos \psi - \beta_{1s} \Omega^2 \sin \psi \quad (18)$$

Next consider the aerodynamic loads, acting on a typical

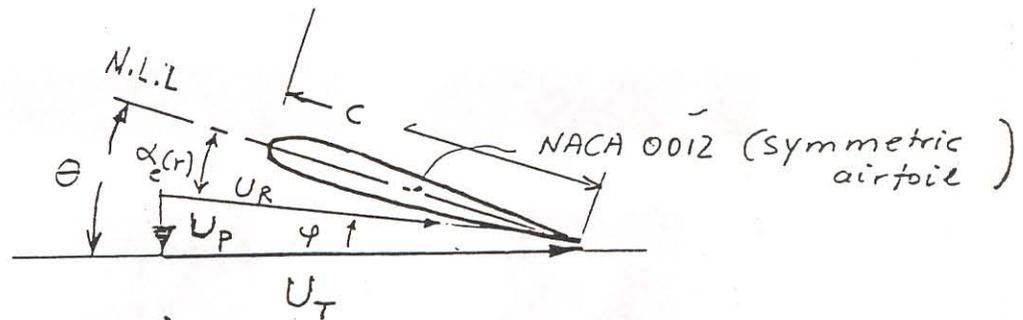


Fig.3 Geometry for blade load calculation, based on blade element theory

airfoil as shown in Fig.3 above, having a pitch setting  $\theta$ .

It is common practice to take components of the velocity in the plane of rotation, denoted by  $U_T$  and perpendicular to the plane of rotation denoted by  $U_P$ . Most helicopter rotors operate at small inflow angles, i.e.

$$\phi = \tan^{-1} \left( \frac{U_P}{U_T} \right) \approx \frac{U_P}{U_T} \quad (19)$$

Therefore the effective angle of attack

$$\alpha_e = \theta - \phi = \theta - \frac{U_P}{U_T} \quad (20)$$

from blade element theory

$$L = \frac{1}{2} \rho a c U_R^2 \alpha_e \quad (21)$$

Where  $\rho$  is the air density,  $a$  - lift curve slope,  $c$  - chord  
 $U_R$  is the resultant velocity

$$U_R^2 = U_P^2 + U_T^2 \quad (22)$$

Combining Eqs (20), (21) and (22)

$$\begin{aligned} L &= \frac{1}{2} \rho a c U_R^2 \alpha_c = \frac{1}{2} \rho a c (U_P^2 + U_T^2) \left( \theta - \frac{U_P}{U_T} \right) \\ &= \frac{1}{2} \rho a c U_T^2 \left( 1 + \frac{U_P^2}{U_T^2} \right) \left( \theta - \frac{U_P}{U_T} \right) \end{aligned} \quad (23)$$

Since we have assumed small inflow angles (Eq. 19)

$$1 + \frac{U_P^2}{U_T^2} \approx 1$$

Thus Eq (23) reduces to

$$L = \frac{1}{2} \rho a c (\theta U_T^2 - U_P U_T) \quad (24)$$

Next determine the velocity components, parallel and perpendicular to the plane of rotation (also known as the hub-plane). Recall that  $\beta$ , the flapping angle was assumed to be small, and use Fig 2a which contains the illustration of the effects due to forward flight  $\mu$ .

76

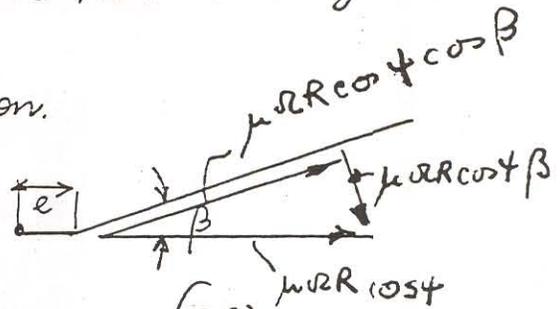
From the geometry

$$u_T = \mu \Omega R \sin \psi + \Omega x \quad (25)$$

where the first term in Eq (25) is due to forward flight and the second term is due to rotation.

Similarly

$$u_p = \lambda \Omega R + (x-e)\dot{\beta} + \mu \Omega R \beta \cos \psi \quad (26)$$



where the first term is due to inflow ratio  $\lambda$ , the second term is due to the flapping motion  $\beta(\psi)$  and the last term of Eq (26) is due to the component of the advance ratio along the blade (see Fig 2a). The inflow ratio  $\lambda$  is defined as

$$\lambda = \frac{v_i + V \sin \alpha}{\Omega R} \quad (27)$$

where  $v_i$  is the induced velocity over the rotor disc, and perpendicular to the disc. (positive down)

Combining Eqs (24), (25), (26), (16)-(18) and (14) the lift per unit span of the blade is given by

77

86 Recall  
 $\sin^2 \psi = \frac{1}{2} - \frac{1}{2} \cos 2\psi$   
 $\cos^2 \psi = \frac{1}{2} + \frac{1}{2} \cos 2\psi$   
 $2 \sin \psi \cos \psi = \sin 2\psi$

$$L = \frac{1}{2} g a c \left\{ (\theta_0 + \theta_{1c} \cos \psi + \theta_{1s} \sin \psi) (\mu^2 \omega^2 R^2 \sin^2 \psi + \omega^2 x^2 + 2 \omega^2 x \mu R \sin \psi) - (\mu \omega R \sin \psi + \omega x) [\lambda \omega R + (x-e) (-\beta_{1c} \omega \sin \psi + \beta_{1s} \omega \cos \psi) + \mu \omega R (\beta_0 + \beta_{1c} \cos \psi + \beta_{1s} \sin \psi) \cos \psi] \right\}$$

Also during the manipulation of Eq (13) it is convenient (28)

to use the identity

$$(x-e)^2 + e(x-e) \equiv x(x-e) \quad (29)$$

Combining Eqs (13), (28), (29), (16) and (18) one can obtain the following relation [where one neglects harmonics above the first]

$$\begin{aligned} \beta_0 (1+G) + \beta_{1c} G \cos \psi + \beta_{1s} G \sin \psi &= [\theta_0 (B + \frac{1}{2} \mu^2 E) - \lambda C \\ &+ C \mu \theta_{1s} - \frac{1}{2} \mu D \beta_{1c}] + [\theta_{1c} (B + \frac{1}{4} \mu^2 E) - \beta_{1s} (A + \frac{1}{4} \mu^2 E) - \mu \beta_0 C] \cos \psi \\ &+ [\theta_{1s} (B + \frac{3}{4} \mu^2 E) + \beta_{1c} (A - \frac{1}{4} \mu^2 E) + 2 \mu \theta_0 C - \mu \lambda E] \sin \psi \quad (30) \end{aligned}$$

where

$$\begin{aligned} A &= \frac{\chi}{2} \left( \frac{1}{4} - \frac{2}{3} \xi + \frac{\xi^2}{2} - \frac{\xi^4}{12} \right) \\ B &= \frac{\chi}{2} \left( \frac{1}{4} - \frac{\xi}{3} + \frac{\xi^4}{12} \right) \\ C &= \frac{\chi}{2} \left( \frac{1}{3} - \frac{\xi}{2} + \frac{\xi^3}{6} \right) \\ D &= \frac{\chi}{2} \left( \frac{\xi}{2} - \xi^2 + \frac{\xi^3}{2} \right) \end{aligned}$$

(31)

AER LOAD  
 SA, B, C, D

$$\left. \begin{aligned} E &= \frac{\gamma}{2} \left( \frac{1}{2} - \xi + \frac{\xi^2}{2} \right) \\ G &= \frac{\xi R^3}{I_1} \int_{\xi}^1 (\bar{x} - \xi) m(\bar{x}) d\bar{x} + \psi_{10}^2 \end{aligned} \right\} (31)$$

$G = \psi_{10}^2 + \text{restoring moment due to centrifugal force}$   
 where  $\xi = \frac{e}{R}$ ;  $\bar{x} = \frac{x}{R}$ ;  $\psi_{10} = \frac{\omega_{10}}{\Omega} = \text{nondimensional freq}$

$\xi = \left(\frac{x}{R}\right)^2$   $I_1 = R^3 \int_{\xi}^1 (\bar{x} - \xi)^2 m(\bar{x}) d\bar{x} = \text{flapping inertia of the blade}$

$\gamma = \frac{\rho a c R^4}{I_1} = \text{Lock number } \checkmark$

Equating the constant and the harmonic terms in Eq (30)

one has

LINKDOWN

$$\beta_0 (1+G) = \theta_0 (B + \frac{1}{2} \mu^2 E) - \lambda C + C \mu \theta_{1s} - \frac{1}{2} \mu D \beta_{1c} \quad (32)$$

$$G \beta_{1c} = \theta_{1c} (B + \frac{1}{4} \mu^2 E) - \beta_{1s} (A + \frac{1}{4} \mu^2 E) - \mu \beta_0 C \quad (33)$$

$$G \beta_{1s} = \theta_{1s} (B + \frac{3}{4} \mu^2 E) + \beta_{1c} (A - \frac{1}{4} \mu^2 E) + 2 \mu \theta_0 C - \mu \lambda E \quad (34)$$

A simultaneous solution of these three equations yields the rotor blade flapping displacements  $\beta_0, \beta_{1c}, \beta_{1s}$  in terms of the control displacements  $\theta_0, \theta_{1c}, \theta_{1s}$  for the flight condition described by the flow quantities  $\mu$  and  $\lambda$ , and the rotor geometry, mass, and elastic characteristics

described by  $\gamma$ ,  $\xi$ ,  $R$ ,  $m(x)$ ,  $I_1$  and  $V_{10}$ .

