① Coupled Rotor / Fuselage Aeromechanical Problems (Derivation .. 2[illegible])

1. Introduction

A rotor mounted on a flexible support, or a rigid fuselage which has a number of rigid body degrees of freedom can have additional instabilities when compared to an isolated blade. When the rotor is on the ground an aeromechanical instability can occur which is known as ground resonance (Refs 1 and 2).

This instability consists primarily of the coupling of the blade inplane, or lead-lag degrees of freedom with the roll motion of the fuselage. This instability is known to be sensitive to the flexibility and damping of the landing gear system. In the past it was usual to deal with this problem as a purely mechanical instability problem, where the aerodynamic loads acting on the blades were neglected. The advent of hingeless rotors has shown that this problem can be influenced by the aerodynamic loads acting on the blade, thus more recent treatments of this problem have taken into account the aerodynamic loads acting on the rotor blades (Refs 3 and 4)

In flight the coupled rotor / fuselage system can experience an aeromechanical instability which is usually denoted as air resonance (Refs 1 and 2). In air resonance the lead-lag degree of freedom of the blades can couple with the pitch or roll degree of freedom of the fuselage to produce this instability.

It is important to note that ground resonance is a violent instability which leads to the destruction of the helicopter in a few seconds. On the other hand air resonance is a relatively mild instability which manifests itself as strong oscillation to the pilot. By appropriate changes in the flight condition (i.e. changing the pitch setting of the blades) the pilot

(2) can avoid this unstable region.

2. Formulation of the Governing Equations of Motion

This formulation is for hover, the case of forward flight is given in Ref. 5

2.1 Assumptions

(1) The rotor blades are assumed to be rigid with equivalent root springs representing the flexibility of the blade, and are attached to the hub with an offset e from the axis of rotation (hub center)

(2) The blade has no precone, droop or sweep. The torque offset is also zero. The built in twist is zero.

(3) The blade cross sectional center of mass, aerodynamic center, elastic axis and tension center coincide.

(4) Structural damping in the blade is assumed to be of a viscous type

(5) The rotor shaft is rigid and the speed of rotation is constant.

(6) The rotor consists of three or more blades

(7) The center of gravity of the fuselage is located on the axis of rotation at a distance h below the hub

(8) Two-dimensional quasi-steady aerodynamic are used to obtain the aerodynamic loads. Compressibility effects are neglected and the helicopter is assumed to be in hover.

(9) Each blade has flap β_k and lag ζ_k degrees of freedom and the fuselage has θ_y (pitch) and θ_x (roll) degrees of freedom.

2.2 Ordering Scheme.

When deriving equations of motion for a coupled rotor/fuselage system a large number of higher order terms has to be considered. Previous research has clearly indicated (Ref. 6) that many higher order terms can be neglected systematically by

(3) using an ordering scheme. The ordering scheme employed here assumes that blade slopes are of order ϵ while fuselage rotations are of order $\epsilon^{3/2}$. The basis of the ordering scheme is a small dimensionless quantity (parameter) ϵ which represents typical blade slopes due to elastic deflections. For helicopter blades it is known that ϵ is in the range of

$$0.1 \leq \epsilon \leq 0.20$$

The ordering scheme is based on the assumption that

$$1 + O(\epsilon^2) \cong 1.0$$

i.e. terms of order $O(\epsilon^2)$ are neglected in comparison with unity.

The various symbols used in this section are defined in Appendix A. The orders of magnitude for the various parameters governing this problem are given below

$$\cos\psi_k,\ \sin\psi_k,\ \frac{X_R}{R},\ \frac{\rho_a bRa}{m},\ \frac{h}{R} = O(1)$$

$$\frac{1}{\Omega}\frac{\partial}{\partial t}(\ ) = \frac{\partial}{\partial\psi}(\ ) = O(1)$$

$$R\frac{\partial}{\partial x_k}(\ ) = \frac{\partial}{\partial \bar{x}_k} = O(1)$$

$$\theta_{0k},\ \theta_{Gk} = O(\epsilon^{1/2})$$

$$\beta_k,\ \zeta_k,\ \frac{e}{R},\ \frac{b}{R},\ \lambda_k = O(\epsilon)$$

$$\frac{C_{do}}{a},\ \theta_y,\ \theta_x = O(\epsilon^{3/2})$$

Application of such an ordering scheme leads to the neglect of numerous higher order terms. However it should be noted that such a scheme is based on a combination of common sense

(4) and experience with practical blade configurations, thus it should be applied with a certain degree of flexibility.

2.3 Coordinate systems

Central to the derivation of the nonlinear blade equations and the matching of rotor forces, moments and displacements at the rotor hub with those of the fuselage, is the definition of the relationship between the various Cartesian coordinate systems required. The transformation relations between quantities referred in the various inertial and noninertial coordinate systems has to be established before deriving the equations of motion.

The transformation between two orthogonal coordinate systems with axes X_i, Y_i, Z_i and X_j, Y_j, Z_j with $\hat{e}_{xi}, \hat{e}_{yi}, \hat{e}_{zi}$ and $\hat{e}_{xj}, \hat{e}_{yj}, \hat{e}_{zj}$ as unit vectors along the various axes, respectively, is given by

$$\begin{Bmatrix} \hat{e}_{xi} \\ \hat{e}_{yi} \\ \hat{e}_{zi} \end{Bmatrix} = [T_{ij}] \begin{Bmatrix} \hat{e}_{xj} \\ \hat{e}_{yj} \\ \hat{e}_{zj} \end{Bmatrix} \qquad (1)$$

where the transformation matrix $[T_{ij}]$ can be found by using the Euler angles required to rotate the j-system into the i-system.

(a) The S system (Fig 1) is an inertial system whose origin O_s is fixed at the mass center of the undisturbed fuselage. Z_s is oriented upwards (vertically), X_s is directed aft and Y_s is as shown. $\hat{e}_{sx}$ lies along the undisturbed roll axis, $\hat{e}_{sy}$ is along the undisturbed pitch axes and $\hat{e}_{sz}$ is along the undisturbed rotor shaft axes

(5) (b) The S1 system is a non inertial, body fixed coordinate system whose origin is also the mass center of the fuselage. This coordinate system moves with the body during perturbational motion. The S-system and S1-system coincide with each other in the unperturbed state of the model

(c) R-system is another inertial system fixed at the center O_H of the unperturbed hub. The directions of the axes of this system are parallel to that of the S1 system.

(d) The 1-system is a body fixed system with its origin fixed at the hub O_H. Prior to perturbational motion the 1-system coincides with the R-system.

(e) The 2k-system is a blade fixed rotating coordinate system which rotates with the k^{th} blade. The 2-k system is rotating from the 1-system by the azimuth angle ψ_k, of the k^{th} blade about the z_1 axis (see Fig. 4.)

(f) The 4k-system is fixed in the cross section of the k^{th} blade. Translating the 2-k system an amount $x_k \hat{e}_{x2k}$ gives the 4k-system at the cross section x_k of the k^{th} blade prior to elastic deformation. During the elastic deformation of the k^{th} blade, the 4k-system translates and rotates with the cross-section. The origin of the 4k system after deformation is given by

$$(x_k + u_k)\,\hat{e}_{x2k} + v_k\,\hat{e}_{y2k} + w_k\,\hat{e}_{z2k} \qquad (2)$$

(6)

Consider next the transformation matrices associated with the various coordinate systems which have been listed above

Assuming a sequence of fuselage rotations consisting of pitch (θ_y) and roll (θ_x)

$$[T_{1R}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -\theta_y \\ \theta_y\theta_x & 1 & \theta_x \\ \theta_y & -\theta_x & 1 \end{bmatrix} \qquad (3) \qquad (\underset{\sim}{T}_{1R} = \underset{\sim}{T}_{1S})$$

where in Eq (3) we have used the fact that since θ_x, θ_y are of order $O(\epsilon^{3/2})$ the sines and cosines can be replaced by $\sin\theta \cong \theta$ $\cos\theta \cong 1.0$.

The relation between $2k$ and $1k$ systems can be written as

$$[T_{21}] = \begin{bmatrix} \cos\psi_k & \sin\psi_k & 0 \\ -\sin\psi_k & \cos\psi_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (4)$$

The rotation of the $4k$-system is obtained by Euler angles corresponding to a flap-lag sequence of rotation, by angles $-\beta_k$ and ζ_k. Thus the transformation matrix $[T_{42}]$ is given by

$$[T_{42}] = \begin{bmatrix} \cos\zeta_k & \sin\zeta_k & 0 \\ -\sin\zeta_k & \cos\zeta_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta_k & 0 & \sin\beta_k \\ 0 & 1 & 0 \\ -\sin\beta_k & 0 & \cos\beta_k \end{bmatrix} \qquad (5)$$

(7) Again since the angles β_k and ζ_k are of order(ϵ) the transformation matrix can be simplified by assuming $\sin\theta \simeq \theta$ $\cos\theta \simeq 1.0$, thus

$$[T_{42}] = \begin{bmatrix} 1 & \zeta_k & \beta_k \\ -\zeta_k & 1 & -\beta_k\zeta_k \\ -\beta_k & 0 & 1 \end{bmatrix} \quad (6)$$

Furthermore since in our model we have assumed a rigid blade with root springs the relation between translation and blade rotation is given by

$$v_k = x_k \zeta_k \quad \text{and} \quad w_k = x_k \beta_k = -(-x_k \beta_k) \quad (7)$$

2.4 Equations of Motion for the Individual Blade

First the dynamic equations of equilibrium of a blade are obtained by using a Newtonian approach. The equations are obtained by combining the structural operator with the inertial, aerodynamic and structural damping loads. Since a rigid, offset hinged, spring restrained model of the blade is used the various distributed loads are integrated over the blade length and then combined together to give the equation of the blade motion. The various distributed loads acting on the blade are described in the following sections.

2.4.1 Distributed Inertia Loads on the Blade

The distributed inertia loads on the k^{th} blade are obtained by first determining the acceleration at a general point 'P' on the blade cross section. The loads per unit volume are found from D'Alembert's principle and they are integrated

(8) over the cross-section to give the distributed blade loads per unit length of the blade.

The absolute acceleration at a point 'P' in a translating and rotating coordinate system, with respect to an inertial reference frame is given by

$$\underset{\sim}{a}_{pk} = \overset{**}{\underset{\sim}{R}_o} + \overset{**}{\underset{\sim}{r}_{pk}} + 2\,\underset{\sim}{\omega}_k \times \overset{*}{\underset{\sim}{r}_{pk}} + \overset{*}{\underset{\sim}{\omega}_k} \times \underset{\sim}{r}_{pk} + \underset{\sim}{\omega}_k \times (\underset{\sim}{\omega}_k \times \underset{\sim}{r}_{pk}) \quad (8)$$

(* - derivative w.r.t. time) → 이계미분   * ~ derivative w.r.t ψ

where $\underset{\sim}{R}_o$ is the position vector of the origin of the moving coordinate system with respect to the inertial frame, and $\underset{\sim}{r}_{pk}$ is the position vector of the point 'P' in the k^{th} blade from the origin of the moving reference frame and $\underset{\sim}{\omega}_k$ is the angular velocity of the moving coordinate system.

