ENGINEERING MATHEMATICS II

010.141

NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS

MODULE 6

INTRODUCTION

- The methods which we will cover are:
  - Methods for First-Order Differential Equation
    - **Euler Method**
    - **Improved Euler Method**
    - **Runge-Kutta**
    - **Adams-Bashforth**
    - **Adams-Moulton**
  - Methods for Higher Order Equations and for Systems

METHODS FOR FIRST-ORDER DIFFERENTIAL EQUATION

In this section, we will assume that the problem to be solved is expressed as:

$$y' = f(x, y) \quad y(x_0) = y_0 \qquad \text{Initial Value Problem}$$

and f is such that the problem has a unique solution.

We shall start with step-by-step methods, i.e.,

$$x_0 \rightarrow x_1 = x_0 + h \rightarrow x_2 = x_0 + 2h, \cdots$$

where "h" is the **step size**.

METHODS FOR FIRST-ORDER DIFFERENTIAL EQUATION (cont)

Step-by-step methods' formulation is derived from Taylor series

$$\boxed{y(x+h) = y(x) + y'(x)h} + y''(x)\frac{h^2}{y} + \cdots$$

EULER'S METHOD

For instance, the Euler method also called Euler-Cauchy is derived directly from the first two terms of the Taylor expansion:

$$y' = f(x, y)$$

$$y(x+h) \approx y(x) + h\,y'(x)$$

$$\approx y(x) + h\,f(x, y)$$

EULER'S METHOD

The iteration process follows:

$$y_0 \rightarrow \text{known} \quad \left(\text{i.e. } y(0) = y_0\right)$$

$$y_1 = y(x_0 + h) = y(x_0) + h\,f(x_0, y_0)$$

$$y_1 = y_0 + h\,f(x_0, y_0)$$

$$y_2 = y_1 + h\,f(x_1, y_1)$$

$$\boxed{y_{n+1} = y_n + h\,f(x_n, y_n)}$$

EULER'S METHOD

ALGORITHM EULER (f, x_0, y_0, h, N)

This algorithm computes the solution of the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ at equidistant points $x_1 = x_0 + h$, $x_2 = x_0 + 2h, \cdots, x_N = x_0 + Nh$; here f is such that this problem has a unique solution on the interval $[x_0, x_N]$ (see Sec. 1.7).

INPUT: Initial values x_0, y_0, step size h, number of steps N

OUTPUT: Approximation y_{n+1} to the solution $y(x_{n+1})$ at $x_{n+1} = x_0 + (n + 1)h$, where $n = 0, \cdots, N - 1$

For $n = 0, 1, \cdots, N - 1$ do:

$$x_{n+1} = x_n + h$$
$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_{n+1}, y_n + k_1)$$
$$y_{n+1} = y_n + \tfrac{1}{2}(k_1 + k_2)$$

OUTPUT x_{n+1}, y_{n+1}

End

Stop

End EULER

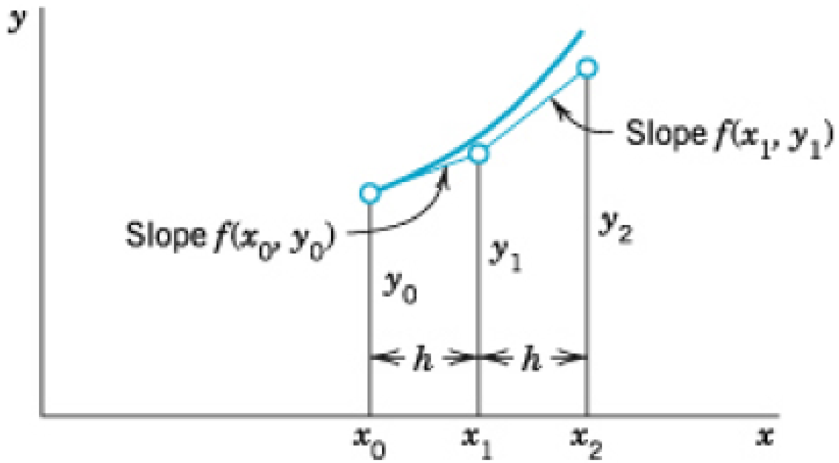

Fig. 448. Euler method

EULER'S METHOD

Figure 449 concerns the initial value problem

$$y' = \left(y - 0.01x^2\right)^2 \sin\left(x^2\right) + 0.02x, \quad y(0) = 0.4 \tag{5}$$

It also shows 80 approximate values for $0 \leqq x \leqq 4$ obtained by the Euler method

$$y_{n+1} = y_n + 0.05\left[\left(y_n - 0.01x_n^2\right)^2 \sin\left(x_n^2\right) + 0.02x_n\right].$$

Although $h = 0.05$ is smaller than

, the accuracy is still not good.