In order to have some physical feel for typical values of the parameters used in the flapping analysis of a typical rotor blade the following information is provided

$$\frac{1}{14} < \frac{c}{R} < \frac{1}{10} \quad \xi \pi < a < 2\pi$$

$$0 < \frac{e}{R} < 0.20 \quad 10 < R < 60 \text{ ft}$$

$$0 < V_{10} < 0.15 \quad \rightarrow W_{FIR} \approx 1 + 0.15 = 1.15$$

KEEP IN MIND  $\downarrow$

$$2 < \gamma < 10$$

$\gamma \uparrow \rightarrow$  Aerodynamic Load  $\uparrow$  or Inertial Load  $\downarrow$

Furthermore, a few additional comments on these equations should be made. Eqs (32)-(34) are based on the assumption of uniform inflow over the rotor disc, i.e.

$$\lambda = \text{const.}$$

For the case of forward flight this inflow can be determined from (See Refs 1 or 2 for the derivation of this Eq)

$$\lambda = \mu \tan \alpha_R + \frac{C_T^2}{2(\mu^2 + \lambda^2)^{1/2}} \quad (35)$$

where  $C_T$  is the thrust coefficient which is closely

related to the weight of the helicopter. The assumption of constant inflow velocity the disc is not a very realistic one, a more reasonable assumption would be

$$\lambda(\bar{x}, \psi) = \lambda_0 + \lambda_{1c} \frac{x}{R} \cos \psi + \lambda_{1s} \frac{x}{R} \sin \psi \quad (36) \checkmark$$

Obviously introducing this assumption into the previous problem would lead to substantial complications even for such a simple problem.

### 3.2 Special Case for Hovering Flight

For hovering flight  $\mu = 0$ , and equations (32) - (34) hinge offset, spring - present reduce to

$$\beta_0 (1+G) = B\theta_0 - C\lambda \quad (37)$$

$$G\beta_{1c} = \theta_{1c} B - A\beta_{1s} \quad (38)$$

$$G\beta_{1s} = \theta_{1s} B + A\beta_{1c} \quad (39)$$

from which 
$$\beta_0 = \frac{B\theta_0 - C\lambda}{1+G} \quad (37a)$$

And solving Eqs (38) and (39) as a pair yields

$$\beta_{1c} = \frac{B(G\theta_{1c} - A\theta_{1s})}{G^2 + A^2} \quad (40)$$

$$\beta_{1s} = \frac{B(G\theta_{1s} + A\theta_{1c})}{G^2 + A^2} \quad (41)$$

• NO HINGE OFFSET  $\xi = 0$ , NO SPRING  $\nu_{10} = 0$

Consider a blade with no hinge offset and without elastic restraint, then ( $\xi = \nu_{10} = 0$ ) and

$$A = \gamma/8, \quad B = \gamma/8, \quad G = 0 \quad \text{i.e. } A = B = \gamma/8$$

$$\beta_{1c} = -\frac{A^2\theta_{1s}}{A^2} = -\theta_{1s} \quad (42) \quad \text{Equivalent of flapping and feathering}$$

$$\beta_{1s} = \frac{A^2\theta_{1c}}{A^2} = \theta_{1c} \quad (43)$$

Which means that blade flapping amplitude is numerically equal to the cyclic pitch amplitude, and flapping lags control by ninety degrees. As hinge offset and/or elastic restraint is increased the amplitude ratio between flap and pitch increases and the phase angle decreases.

### 3.3 Special Case for Forward Flight

Consider again the case of zero hinge offset and

absence of elastic restraint (i.e. articulated rotor), using NO SPRING

Eqs (31) one has, and  $\mu \neq 0$

$$A = B = \gamma/8, \quad G = 0, \quad C = \gamma/6$$

$D = 0, \quad E = \gamma/4$  and Eqs (32) - (34) simplify to

$$\beta_0 = \theta_0 \left( \frac{\gamma}{8} + \frac{\mu^2}{8} \gamma \right) - \frac{\lambda}{6} \gamma + \gamma \mu \frac{\theta_{1s}}{6} \quad \beta_0 \sim \mu, \theta_{1s}, \lambda$$

$$= \frac{\gamma}{2} \left[ \frac{\theta_0}{4} (1 + \mu^2) + \mu \frac{\theta_{1s}}{3} - \frac{\lambda}{3} \right] \quad (44)$$

← Reference plane was hub plane

$$0 = \theta_{1c} \left( \frac{\gamma}{8} + \frac{1}{4} \mu^2 \frac{\gamma}{4} \right) - \beta_{1s} \left( \frac{\gamma}{8} + \frac{1}{4} \mu^2 \frac{\gamma}{4} \right) - \mu \beta_0 \frac{\gamma}{6}$$

$$\beta_{1s} - \theta_{1c} = -\mu \beta_0 \frac{\gamma}{6} \frac{1}{\frac{\gamma}{8} (1 + \frac{1}{2} \mu^2)} = -\frac{4/3 \mu \beta_0}{1 + \frac{1}{2} \mu^2} \quad (45)$$

$$0 = \theta_{1s} \left( \frac{\gamma}{8} + \frac{3}{4} \mu^2 \frac{\gamma}{4} \right) + \beta_{1c} \left( \frac{\gamma}{8} - \frac{1}{4} \mu^2 \frac{\gamma}{4} \right) + 2\mu \theta_0 \frac{\gamma}{6} - \mu \lambda \frac{\gamma}{4}$$

$$\beta_{1c} \frac{\gamma}{8} \left( 1 - \frac{1}{2} \mu^2 \right) + \theta_{1s} \frac{\gamma}{8} \left( 1 + \frac{3}{2} \mu^2 \right) = -\mu \theta_0 \frac{\gamma}{3} + \mu \lambda \frac{\gamma}{4}$$

$$\beta_{1c} \left( 1 - \frac{1}{2} \mu^2 \right) + \theta_{1s} \left( 1 + \frac{3}{2} \mu^2 \right) = -\frac{8}{3} \mu \theta_0 + 2\mu \lambda \quad (46)$$

A more convenient form of Eq (46) is obtained if  $\lambda$

is referred to the tip path plane rather than the hub plane,

↳ transform Eqns to TPP

using the relation

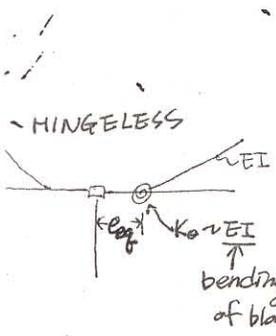
$$\lambda \approx \lambda_{TPP} - \mu \beta_{1c} \quad (47)$$

It can be shown that for this case Eq (46) becomes

$$\beta_{1c} + \theta_{1s} = \frac{-\frac{8}{3} \mu \left( \theta_0 - \frac{3}{4} \lambda_{TPP} \right)}{1 + \frac{3}{2} \mu^2} \quad (48)$$

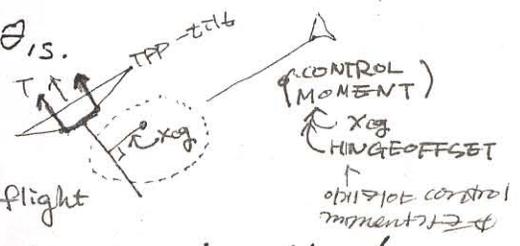
Note that for a given advance ratio  $\mu$ , the

combinations  $(\beta_{1c} + \theta_{1s})$  and  $(\beta_{1s} - \theta_{1c})$  are prescribed. This means that the rotor orientation  $\beta_{1c}$  and  $\beta_{1s}$  with respect to the helicopter center of gravity required for trimmed flight at the given value of  $\mu$  can be achieved by appropriate control of  $\theta_{1c}$  and  $\theta_{1s}$ .



3.4 Helicopter Control Moments

To control a helicopter in flight



Control moments acting on the helicopter are a result of

the following physical effects:

- (1) Thrust vector tilt with respect to the shaft axis
- (2) Hub moments due to off set hinges and or elastic restraint.

The effective tilt of the thrust vector is given by  $\beta_{1c}$  and  $\beta_{1s}$  as determined by the equations which have been derived in the previous sections.

It can be shown that the hub moment can be approximated

by the expression  $\xi = \frac{e}{R}, \bar{x} = \frac{rc}{R}$

$$M_{HUB} \approx \frac{bR^3}{2} (\omega_1^2 - \Omega^2) \int_{\xi}^1 m(\bar{x}) \bar{x}(\bar{x} - \xi) d\bar{x} (\beta_{1c} \cos \psi + \beta_{1s} \sin \psi) \quad (49)$$

\* of blades

HINGELESS가  
타원형 control  
반응한다

steady state!

$\beta_0, \beta_{1c}, \beta_{1s} \rightarrow$  ASSUMED "CONSTANT"  $\rightarrow$   $\rightarrow$  unsteady  $\rightarrow \beta_0(\psi), \beta_{1c}(\psi), \beta_{1s}(\psi)$

where  $b =$  number of blades

$\omega_1 =$  rotating fundamental natural frequency of the blade for bending out of the plane of rotation (or flapwise frequency)  $= \sqrt{1+G} \Omega$

Note that for  $\omega_1 = \Omega$ , i.e. no hinge offset or elastic restraint, the hub moment is zero, and control moments are achieved entirely by thrust vector tilt. Increasing hinge offset and/or elastic restraint increases  $\omega_1$ , hence the hub moment increases, with a corresponding increase in control power.