In the present case the inertial reference frame is the R-system, whose origin is fixed at the undeformed hub and the moving reference frame is the 2k-system whose origin moves and rotates with the blade. The position vector of a point 'P' in the cross section of the k^{th} blade is given by

$$\underset{\sim}{r}_{p2k} = \hat{e}_{x2k}(x_k + e + u_k) + v_k\,\hat{e}_{y2k} + w_k\,\hat{e}_{z2k} + y_{ok}\,\hat{e}_{y4k} + z_{ok}\,\hat{e}_{z4k} \quad (9)$$

Transforming all unit vectors to the 2k-system

$$\underset{\sim}{r}_{p2k} = \hat{e}_{x2k}\left[(e + x_k + u_k) - y_{ok}\,\zeta_k - z_{ok}\,\beta_k\right] + \hat{e}_{y2k}(v_k + y_{ok}) + \hat{e}_{z2k}(w_k - y_{ok}\,\zeta_k(\beta_k + z_{ok}) \quad (10)$$

where y_{ok} and z_{ok} are related to the principal

(9) axes of the cross section by

$$\begin{Bmatrix} y_{0k} \\ z_{0k} \end{Bmatrix} = \begin{bmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix} \begin{Bmatrix} \eta_{0k} \\ \xi_{0k} \end{Bmatrix} \quad (11)$$

where θ_k is the pitch setting on the k^{th} blade.

Applying the ordering scheme and substituting $w_k = x_k \beta_k$ and $v_k = x_k \beta_k$, Eq (11) becomes

$$\underset{\sim}{r}_{pk} = \hat{e}_{x2k}\left[e + x_k + u_k - y_{0k}\zeta_k - z_{0k}\beta_k\right] + \hat{e}_{y2k}\left(x_k\zeta_k + y_{0k}\right) + \hat{e}_{z2k}\left(\beta_k x_k - y_{0k}\zeta_k\beta_k + z_{0k}\right) \quad (12)$$

Taking the first and second derivative of $\underset{\sim}{r}_{p2k}$ and applying the ordering scheme yields (note dot is derivative with respect to ψ, $\psi = \Omega t$)

$$\dot{\underset{\sim}{r}}_{pk} = \Omega\hat{e}_{x2k}\left(\dot{u}_k - y_{0k}\dot{\zeta}_k - z_{0k}\dot{\beta}_k\right) + \Omega\hat{e}_{y2k}\left(x_k\dot{\zeta}_k\right) + \Omega\hat{e}_{z2k}\left[\dot{\beta}_k x_k - y_{0k}\left(\dot{\zeta}_k\beta_k + \zeta_k\dot{\beta}_k\right)\right] \quad (13)$$

$$\ddot{\underset{\sim}{r}}_{pk} = \Omega^2\hat{e}_{x2k}\left(\ddot{u}_k - y_{0k}\ddot{\zeta}_k - z_{0k}\ddot{\beta}_k\right) + \Omega^2\hat{e}_{y2k}\left(x_k\ddot{\zeta}_k\right) + \Omega\hat{e}_{z2k}\left[\ddot{\beta}_k x_k - y_{0k}\left(\ddot{\zeta}_k\beta_k + 2\dot{\zeta}_k\dot{\beta}_k + \zeta_k\ddot{\beta}_k\right)\right] \quad (14)$$

The angular velocity of the k^{th} blade is given by

중요!

$$\underset{\sim}{\omega}_k = \Omega\hat{e}_{z1} + \underset{\sim}{\omega}_{0H} \quad (15)$$

$\dot{\omega}_k \neq 0$ — 중요한 차이점 — $\dot{\omega}_k = 0$

cf.) $\underset{\sim}{\omega}_R = \Omega\hat{e}_{z1}$ in isolated blade

where $\underset{\sim}{\omega}_{0H}$ – is the angular velocity of the hub center C_F due to fuselage motion and it's given by

(10) $$\underset{\sim}{\omega}_{OH} = \Omega(\dot{\theta}_x)\hat{e}_{xs} + \Omega\,\dot{\theta}_y\,\hat{e}_{ys} - \dot{\theta}_x\,\theta_y\,\hat{e}_{zs} \quad (16)$$

This expression is due to the fact that the rigid body angular velocities are $\Omega\dot{\theta}_y$ about the Y_s axis and $\dot{\theta}_x\,\Omega$ about the pitched X_s axis.

The angular velocity of the k^{th} blade in the $2k$ system is obtained by using Eq (16) and noting that the k and s systems are parallel and inertial systems

$$\underset{\sim}{\omega}_{2k} = \Omega\hat{e}_{x2k}\left[\cos\psi_k\,\dot{\theta}_x + \sin\psi_k\,\dot{\theta}_y\right]$$

$$+ \Omega\hat{e}_{y2k}\left[\dot{\theta}_y\cos\psi_k - \dot{\theta}_x\sin\psi_k\right] + \Omega\hat{e}_{z2k}$$

$$= \Omega\left[\omega_x\,\hat{e}_{x2k} + \omega_y\,\hat{e}_{y2k} + \hat{e}_{z2k}\right] \quad (17)$$

where the definitions of ω_x and ω_y are clear from the last equation.

the angular acceleration $\dot{\underset{\sim}{\omega}}_{2k}$ is given by

$$\dot{\underset{\sim}{\omega}}_{2k} = \Omega^2{}_{x2k}\left[\cos\psi_k(\ddot{\theta}_x + \dot{\theta}_y) + \sin\psi_k(\ddot{\theta}_y - \dot{\theta}_x)\right]$$

$$+ \Omega^2\hat{e}_{y2k}\left[\cos\psi_k\,(\ddot{\theta}_y - \dot{\theta}_x) + \sin\psi_k(-\ddot{\theta}_x - \dot{\theta}_y)\right]$$

$$+ \Omega^2\hat{e}_{z2k}\left(-\dot{\theta}_x\,\dot{\theta}_y - \theta_x\,\ddot{\theta}_y\right) = \quad (18)$$

$$= \Omega^2\left[\dot{\omega}_x\,\hat{e}_{x2k} + \dot{\omega}_y\,\hat{e}_{y2k} + \dot{\omega}_z\,\hat{e}_{z2k}\right]$$

Where again the definitions of $\dot{\omega}_x$, $\dot{\omega}_y$ and $\dot{\omega}_z$ are clear from the last equation

(11) Using Eqs (18) and (10) one can write the angular acceleration term in Eq (8)

$$\dot{\underset{\sim}{\omega}}_{2k} \times \underset{\sim}{r}_{p2k} = \Omega^2 \hat{e}_{x2k}\left(-\dot{\omega}_z x_k \zeta_k + \dot{\omega}_x x_k \zeta_k - y_{ok}\dot{\omega}_z + z_{ok}\dot{\omega}_y\right)$$
$$+\Omega^2 \hat{e}_{y2k}\left(\dot{\omega}_z e + \dot{\omega}_z x_k - \dot{\omega}_x x_k \beta_k - y_{ok}\dot{\omega}_z \zeta_k + \dot{\omega}_x \zeta_k \beta_k y_{ok} - z_{ok}\dot{\omega}_z \beta_k - z_{ok}\dot{\omega}_x\right)$$
$$+\Omega^2 \hat{e}_{z2k}\left[\dot{\omega}_x x_k \zeta_k - \dot{\omega}_y e - \dot{\omega}_y x_k + y_{ok}(\dot{\omega}_x + \dot{\omega}_y \zeta_k) + z_{ok}(+\dot{\omega}_y \beta_k)\right] \quad (19)$$

In a similar manner the coriolis term is obtained by Eqs (7) and (13) and the application of the ordering scheme

$$2\underset{\sim}{\omega}_{2k} \times \dot{\underset{\sim}{r}}_{p2k} = 2\Omega^2 \hat{e}_{x2k}\left[\omega_y x_k \dot{\beta}_k - x_k \dot{\zeta}_k + y_{ok}(-\omega_y \dot{\zeta}_k \beta_k - \zeta_k \dot{\beta}_k)\right]$$
$$+2\Omega^2 \hat{e}_{y2k}\left[\dot{u}_k + \quad - \omega_x x_k \dot{\beta}_k + y_{ok}(-\dot{\zeta}_k) + z_{ok}(-\dot{\beta}_k)\right]$$
$$+2\Omega^2 \hat{e}_{z2k}\left[\omega_x x_k \dot{\zeta}_k - \omega_y v_k + y_{ok}(+\omega_y \dot{\zeta}_k) + z_{ok}(+\omega_y \dot{\beta}_k)\right] \quad (20)$$

The centripetal acceleration term is given by

$$\underset{\sim}{\omega}_{2k} \times (\underset{\sim}{\omega}_{2k} \times \underset{\sim}{r}_{p2k}) = \Omega^2 \hat{e}_{x2k}\left[-(x_k + e + 2x_k\omega_z) + y_{ok}(\zeta_k + \omega_z \zeta_k) + z_{ok}(\beta_k + \omega_z \beta_k + \omega_x + \omega_z \beta_k)\right]$$
$$+\Omega^2 \hat{e}_{y2k}\left[-x_k\zeta_k - \omega_z x_k \zeta_k + \omega_y x_k \beta_k - \omega_z x_k \zeta_k + y_{ok}(-1 - 2\omega_z) + z_{ok}(\omega_y)\right]$$
$$+\Omega^2 \hat{e}_{z2k}\left[\omega_x x_k + \omega_x e + \omega_y x_k \zeta_k + y_{ok}(\omega_y - \omega_x \zeta_k) + z_{ok}(-\beta_k \omega_x - \omega_x^2 - \omega_y^2)\right] \quad (21)$$

Finally note that $\ddot{\underset{\sim}{R}}_O \equiv 0$ when the fuselage has only rotations and no translation.

(12)

Combining the various terms in Eq(8), i.e. Eqs (14), (19), (20) and (21) yields the three components of the acceleration which can be written as

$$a_{p_{x2k}} = \Omega^2\{-x_k - e - 2x_k\dot{\zeta}_k + y_{0k}[\zeta_k - \ddot{\zeta}_k] + z_{0k}[\beta_k + \omega_x + \dot{\omega}_y - \ddot{\beta}_k]\}$$

$$= \Omega^2[a^{(c)}_{p_{x2k}} + y_{0k}\, a^{(y)}_{p_{x2k}} + z_{0k}\, a^{(z)}_{p_{x2k}}] \qquad (22)$$

where the three terms correspond respectively to (1) the constant part of the x-component of the acceleration over the cross section of the blade at a distance x_k, (2) the part dependent on y_{0k} and (3) the term dependent on z_{0k} respectively. This separation facilitates integration of those terms over the cross section of the blade to yield inertia forces. It also helps in identifying orders of magnitude of the various terms.