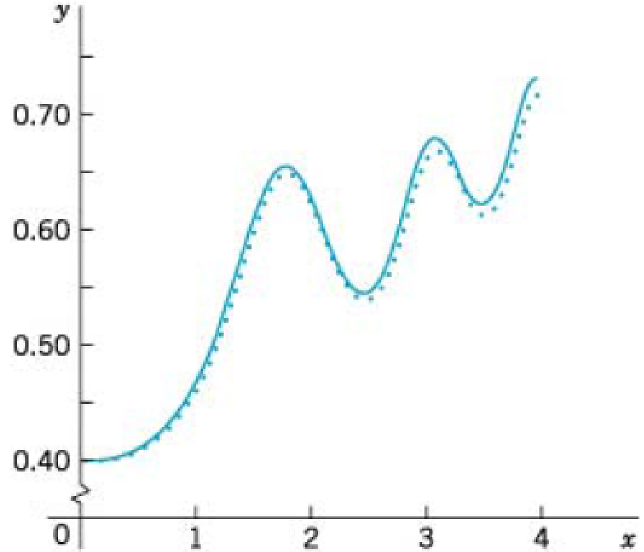

Fig. 449. Solution curve and Euler approximation in Example 2

EULER

Example

Consider the D.E.

wyth i.c. $y(0) = 0$

$$\frac{dy}{dx} = x + y$$

This D.E. has solution:

$$y(x) = e^x - x - 1$$

EULER: TABLE 19.1 h = 0.2

TABLE 21.1 Euler Method Applied to (4) in Example 1 and Error

n	x_n	y_n	$0.2(x_n + y_n)$	Exact Values	Error ϵ_n
0	0.0	0.000	0.000	0.000	0.000
1	0.2	0.000	0.040	0.021	0.021
2	0.4	0.040	0.088	0.092	0.052
3	0.6	0.128	0.146	0.222	0.094
4	0.8	0.274	0.215	0.426	0.152
5	1.0	0.489		0.718	0.229

Note: exact solution $y = e^1 - 1 - 1 = 0.718$

Approx. is 0.489

error 0.229 $\rightarrow$ significant

IMPROVED EULER (HEUN'S METHOD)

A priori, one would want to create methods which use more terms of the Taylor series

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2} y''(x) + \cdots$$

$$= y(x) + hf(x, y) + \frac{h^2}{2} f'(x, y) + \cdots$$

where

$$f'(x,y) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial x}$$

$$= f_x + f_y \cdot f$$

IMPROVED EULER METHOD

Obtaining the derivatives of f is time consuming and cumbersome.

Instead an another approach is preferred where intermediate values are introduced to increase accuracy.

Uses Predictor - Corrector Method

Predictor:

$$y^*_{n+1} = y_n + h\,f(x_n, y_n)$$

Corrector:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_{n+1}, y^*_{n+1}\right)\right]$$

IMPROVED EULER METHOD

Improved Euler is called a **predictor-corrector method**, i.e., one step predicts, one step predicts.

Example: Consider:

$$\frac{dy}{dx} = x + y$$
$$y(0) = 0$$

IMPROVED EULER METHOD (cont)

$$y_{n+1} = y_n + \tfrac{h}{2}\left[x_n + y_n + x_{n+1} + y^*_{n+1}\right], \qquad y' = x + y$$

$$= y_n + \tfrac{h}{2}\left[x_n + y_n + (x_n + h + y_n) + h(x_n + y_n)\right]$$

$$= y_n + \tfrac{h}{2}\left[x_n + y_n + x_n + y_n + 0.2x_n + 0.2y_n\right], \qquad h = 0.2$$

$$= y_n + \tfrac{h}{2}\left[2.2x_n + 2.2y_n + 0.2\right]$$

$$= y_n + 0.22x_n + 0.22y_n + 0.02$$

$$= y_n + 0.22(x_n + y_n) + 0.02 \qquad \text{Improved Euler}$$

$$y_{n+1} = y_n + 0.2(x_n + y_n) \qquad \text{Euler}$$

ILLUSTRATION – TABLE 19.3

Define:

$$y^*_{n+1} = y_n + h f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_{n+1}, y^*_{n+1}\right)\right]$$