### 3.5 Application to the Hingeless Rotor

Practical hingeless rotor blades are very flexible and their fundamental mode flapwise bending frequency is only slightly greater than rotor rotational frequency, i.e.

$1.0 < \frac{\omega_1}{\Omega} < 1.15$ . Therefore the response of the hingeless

blade to cyclic control input at the rotor rotational

frequency is almost entirely in the fundamental mode. It

can be shown that the fundamental hingeless blade

25

mode shape can be closely approximated by that of a blade with an offset flapping hinge and elastic restraint, as shown in Fig. 1. Then the analytical results of the preceding sections become directly applicable to hingeless rotors.

For a typical hingeless rotor blade

$$\xi = 0.20$$

$$G = 0.20$$

$$\omega_1/\Omega = \sqrt{1+G} \approx 1.1$$

$$\frac{\omega_2}{\Omega} = 1.5 \approx 2.0$$

$$\frac{\omega_3}{\Omega} = 1.4 \approx 2.0$$

Finally it should be noted that the most convenient reference plane for dealing with the hingeless rotor is the hub plane. Recent research has also indicated that for dynamics, stability and control of hingeless rotored helicopters the second bending mode can also have noticeable influence, in certain flight conditions. Clearly for such cases the simple model, shown in Fig 1, is inadequate. Furthermore note that for hingeless rotors the lag mode, or inplane dynamics are also very important.

References

1. Johnson, W., Helicopter Theory, Princeton University Press, 1980
2. Geosow, A. and Myers, G.C., Aerodynamics of the Helicopter, Frederick Ungar Publishing Co., 1967 (originally 1952)
3. Bramwell, A.R.S., Helicopter Dynamics, 1976

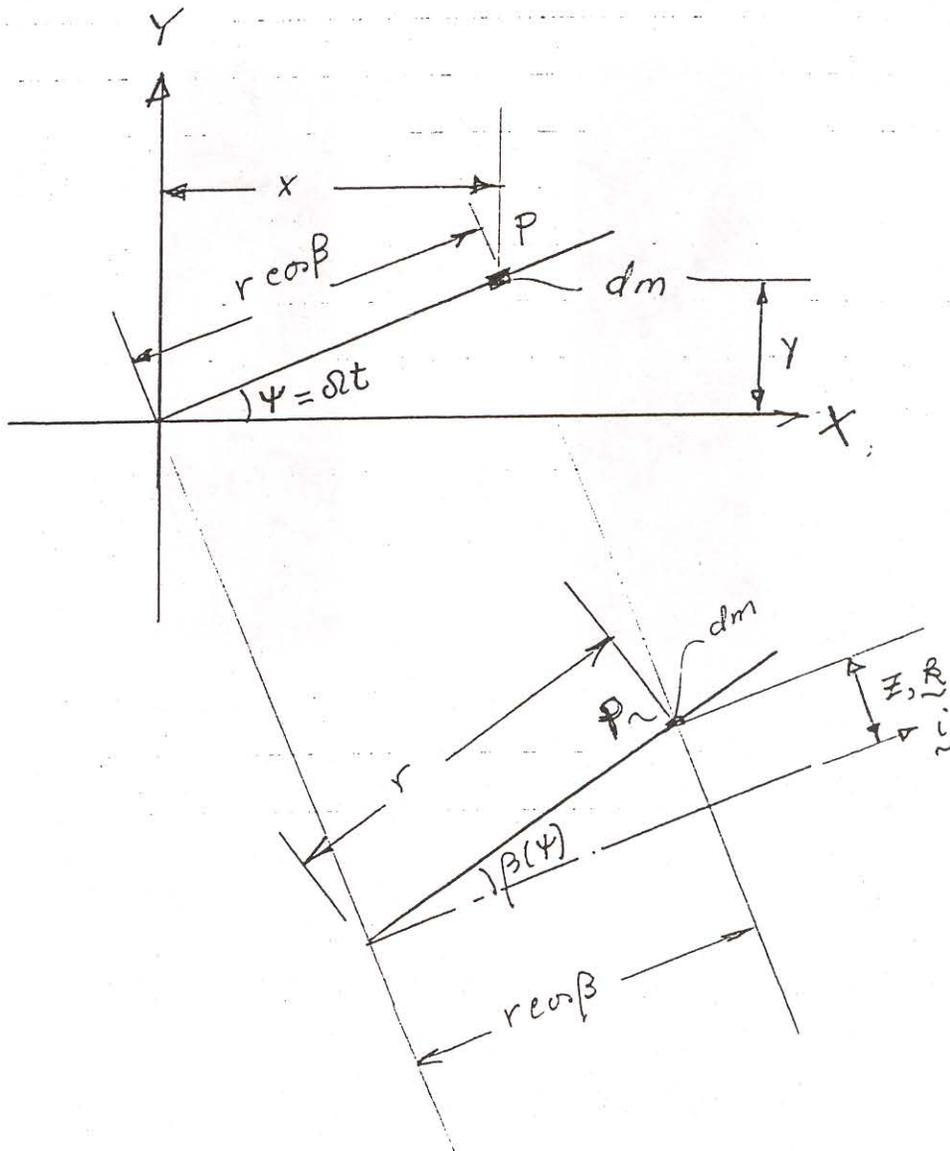
96

Assumption - Blade element theory is good representation for the unsteady aerodynamic loading on the blade adequate

## Blade Flapping Dynamics (continued) UNSTEADY FLIGHT

Consider the case of blade flapping dynamics in accelerated flight and use an inertial axis system; however neglect linear acceleration of the hub

### 1. Inertia Moments



The absolute acceleration of a point P on the blade associated with a translating and rotating coordinate system, with respect to an inertial coordinate system is given by

$$\underset{\sim}{a}_{PK} = \underset{\sim}{\ddot{R}}_0 + \underset{\sim}{\ddot{r}}_{PK} + 2\underset{\sim}{\omega} \times \underset{\sim}{\dot{r}}_{PK} + \underset{\sim}{\dot{\omega}} \times \underset{\sim}{r}_{PK} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}_{PK}) \quad (1)$$

where  $\vec{R}_0$  is the position vector of the origin of the moving coordinate system with respect to the inertial coordinate system. The quantity  $\vec{r}_{Pk}$  is the position vector of the point P in the  $k^{\text{th}}$  blade from the origin of the moving reference system and  $\vec{\omega}_k$  is the angular velocity of the moving coordinate system.

For our case assume  $\vec{R}_0 = 0$  and  $\vec{\omega}_k = \vec{k} \Omega$ , then

$$\vec{r}_{Pk} = \vec{i} r \cos \beta + \vec{k} r \sin \beta, \quad (2)$$

$$\vec{\omega}_k = \Omega \vec{k}$$

$$\vec{a}_{Pk} = \ddot{\vec{r}}_{Pk} + 2\vec{\omega}_k \times \dot{\vec{r}}_{Pk} + \vec{\omega}_k \times (\vec{\omega}_k \times \vec{r}_{Pk}) \quad (3)$$

$$\dot{\vec{r}}_{Pk} = -\vec{i} r \sin \beta \dot{\beta} + \vec{k} r \cos \beta \dot{\beta} \quad (4)$$

$$\ddot{\vec{r}}_{Pk} = -\vec{i} r \cos \beta \dot{\beta}^2 - \vec{i} r \sin \beta \ddot{\beta} - \vec{k} r \sin \beta \dot{\beta}^2 + \vec{k} r \cos \beta \ddot{\beta}$$

$$= -\vec{i} (r \cos \beta \dot{\beta}^2 + r \sin \beta \ddot{\beta}) + \vec{k} r (-\sin \beta \dot{\beta}^2 + \cos \beta \ddot{\beta}) \quad (5)$$