In a similar manner the y and z components of the acceleration are given by

$$a_{p_{y2k}} = \Omega^2[-x_k\zeta_k + \omega_y\beta_k x_k - \dot{\omega}_x\beta_k x_k + \ddot{\zeta}_k x_k + 2\dot{v}_k - 2\omega_x x_k\dot{\beta}_k$$

$$- y_{0k}(1 + 2\dot{\zeta}_k) + z_{0k}(\omega_y - \dot{\omega}_x - 2\dot{\beta}_k)]$$

$$= \Omega^2[a^{(c)}_{p_{y2k}} + y_{0k}\, a^{(y)}_{p_{y2k}} + z_{0k}\, a^{(z)}_{p_{y2k}}] \qquad (23)$$

$$a_{p_{z2k}} = \Omega^2[\omega_x(x_k + e) + \omega_y x_k\zeta_k + \dot{\omega}_x x_k\zeta_k - \omega_y(e + x_k)$$

$$+ 2\omega_x x_k\dot{\zeta}_k + \ddot{\beta}_k x_k + y_{0k}(-\ddot{\zeta}_k\beta_k - 2\dot{\zeta}_k\dot{\beta}_k - \zeta_k\ddot{\beta}_k$$

$$+ 2\omega_y\dot{\zeta}_k + \dot{\omega}_x + \dot{\omega}_y\zeta_k + \omega_y - \omega_x\zeta_k) +$$

(13) $+ z_{0k}\left(+ 2\omega_y \mathring{\beta}_k + \dot{\omega}_y \beta_k - \omega_x \beta_k - \omega_x^2 - \omega_y^2\right)\Big]$

$$= \Omega^2 \left[a^{(c)}_{pzzk} + y_{0k}\, a^{(y)}_{pzzk} + z_{0k}\, a^{(z)}_{pzzk} \right] \qquad (24)$$

Before evaluation the distributed inertia forces and moments, it is worth noting down the relative orders of magnitude of the leading terms in the various acceleration components. They are

$$a^{(c)}_{pxzk} = O(1) \;;\quad a^{(y)}_{pxzk} = O(\epsilon) \;;\quad a^{(z)}_{pxzk} = O(\epsilon)$$

$$a^{(c)}_{pyzk} = O(\epsilon) \;;\quad a^{(y)}_{pyzk} = O(1) \;;\quad a^{(z)}_{pyzk} = O(\epsilon)$$

$$a^{(c)}_{pzzk} = O(\epsilon) \;;\quad a^{(y)}_{pzzk} = O(\epsilon) \;;\quad a^{(z)}_{pzzk} = O(\epsilon^2)$$

This information is useful in neglecting higher order terms even before evaluating the integrals to obtain inertia forces and moments

Consider next the distributed inertia force per unit length obtained from applying D'Alemberts, for the k^{th} blade

Acceleration vector

$$\underset{\sim}{p}_{Ik} = \iint_{A_T} \left(-\rho\, \underset{\sim}{a}_{pk}\right) dA \qquad (25)$$

where ρ is the material density of the blade and A_T is the total cross sectional area.

In substituting the expressions for the acceleration components and carrying out the various integration a number of integrals associated with the mass properties of the blade need to be defined. These expressions are presented below, together with some related material which is also needed.

(14)
$$\begin{Bmatrix} y_{oR} \\ z_{oR} \end{Bmatrix} = \begin{bmatrix} \cos\theta_{GR} & -\sin\theta_{GR} \\ \sin\theta_{GR} & \cos\theta_{GR} \end{bmatrix} \begin{Bmatrix} \eta_{oR} \\ \xi_{oR} \end{Bmatrix} \qquad (26)$$

$$\iint_{A_T} \rho\, dA = m; \quad \iint_{A_T} \rho\, \eta_{oR}\, dA = m x_I = 0; \quad \iint_{A_T} \rho\, \xi_{oR}\, dA = 0 \qquad (27a)$$

$$\iint_{A_T} \rho\, \eta_{oR}^2\, dA = I_{MB3}; \quad \iint_{A_T} \rho\, \xi_{oR}^2\, dA = I_{MB2}; \quad \iint_{A_T} \rho\, \eta_{oR}\, \xi_{oR}\, dA = 0$$

from these it also follows that

$$\iint_{A_T} \rho\, y_{oR}\, dA = m x_I \cos\theta_{GR} = 0; \quad \iint_{A_T} \rho\, z_{oR}\, dA = m x_I \sin\theta_{GR} = 0$$

$$\iint_{A_T} \rho\, y_{oR}^2\, dA = I_{MB3} \cos^2\theta_{GR} + I_{MB2} \sin^2\theta_{GR} \qquad (27b)$$

$$\iint_{A_T} \rho\, z_{oR}^2\, dA = I_{MB3} \sin^2\theta_{GR} + I_{MB2} \cos^2\theta_{GR}$$

$$\iint_{A_T} \rho\, z_{oR}\, y_{oR}\, dA = (I_{MB3} - I_{MB2}) \sin\theta_{GR} \cos\theta_{GR}$$

On some of these quantities the following order of magnitude assumptions are also introduced

$$\frac{x_I}{R} = O(\epsilon^2); \quad \frac{I_{MB3}}{mR^2} = O(\epsilon^3); \quad \frac{I_{MB2}}{mR^2} = O(\epsilon^7)$$

With these expressions the distributed inertia loads can be written as

$$p_{IX2R} = m\Omega^2 \left[x_R + e + 2 x_R \dot{\zeta}_R \right] \qquad (28)$$

inertial loads / length

$$p_{IY2R} = m\Omega^2 \Big[x_R \zeta_R - x_R \ddot{\zeta}_R - 2\dot{u}_R + \cos\psi_R \left(x_R \beta_R \ddot{\theta}_x + 2 x_R \dot{\beta}_R \dot{\theta}_x \right) + \sin\psi_R \left(x_R \beta_R \ddot{\theta}_y + 2 x_R \dot{\beta}_R \dot{\theta}_y \right) \Big] \qquad (29)$$

$\overset{\circ}{( )} = \frac{d}{d\psi}$

(15)

$$P_{Iz2k} = m\Omega^2 \Big\{ -x_k \overset{\circ\circ}{\beta}_k + \cos\psi \Big[-2x_k \overset{\circ}{\zeta}_k \overset{\circ}{\theta}_x + (x_k + e)(\overset{\circ\circ}{\theta}_y - 2\overset{\circ}{\theta}_x) - 2x_k \zeta_k (\overset{\circ\circ}{\theta}_x + 2\overset{\circ}{\theta}_y) \Big]$$

$$+ \sin\psi_k \Big[-2x_k \overset{\circ}{\zeta}_k \overset{\circ}{\theta}_y - (x_k + e)(\overset{\circ\circ}{\theta}_x + 2\overset{\circ}{\theta}_y) - x_k \zeta_k (\overset{\circ\circ}{\theta}_y - 2\overset{\circ}{\theta}_x) \Big] \Big\} \quad ($$

The distributed inertia moments per unit length of the k th blade are also obtained by D'Alemberts principle by taking the integral of the vector product given below

$$\underset{\sim}{q}_{Ik} = \iint_{A_T} \left[-\rho \left(y_{0k} \hat{e}_{y4k} + z_{0k} \hat{e}_{z4k} \right) \times \underset{\sim}{a}_{pk} \right] dA \qquad (31)$$

These moments in the component form in the $2k$-system are given below

$$q_{Ix2k} = \iint_{A_T} -\rho \left[y_{0k} a_{pz2k} - (z_{0k} - y_{0k} \beta_k \zeta_k) a_{py2k} \right] dA =$$

$$= \Omega^2 \Big\{ (I_{MB3} \cos^2\theta_{Gk} + I_{MB2} \sin^2\theta_{Gk}) \Big[\overset{\circ\circ}{\zeta}_k \beta_k + 2\overset{\circ}{\zeta}_k \overset{\circ}{\beta}_k + \zeta_k \overset{\circ\circ}{\beta}_k + \beta_k \zeta_k + \cos\psi_k (-\overset{\circ\circ}{\theta}_x + 2\overset{\circ}{\theta}_y + \zeta_k (\overset{\circ\circ}{\theta}_y - 2\overset{\circ}{\theta}_x) + 2\overset{\circ}{\zeta}_k \overset{\circ}{\theta}_y)$$

$$+ \sin\psi_k (-\overset{\circ\circ}{\theta}_y + 2\overset{\circ}{\theta}_x + \zeta_k (\overset{\circ\circ}{\theta}_x + 2\overset{\circ}{\theta}_y) - 2\overset{\circ}{\zeta}_k \overset{\circ}{\theta}_x) \Big]$$

$$+ (I_{MB3} \sin^2\theta_{Gk} + I_{MB2} \cos^2\theta_{Gk}) \left[-2\overset{\circ}{\beta}_k - \overset{\circ\circ}{\theta}_x \cos\psi_k - \overset{\circ\circ}{\theta}_y \sin\psi_k \right]$$

$$+ (I_{MB3} - I_{MB2}) \sin\theta_{Gk} \cos\theta_{Gk} (-1 - 2\overset{\circ}{\zeta}_k) \Big\} \qquad (32) \checkmark$$

$$q_{Iy2k} = \iint_{A_T} -\rho \Big[(z_{0k} - y_{0k} \beta_k \zeta_k) a_{px2k}$$

$$- (y_{0k} \zeta_k - z_{0k} \beta_k) a_{pz2k} \Big] dA =$$

$$= \Omega^2 \Big\{ (I_{MB3} \cos^2\theta_{Gk} + I_{MB2} \sin^2\theta_{Gk}) \Big[\beta_k \zeta_k^2 + 2\zeta_k \overset{\circ}{\zeta}_k \overset{\circ}{\beta}_k + \zeta_k \overset{\circ\circ}{\beta}_k$$

(16)

$$+\cos\psi_k\left\langle -\zeta_k(\ddot{\theta}_x + 2\dot{\theta}_y + \zeta_k(\ddot{\theta}_y - 2\dot{\theta}_x) + 2\dot{\zeta}_k\dot{\theta}_y)\right\rangle$$

$$+\sin\psi_k\left\langle -\zeta_k(\ddot{\theta}_y - 2\dot{\theta}_x - \zeta_k(\ddot{\theta}_x + 2\dot{\theta}_y) - 2\dot{\zeta}_k\dot{\theta}_x\right\rangle\Big]$$

$$+(I_{MB3}\sin^2\theta_{GR} + I_{MB2}\cos^2\theta_{GR})\left[\ddot{\beta}_k - \beta_k - \ddot{\theta}_y\cos\psi_k + \ddot{\theta}_x\sin\psi_k\right]$$

$$+(I_{MB3} - I_{MB2})\sin\theta_{GR}\cos\theta_{GR}\left[-\zeta_k - \ddot{\zeta}_k - \beta_k(\ddot{\theta}_x + 2\dot{\theta}_y)\cos\psi_k\right.$$

$$\left.-\sin\psi_k\,\beta_k(\ddot{\theta}_y - 2\dot{\theta}_x)\right]\Big\} \qquad (33)$$

$$q_{IZ2k} = \iint_{A_T} -\rho\left[(-y_{0k}\zeta_k - z_{0k}\beta_k)a_{py2} - y_{0k}a_{px2k}\right]dA =$$

$$= \Omega^2\Big\{(I_{MB3}\cos^2\theta_{GR} + I_{MB2}\sin^2\theta_{GR})(-\ddot{\zeta}_k - 2\zeta_k\dot{\zeta}_k)$$

$$+(I_{MB3}\sin^2\theta_{GR} + I_{MB2}\cos^2\theta_{GR})\left[-\beta_k 2\dot{\beta}_k - \beta_k\ddot{\theta}_x\cos\psi_k - \beta_k\ddot{\theta}_y\sin\psi_k\right.$$

$$+(I_{MB3} - I_{MB2})\cos\theta_{GR}\sin\theta_{GR}\left[-\ddot{\beta}_k - 2\zeta_k\dot{\beta}_k - 2\dot{\zeta}_k\beta_k +\right.$$

$$\left.+\cos\psi_k(\ddot{\theta}_y - \ddot{\theta}_x\zeta_k) + \sin\psi_k(-\ddot{\theta}_x - \ddot{\theta}_y\zeta_k)\right]\Big\} \qquad (34)$$

2.4.2 Distributed Aerodynamic Loads

Greenberg (Ref. 7) has derived expressions for unsteady lift and moment on a two dimensional airfoil executing harmonic motion, in pitch and plunge in a pulsating stream of incompressible fluid. Greenberg's theory represents an extension of Theodorsen's unsteady aerodynamic theory (Ref. 8) to account for constant angle of attack on pulsating oncoming velocity. The lift and moment expressions consist of two contributions,

기본적으로 예전과 같지만! ~~[illegible]~~ θx θy term 에 유의하라

Ground Resonance 는 Aero term을 다 무시할수 있다.