TABLE 21.3 Improved Euler Method Applied to (4) and Error

n	x_n	y_n	$0.22(x_n + y_n) + 0.02$	Exact Values (4D)	Error
0	0.0	0.0000	0.0200	0.0000	0.0000
1	0.2	0.0200	0.0684	0.0214	0.0014
2	0.4	0.0884	0.1274	0.0918	0.0034
3	0.6	0.2158	0.1995	0.2221	0.0063
4	0.8	0.4153	0.2874	0.4255	0.0102
5	1.0	0.7027		0.7183	0.0156

ILLUSTRATION – TABLE 19.3 (cont)

$$y_{n+1} = y_n + \tfrac{h}{2}\left[x_n + y_n + \left(x_n + h + y_n\right) + h\left(x_n + y_n\right)\right]$$

$$= y_n + 0.1\left[2.2x_n + 2.2y_n + 0.2\right]$$

$$= y_n + 0.22\left(x_n + y_n\right) + 0.02$$

RUNGE-KUTTA

$$y' = f(x, y); \quad x = x_0, \quad y = y_0$$

$$at \quad x_{n+1} = x_0 + (n+1)h$$

Find y_{n+1}

Define :

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$

$$k_3 = h f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right)$$

$$k_4 = h f(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

RUNGE-KUTTA (Cont.)

Again $y' = x + y$

$h = 0.2$, 5 steps, from 0 to 1

$$k_1 = 0.2(x_n, y_n)$$

$$k_2 = 0.2(x_n + 0.1 + y_n + 0.5k_1)$$

$$k_3 = 0.1(x_n + 0.1 + y_n + 0.5k_2)$$

$$k_4 = 0.2(x_n + 0.2 + y_n + k_3)$$

After simplification:

$$y_{n+1} = y_n + 0.2214(x_n, y_n) + 0.0214$$

at n=5: $x_5=1$

$y_n = 0.718215 \rightarrow$ very close

$$y_{n+1} = y_n + 0.22(x_n, y_n) + 0.02$$

Euler: 0.489

IE: 0.7027

RK: 0.718215

True : 0.718282

MULTI-STEP METHOD (*Optional*)

Multi-step method are methods that use information from more than one of the preceding steps, thereby increasing accuracy.

ADAMS-BASFORTH METHOD (*Optional*)

Consider

$$y' = f(x, y), \quad y(x_0) = y_0$$

where f such that there is a **unique solution**

$$\int_{x_n}^{x_{n+1}} y'(x)dx = (y]_{x_n}^{x_{n+1}} = y(x_{n+1}) - y(x_n)$$

$$= \int_{x_n}^{x_{n+1}} f(x, y(x))dx$$

IDEA!! Replace f(x, y(x)) by an **interpolation polynomial** p(x).

ADAMS-BASFORTH METHOD (*Optional*)

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} p(x)\,dx$$

The method is explained by taking a polynomial of order 3, $p_3(x)$ such that

$$f_n = f(x_n, y_n)$$
$$f_{n-1} = f(x_{n-1}, y_{n-1})$$
$$f_{n-2} = f(x_{n-2}, y_{n-2})$$
$$f_{n-3} = f(x_{n-3}, y_{n-3})$$

ADAMS-BASFORTH METHOD (*Optional*)

$p_3(x)$ can for instance be given by the **Newton-backward difference formula (p807)**

$$p_3(x) = f_n + r\nabla f_n + \frac{r(r+1)}{2}\nabla^2 f_n + \frac{1}{6}r(r+1)(r+2)\nabla^3 f_n$$

where

$$\nabla f_n = f_n - f_{n-1}$$
$$\nabla^k f_n = \nabla^{k-1} f_n - \nabla^{k-1} f_{n-1}$$