Coriolis  
Effect

$$2\vec{\omega}_k \times \dot{\vec{r}}_{Pk} = 2 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \Omega \\ -r \sin \beta \dot{\beta} & 0 & r \cos \beta \dot{\beta} \end{vmatrix} =$$

$$= 2 \left[ -\vec{j} (r \sin \beta \dot{\beta} \Omega) \right] = -2\vec{j} r \sin \beta \dot{\beta} \Omega \quad (6)$$

$$\vec{\omega}_k \times \vec{r}_{Pk} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \Omega \\ r \cos \beta & 0 & r \sin \beta \end{vmatrix} = -\vec{j} (-r \cos \beta \Omega)$$

$$= \vec{j} r \Omega \cos \beta \quad (7)$$

$$\begin{aligned} \vec{\omega}_K \times \vec{\omega}_R \times \vec{r}_{PK} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \Omega \\ 0 & r \Omega \cos \beta & 0 \end{vmatrix} \\ &= -\hat{i} \Omega^2 r \cos \beta \quad (8) \end{aligned}$$

$$\begin{aligned} \vec{a}_{PK} &= -\hat{i} (r \cos \beta \ddot{\beta}^2 + r \sin \beta \ddot{\beta}) + \hat{k} r (-\sin \beta \dot{\beta}^2 + \cos \beta \ddot{\beta}) \\ &\quad - 2 \hat{j} r \sin \beta \dot{\beta} \Omega - \hat{i} \Omega^2 r \cos \beta \\ &= \hat{i} (-r \cos \beta \dot{\beta}^2 - r \sin \beta \ddot{\beta} - \Omega^2 r \cos \beta) - 2 \hat{j} r \sin \beta \dot{\beta} \Omega \\ &\quad + \hat{k} r (-\sin \beta \dot{\beta}^2 + \cos \beta \ddot{\beta}) \quad (9) \end{aligned}$$

$$\begin{aligned} d\vec{p}_{KI} &= -a_{PK} dm = dm \left[ \hat{i} (r \cos \beta \dot{\beta}^2 + r \sin \beta \ddot{\beta} + \Omega^2 r \cos \beta) \right. \\ &\quad \left. + 2 \hat{j} r \sin \beta \dot{\beta} \Omega + \hat{k} r (\sin \beta \dot{\beta}^2 - \cos \beta \ddot{\beta}) \right] \quad (10) \end{aligned}$$

$$dM_I = \vec{r}_{PK} \times d\vec{p}_{KI} =$$

$$\begin{aligned} &\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r \cos \beta & 0 & r \sin \beta \\ dm (r \cos \beta \dot{\beta}^2 + r \sin \beta \ddot{\beta} + \Omega^2 r \cos \beta) & 2 r \sin \beta \dot{\beta} \Omega dm & dm r (\sin \beta \dot{\beta}^2 - \cos \beta \ddot{\beta}) \end{vmatrix} \\ &= \hat{i} (-2 r^2 \sin^2 \beta \dot{\beta} \Omega) dm \\ &\quad - \hat{j} dm \left[ r^2 \sin \beta \cos \beta \dot{\beta}^2 - r^2 \cos^2 \beta \ddot{\beta} - r^2 \cos \beta \sin \beta \dot{\beta}^2 - r^2 \sin^2 \beta \ddot{\beta} - \Omega^2 r^2 \sin \beta \cos \beta \right] \\ &\quad + \hat{k} dm \left[ r^2 \sin \beta \cos \beta \dot{\beta}^2 - \cos^2 \beta r^2 \ddot{\beta} \right] \quad (11) \end{aligned}$$

The moment about the flapping hinge is the  $\underline{z}$ -component, thus

$$M_I = - \int_0^R \left[ - r^2 (\cos^2 \beta + \sin^2 \beta) \ddot{\beta} - \Omega^2 r^2 \sin \beta \cos \beta \right] dm$$

$$= + \int_0^R (\ddot{\beta} + \Omega^2 \sin \beta \cos \beta) r^2 dm = I_1 (\ddot{\beta} + \Omega^2 \sin \beta \cos \beta) \quad (1)$$

where  $I_1 = \int_0^R r^2 dm$  flapping (inertia)

assume  $\beta$ -small  $\sin \beta = \beta$ ;  $\cos \beta = 1$

$$M_I = I_1 (\ddot{\beta} + \Omega^2 \beta) \quad (13) \quad \begin{array}{l} \text{Inertia Moment} = \int I_1 (\ddot{\beta} + \Omega^2 \sin \beta \cos \beta) \\ M_I = \text{about the hinge} \end{array} = \int (\ddot{\beta} + \Omega^2 \beta)$$

Flapping motion of the blade can be assumed to be

$$\beta(\psi) = \beta_0 + \beta_{1c} \cos \psi + \beta_{1s} \sin \psi \quad (14) \quad \begin{array}{l} \text{Unsteady Conditions} \\ \beta_0(\psi), \beta_{1c}(\psi), \beta_{1s}(\psi) \end{array}$$

where in general accelerated flight  $\beta_0, \beta_{1c}, \beta_{1s}$  can be assumed to be functions of time, therefore

$$\dot{\beta} = \dot{\beta}_0 + \dot{\beta}_{1c} \cos \psi - \dot{\beta}_{1c} \sin \psi \Omega + \dot{\beta}_{1s} \sin \psi + \dot{\beta}_{1s} \Omega \cos \psi \quad (14a)$$

$$\ddot{\beta} = \ddot{\beta}_0 + \ddot{\beta}_{1c} \cos \psi - \ddot{\beta}_{1c} \Omega \sin \psi - \ddot{\beta}_{1c} \sin \psi \Omega -$$

$$- \dot{\beta}_{1c} \cos \psi \Omega^2 + \ddot{\beta}_{1s} \sin \psi + \dot{\beta}_{1s} \cos \psi \Omega + \dot{\beta}_{1s} \Omega \cos \psi$$

$$+ \dot{\beta}_{1s} \Omega^2 \sin \psi$$

$$\ddot{\beta} = \ddot{\beta}_0 + \ddot{\beta}_{1c} \cos \psi - 2 \dot{\beta}_{1c} \Omega \sin \psi - \dot{\beta}_{1c} \Omega^2 \cos \psi +$$

$$\ddot{\beta}_{1s} \sin \psi + 2 \dot{\beta}_{1s} \Omega \cos \psi - \dot{\beta}_{1s} \Omega^2 \sin \psi \quad (15)$$

Combining Eqs (13), (14) and (15) we can write

$$M_I = M_{I0} + M_{I1c} \cos \psi + M_{I1s} \sin \psi \quad (16)$$

where  $M_{I1c}$ ,  $M_{I1s}$  are the cyclic components of the inertia moment

$$M_I = I_1 \left( \ddot{\beta}_0 + \ddot{\beta}_{1c} \cos \psi - 2\dot{\beta}_{1c} \Omega \sin \psi - \beta_{1c} \Omega^2 \cos \psi + \ddot{\beta}_{1s} \sin \psi + 2\dot{\beta}_{1s} \Omega \cos \psi - \beta_{1s} \Omega^2 \sin \psi \right) + I_1 \Omega^2 (\beta_0 + \beta_{1c} \cos \psi + \beta_{1s} \sin \psi) \quad (17)$$

Thus, from Eqs (16) and (17) one has

$$M_{I0} = I_1 \ddot{\beta}_0 + I_1 \Omega^2 \beta_0 \quad (18) \checkmark$$

$$M_{I1c} = I_1 \left( \ddot{\beta}_{1c} - \beta_{1c} \Omega^2 + 2\dot{\beta}_{1s} \Omega \right) + I_1 \Omega^2 \beta_{1c}$$

$$M_{I1c} = (2 I_1 \dot{\beta}_{1s} \Omega + I_1 \ddot{\beta}_{1c}) \quad (19) \checkmark$$

$$M_{I1s} = I_1 \left( -2\dot{\beta}_{1c} \Omega + \ddot{\beta}_{1s} - \beta_{1s} \Omega^2 + \beta_{1s} \Omega^2 \right)$$

$$M_{I1s} = (-I_1 \Omega \dot{\beta}_{1c} + I_1 \ddot{\beta}_{1s}) \quad (20) \checkmark$$

2. Aerodynamic Moment in Hover 문제를 계산하기 위해 Hover는 Assumed

$$M_A \approx -\frac{1}{2} \rho a c \int_0^R U_T^2 \left( \theta - \frac{U_T}{U_T} \right) r dr \quad (21)$$

where we have used blade element theory to approximate the unsteady aerodynamic loads

Inertial term  
(gives restoring moment)

For hover  $u_T = \Omega r$  (22)

$$u_p = \lambda \Omega R + \dot{z}$$

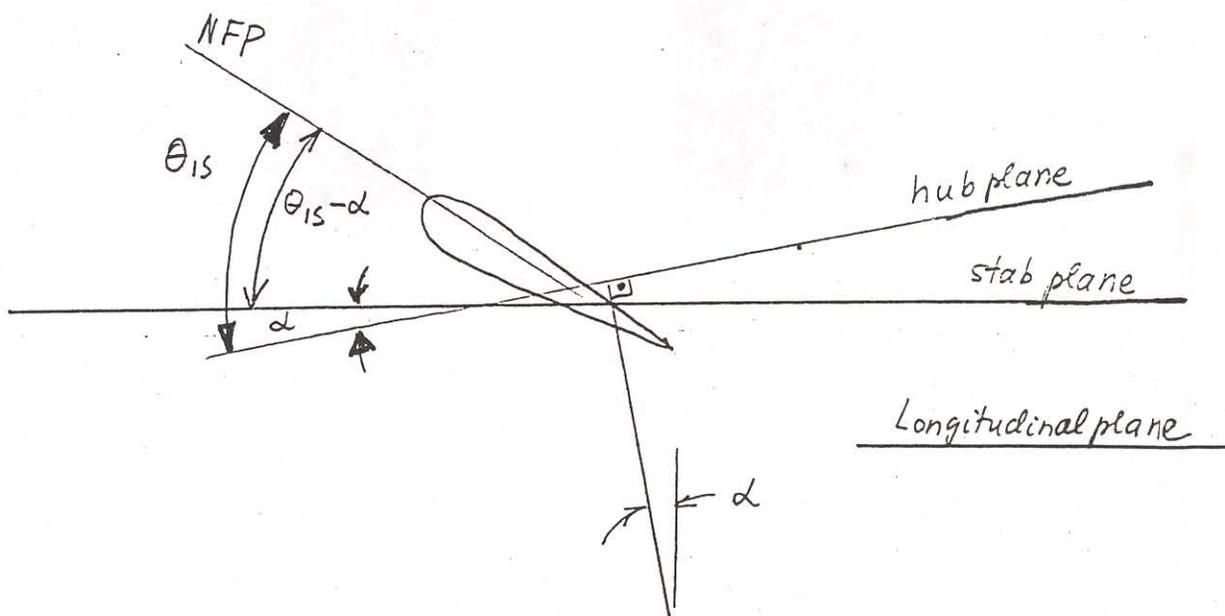
$$M_A = \frac{1}{2} \rho a e \int_0^R (\Omega u_T^2 + u_p v_e) r dr = \frac{1}{2} \rho a e \int_0^R (\Omega R^2 r^2 + \Omega^2 \lambda r^2 R + \Omega^2 r^3) r dr$$

$$z = r \sin \beta; \quad \dot{z} = r \cos \beta \dot{\beta} \cong r \dot{\beta}$$

$$u_p \cong \lambda \Omega R + r \dot{\beta} \quad (23)$$

$$\theta = \theta_0 + \underbrace{(\theta_{1c} - \phi)}_{\text{roll angle}} \cos \psi + \underbrace{(\theta_{1s} - \alpha)}_{\text{tilt in vertical plane}} \sin \psi \quad (24)$$