(17) a circulatory part and a noncirculatory part. Greenberg has assumed that both circulatory and non circulatory parts of the lift are acting in the same direction, i.e. normal to the resultant flow. However other researchers using this theory have introduced their own interpretations. For example Hodges and Ormiston (Ref 9) assumed that the circulatory lift acts normal to the resultant flow and the noncirculatory lift acts normal to the blade chord. An examination of the alternative mathematical expressions for unsteady lift indicates that assuming the noncirculatory part of the lift to be perpendicular to the blade chord is somewhat more convenient. In this derivation it is assumed that the circulatory lift acts normal to the resultant flow and the noncirculatory lift acts normal to the blade chord, for mathematical convenience.

The lift and moment expressions given by Greenberg are (see Fig 10)

$$L_C = 2\pi\rho_A bV\{ V_0\alpha + \sigma V_0\alpha\, C(k_v)\, e^{i\omega_v t} + [b(\tfrac{1}{2}-a)\dot\beta + V_0\beta]\, C(k_\beta) + \dot h\, C(k_h) + \sigma V_0\beta\, C(k_{v+\beta})\, e^{i\omega_v t} \} \quad (35)$$

$$L_{NC} = \pi\rho_A b^2 [\ddot h + V\dot\beta + \dot V(\alpha+\beta) - ab\ddot\beta] \quad (36)$$

where L_C is the circulatory lift and L_{NC} is the noncirculatory lift

Referring to Fig - 11

h - is the vertical displacement of the axis of rotation (positive downward)

α - is the constant part of the angle of attack

β - is the time varying part of the angle of attack

V_0 - is the constant part of the free stream velocity

$\sigma V_0 e^{i\omega_v t}$ - is the pulsating part of the free stream velocity and $V = V_0(1+\sigma e^{i\omega_v t})$

(18) ba – is the position of the torsion axis (axis of rotation of the airfoil) measured from the center of the airfoil section

The total moment due to both circulatory and noncirculatory parts is given by

$$M = \pi\rho_A b^2 [ba\ddot{V} + \dot{V}ba(\alpha+\beta) - Vb(\tfrac{1}{2}-a)\dot{\beta} - b^2(\tfrac{1}{8}+a^2)\ddot{\beta}]$$

$$+2\pi\rho_A Vb^2(a+\tfrac{1}{2})\{V_0\alpha + \sigma V_0\alpha\, C(k_v)e^{i\omega_v t} + [b(\tfrac{1}{2}-a)\dot{\beta}+V_0\beta]C(k$$

$$+\dot{h}\,C(k_h) + \sigma V_0\beta\, C(k_{v+\beta})e^{i\omega_v t}\} \qquad (37)$$

where M is the pitching moment about the axis of rotation (positive nose up).

For low frequency oscillations of rotor blades, such as encountered in aeromechanical problems, the reduced frequency k is low and one can introduce the quasi-steady assumption that $C(k) \cong 1.0$. Furthermore from Figs 3 and 10 one has

$$ba = -b + x_A + \frac{b}{2} = x_A - \frac{b}{2} \qquad (38)$$

Using Eq (38) and replacing 2π by the incompressible lift curve slope a_i, and replacing $\alpha+\beta = \bar{\alpha}$ one can rewrite Eqs (35), (36) and (37) as

$$L_C = \rho_A a_i bV[\dot{h} + V\bar{\alpha} + (b - x_A)\dot{\bar{\alpha}}] \qquad (39)$$

$$L_{NC} = \rho_A a_i b^2[\ddot{h} + V\dot{\bar{\alpha}} + \dot{V}\bar{\alpha} - (x_A - \tfrac{b}{2})\ddot{\bar{\alpha}}] \qquad (40)$$

$$M = \tfrac{1}{2}\rho_A a_i b^2\{(x_A - \tfrac{b}{2})[\ddot{h} + \dot{V}\bar{\alpha} + V\dot{\bar{\alpha}} - (x_A - \tfrac{b}{2})\ddot{\bar{\alpha}}] - \frac{Vb}{2}\dot{\bar{\alpha}} - \frac{1}{8}b^2\ddot{\bar{\alpha}}\}$$
$$+ a_i\rho_A bVx_A[\dot{h} + V\bar{\alpha} + (b - x_A)\dot{\bar{\alpha}}] \qquad (41)$$

where now $\dot{\bar{\alpha}} = \dot{\beta}$ and $\ddot{\bar{\alpha}} = \ddot{\beta}$.

(19) Next the various velocity components, relative to the oscillating rotor blade have to be identified. Let $\underset{\sim}{V}_{AR}$ be the free stream velocity vector and $\underset{\sim}{V}_{ECk}$ be the velocity vector at any point on the elastic axis of the k^{th} blade due to its oscillation, the net air flow velocity for the k^{th} blade is then

$$\underset{\sim}{V}_k = \underset{\sim}{V}_{Ak} - \underset{\sim}{V}_{ECk} \qquad (42)$$

for a rotor blade in hover the only contribution to $\underset{\sim}{V}_{Ak}$ is due to inflow thus

$$\underset{\sim}{V}_{Ak} = -\Omega R \lambda_k \hat{e}_{RZ} \qquad (43)$$

This velocity can be written in terms of components along the Zk syste

$$\underset{\sim}{V}_{AZk} = \Omega R \{ (\lambda_k \theta_y \cos\psi_k - \lambda_k \theta_x \sin\psi_k) \hat{e}_{xZk}$$

$$+ (-\lambda_k \theta_x \cos\psi_k + \lambda_k \theta_y \sin\psi_k) \hat{e}_{yZk} - \lambda_k \hat{e}_{zZk} \} \qquad (44)$$

The velocity at any point on the elastic axis due to blade deformation is given by

$$\underset{\sim}{V}_{ECk} = \dot{\underset{\sim}{R}}_{OH} + \dot{\underset{\sim}{r}}_{pk} + \underset{\sim}{\omega}_k \times \underset{\sim}{r}_{pk} \qquad (45)$$

where $\dot{\underset{\sim}{R}}_{OH}$ is the velocity at the hub center, $\dot{\underset{\sim}{r}}_{pk}$ is the velocity of the point 'p' on the elastic axis of the blade in the rotating reference frame and $\underset{\sim}{\omega}_k$ is the angular velocity vector of the rotating reference frame recall

$$\underset{\sim}{r}_{pk} = (x_k + e + u_k)\hat{e}_{xZk} + w_k \hat{e}_{zZk} + v_k \hat{e}_{yZk} \qquad (46)$$

(20)

Using the rigid blade assumption

$w_k = x_k \beta_k \quad v_k = \zeta_k x_k$ and

$$\dot{r}_{pk} = \Omega \left[\dot{u}_k \hat{e}_{x2k} + x_k \dot{\zeta}_k \hat{e}_{y2k} + x_k \dot{\beta}_k \hat{e}_{z2k} \right] \quad (47)$$

Since we only consider the fuselage rotations θ_x, θ_y this implies that $\dot{R}_{OH} = 0$

$$\omega_k \times r_{p2k} = \Omega \{ \hat{e}_{x2k} [-x_k \zeta_k + \dot{\theta}_y \beta_k x_k \cos\psi_k - $$
$$ - \dot{\theta}_x \beta_k x_k \sin\psi_k] + \hat{e}_{y2k} (x_k + e) + \hat{e}_{z2k} [x_k \zeta_k (\dot{\theta}_x \cos\psi_k $$
$$ + \dot{\theta}_y \sin\psi_k) - (x_k + e)(\dot{\theta}_y \cos\psi_k - \dot{\theta}_x \sin\psi_k)] \quad (48)$$

Substituting Eqs (47) and (48) into Eq. (45)

$$V_{ECk} = \Omega \hat{e}_{x2k} \left[-x_k \zeta_k + \dot{u}_k + \cos\psi_k \, x_k \beta_k - \dot{\theta}_x x_k \beta_k \sin\psi_k \right]$$
$$ + \Omega \hat{e}_{y2k} \left[x_k \dot{\zeta}_k + x_k + e \right] + \Omega \hat{e}_{z2k} \left[x_k \dot{\beta}_k - (e + x_k)(\dot{\theta}_y \cos\psi_k \right.$$
$$ \left. - \dot{\theta}_x \sin\psi_k) + x_k \zeta_k (\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k) \right] \quad (49)$$

Substituting Eqs (44) and (49) into Eq (42) yields the velocity components in the 2k-system, after the application of the ordering scheme

$$V_{2k} = \Omega \{ \hat{e}_{x2k} [x_k \zeta_k - \dot{u}_k + \sin\psi_k (-R \lambda_k \theta_x + x_k \dot{\theta}_x \beta_k)$$
$$ + \hat{e}_{y2k} [-x_k - e - x_k \dot{\zeta}_k - \lambda_k R \theta_x \cos\psi_k] + $$

(21)

$$+\hat{e}_{zzk}\left[-x_k\dot{\beta}_k-\lambda_k R-\cos\psi_k\left(x_k \zeta_k \dot{\theta}_x-(x_k+e)\dot{\theta}_y\right)-\sin\psi_k\left(x_k\zeta_k\dot{\theta}_y+(x_k+e)\dot{\theta}_x\right)\right] \quad (50)$$

This velocity has to be transformed into the 4k-system whose origin is fixed at the deformed blade elastic axis. In the 4k-system the velocity components are

$$V_{x4k}=\Omega\left[e\zeta_k-\mathring{U}_k-x_k\beta_k\mathring{\beta}_k-\beta_k\lambda_k R-x_k\zeta_k\dot{\zeta}_k-\lambda_k R\theta_x\sin\psi_k\right] \quad ($$

$$V_{y4k}=\Omega\left[-x_k-e-x_k\dot{\zeta}_k\right] \quad (52)$$

$$V_{z4k}=\Omega\left\{-x_k\dot{\beta}_k-\lambda_k R-x_k\zeta_k\beta_k-\cos\psi_k\left[x_k\zeta_k\dot{\theta}_x-(e+x_k)\dot{\theta}_y\right]-\sin\psi_k\left[x_k\zeta_k\dot{\theta}_y+(e+x_k)\dot{\theta}_x\right]\right\} \quad (53)$$

For the evaluation of the unsteady aerodynamic forces and moments, the various velocity components in Eqs.(39)–(40) have to be identified.