ADAMS-BASFORTH METHOD (*Optional*)

$$\nabla f_n = f_n - f_{n-1}$$

$$\nabla^2 f_n = \nabla f_n - \nabla f_{n-1}$$

$$= f_n - f_{n-1} - \left(f_{n-1} - f_{n-2}\right) = f_n - 2f_{n-1} + f_{n-2}$$

$$\nabla^3 f_n = \nabla^2 f_n - \nabla^2 f_{n-1}$$

$$\nabla^2 f_{n-1} = \nabla f_{n-1} - \nabla f_{n-2}$$

$$= f_{n-1} - 2f_{n-2} + f_{n-3}$$

ADAMS-BASFORTH METHOD (*Optional*)

$$p_3(x) = f_n + r\left(f_n - f_{n-1}\right) + \frac{r(r+1)}{2}\left(f_n - 2f_{n-1} + f_{n-2}\right)$$

$$+\frac{1}{6} r\left(r+1\right)\left(r+2\right)\left(f_n - 2f_{n-1} + f_{n-2} - f_{n-1} + 2f_{n-2} - f_{n-3}\right)$$

$$r = \frac{x - x_n}{h}$$

where r varies between 0 and 1

h dr = dx

ADAMS-BASFORTH METHOD (*Optional*)

$$\int_{x_n}^{x_{n+1}} p_3(x)\,dx = \int_0^1 h\,dr\,[\cdots]$$

$$= h\,f_n \int_0^1 dr$$

$$+\,h\left(f_n - f_{n-1}\right) \int_0^1 r\,dr$$

$$+\,h\left(f_n - 2f_{n-1} + f_{n-2}\right) \int_0^1 \frac{r^2 + r}{2}\,dr$$

$$+\,h\left(f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3}\right) \frac{1}{6} \int_0^1 r\left(r^2 + 3r + 2\right) dr$$

ADAMS-BASFORTH METHOD (*Optional*)

$$\int_{x_n}^{x_{n+1}} p_3(x)\,dx = ?$$

$$= h\,f_n\left(r\right]_0^1 + h\left(f_n - f_{n-1}\right)\left(\frac{r^2}{2}\right]_0^1$$

$$+ h\left(f_n - 2f_{n-1} + f_{n-2}\right)\left(\frac{r^3}{6} + \frac{r^2}{4}\right]_0^1$$

$$+ h\left(f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3}\right)\frac{1}{6}\left(\frac{r^4}{4} + 3\frac{r^3}{3} + 2\frac{r^2}{2}\right]$$

ADAMS-BASFORTH METHOD (*Optional*)

$$\int_{x_n}^{x_{n+1}} p_3(x)\,dx = h\,f_n + \frac{h}{2}\left(f_n - f_{n-1}\right)$$

$$+h\left(f_n - 2f_{n-1} + f_{n-2}\right)\left(\frac{1}{6}+\frac{1}{4}\right)$$

$$+h\left(f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3}\right)\frac{1}{6}\left(\frac{1}{4}+1+1\right)$$

$$= h\,f_n\left(1+\frac{1}{2}+\frac{1}{6}+\frac{1}{4}+\frac{1}{24}+\frac{4}{24}+\frac{4}{24}\right)$$

$$+h\,f_{n-1}\left[-\frac{1}{2}-2\left(\frac{1}{6}+\frac{1}{4}\right)-\frac{3}{6}\left(\frac{1}{4}+2\right)\right]+\cdots$$

ADAMS-BASFORTH METHOD (*Optional*)

$$\int_{x_n}^{x_{n+1}} p_3(x)\,dx = \cdots + h\,f_{n-2}\left[\frac{1}{6}+\frac{1}{4}+\frac{3}{6}\left(\frac{1}{4}+2\right)\right] + h\,f_{n-3}\left[-\frac{1}{6}\left(\frac{1}{4}+2\right)\right]$$

$$= \frac{24+12+4+6+1}{24} h\,f_n$$

$$+ h\,f_{n-1}\left(\frac{-12-8-12-3-24}{24}\right)$$

$$+ h\,f_{n-2}\left(\frac{4+6+3+24}{24}\right)$$

$$+ h\,f_{n-3}\left(\frac{-1-8}{24}\right)$$

ADAMS-BASFORTH METHOD (*Optional*)

$$\int_{x_n}^{x_{n+1}} p_3(x)\,dx = \frac{55}{24} h\,f_n + h\,f_{n-1}\left(\frac{-59}{24}\right) + h\,f_{n-2}\left(\frac{37}{24}\right) + h\,f_{n-3}\left(\frac{-9}{24}\right)$$

$$y_{n+1} - y_n = \frac{55}{24} h\,f_n - \frac{59}{24} h\,f_{n-1} + \frac{37}{24} h\,f_{n-2} - \frac{9}{24} h\,f_{n-3}$$