Note that this special expression for the cyclic pitch is due to the geometry shown below



Same geometry can be depicted in the lateral plane where the roll angle  $\phi$  is introduced, both effects are present in Eq (24)

Again we assume that

$$\beta = \beta_0 + \beta_{1c} \cos \psi + \beta_{1s} \sin \psi \quad (25)$$

and the disc of the rotor can move around in

inertial space such that  $\beta_0(t)$ ,  $\beta_{1c}(t)$ ,  $\beta_{1s}(t)$ , and  $\beta$ ,  $\dot{\beta}$  and  $\ddot{\beta}$  are given by Eqs (14), (14a) and (15). Next  $v_p$  can be written as

$$v_p = \lambda \dot{\Omega} R + r \dot{\beta} = \lambda \dot{\Omega} R + r (\dot{\beta}_0 + \dot{\beta}_{1c} \cos \psi - \dot{\beta}_{1c} \sin \psi \Omega + \dot{\beta}_{1s} \sin \psi + \dot{\beta}_{1s} \Omega \cos \psi) \quad (26)$$

Combining Eqs (21), (22), (24) and (26) yields

$$M_A = -\frac{1}{2} \rho a e \int_0^R \left\{ \Omega^2 r^3 [\theta_0 + (\theta_{1c} - \phi) \cos \psi + (\theta_{1s} - \alpha) \sin \psi] - \Omega r^2 [\lambda \dot{\Omega} R + r (\dot{\beta}_0 + \dot{\beta}_{1c} \cos \psi - \dot{\beta}_{1c} \Omega \sin \psi + \dot{\beta}_{1s} \sin \psi + \dot{\beta}_{1s} \Omega \cos \psi)] \right\} dr \quad (27)$$

which can be also written as

$$\underline{M_A = M_{A0} + M_{A1c} \cos \psi + M_{A1s} \sin \psi} \quad (28)$$

From Eqs (27) and (28) one has

$$M_{A0} = -\frac{1}{2} \rho a e \Omega^2 R^4 \left[ \frac{\theta_0}{4} - \frac{\lambda}{3} - \frac{1}{4} \frac{\dot{\beta}_0}{\Omega} \right] \quad (29)$$

$$M_{A1c} = -\frac{1}{2} \rho a e \Omega^2 R^4 \left[ \frac{1}{4} (\theta_{1c} - \phi) - \frac{1}{4} \frac{\dot{\beta}_{1c}}{\Omega} - \frac{1}{4} \dot{\beta}_{1s} \right] \quad (30)$$

$$M_{A1s} = -\frac{1}{2} \rho a e \Omega^2 R^4 \left[ \frac{1}{4} (\theta_{1s} - \alpha) - \frac{1}{4} \left( \frac{\dot{\beta}_{1s}}{\Omega} + \frac{1}{4} \dot{\beta}_{1c} \right) \right] \quad (31)$$

### 3. Equations of Motion in Hover

- Rotor tip path equilibrium (dynamic) in the stability plane

Summing moments about the flapping hinge

$$\sum M_y = 0 = M_A + M_I = 0 \quad (32) \text{ Moment Equilibrium about hinge point}$$

Equating coefficients of the appropriate harmonics yields

$$I_1 \ddot{\beta}_0 + I_1 \omega^2 \beta_0 = \frac{1}{2} \rho a c \omega^2 R^4 \left[ \frac{\theta_0}{4} - \frac{\lambda}{3} - \frac{1}{4} \frac{\dot{\beta}_0}{\omega} \right]$$

$$\text{or } \boxed{\frac{\ddot{\beta}_0}{\omega^2} + \frac{\dot{\beta}_0}{8\omega} + \beta_0 = \frac{\chi}{2} \left[ \frac{\theta_0}{4} - \frac{\lambda}{3} \right]} \quad (33) \checkmark$$

for  $\boxed{\ddot{\beta}_0 = \dot{\beta}_0 = 0}$  ← recover the previous which described steady state response

$$\beta_0 = \frac{\chi}{2} \left[ \frac{\theta_0}{4} - \frac{\lambda}{3} \right] \quad \frac{1}{2} \rho a c \int_0^R r [u^2 \theta + u_p u_T] r dr$$

$\theta = \theta_0 + (\theta_{1c} - \phi) \cos$

as before,

$$\beta(\psi) = \beta_0 + \beta_{1c} \cos \psi + \beta_{1s} \sin \psi$$

You get no higher harmonic terms

$$I_1 \ddot{\beta}_{1c} + 2 I_1 \omega \dot{\beta}_{1s} = \frac{1}{2} \rho a c \omega^2 R^4 \left[ \frac{1}{4} (\theta_{1c} - \phi) - \frac{1}{4} \frac{\dot{\beta}_{1c}}{\omega} - \frac{1}{4} \beta_{1s} \right]$$

$$\boxed{\frac{\ddot{\beta}_{1c}}{\omega^2} + \frac{\dot{\beta}_{1c}}{8\omega} + 2 \frac{\dot{\beta}_{1s}}{\omega} + \frac{\beta_{1s}}{8} = \frac{\chi}{8} (\theta_{1c} - \phi)} \quad (34) \checkmark$$

$$I_1 \ddot{\beta}_{1s} - 2 I_1 \omega \dot{\beta}_{1c} = \frac{1}{2} \rho a c \omega^2 R^4 \left[ \frac{1}{4} (\theta_{1s} - \alpha) - \frac{1}{4} \frac{\dot{\beta}_{1s}}{\omega} + \frac{1}{4} \beta_{1c} \right]$$

$$\boxed{\frac{\ddot{\beta}_{1s}}{\omega^2} + \frac{\dot{\beta}_{1s}}{8\omega} - 2 \frac{\dot{\beta}_{1c}}{\omega} - \frac{\beta_{1c}}{8} = \frac{\chi}{8} (\theta_{1s} - \alpha)} \quad (35) \leftarrow$$

Note that when all time dependent terms are set equal to zero, one has

$$\left. \begin{aligned} \beta_{1s} - \phi - \theta_{1c} &= 0 \\ \beta_{1c} - \alpha + \theta_{1s} &= 0 \end{aligned} \right\} (36)$$

i.e. all  $(\cdot)$ ,  $(\ddot{\cdot})$  terms have been set equal to zero.

Response to the input

$\theta_{1s} = \theta_{1c} = 0$  Consider the input  $\alpha = \alpha(t)$  and apply the Laplace transform. Introducing the notation  $\mathcal{L}(\alpha) = \bar{\alpha}$  and

$\dot{v} = \frac{s}{\rho}$  we have from Eqs (35) and (34)

$$\left. \begin{aligned} (v^2 + \frac{\gamma}{\rho} v) \bar{\beta}_{1c} + (2v + \frac{\gamma}{\rho}) \bar{\beta}_{1s} &= 0 \\ -(2v + \frac{\gamma}{\rho}) \bar{\beta}_{1c} + (v^2 + \frac{\gamma}{\rho} v) \bar{\beta}_{1s} &= -\frac{\gamma}{\rho} \bar{\alpha} \end{aligned} \right\} (37)$$

From which we can write

$$\begin{aligned} \frac{\bar{\beta}_{1s}}{\bar{\alpha}} &= \frac{\begin{vmatrix} v^2 + \frac{\gamma}{\rho} v & 0 \\ -(2v + \frac{\gamma}{\rho}) & -\frac{\gamma}{\rho} \end{vmatrix}}{\begin{vmatrix} (v^2 + \frac{\gamma}{\rho} v) & (2v + \frac{\gamma}{\rho}) \\ -(2v + \frac{\gamma}{\rho}) & (v^2 + \frac{\gamma}{\rho} v) \end{vmatrix}} = \text{transfer function} \\ &= \frac{-\frac{\gamma}{\rho} (v^2 + \frac{\gamma}{\rho} v)}{\left[ (v^2 + \frac{\gamma}{\rho} v) (v^2 + \frac{\gamma}{\rho} v) \right] + \left[ 2v + \frac{\gamma}{\rho} \right]^2} \\ &= \frac{-\frac{\gamma}{\rho} (v^2 + \frac{\gamma}{\rho} v)}{v^4 + \frac{\gamma}{\rho} v^3 + v^3 \frac{\gamma}{\rho} + \left( \frac{\gamma}{\rho} v \right)^2 + 4v^2 + \frac{1}{2} v \gamma + \left( \frac{\gamma}{\rho} \right)^2} \end{aligned}$$

$$\frac{\beta_{1s}}{\alpha} = \frac{-\frac{\gamma}{8} (v^2 + \frac{\gamma}{8} v)}{v^4 + 2 \left(\frac{\gamma}{8}\right) v^3 + \left[\left(\frac{\gamma}{8}\right)^2 + 4\right] v^2 + 4 \left(\frac{\gamma}{8}\right) v + \left(\frac{\gamma}{8}\right)^2} \quad (38)$$