(11) From Fig .... is absence of the elastic torsion ($\phi_k=0$)

$$V=-V_{y4k}\;;\quad \dot{h}=V_{z4k}\;;\quad \bar{\alpha}=\theta_{Gk}\;;\quad (\beta=\phi_k=0) \quad (54$$

Substituting these in the expressions for lift (moment is not needed since torsional degree of freedom is excluded)

$$L_{Ck}=\rho_A ab\,V_{y4k}\left[-V_{z4k}+V_{y4k}\theta_{Gk}\right] \quad (55)$$

$$L_{NCk}=\tfrac{1}{2}\rho_A ab^2\left[\dot{V}_{z4k}-\dot{V}_{y4k}\theta_{Gk}\right] \quad (56)$$

and the drag force is given by

$$D_k=\rho_A ab\left(\frac{C_{do}}{a}\right)\left[V_{y4k}^2+V_{z4k}^2\right] \quad (57)$$

(22) The inflow angle ϕ_{ik} is given by

$$\phi_{ik} = \tan^{-1}\left(\frac{V_{Z4k}}{V_{Y4k}}\right) \qquad (58)$$

According to the assumption made previously, the circulatory lift acts normal to the resultant flow and the non circulatory lift acts normal to the blade chord. Resolving the lift and drag

(11) along the 4k system (Fig. . .) the aerodynamic loads per unit length are given by

$$P_{A_{Y4k}} = -\rho_A V_{Y4k} ab\left[-V_{Z4k} + V_{Y4k}\theta_{Gk}\right]\sin\phi_{ik}$$

$$-\tfrac{1}{2}\rho_A ab^2\left[\dot{V}_{Z4k} - \dot{V}_{Y4k}\theta_{Gk}\right]\sin\theta_{Gk}$$

$$-\rho_A ab\left(\frac{C_{do}}{a}\right)\left[V_{Y4k}^2 + V_{Z4k}^2\right]\cos\phi_{ik} \qquad (59)$$

$$P_{AZ4k} = \rho_A ab V_{Y4k}\left[-V_{Z4k} + V_{Y4k}\theta_{Gk}\right]\cos\phi_{ik}$$

$$+\tfrac{1}{2}\rho_A ab^2\left[\dot{V}_{Z4k} - \dot{V}_{Y4k}\theta_{Gk}\right]\cos\theta_k - \rho_A ab\left(\frac{C_{do}}{a}\right)\left[V_{Y4k}^2 + V_{Z4k}^2\right]\sin \qquad (60$$

$\sin\phi_{ik}$ is small the following approximation can be made

$\sin\phi_{ik} \cong \phi_{ik} = V_{Z4k}/V_{Y4k}$ and $\cos\phi_{ik} = 1.0$

Recall that based on the ordering scheme

$$\frac{V_{Z4k}}{V_{Y4k}} = O(\epsilon) \;;\; \phi_{ik} = O(\epsilon) \;;\; \theta_{Gk} = O(\epsilon^{1/2}) \;;\; \frac{C_{do}}{a} = O(\epsilon^{3/2})$$

Using these approximation the aerodynamic loads per unit length can be written as

(23)

$$P_{AY4k} = \rho_A ab\, V_{Z4k}\left[V_{Z4k} - V_{Y4k}\,\theta_{Gk} \right] - \rho_A ab\left(\frac{C_{do}}{a}\right) V_{Y4k}^2$$

$$- \rho_A ab^2 \left[\dot{V}_{Z4k} - \dot{V}_{Y4k}\,\theta_{Gk} \right] \qquad (61)$$

$$P_{AZ4k} = \rho_A ab\, V_{Y4k}\left[-V_{Z4k} + V_{Y4k}\,\theta_{Gk} \right]$$

$$+ \frac{1}{2}\rho_A ab^2 \left[\dot{V}_{Z4k} - \dot{V}_{Y4k}\,\theta_{Gk} \right] \qquad (62)$$

The second underlined terms in Eqs (61) and (62) are the noncirculatory or apparent mass terms and these can be usually neglected in coupled rotor fuselage analyses. To obtain the loads the appropriate velocity components given by Eqs (51) – (53) have to be substituted into these equations.

$$P_{AY4k} = \rho_A b\Omega^2 \Big\{ \frac{C_{do}}{a}\left[x_k^2 (1 + 2\dot{\zeta}_k) + 2x_k e \right] - x_k^2 \Big[\dot{\beta}_k \big(\theta_{Gk} - \dot{\beta}_k + \dot{\zeta}_k \theta_{Gk} - \zeta_k \beta_k \big) + \zeta_k \beta_k (\theta_{Gk} - \dot{\beta}_k) \Big] - x_k \Big[\dot{\beta}_k (-\lambda_k R + e\theta_{Gk})$$

$$+ \lambda_k R (\theta_{Gk} - \dot{\beta}_k + \dot{\zeta}_k \theta_{Gk} - \zeta_k \beta_k) + \zeta_k \beta_k (-\lambda_k R) \Big] - R\lambda_k (-R\lambda_k + e\theta_k)$$

$$- x_k^2 \left[-\dot{\beta}_k \dot{\theta}_y - \zeta_k \dot{\theta}_x \theta_{Gk} + \dot{\theta}_y (\theta_{Gk} - \dot{\beta}_k) + \dot{\zeta}_k \theta_{Gk} \dot{\theta}_y \right] - x_k \Big[-\dot{\theta}_y \lambda_k R$$

$$+ \theta_{Gk} e \dot{\theta}_y - \lambda_k R \dot{\theta}_y + \dot{\theta}_y e \theta_{Gk} \Big] - x_k^2 \Big[\dot{\beta}_k \dot{\theta}_x - \theta_{Gk} \zeta_k \dot{\theta}_y -$$

$$- \dot{\theta}_x (\theta_{Gk} - \dot{\beta}_k) - \theta_{Gk} \dot{\zeta}_k \dot{\theta}_x \Big] - x_k \Big[\dot{\theta}_x \lambda_k R - \theta_{Gk} e \dot{\theta}_x + \lambda_k R \dot{\theta}_x - \dot{\theta}_x e \theta_{G}$$

$$- x_k^2\, 2\dot{\theta}_y \dot{\theta}_x + x_k^2 \dot{\theta}_y^2 \cos^2 \psi_k + x_k^2 \dot{\theta}_x^2 \sin^2 \psi_k \Big\} \qquad (63)$$

$$P_{AZ4k} = \rho_A ab\Omega^2 \Big\{ x_k^2 \Big[\theta_{Gk} - \dot{\beta}_k + \dot{\zeta}_k \theta_{Gk} - \zeta_k \beta_k + \dot{\zeta}_k (\theta_{Gk} - \dot{\beta}_k) \Big]$$

(24) $+x_k\left[-\lambda_k R + e\theta_{Gk} - \lambda_k R\dot{\zeta}_k + e(\theta_{Gk} - \dot{\beta}_k)\right] - e\lambda_k R + x_k^2\dot{\theta}_y \cos\psi_k$

$$- x_k^2\dot{\theta}_x \sin\psi_k \} \qquad (64)$$

These aerodynamic loads are transformed into components which are in the 2k system because the blade dynamic equations of equilibrium are written in the 2k system. The appropriate components are given below

$$p_{Ax2k} = -\zeta_k p_{Ay4k} - \beta_k p_{Az4k} \qquad (65)$$

$$p_{Ay2k} = p_{Ay4k} \qquad (66)$$

$$p_{Az2k} = p_{Az4k} - \beta_k \zeta_k p_{Ay4k} = p_{Az4k} \qquad (67)$$

$$q_{Ax2k} = q_{Ax4k} \; ; \; q_{Ay2k} = \zeta_k q_{Ax4k} \; ; \; q_{Az2k} = \beta_k q_{Ax4k} \qquad (68)$$

2.4.3 Distributed Structural Damping Loads

The structural damping incorporated in the analysis is of a viscous equivalent type. The damping forces per unit length of the blade for flap and lag can be written as

$$\text{flap} \qquad p_{Dz2k} = \Omega x_k \dot{\beta}_k g_{SF} \qquad (69)$$

$$\text{lead-lag} \qquad p_{Dy2k} = -\Omega x_k \dot{\zeta}_k g_{SL} \qquad (70)$$

25 2.4.4 Rotor Blade Equations of Motion

In this section the individual blade equations of motion are presented. For the rigid, offset hinged, spring restrained blade model used in this study, the distributed inertia, aerodynamic and structural damping loads are integrated over the length of the blade and moment equilibrium at the spring restrained hinge is enforced. The expressions for the loads integrated over the span are provided below.

The forces and moments at the root of the blade, due to the distributed inertia loads are given by

$$\underset{\sim}{P}_{I2k} = \int_0^{R-e} \underset{\sim}{p}_{I2k}\, dx_k \qquad (71)$$

$$\underset{\sim}{Q}_{I2k} = \int_0^{R-e} \left(\underset{\sim}{q}_{I2k} + \underset{\sim}{r}_{p2k} \times \underset{\sim}{p}_{I2k} \right) dx_k \qquad (72)$$

where $\underset{\sim}{p}_{I2k}$ are the distributed inertia loads on the k^{th} blade (per unit length) and $\underset{\sim}{q}_{I3k}$ is the distributed inertia moment about the elastic axis, given by Eqs (28)–(30) and (32)–(34) respectively. Also recall that

$$\underset{\sim}{r}_{p2k} = (x_k + u_k)\,\hat{e}_{x2k} + x_k \zeta_k\, \hat{e}_{y2k} + x_k \beta_k\, \hat{e}_{z2k} \qquad (73)$$

and u_k is the axial displacement, primarily due to geometric shortening

$$u_k = -\frac{1}{2} \int_0^{x_k} (\zeta_k^2 + \beta_k^2)\, dx_k = -\frac{1}{2} x_k (\zeta_k^2 + \beta_k^2) \qquad (74)$$

and

$$\dot{u}_k = -x_k (\zeta_k \dot{\zeta}_k + \beta_k \dot{\beta}_k) \qquad (75)$$

It is assumed that inflow is constant over the disk, and the pitch setting is constant and independent of x_k. Mass per unit length of the blade is also assumed to be a constant. Thus when performing integrations over the blade span these quantities are considered as constants.

The components of the inertia forces at the root of the blade in the 2k-system are given by

$$p_{IX2k} = m\Omega^2 \left[\frac{(R-e)^2}{2} + e(R-e) + \frac{(R-e)^2}{2} \dot{\zeta}_k^2 \right] \qquad (76)$$

$$p_{Iy2k} = m\Omega^2 \Big[\frac{(R-e)^2}{2}(\zeta_k - \ddot{\zeta}_k) + \frac{2(R-e)^2}{2}(\zeta_k\dot{\zeta}_k + \beta_k\dot{\beta}_k) + \cos\psi_k \frac{(R-e)^2}{2}(\beta_k\ddot{\theta}_x + 2\dot{\beta}_k\dot{\theta}_x) + \sin\psi_k \frac{(R-e)^2}{2} 2\dot{\beta}_k\dot{\theta}_y \Big] \qquad (77)$$

$$p_{Iz2k} = m\Omega^2 \Big\{ -\frac{(R-e)^2}{2}\ddot{\beta}_k + \cos\psi_k \Big\langle (R-e)e(\ddot{\theta}_y - 2\dot{\theta}_x) - \frac{(R-e)^2}{2}\big[2\dot{\zeta}_k\dot{\theta}_x - \ddot{\theta}_y + 2\dot{\theta}_x + \zeta_k(\ddot{\theta}_x + 2\dot{\theta}_y)\big] \Big\rangle + \sin\psi_k \Big\langle -(R-e)e(\ddot{\theta}_x + 2\dot{\theta}_y) - \frac{(R-e)^2}{2}\big[2\dot{\zeta}_k\dot{\theta}_y + (\ddot{\theta}_x + 2\dot{\theta}_y) + \zeta_k(\ddot{\theta}_y - 2\dot{\theta}_x)\big] \Big\rangle \Big\} \qquad (78)$$