For  $|v| = \frac{\omega}{\Omega} \approx \frac{0.5}{25} = 0.02$  and  $\frac{\gamma}{8} \approx 1$   $2 < \gamma < 10$

Roll coupling  $= \frac{\beta_{1s}}{\alpha} \approx 0.02$  = roll coupling in hover can be neglected

$$\theta = \theta_0 + (\theta_{1c} - \phi) \cos \psi + (\theta_{1s} - \alpha) \sin \psi$$

$$\beta_{1s} - \phi = \theta_{1c}$$

$$\beta_{1c} - \alpha = \theta_{1s}$$

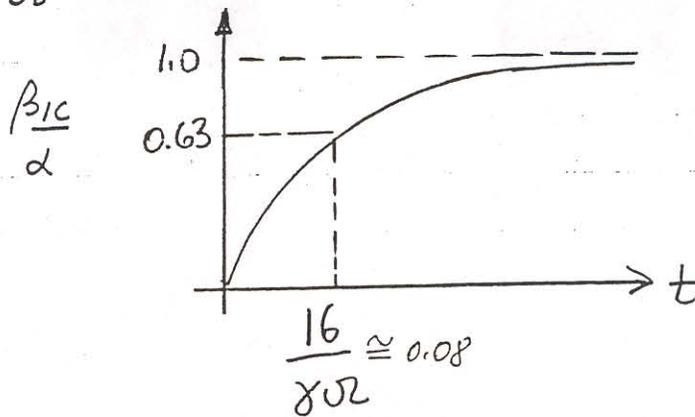
Also  $\frac{\beta_{1c}}{\alpha} = \frac{\frac{\gamma}{8} (2v + \frac{\gamma}{8})}{\Delta} \approx 1$

where  $\Delta$  is the denominator in Eq(38). Therefore one can neglect roll coupling, which implies that  $\beta_{1s}, \beta_{1c}$  terms in Eq(35) can be neglected and then  $\beta_{1s}, \beta_{1c} \rightarrow \text{set } = c$

$$-2 \frac{\beta_{1c}}{\Omega} - \frac{\gamma}{8} \beta_{1c} = \frac{\gamma}{8} (\theta_{1s} - \alpha)$$

(39) Flapping Response to low frequency shaft tilt

$$\frac{16}{\gamma \Omega} \beta_{1c} + \beta_{1c} \approx \alpha - \theta_{1s} \quad (40)$$



The plot above represents flapping response to low frequency shaft movement, when  $\beta_{1s}$  coupling is neglected. Note that

$$\frac{16}{\gamma \Omega} \approx \frac{16}{8 \times 25} = 0.08 \text{ sec}$$

HOVER

As a result of this plot one often makes the approximation that

$\beta_{1c} \neq \alpha$  in studying vehicle motion. It is known as the quasi-steady approximation. Note, also that  $\beta_{1c} \neq \alpha$  during transient motion. However  $\beta_{1c} \approx \alpha$

Rotor disc tends to resist pitching forward velocity

Finally it should be noted when the decoupling between  $\beta_{1c}$  and  $\beta_{1s}$  is not valid:

- (1) For large hinge offset or spring restraint ( $e > 0.10R$ ), which implies all hingeless rotors.
- (2) Small  $\gamma_1$  ( $\gamma_1 \rightarrow 1$ )
- (3) "Tight" autostabilizers (then  $\omega \cong \Omega$ )

이전까지의 결과는 centrally hinged blade



$$dL = \frac{1}{2} \rho a c v_T^2 \left[ \theta - \frac{v_p}{v_T} \right] dr$$

$$\theta v_T^2 = (a r^2 + 2 r \mu v_R + \mu^2 v_R^2 \sin^2 \psi) (\beta_0 + \beta_{1c} \cos \psi + \beta_{1s} \sin \psi)$$

$$\sin^2 \psi \cos \psi \rightarrow h \cdot h$$

$$\sin^2 \psi \sin \psi$$

Higher Harmonics  
of the magnet

$$\sin^3 \psi = \frac{3}{4} \sin \psi - \frac{1}{4} \sin 3\psi$$

$$\sin^2 \psi \cos \psi = \frac{1}{4} \sin \psi + \frac{1}{4} \sin 3\psi$$

$$\frac{1}{2} - \frac{1}{2} \cos 2\psi$$

$$v_T = v_R + \mu v_R \sin \psi$$

$$v_p = \lambda v_R + r \dot{\beta} + \mu v_R \beta \cos \psi$$

$$M = - \int_0^R [\theta (v_R^2 + 2 r \mu v_R \sin \psi + \mu^2 v_R^2 \sin^2 \psi) - (v_R r + r \dot{\beta} + \mu v_R \beta \cos \psi)] \frac{1}{2} \rho a c r dr \quad (46)$$

Also introduce  $\theta = \theta_0 + \theta_{1c} \cos \psi + \theta_{1s} \sin \psi$  (46a)

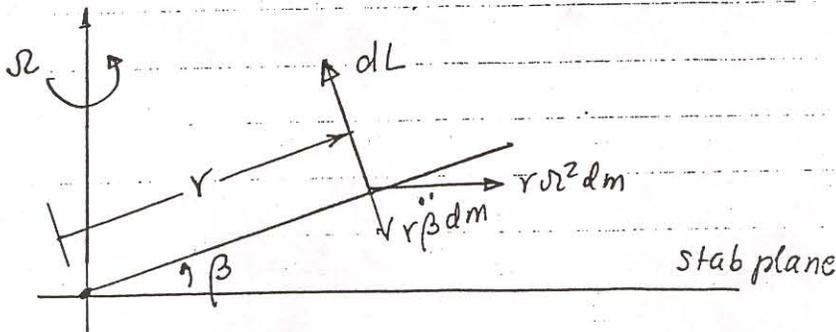
and  $\beta = \beta_0 + \beta_{1c} \cos \psi + \beta_{1s} \sin \psi$  (47)

$$\beta = \beta_0 + \beta_{1c} \cos \psi - \beta_{1c} v_R \sin \psi + \beta_{1s} \sin \psi + \beta_{1s} v_R \cos \psi \quad (48)$$

5, Blade Dynamics in Forward Flight

$M_T = (I \ddot{\beta} + \Omega^2 I \beta)$  Hover  $\Omega = 0$

Stability Convention for forces and moments: Forces and moments opposing disturbed motion are positive



Aerodynamic moment about flapping hinge

$M_A = - \int_0^R r dL \Rightarrow$  difference because  $U_P > U_T$  are different  
 $= - \frac{1}{2} \rho a c \int_0^R [U_T^2 - U_P U_T] r dr$

$dL = \frac{1}{2} \rho a c U_T^2 \left[ \theta - \frac{U_P}{U_T} \right] dr$

$U_T = \Omega r + \mu \Omega R \sin \psi$

$U_P = \lambda \Omega R + r \dot{\beta} + \mu \Omega R \beta \cos \psi$

$U_T^2 = (\Omega^2 r^2 + 2 \Omega r \mu \Omega R \sin \psi + \mu^2 \Omega^2 R^2 \sin^2 \psi) (\theta_0 + \theta_{1c} \cos \psi + \theta_{1s} \sin \psi)$   
 $\sin^2 \psi \cos \psi \rightarrow h.h$   
 $\sin^2 \psi \sin \psi$   
 $\sin^3 \psi = \frac{3}{4} \sin \psi - \frac{1}{4} \sin 3\psi$  Higher Harmonic all  $\frac{1}{4}$  neglect  
 $\sin^2 \psi \cos \psi = \frac{1}{4} \sin \psi + \frac{1}{4} \sin 3\psi$   
 $\frac{1}{2} - \frac{1}{2} \cos 2\psi$

$M = - \int_0^R \left[ \theta (\Omega^2 r^2 + 2 \Omega r \mu \Omega R \sin \psi + \mu^2 \Omega^2 R^2 \sin^2 \psi) - (\Omega r + \mu \Omega R \sin \psi) (\lambda \Omega R + r \dot{\beta} + \mu \Omega R \beta \cos \psi) \right] \frac{1}{2} \rho a c r dr \quad (46)$

Also introduce  $\theta = \theta_0 + \theta_{1c} \cos \psi + \theta_{1s} \sin \psi$  (46a)

and  $\beta = \beta_0 + \beta_{1c} \cos \psi + \beta_{1s} \sin \psi$  (47)