After applying the ordering scheme, the inertia moment components are given below

$$Q_{IX2k} = \int_0^{R-e} \left(q_{I2k} + x_k\zeta_k\, p_{Iz2k} - x_k\beta_k\, p_{Iy2k} \right) dx_k =$$

$$= \zeta_k\Big\{ m\Omega^2\Big[-\frac{(R-e)^3}{3}\ddot{\beta}_k + \cos\psi_k \Big\langle -\frac{(R-e)^3}{3} 2\dot{\zeta}_k\dot{\theta}_x + \frac{(R-e)^3}{3}(\ddot{\theta}_x + 2\dot{\theta}_y) - \frac{(R-e)^2}{2}e(\ddot{\theta}_x + 2\dot{\theta}_y) - \frac{(R-e)^3}{3}\zeta_k(\ddot{\theta}_y - 2\dot{\theta}_x) \Big\rangle \Big] \Big\}$$

$$-\beta_k\Big\{ m\Omega^2 \Big[\frac{(R-e)^3}{3}\zeta_k - \frac{(R-e)^3}{3}\ddot{\zeta}_k + \frac{(R-e)^3}{3} 2(\zeta_k\dot{\zeta}_k + \beta_k\dot{\beta}_k) +$$

$$+ \cos\psi_k \left\langle \frac{(R-e)^3}{3}\beta_k \ddot{\theta}_x + \frac{(R-e)^3}{3} 2\dot{\beta}_k \dot{\theta}_x \right\rangle + \sin\psi_k \left\langle \frac{(R-e)^3}{3}\beta_k \ddot{\theta}_y \right.$$

$$\left. + \frac{(R-e)^3}{3} 2\dot{\beta}_k \dot{\theta}_y \right\rangle\Big]\Big\} + \Omega^2 (R-e)\Big\{ \left(I_{MB3}\cos^2\theta_{Gk} + I_{MB2}\sin^2\theta_{Gk} \right) \Big[\ddot{\zeta}_k \beta_k$$

$$+ 2\dot{\zeta}_k \dot{\beta}_k + \zeta_k \ddot{\beta}_k + \beta_k \zeta_k + \cos\psi_k \left\langle -\left(\ddot{\theta}_x + 2\dot{\theta}_y + \zeta_k(\ddot{\theta}_y - 2\dot{\theta}_x) + 2\dot{\zeta}_k \dot{\theta}_y \right) \right\rangle$$

$$+ \sin\psi_k \left\langle -\left(\ddot{\theta}_y - 2\dot{\theta}_x - \zeta_k(\ddot{\theta}_x + 2\dot{\theta}_y) - 2\dot{\zeta}_k \dot{\theta}_x \right) \right\rangle \Big]$$

$$+ \left(I_{MB3}\sin^2\theta_{Gk} + I_{MB2}\cos^2\theta_{Gk} \right) \left[-2\dot{\beta}_k - \ddot{\theta}_x \cos\psi_k - \ddot{\theta}_y \sin\psi_k \right]$$

$$+ \left(I_{MB3} - I_{MB2} \right) \sin\theta_{Gk} \cos\theta_{Gk} \left(1 - 2\dot{\zeta}_k \right) \Big\} \qquad (79)$$

$$Q_{IY2k} = \int_0^{R-e} \left[q_{IY2k} + x_k \beta_k P_{IX2k} - (x_k + u_k) P_{IZ2k} \right] dx_k =$$

$$= \beta_k \, m\Omega^2 \left[\frac{(R-e)^3}{3} + \frac{(R-e)^2}{2} e + \frac{(R-e)^3}{3} 2\dot{\zeta}_k \right]$$

$$- m\Omega^2 \Big[-\frac{(R-e)^3}{3} \ddot{\beta}_k + \cos\psi_k \Big\langle -\frac{(R-e)^3}{3} 2\dot{\zeta}_k \dot{\theta}_x + \frac{(R-e)^3}{3} (\ddot{\theta}_y - 2\dot{\theta}_x)$$

$$+ \frac{(R-e)^2}{2} e (\ddot{\theta}_y - 2\dot{\theta}_x) - \frac{(R-e)^3}{3} \zeta_k (\ddot{\theta}_x + 2\dot{\theta}_y) \Big\rangle$$

$$+ \sin\psi_k \Big\langle -\frac{(R-e)^3}{3} 2\dot{\zeta}_k \dot{\theta}_y - \frac{(R-e)^3}{3} (\ddot{\theta}_x + 2\dot{\theta}_y) - \frac{(R-e)^2}{2} e (\ddot{\theta}_x - 2\dot{\theta}_y)$$

$$- \frac{(R-e)^3}{3} \zeta_k (\ddot{\theta}_y - 2\dot{\theta}_x) \Big\rangle \Big] \qquad (80)$$

$$Q_{IZ2k} = \int_0^{R-e} \left[q_{IZ2k} + (x_k + u_k) P_{IY2k} - x_k \zeta_k P_{IX2k} \right] dx_k$$

$$= m\Omega^2 \Big[\frac{(R-e)^3}{3} \zeta_k - \frac{(R-e)^3}{3} \ddot{\zeta}_k + \frac{(R-e)^3}{3} 2(\zeta_k \dot{\zeta}_k + \beta_k \dot{\beta}_k) +$$

$$+ \cos\psi_k \left\langle \frac{(R-e)^3}{3} \beta_k \ddot{\theta}_x + \frac{(R-e)^3}{3} 2\dot{\beta}_k \dot{\theta}_x \right\rangle + \sin\psi_k \left\langle \frac{(R-e)^3}{3} \beta_k \ddot{\theta}_y + \right.$$

28 $$+\frac{(R-e)^3}{3} 2\dot{\beta}_k \dot{\theta}_y \rangle \Big] - z_k \Big\{ m\Omega^2 \Big[\frac{(R-e)^3}{3} + \frac{(R-e)^2}{2} e + \frac{(R-e)^3}{3} 2\dot{z}_k \Big\} \qquad (81)$$

The order of magnitude of the leading terms in these expressions for the loads are listed below for convenience

$$P_{IX2k} = O(1) \; ; \; P_{IY2k} = O(\epsilon) \; ; \; P_{IZ2k} = O(\epsilon)$$

$$Q_{IX2k} = O(\epsilon^2) ; \; Q_{IY2k} = O(\epsilon) \; ; \; Q_{IZ2k} = O(\epsilon)$$

The loads at the root of the blade due to distributed aerodynamic loads are

$$\underset{\sim}{P}_{A2k} = \int_0^{R-e} \underset{\sim}{P}_{A2k} \, dx_k \qquad (82)$$

and $$\underset{\sim}{Q}_{A2k} = \int_0^{R-e} \left(\underset{\sim}{q}_{A2k} + \underset{\sim}{r}_{p2k} \times \underset{\sim}{P}_{A2k} \right) dx_k \qquad (83)$$

The components of the force and moment vector at the root are given below

$$P_{AX2k} = -\rho_A ab\Omega^2 \beta_k \Big\{ \frac{(R-e)^3}{3} \Big[\theta_{Gk} - \dot{\beta}_k + \dot{z}_k \theta_{Gk} - z_k \beta_k + \dot{z}_k (\theta_{Gk} - \dot{\beta}_k) \Big]$$
$$+ \frac{(R-e)^2}{2} \Big[-\lambda_k R + e\theta_{Gk} - \lambda_k R \dot{z}_k + e(\theta_{Gk} - \dot{\beta}_k) \Big] - (R-e) e \lambda_k R$$
$$- \cos\psi_k \Big[-\frac{(R-e)^3}{3} \dot{\theta}_y \Big] - \sin\psi_k \Big[\frac{(R-e)^3}{3} \dot{\theta}_x \Big] \Big\}$$
$$- \rho_A ab\Omega^2 z_k \Big\{ -\frac{C_{d0}}{a} \Big[\frac{(R-e)^3}{3} \Big] - \frac{(R-e)^3}{3} \dot{\beta}_k (\theta_G - \dot{\beta}_k)$$
$$- \frac{(R-e)^2}{2} \Big[\dot{\beta}_k (-\lambda_k R) + \lambda_k R (\theta_{Gk} - \dot{\beta}_k) \Big] + (R-e)(\lambda_k R)^2 -$$
$$- \cos\psi_k \Big[-\frac{(R-e)^3}{3} \dot{\theta}_y \theta_{Gk} \Big] - \sin\psi_k \Big[-\frac{(R-e)^3}{3} (-\dot{\theta}_x \theta_{Gk}) \Big] \Big\} \qquad (84)$$

(29) $$P_{AY2k} = \rho_A ab\Omega^2\Big\{-\frac{e\delta_0}{a}\Big[\frac{(R-e)^3}{3}(1+2\dot{\zeta}_k) + 2\frac{(R-e)^2}{2}e\Big]$$

$$-\frac{(R-e)^3}{3}\Big[\dot{\beta}_k\langle\theta_{Gk} - \dot{\beta}_k + \dot{\zeta}_k\theta_{Gk} - \zeta_k\beta_k\rangle + \zeta_k\beta_k(\theta_{Gk} - \dot{\beta}_k)\Big]$$

$$-\frac{(R-e)^2}{2}\Big[\dot{\beta}_k\langle-\lambda_k R + e\theta_{Gk}\rangle + \lambda_k R\langle\theta_{Gk} - \dot{\beta}_k + \dot{\zeta}_k\theta_{Gk} - \zeta_k\beta_k\rangle$$

$$-\zeta_k\beta_k(\lambda_k R)\Big] + (R-e)\Big[+(\lambda_k R)^2 - \lambda_k Re\,\theta_{Gk}\Big] - \frac{(R-e)^3}{3}\Big[-\dot{\beta}_k\dot{\theta}_y -$$

$$-\zeta_k\dot{\theta}_x\theta_{Gk} + \dot{\theta}_y(\theta_{Gk} - \dot{\beta}_k) + \dot{\zeta}_k\theta_{Gk}\dot{\theta}_y\Big] - \frac{(R-e)^2}{2}\Big[-\dot{\theta}_y\lambda_k R + \theta_{Gk}e\dot{\theta}_y$$

$$-\lambda_k R\dot{\theta}_y + \dot{\theta}_y e\theta_{Gk}\Big] + (R-e)\;e\,\theta_{Gk} - \frac{(R-e)^3}{3}\Big[\dot{\beta}_k\dot{\theta}_x - \theta_{Gk}\zeta_k\dot{\theta}_y$$

$$-\dot{\theta}_x(\theta_{Gk} - \dot{\beta}_k) - \theta_{Gk}\dot{\zeta}_k\dot{\theta}_x\Big] - \frac{(R-e)^2}{2}\Big[\dot{\theta}_x\lambda_k R - \theta_{Gk}e\dot{\theta}_x$$

$$+\lambda_k R\dot{\theta}_x - \dot{\theta}_x e\theta_{Gk}\Big] - \frac{(R-e)^3}{3}2\dot{\theta}_y\dot{\theta}_x + \cos^2\psi_k\frac{(R-e)^3}{3}\dot{\theta}_y^2$$

$$+\sin^2\psi_k\frac{(R-e)^3}{3}\dot{\theta}_x^2\Big\} \qquad (85)$$

$$P_{AZ2k} = \rho_A ab\Omega^2\Big\{\frac{(R-e)^3}{3}\Big[\theta_{Gk} - \dot{\beta}_k + \dot{\zeta}_k\theta_{Gk} - \zeta_k\beta_k + \dot{\zeta}_k(\theta_{Gk} - \dot{\beta}_k)\Big]$$

$$+\frac{(R-e)^2}{2}\Big[-\lambda_k R + e\theta_{Gk} - \lambda_k R\dot{\zeta}_k + e(\theta_{Gk} - \dot{\beta}_k)\Big] - (R-e)e\lambda_k R$$

$$-\cos\psi_k\Big[-\frac{(R-e)^3}{3}\dot{\theta}_y\Big] - \sin\psi_k\frac{(R-e)^3}{3}\dot{\theta}_x \qquad (86)$$

The aerodynamic moments at the blade root are

$$Q_{AX2k} = \int_0^{R-e}(q_{AX2k} + x_k\zeta_k P_{AZ2k} - x_k\beta_k P_{AY2k})\,dx_k =$$

$$= \rho_A ab\Omega^2\zeta_k\Big\{\frac{(R-e)^4}{4}\Big[\theta_{Gk} - \dot{\beta}_k + \dot{\zeta}_k\theta_{Gk} - \zeta_k\beta_k + \dot{\zeta}_k(\theta_{Gk} - \dot{\beta})\Big]$$

(30)

$$+\frac{(R-e)^3}{3}\left[-\lambda_k R + e\,\theta_{Gk} - \lambda_k R\mathring{\zeta}_k + e(\theta_{Gk} - \mathring{\beta}_k)\right] + \frac{(R-e)^2}{2} e(-\lambda_k R)$$

$$+\cos\psi_k \frac{(R-e)^4}{4}\dot{\theta}_y - \sin\psi_k \frac{(R-e)^4}{4}\dot{\theta}_x \Big\}$$

$$-\rho_A a b \Omega^2 \beta_k (1+\zeta_k^2)\Big\{-\frac{C_{do}}{a}\left[\frac{(R-e)^4}{4}(1+2\mathring{\zeta}_k) + \frac{(R-e)^3}{3} 2e\right]$$

$$-\frac{(R-e)^4}{4}\left[\mathring{\beta}_k \langle \theta_{Gk} - \mathring{\beta}_k + \mathring{\zeta}_k\theta_{Gk} - \zeta_k\beta_k\rangle + \zeta_k\beta_k(\theta_{Gk} - \mathring{\beta}_k)\right]$$

$$-\frac{(R-e)^3}{3}\Big[+\mathring{\beta}_k(-\lambda_k R + e\,\theta_{Gk}) + \lambda_k R(\theta_{Gk} - \mathring{\beta}_k + \mathring{\zeta}_k\theta_{Gk} - \zeta_k\beta_k)$$

$$+\zeta_k\beta_k(-\lambda_k R)\Big] + \frac{(R-e)^2}{2}\left[-\lambda_k R(-\lambda_k R + e\,\theta_{Gk})\right]$$

$$-\frac{(R-e)^4}{4}\left[-\mathring{\beta}_k\dot{\theta}_y - \zeta_k\dot{\theta}_x\theta_{Gk} + \dot{\theta}_y(\theta_{Gk} - \mathring{\beta}_k) + \mathring{\zeta}_k\theta_{Gk}\dot{\theta}_y\right]$$

$$-\frac{(R-e)^3}{3}\left[\lambda_k R(-\dot{\theta}_y) + \theta_{Gk} e\dot{\theta}_y - \lambda_k R\dot{\theta}_y + \dot{\theta}_y e\,\theta_{Gk}\right]$$

$$-\frac{(R-e)^4}{4}\left[\mathring{\beta}_k\dot{\theta}_x - \theta_{Gk}\zeta_k\dot{\theta}_y - \dot{\theta}_x(\theta_{Gk} - \mathring{\beta}_k) - \theta_{Gk}\mathring{\zeta}_k\dot{\theta}_x\right]$$

$$-\frac{(R-e)^3}{3}\left[\dot{\theta}_x\lambda_k R - \theta_{Gk} e\dot{\theta}_x + \lambda_k R\dot{\theta}_x - \dot{\theta}_x e\,\theta_{Gk}\right] - \frac{(R-e)^4}{4} 2\dot{\theta}_y\dot{\theta}_x$$

$$+\cos^2\psi_k \frac{(R-e)^4}{4}\dot{\theta}_y^2 + \sin^2\psi_k \frac{(R-e)^4}{4}\dot{\theta}_x^2\Big\} \qquad (87)$$

$$Q_{Ay2k} = \int_0^{R-e}\left[q_{Ay2k} + x_k\beta_k p_{Ax2k} - (x_k + v_k)p_{Az2k}\right]dx_k$$

after neglecting the higher order terms

$$Q_{Ay2k} = \int_0^{R-e}(-x_k p_{Az2k})dx_k =$$

$$= -\rho_A a b \Omega^2\Big\{\frac{(R-e)^4}{4}\left[\theta_{Gk} - \mathring{\beta}_k + \mathring{\zeta}_k\theta_{Gk} - \zeta_k\beta_k + \mathring{\zeta}_k(\theta_{Gk} - \mathring{\beta}_k\right] -$$

$$+\frac{(R-e)^3}{3}\left[-\lambda_k R + e\,\theta_{Gk} - \lambda_k R\mathring{\zeta}_k + e(\theta_{Gk} - \mathring{\beta}_k)\right] + \frac{(R-e)^2}{2} e(-\lambda_k R)$$

$$+\cos\psi_k \frac{(R-e)^4}{4}\mathring{\theta}_Y - \sin\psi_k \frac{(R-e)^4}{4}\mathring{\theta}_X \Big\} \qquad (88)$$

$$Q_{AZ2k} = \int_0^{R-e}\left[q_{AZ2k} + (x_k + v_k)\,p_{Ay2k} - x_k\zeta_k p_{Ax2k}\right] dx_k$$

Neglecting higher order terms

$$Q_{AZ2k} = \int_0^{R-e}\left(x_k p_{Ay2k} - x_k \zeta_k p_{Ax2k}\right) dx_k$$

where $p_{Ax2k} = -\zeta_k p_{Ay4k} - \beta_k p_{Az4k} = -\zeta_k p_{Ay2k} - \beta_k p_{Az2k}$

thus

$$Q_{AZ2k} = \int_0^{R-e}\left(x_k p_{Ay2k} + x_k\beta_k\zeta_k p_{Az2k}\right) dx_k =$$

$$= \rho_A a b \Omega^2 \Big\{ -\frac{C_{do}}{a}\left[\frac{(R-e)^4}{4}(1+2\mathring{\zeta}_k) + \frac{(R-e)^3}{3} 2e\right]$$

$$-\frac{(R-e)^4}{4}\left[\mathring{\beta}_k(\theta_{Gk} - \mathring{\beta}_k + \mathring{\zeta}_k\theta_{Gk} - \zeta_k\beta_k) + \zeta_k\beta_k(\theta_{Gk} - \mathring{\beta}_k)\right]$$

$$-\frac{(R-e)^3}{3}\Big[\mathring{\beta}_k(-\lambda_k R + e\,\theta_{Gk}) + \lambda_k R(\theta_{Gk} - \mathring{\beta}_k + \mathring{\zeta}_k\theta_{Gk} - \zeta_k\beta_k)$$

$$+\zeta_k\beta_k(-\lambda_k R)\Big] + \frac{(R-e)^2}{2}(-\lambda_k R)(-\lambda_k R + e\,\theta_{Gk})$$

$$-\frac{(R-e)^4}{4}\left[-\mathring{\beta}_k\mathring{\theta}_Y - \zeta_k\mathring{\theta}_X\theta_{Gk} + \mathring{\theta}_Y(\theta_{Gk} - \mathring{\beta}_k) + \mathring{\zeta}_k\theta_{Gk}\mathring{\theta}_Y\right]$$

$$-\frac{(R-e)^3}{3}\left[(-\lambda_k R\mathring{\theta}_Y) + \theta_{Gk} e\,\mathring{\theta}_Y - \lambda_k R\mathring{\theta}_Y + \mathring{\theta}_Y e\,\theta_{Gk}\right]$$

$$-\frac{(R-e)^4}{4}\left[\mathring{\beta}_k\mathring{\theta}_X - \theta_{Gk}\zeta_k\mathring{\theta}_Y - \mathring{\theta}_X(\theta_{Gk} - \mathring{\beta}_k) - \theta_{Gk}\mathring{\zeta}_k\mathring{\theta}_X\right]$$

$$-\frac{(R-e)^3}{3}\left[\mathring{\theta}_X\lambda_k R - \theta_{Gk} e\,\mathring{\theta}_X + \lambda_k R\mathring{\theta}_X - \mathring{\theta}_X e\,\theta_{Gk}\right]$$

(32) $- \frac{(R-e)^4}{4} 2\dot{\theta}_Y\dot{\theta}_X + \cos^2\psi_k \frac{(R-e)^4}{4} \dot{\theta}_Y^2 + \sin^2\psi_k \frac{(R-e)^4}{4} \dot{\theta}_X^2 \Big\}$

$$+ \rho_A a b \Omega^2 \beta_k \zeta_k \left\{ \frac{(R-e)^4}{4} (\theta_{Gk} - \dot{\beta}_k) + \frac{(R-e)^3}{3} (-\lambda_k R) \right\} \qquad (89)$$

The orders of magnitude of the leading aerodynamic terms in the various aerodynamic load expressions are given for convenience

$$p_{AX2k} = O(\epsilon^{3/2}) \ ; \ p_{AY2k} = O(\epsilon^{3/2}) \ ; \ p_{AZ2k} = O(\epsilon^{1/2})$$

$$Q_{AX2k} = O(\epsilon^{3/2}) \ ; \ Q_{AY2k} = O(\epsilon^{1/2}) \ ; \ Q_{AZ2k} = O(\epsilon^{3/2})$$

Next we consider the damping force. Instead of assuming a distributed damping force representing the structural damping of the blade, one can assume a damping force proportional to the velocity, which is provided at the root of the blade. This damping force can be written as

$$\text{flap:} \quad Q_{DY2k} = \Omega \dot{\beta}_k g_{SF}$$
$$\text{Lag:} \quad Q_{DZ2k} = -\Omega \dot{\zeta}_k g_{SL} \qquad (90)$$

These damping forces are consistent with assumption of an offset hinged, spring restrained blade

Using the moment equilibrium conditions at the blade root, the equation of motion for the k^{th} blade can be written, symbolically, as follows

$$\text{Flap:} \quad M_{\beta k} + Q_{IY2k} + Q_{AY2k} + Q_{DY2k} = 0 \qquad (91)$$

$$\text{Lag:} \quad M_{\zeta k} + Q_{IZ2k} + Q_{AZ2k} + Q_{DZ2k} = 0 \qquad (9?)$$

(33) The elastic restoring moments due to the root springs can be found in Refs (3) and (5) and are given by the following expressions

$$M_{\beta k} = \beta_k \left[K_\beta + (K_\zeta - K_\beta)\sin^2\theta_{GR} \right] + \zeta_k (K_\zeta - K_\beta)\sin\theta_{GR}\cos\theta_{GR} \quad (93)$$

$$M_{\zeta k} = -\zeta_k \left[K_\zeta - (K_\zeta - K_\beta)\sin^2\theta_{GR} \right] - \beta_k (K_\zeta - K_\beta)\sin\theta_{GR}\cos\theta_{GR} \quad (94)$$

To solve the coupled rotor fuselage problem the blade equations have to be coupled with the fuselage equations and then the complete system equations can be solved as described in the paper which was handed out in the class.

(34)

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Appendix A — Notation

A_T – total cross-sectional area of blade
a – lift curve slope
a_{pk} – acceleration of a point p on the k^{th} blade
b – blade semichord
C_{do} – drag coefficient of the blade
$C(k)$ – Theodorsen's lift deficiency function
D_k – Drag force per unit length on the k^{th} blade
e – blade offset
$\hat{e}_x, \hat{e}_y, \hat{e}_z$ – unit vectors along x, y, z axes
g_{SF}, g_{SL} – damping coefficients
I_{MB2}, I_{MB3} – principal moments of inertia per unit length of the blade about cross sectional axes.
K_β, K_ζ – root spring stiffness in flap and lag, representing blade stiffness.
L_C – circulatory part of lift
L_{NC} – noncirculatory part of lift
m – mass per unit length of blade
p_{Ik}, p_{Ak}, p_{Dk} – distributed blade inertia, aerodynamic and damping forces
q_{Ik}, q_{Ak}, q_{Dk} – distributed blade inertia, aerodynamic and damping moments
P_{Ik}, P_{Ak} – inertia, aerodynamic forces of the blade
Q – moment
Q_{Ik}, Q_{Ak}, Q_{Dk} – inertia, aerodynamic and damping moments of the rotor blade
R – rotor radius
r_{pk} – position vector of a point 'p' on the kth blade
t – time
u_k, v_k, w_k – k^{th} blade deformation in axial, lead-lag and flap directions.