$\dot{\beta} = \dot{\beta}_0 + \dot{\beta}_{1c} \cos \psi - \beta_{1c} \Omega \sin \psi + \dot{\beta}_{1s} \sin \psi + \beta_{1s} \Omega \cos \psi$  (48)

$$M_A = -\frac{1}{2} \rho a c \int_0^R r \left[ \theta (\omega^2 r^2 + 2\omega^2 r R \mu \sin \psi + \mu^2 \omega^2 R^2 \sin^2 \psi) \right. \\ \left. - (\lambda \omega^2 R r + \lambda \mu \omega^2 R^2 \sin \psi + \omega r^2 \dot{\beta} + \mu \dot{\beta} \mu \omega R \sin \psi \right. \\ \left. + \mu \omega^2 R r \beta \cos \psi + \mu^2 \omega^2 R^2 \beta \sin \psi \cos \psi) \right] dr \quad (49)$$

Combining (46a), (47), (48) and (49) yields

$$M_A = -\frac{1}{2} \rho a c \int_0^R r \left\{ (\theta_0 + \theta_{1c} \cos \psi + \theta_{1s} \sin \psi) (\omega^2 r^2 + 2\omega^2 r R \mu \sin \psi + \mu^2 \omega^2 R^2 \sin^2 \psi) \right. \\ \left. - [\lambda \omega^2 R r + \lambda \mu \omega^2 R^2 \sin \psi + \omega r^2 (\dot{\beta}_0 + \beta_{1c} \cos \psi - \beta_{1c} \omega \sin \psi + \beta_{1s} \sin \psi + \beta_{1s} \omega \cos \psi) + \mu \omega R r \sin \psi (\dot{\beta}_0 + \beta_{1c} \cos \psi - \beta_{1c} \omega \sin \psi + \beta_{1s} \sin \psi + \beta_{1s} \omega \cos \psi) + \mu^2 \omega^2 R^2 \sin \psi \cos \psi (\dot{\beta}_0 + \beta_{1c} \cos \psi + \beta_{1s} \sin \psi)] \right\} dr$$

$$M_A = -\frac{1}{2} \rho a c \omega^2 R^4 \left\{ \frac{\theta_0}{4} + \frac{\theta_{1c} \cos \psi}{4} + \frac{\theta_{1s} \sin \psi}{4} + \frac{2}{3} \mu (\theta_0 \sin \psi + \theta_{1c} \cos \psi \sin \psi + \theta_{1s} \sin^2 \psi) + \frac{\mu^2}{2} (\theta_0 \sin^2 \psi + \theta_{1c} \cos \psi \sin^2 \psi + \theta_{1s} \sin^3 \psi) \right. \\ \left. - \left[ \frac{\lambda}{3} + \frac{\lambda}{2} \mu \sin \psi + \frac{1}{4} \left( \frac{\dot{\beta}_0}{\omega} + \frac{\dot{\beta}_{1c} \cos \psi}{\omega} - \beta_{1c} \sin \psi + \frac{\dot{\beta}_{1s} \sin \psi}{\omega} + \beta_{1s} \cos \psi \right) \right. \right. \\ \left. + \frac{\mu}{3} \left( \frac{\dot{\beta}_0 \sin \psi}{\omega} + \frac{\dot{\beta}_{1c} \sin \psi \cos \psi}{\omega} - \beta_{1c} \sin^2 \psi + \frac{\dot{\beta}_{1s} \sin^2 \psi}{\omega} + \beta_{1s} \sin \psi \cos \psi \right) \right. \\ \left. + \frac{\mu}{3} (\beta_0 \cos \psi + \beta_{1c} \cos^2 \psi + \beta_{1s} \sin \psi \cos \psi) + \frac{\mu^2}{2} (\beta_0 \sin \psi \cos \psi + \beta_{1c} \sin \psi \cos^2 \psi + \beta_{1s} \sin^2 \psi \cos \psi) \right] \right\} \quad (50)$$

Consider the longitudinal flapping component  $M_{A15}$ , and recall

$$\left. \begin{aligned} \cos\psi \sin^2\psi &= \frac{1}{4} \cos\psi - \frac{1}{4} \cos 3\psi \\ \sin^3\psi &= \frac{3}{4} \sin\psi - \frac{1}{4} \sin 3\psi \\ \sin\psi \cos^2\psi &= \frac{1}{4} \sin\psi + \frac{1}{4} \sin 3\psi \end{aligned} \right\} (51)$$

Combining Eqs (50) and (51)

$$\begin{aligned} M_A &= M_0 + M_{A15} \sin\psi + M_{A3} \cos\psi \\ M_{A15} &= -\frac{1}{2} \rho a c \Omega^2 R^4 \left\{ \frac{\theta_{15}}{4} + \frac{\mu^2}{2} \left( \frac{3}{4} \right) \theta_{15} \left[ \frac{\lambda}{2} \mu - \frac{1}{4} \beta_{1c} + \frac{\dot{\beta}_{15}}{\Omega} \frac{1}{4} \right. \right. \\ &\quad \left. \left. + \frac{\mu}{3} \frac{\dot{\beta}_0}{\Omega} + \frac{\mu^2}{2} \beta_{1c} \frac{1}{4} \right] + \frac{2}{3} \mu \theta_0 \right\} \\ &= -\frac{1}{2} \rho a c \Omega^2 R^4 \left\{ \frac{\theta_{15}}{4} \left( 1 + \frac{3}{2} \mu^2 \right) + \frac{2}{3} \mu \theta_0 - \frac{1}{2} \mu \lambda + \frac{1}{4} \beta_{1c} \right. \\ &\quad \left. - \frac{1}{4} \beta_{1c} \frac{\mu^2}{2} - \frac{1}{3} \mu \frac{\dot{\beta}_0}{\Omega} - \frac{1}{4} \frac{\dot{\beta}_{15}}{\Omega} \right\} \end{aligned}$$

$$\begin{aligned} * M_{A15} &= -\frac{1}{2} \rho a c \Omega^2 R^4 \left\{ \frac{\theta_{15}}{4} \left( 1 + \frac{3}{2} \mu^2 \right) + \frac{2}{3} \mu \theta_0 - \frac{1}{2} \mu \lambda + \right. \\ &\quad \left. + \frac{1}{4} \beta_{1c} \left( 1 - \frac{\mu^2}{2} \right) - \frac{1}{3} \mu \frac{\dot{\beta}_0}{\Omega} - \frac{1}{4} \frac{\dot{\beta}_{15}}{\Omega} \right\} \quad (52) \end{aligned}$$

these coupling terms are often considered to be negligible and are neglected

Next recall, that the cyclic pitch when referred to the stability plane is given by

$$\left[ \theta(\psi) = \theta_0 + (\theta_{1c} - \phi) \cos\psi + (\theta_{1s} - \alpha) \sin\psi \right] (53)$$

roll angle of shaft
inclination angle of shaft

So whenever one has  $\theta_{1s}$  in Eq (52) it should be

replaced

by  $\theta_{1s} - \alpha$ , thus

$$M_{A1S} = -\frac{1}{2} \rho a c \Omega^2 R^4 \left\{ \frac{(\theta_{1s} - \alpha) (1 + \frac{3}{2} \mu^2)}{4} + \frac{2}{3} \mu \theta_0 - \frac{1}{2} \mu \lambda + \frac{1}{4} \beta_{1c} (1 - \frac{\mu^2}{2}) - \frac{1}{3} \frac{\mu \beta_0}{\Omega} - \frac{1}{4} \frac{\beta_{1s}}{\Omega} \right\}$$

neglect

$$= -\frac{1}{2} \rho a c \Omega^2 R^4 \left\{ \frac{1}{4} (\beta_{1c} - \alpha + \theta_{1s}) (1 + \frac{3}{2} \mu^2) - \frac{1}{4} \beta_{1c} \frac{3}{2} \mu^2 + \frac{2}{3} \mu \theta_0 - \frac{1}{2} \mu \lambda - \frac{1}{4} \beta_{1c} \frac{\mu^2}{2} \right\} =$$

$$M_{A1S} = -\frac{1}{2} \rho a c \Omega^2 R^4 \left\{ \frac{1}{4} (\beta_{1c} - \alpha + \theta_{1s}) (1 + \frac{3}{2} \mu^2) + \frac{2}{3} \mu \theta_0 - \frac{1}{2} \mu \lambda - \frac{1}{4} \beta_{1c} \left( \frac{\mu^2}{2} + \frac{3}{2} \mu^2 \right) \right\} \quad (54)$$

and one can also introduce sp notation on  $\mu$  and  $\lambda$  to indicate that quantities are evaluated in the stability plane, thus

$$M_{A1S} = -\frac{1}{2} \rho a c \Omega^2 R^4 \left\{ \frac{1}{4} (\beta_{1c} - \alpha + \theta_{1s}) \left( 1 + \frac{3}{2} \mu_{sp}^2 \right) + \frac{2}{3} \mu_{sp} \theta_0 - \frac{1}{2} \mu_{sp} \lambda_{sp} - \frac{1}{2} \beta_{1c} \mu_{sp}^2 \right\} \quad (55)$$

$\mu_{sp}$   
 $\lambda_{sp}$

and recall Eq(20) from p. 96, for the  $\sin \psi$  component of the inertia moment about the flapping hinge

$$M_{I1S} = -I_1 \Omega \beta_{1c} 2 + I_1 \ddot{\beta}_{1s} \quad (21)$$

Taking the first harmonic total moment equilibrium

about the flapping hinge

$$\sum M_{flap} = 0 = M_{A1S} + M_{I1S} = 0 \quad \checkmark$$

$$-\frac{1}{2} \rho a c \Omega^2 R^4 \left\{ \frac{1}{4} (\beta_{ic} - \alpha + \theta_{1s}) \left( 1 + \frac{3}{2} \mu_{sp}^2 \right) + \frac{2}{3} \mu_{sp} \theta_0 \right. \\ \left. - \frac{1}{2} \mu_{sp} \lambda_{sp} - \frac{\beta_{ic}}{2} \mu_{sp}^2 \right\} - I_1 \Omega \dot{\beta}_{ic} + I_1 \ddot{\beta}_{1s} \quad (56)$$

neglect lateral coupling

and one can also use

Stability Derivatives for a Rotor in Fwd Flight

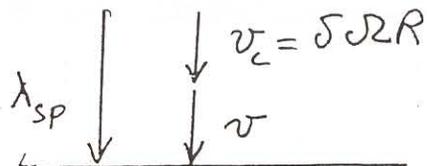
$$\lambda_{sp} = \lambda_{TPP} - \mu \beta_{ic}$$

$$-2 \frac{\beta_{ic}}{\Omega} \approx \frac{\gamma}{\Omega} \left\{ \frac{1}{4} (\beta_{ic} - \alpha + \theta_{1s}) \left( 1 + \frac{3}{2} \mu_{sp}^2 \right) + \frac{2}{3} \mu_{sp} \theta_0 \right. \\ \left. - \frac{1}{2} \mu_{sp} \lambda_{TPP} \right\} \quad (57) \quad \leftarrow$$

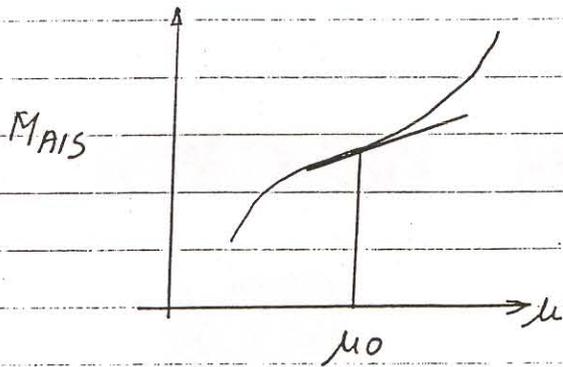
Equation (57) is nonlinear in terms of advance ratio, and it can be linearized for small displacements of the helicopter from equilibrium, assuming  $\theta_0$ ,  $\Omega$  and  $v$  constant. (valid for initial response)

$$\Delta M_{A1S} = \left( \frac{\partial M_{A1S}}{\partial \mu} \right)_0 \Delta \mu + \left( \frac{\partial M_{A1S}}{\partial \alpha} \right) \Delta \alpha + \left( \frac{\partial M_{A1S}}{\partial \delta} \right) \Delta \delta \\ + \left( \frac{\partial M_{A1S}}{\partial \beta_{ic}} \right) \Delta \beta_{ic} + \left( \frac{\partial M_{A1S}}{\partial \theta_{1s}} \right) \Delta \theta_{1s} \quad (58)$$

Also note  $\lambda_{sp} = \frac{v}{\Omega R} + \delta$



The quantities in the brackets in Eq (58) are known as stability derivatives



Note also that

$$\frac{\partial}{\partial \mu} \left( \frac{1}{2} \mu \lambda_{TPP} \right) = \frac{1}{2} \lambda_{TPP0} + \frac{1}{2} \mu_0 \frac{\partial \lambda_{TPP}}{\partial \mu} = \frac{1}{2} \lambda_{TPP0} + \frac{1}{2} \mu_0 \beta_{ic} \quad (59)$$

because  $\lambda_{TPP} = \lambda_{SP} + \mu \beta_{ic}$

$$\begin{aligned} \frac{\partial MAIS}{\partial \mu} &= -\frac{1}{2} \rho a c \omega^2 R^4 \left\{ \frac{2}{3} \theta_0 + \frac{3}{4} \mu_0 (\beta_{ic0} - \alpha_0 + \theta_{is0}) - \right. \\ &\quad \left. - \frac{1}{2} \lambda_{TPP0} - \frac{1}{2} \mu_0 \beta_{ic0} \right\} \\ &= -\frac{1}{2} \rho a c \omega^2 R^4 \left\{ \frac{2}{3} (\theta_0 - \frac{3}{4} \lambda_{TPP0}) + \frac{3}{4} \mu_0 (\beta_{ic0} - \alpha_0 + \theta_{is0}) - \right. \\ &\quad \left. - \frac{1}{2} \mu_0 \beta_{ic0} \right\} \quad (60) \end{aligned}$$

$$\frac{\partial MAIS}{\partial \alpha} = -\frac{1}{2} \rho a c \omega^2 R^4 \left[ -\frac{1}{4} \left( 1 + \frac{3}{2} \mu_0^2 \right) \right] \quad (61)$$

$$\frac{\partial MAIS}{\partial \beta_{ic}} = -\frac{1}{2} \rho a c \omega^2 R^4 \left[ \frac{1}{4} \left( 1 + \frac{3}{2} \mu_0^2 \right) - \frac{1}{2} \mu_0^2 \right] \quad (62)$$

$$\frac{\partial MAIS}{\partial \delta} = -\frac{1}{2} \rho a c \omega^2 R^4 \left[ -\frac{1}{2} \mu_0 \right] \quad (63)$$

$$\frac{\partial MAIS}{\partial \theta_{15}} = -\frac{1}{2} \rho a c \Omega^2 R^4 \left[ \frac{1}{4} \left( 1 + \frac{3}{2} \mu_0^2 \right) \right] \quad (64)$$

Also note

$$\lambda_{TRP} = \mu \beta_{1c} + \frac{c_T + \delta}{2 \mu_0}$$

Define

$$P_0 = \frac{\gamma}{8} \left( 1 + \frac{3}{2} \mu_0^2 \right) \quad (65)$$

Then combining Eq(57) with Eqs (60)-(65) yields

$$-2 \frac{\dot{\beta}_{1c}}{\Omega} = -\frac{\Delta MAIS}{I_1 \Omega^2} = N_0 \Delta \mu - P_0 \Delta \alpha + P_0 \Delta \beta_{1c} - \frac{\gamma}{4} \mu_0^2 \beta_{1c} - \frac{\gamma}{4} \mu_0 \Delta \delta + P_0 \Delta \theta_{15} \quad (66)$$

(67)

where  $N_0 = -\frac{\gamma}{2} \left[ \frac{2}{3} \theta_0 + \frac{(\theta_{150} - \alpha_0)^3}{4} - \frac{1}{4} \beta_{1c0} \mu_0^{-1} \lambda_{TRP0} \right]$

Rewrite Eq(66) as; with  $(\Delta \mu \rightarrow \mu)$    
  $\Delta$  are dropped

$$-2 \frac{\dot{\beta}_{1c}}{\Omega} = \frac{N_0}{P_0} \mu - \alpha + \beta_{1c} - \frac{\gamma}{4} \mu_0^2 \beta_{1c} - \frac{\gamma}{4} \frac{\mu_0^2}{\mu_0} \delta + P_0 \theta_{15}$$

오류를 움직여야  
 자세를 정정하는  $\beta_{1c}$ 를  
 바꿀수 있다  
 하강이르면  
 tilt ↑  
 forward ↑  
 fast

$$\alpha - \beta_{1c} - \theta_{15} = \frac{N_0}{P_0} \mu + \left( \frac{2}{P_0} \frac{\beta_{1c}}{\Omega} - \frac{\gamma}{4} \frac{\mu_0^2}{P_0} \left( \beta_{1c} + \frac{\delta}{\mu_0} \right) \right) \quad (68)$$

$\alpha - \beta_{1c} =$  restoring moment of thrust, positive terms   
 (unstable term associate with angle of attack)

in Eq(68) are stabilizing and negative terms as

destabilizing. The coefficient of the  $\mu$ -term

- ⊕ - terms stabilizing
- ⊖ - destabilizing

110

represented by  $\left(\frac{N_0}{P_0}\right)$  - represents the speed stability of the rotor, This effect tilts the thrust back and decreases the velocity; i.e.  $\Delta\mu > 0$  gives slowdown and represents a stabilizing effect.

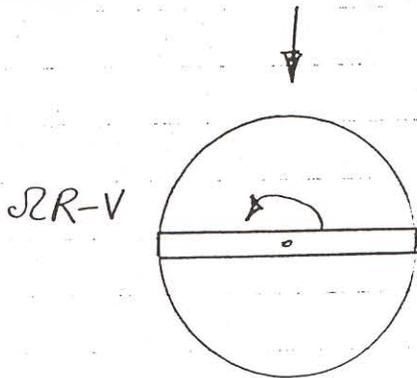
$\alpha - \beta_{ic} \Rightarrow$  cyclic pitch gives rolling moment which creates disc precession, resulting in nose up

$\frac{2}{\beta_0} \frac{\beta_{ic}}{\Omega} = \text{gyro damping}$  (\*)

The term  $-\frac{\gamma}{4} \frac{\mu_0^2}{P_0} \left(\beta_{ic} + \frac{\delta}{\mu_0}\right)$  (\*\*) represents instability w.r.t. angle of attack, tilt of  $\alpha$  forward, process increase of

inflow

(\*)



Response to this 90° later by flapping up, response to increasing and flapping down 90° after  $\Omega R - V$

(\*\*)