V_{Ak} - free stream velocity
X_A, X_I, X_T - offsets (all zero)
y_{ok}, z_{ok} - blade cross sectional coordinate
β_k - flap angle k^{th} blade
ζ_k - lead lag angle k^{th} blade
ϵ - basis for ordering scheme
η_{ok}, ξ_{ok} - blade crosssectional principal axis coordinate
θ_{Gk} - collective pitch setting on blade
θ_Y, θ_X - rigid body pitch and roll angles (perturbations)
λ_k - inflow ratio of k^{th} blade
ρ - density of blade material
ρ_A - density of air
ϕ_{ik} - inflow angle k^{th} blade
ψ_k - azimuth angle k^{th} blade
Ω = rotor R.P.M

Symbols - special

$\overset{\circ}{(\ )} = \frac{\partial}{\partial \psi}$

$\underset{\sim}{(\ )}$ - vector

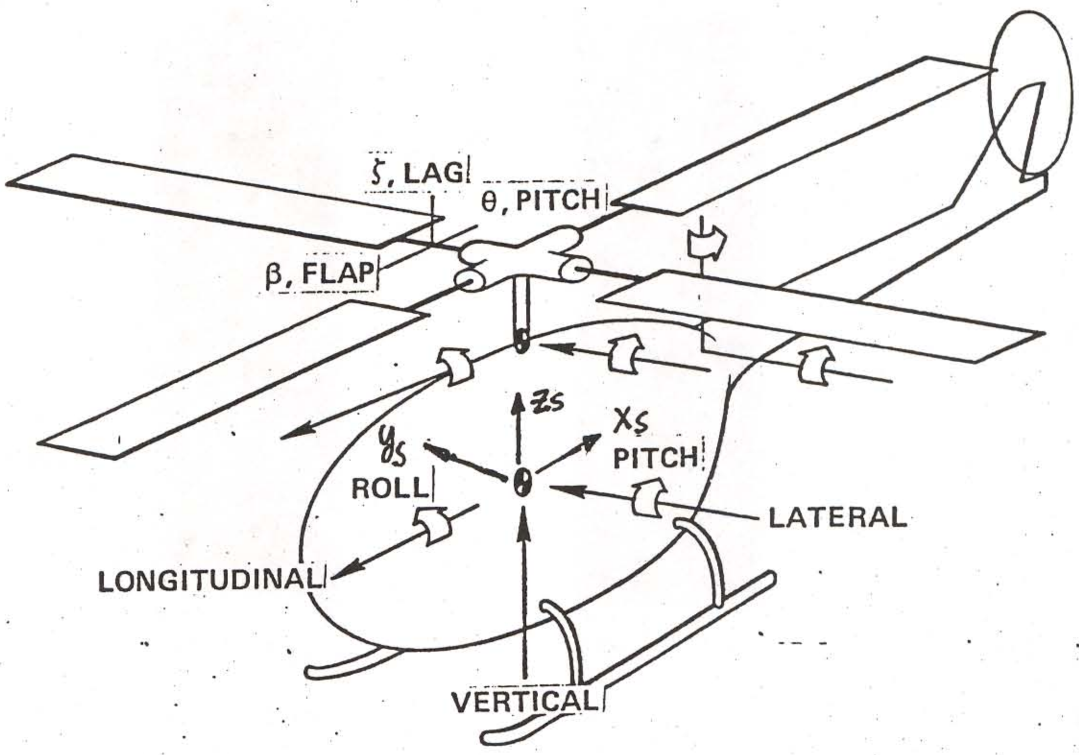


FIG. 1 COUPLED ROTOR/FUSELAGE GEOMETRY

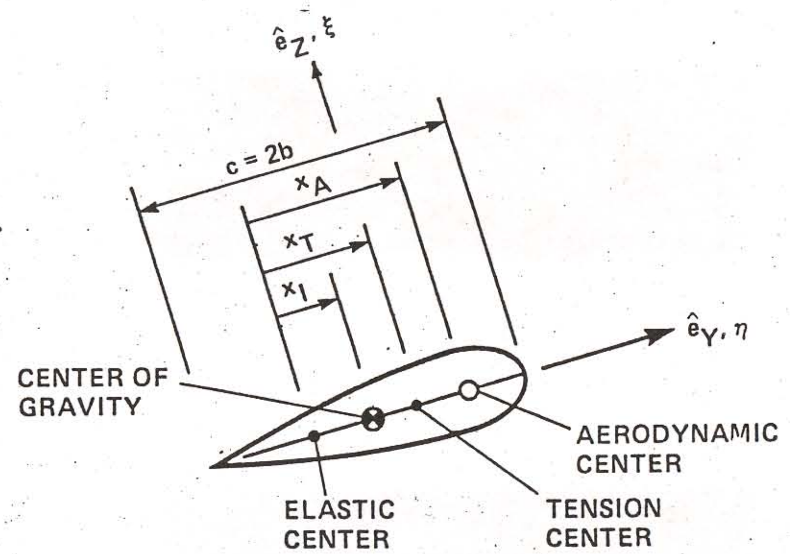


Figure 3. Blade Cross-Section Configuration

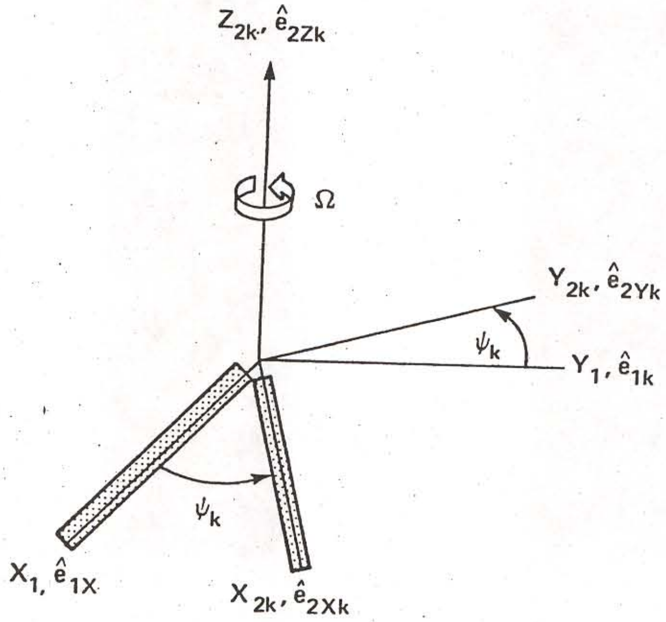


Figure 4. Disturbed Rotor Hub and Rotor Blade Coordinate Systems

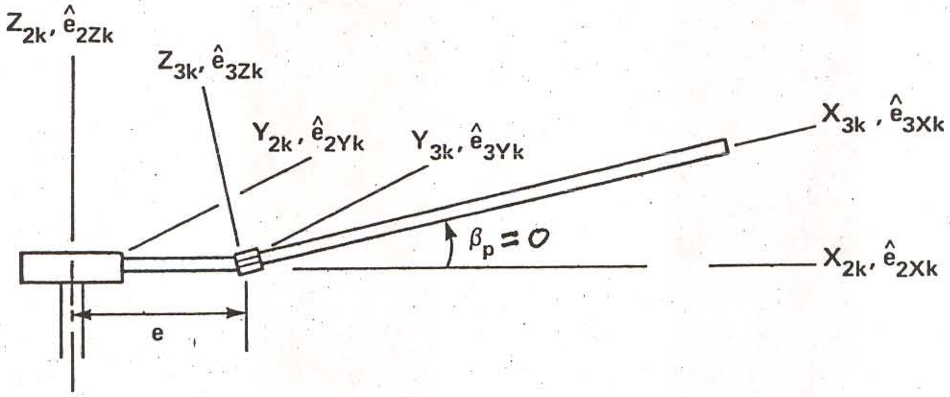


Figure 5. Rotor Blade Coordinate Systems

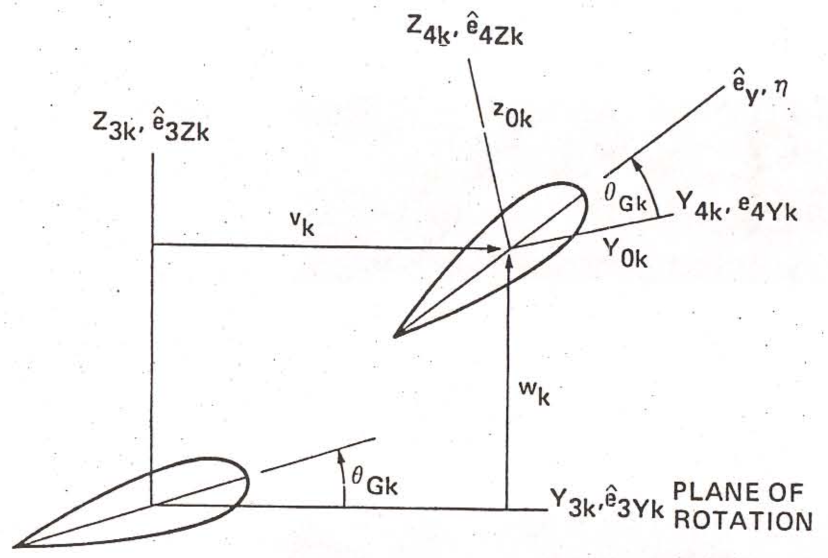


Figure 6. Undeformed and Deformed k^{th} Blade Cross-Sections

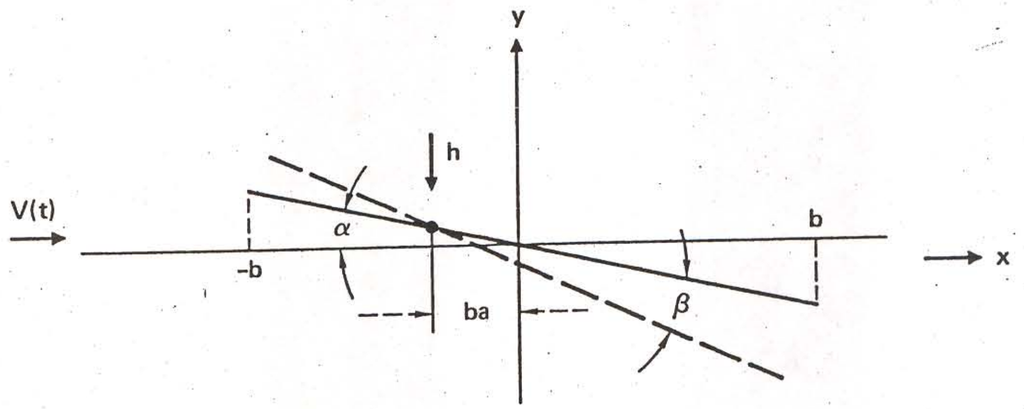
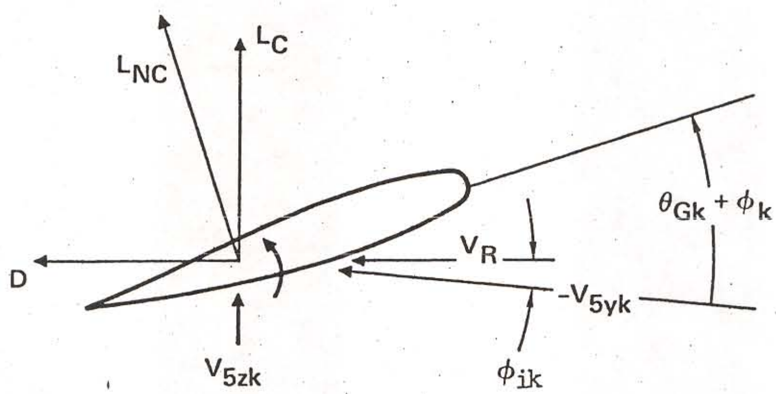


Figure 10. Geometry for Oscillating Airfoil in Pulsating Flow



V_R = RESULTANT VELOCITY

Figure 11. Relative Flow Velocities
L_C - Circulatory Lift Normal to the Resultant Flow
L_{NC} - Noncirculatory Lift Normal to the Blade Chord